Some Suboptimal Strategies for Numerical Realisation of Large Scale Optimal Control Problems

Karl Kunisch[1]

January 2001

Dedicated to Prof. H.T. Banks on the Occasion of his sixteeth Birthday

[1]Institut für Mathematik, Karl-Franzens-Universität Graz, A-8010 Graz, Austria, supported in part by the Fonds zur Förderung der wissenschaftlichen Forschung under SFB 03 ,,Optimierung und Kontrolle".

Abstract

A brief introduction to selected topics of suboptimal strategies for numerical realisation of large scale optimal control problems is given. Receding horizon strategies, reduced order modelling methods as well as suboptimal methods to solve certain Hamilton–Jacobi–Bellman equations are discussed.

1 Introduction

We survey some of the techniques developed and modified within the last decade on numerical realisation of optimal control problems governed by large scale partial differential equations. Large scale is a vague term, of course, depending on the available resources in manpower, hard– and software. What may appear to be large scale at a certain instance of time can become quite tractable soon thereafter. The study of suboptimal techniques, nevertheless, is a viable one. First because as resources increase, the models become increasingly more complex. Second the interest in suboptimal strategies is not only motivated by making large scale problems feasible, but also by reducing computing time for smaller problems and by systems–theoretic questions which go beyond optimal control, be it open or closed loop control.

In Section 6 we state a model problem from fluid dynamics which serves both as motivation and as reference in the following sections. Section 2 is devoted to the instantaneous control technique which can be considered as a specical case of a receding horizon strategy. Reduced order techniques are the subject of Section 3. We address both order reduction by proper orthogonal decomposition and by the reduced basis method. In Section 4 we give a very brief account of methods that can be utilized to obtain suboptimal solutions to the Hamilton–Jacobi–Bellman equation.

The fields of suboptimal strategies and of optimal control of fluids are growing rapidly. We have not made an attempt to give complete lists of methods let alone authors who contributed to these topics. We hope, however, that the interested reader will find many relevant issues and that the references serve as an adequat introduction to the topics addressed.

2 A model problem

To explain some of the concepts for suboptimal control we shall repeatedly return to a model problem in fluid mechanics. For this purpose let Ω be a bounded domain in $\mathbb{R}^2$, let $T > 0$, and set $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \partial\Omega$, where $\partial\Omega$ denotes the boundary of Ω. We consider the controlled unstationary Navier–Stokes equations

$$(2.1) \qquad \begin{cases} y_t(t) - \frac{1}{Re}\Delta y(t) + (y(t)\cdot\nabla)y + \nabla p(t) = B\,u(t) \;\text{ in }\; Q, \\ -\text{div}\; y(t) = 0 \;\text{ in }\; Q, \\ y(0,\cdot) = y_0 \;\text{ in }\; \Omega, \\ y(t,\cdot) = g(t) \;\text{ on }\; \Sigma, \end{cases}$$

where $Re > 0$, $g \in W^{1,2}(0,T; H_0^{1/2}(\partial\Omega))$, and $y_0 \in L^2(\Omega)$, div $y_0 = 0$, are given, $B \in \mathcal{L}(U, L^2(\Omega))$ and $u \in L^2(0,T;U)$. Here U denotes the control space, which is assumed to be a Hilbert space with inner product $(\cdot,\cdot)_U$. Further $y(t) = y(t,x) \in \mathbb{R}^2$ and $((y \cdot$

$\nabla)y)_j = (\sum_{i=1}^{2} y_i \partial_{x_i})\, y_j, j = 1,2$, and $H_0^{1/2}(\partial\Omega) = \{\varphi \in H^{1/2}(\partial\Omega) \colon \int_{\partial\Omega} \varphi \cdot n\, dx = 0\}$, where n denotes the unit outer normal to $\partial\Omega$. For a function space treatment of (2.1) we refer to [CF, GR, T], for example. Associated to (2.1) we consider the optimal control problem

$$
\text{(P)} \qquad \begin{cases} \min J(y,u) &= \displaystyle\int_0^T \int_\Omega F(y) dx\, dt \; + \; \frac{\beta}{2} \int_0^T |u(t)|_U^2 dt \\[2ex] \quad u(t) \in U, & \text{subject to } \; (2.1), \end{cases}
$$

where F is a smooth real–valued functional that is bounded from below. Typical choices for F are

$$
F(y(t,x)) = \tfrac{1}{2} |y(t,x) - z(t,x)|^2
$$

and

$$
F(y(t,x)) = \tfrac{1}{2} |\mathrm{curl}(t,x)|^2,
$$

where z is a fixed control target. Let us assume that (P) admits an optimal control $u^* \in U$ with associated velocities and pressure $(y^*, p^*) = (y(u^*), p(u^*))$. To formally derive the optimality condition for (P) it is convenient to introduce the Lagrangian

$$
\text{(2.2)} \qquad \begin{aligned} L(y,p,u,\xi,\pi) &= \int_Q F(y) dx\, dt \; + \; \frac{\beta}{2} \int_0^T |u(t)|_U^2 dt \\ &+ \int_Q \big((y_t - \tfrac{1}{Re} \Delta y + (y \cdot \nabla) y + \nabla p) - B\, u(t) \big)\, \xi \, dx\, dt \; - \; \int_Q \pi \;\mathrm{div}\; y \, dx\, dt. \end{aligned}
$$

Here $y(0,\cdot) = y_0$ in Ω is kept as explicit constraint. Setting the partial derivatives of L with respect to y, p, and u equal to zero we obtain the equations for (ξ, π):

$$
\text{(2.3)} \qquad \begin{cases} -\xi_t - \frac{1}{Re} \Delta \xi + (\nabla y)^T \xi - (y \cdot \nabla) \xi + \nabla \pi = -F'(y) \;\text{ in }\; Q, \\[1ex] -\mathrm{div}\; \xi = 0 \;\text{ in }\; Q, \\[1ex] \xi(T, \cdot) = 0 \;\text{ in }\; \Omega, \\[1ex] \xi = 0 \;\text{ on }\; \Sigma, \end{cases}
$$

$$
\text{(2.4)} \qquad \beta\, u(t) = B^* \xi(t) \quad \text{in} \quad Q.
$$

Equation (2.3) is refered to as the adjoint equation, (2.4) is the optimality condition. Combined (2.1), (2.3), (2.4) are called the optimality system.

Due to the forward–backward nature of the primal equation (2.1) and the adjoint equation (2.3), as well as the strong coupling of the primal variables (y, p, u) and the adjoint variables (ξ, π) the efficient numerical solution of the optimality system is a challenging task. If (P) is posed in 3–D or if it involves further coupling e.g. with thermal, chemical or mechanical processes [KTB, LT, LT1] then it may become an almost impossible task to efficiently solve the optimality system directly. This is one of the reasons why suboptimal strategies are important. Another motivation is the reduction of computing time. We shall address suboptimal schemes in Sections 2–4.

It will be useful to characterize the gradient of J at a control u in direction δu. This can be achieved efficiently by means of the Lagrangian. Let us proceed formally by expressing (P) as

$$\begin{cases} \min J(u) = \hat{F}(y) + \frac{\beta}{2} \int_0^T |u(t)|_U^2 \, dt \\ \\ u(t) \in U, \quad \text{subject to} \quad e(x) = 0, \end{cases} \tag{2.5}$$

where $x = (y, p, u)$ and $\hat{F}(y) = \displaystyle\int_Q F(y) dQ$. Here $e(x) = 0$ represents the equality constraint given by (2.1). We have

$$J'(u)\delta u = (e_u^* \lambda + \beta u, \delta u)_{L^2(0,T;U)}, \tag{2.6}$$

where λ satisfies

$$e_y^* \lambda = -\hat{F}'(y). \tag{2.7}$$

Here e_u denotes the derivative of e with respect to u, e_u^* stands for the adjoint of e_u and analogous notation is used for e_y^*. The Lagrangian associated to (2.5) is given by

$$\hat{L}(x, \lambda) = \hat{F}(y) + \tfrac{\beta}{2} |u|^2_{L^2(0,T;U)} + \langle \lambda, e(x) \rangle,$$

with $\langle \cdot, \cdot \rangle$ the inner product in the range space of e. The condition $\hat{L}_y = 0$ is equivalent to

$$\hat{L}_y = e_y^* \lambda + \hat{F}'(y) = 0,$$

which is (2.7), and further

$$\hat{L}_u(u)\delta u = (e_u^* \lambda + \beta u, \delta u)_{L^2(0,T;U)}.$$

Hence by (2.6) we have

$$J'(u)\delta u = \hat{L}_u(u)\delta u. \tag{2.8}$$

Applying (2.8) to (2.2) we find for the Riesz representation of the gradient of J in (P)

$$J'(u) = \beta u - B^* \xi. \tag{2.9}$$

The functional analytical framework for optimality systems related to optimal control of flow phenomena such as (P) has been investigated in several papers: we refer to [AT, GM1, GM2] and the references given there.

If the distributed control term in (2.1) is replaced by boundary control

$$y(t,\cdot) = Bu(t) \ \text{ on } \ \Sigma,$$

then (2.4) becomes

$$\beta u + B^*(\tfrac{1}{Re}\,\tfrac{\partial \xi}{\partial n} - \pi\, n) = 0 \ \text{ on } \ \Sigma \tag{2.10}$$

and

$$J'(u) = \beta u + B^*(\tfrac{1}{Re}\,\tfrac{\partial \xi}{\partial n} - \pi\, n).$$

The correct functional analytic setting of boundary control problems requires time–derivative bounds for the controls, see [GM2], for example. In our case this could be realized by choosing $\frac{\beta}{2}\int_0^T (|u(t)|_U^2 + |\frac{d}{dt}\,u(t)|_U^2)dt$ as control–cost. This would result in the extra term $-\beta(\frac{d}{dt})^2 u(t)$ plus boundary conditions in (2.10). An alternative approach, approximating Dirichlet by penalized Neumann boundary conditions was pursued in [HR].

3 Instantaneous control–receding horizon control

To explain the approach let $m > 1$ be fixed and set $\delta t = T/m$, $t_i = i\,\delta t$, for $i = 0, \cdots, m$. As a first step in the presentation let us consider the case where the Navier–Stokes equations (2.1) are approximated by a Crank–Nicolson scheme. At the i–th level of the instantaneous control method one solves the following stationary optimal control problem, where the variables (y, p, u) correspond to $(y(t_i), p(t_i), u(t_i))$:

$$\begin{cases} \min\ \hat{J}(u) = \displaystyle\int_\Omega F(y)dx + \tfrac{\beta}{2}|u|_U^2 \\ \text{over } \ u \in U, \quad \text{subject to} \\ \tfrac{1}{\delta t}y + \tfrac{1}{2}(y\cdot\nabla)y - \tfrac{1}{2Re}\Delta y + \nabla p = R + Bu \ \text{ in } \ \Omega, \\ -\text{div}\ y = 0 \ \text{ in } \ \Omega, \\ y = 0 \ \text{ on } \ \delta\Omega, \end{cases} \tag{3.1}$$

where

$$R = \tfrac{1}{\delta t}\, y(t_{i-1}) + \tfrac{1}{2Re}\,\Delta y(t_{i-1}) - \tfrac{1}{2}(y(t_{i-1})\cdot\nabla)y(t_{i-1}) + f(t_{i-1/2}),$$

is a known forcing term. Let $u_i = u(t_i)$ denote a solution to (3.1) and set $(y_i, p_i) = (y(u_i), p(u_i))$. Then (y_i, p_i, u_i) satisfy the optimality system for (3.1) consisting in the equality constraints in (3.1), together with the adjoint equations:

$$
(3.2) \qquad \begin{cases} \frac{1}{\delta t}\xi_i - \frac{1}{2Re}\Delta\xi_i + \frac{1}{2}(\nabla y_i)^T\xi_i - \frac{1}{2}(y_i \cdot \nabla)\xi_i + \nabla\pi_i = -F'(y_i, \cdot) \ \text{ in } \ \Omega, \\ -\text{div} \ \ \xi_i = 0 \ \text{ in } \ \Omega, \\ \xi_i = 0 \ \text{ on } \ \partial\Omega, \end{cases}
$$

and the optimality condition

$$
(3.3) \qquad \beta u_i = B^*\xi_i \ \text{ in } \ \Omega.
$$

Note that y_i enters into (3.2) through F as well as through the linearization terms. Even though F may depend on y only in a subdomain $\widetilde{\Omega} \subset \Omega$ corresponding to locations where observations are available, (3.2) requires y_i throughout Ω due to the contribution of the linearized convection terms. If instead of the semi–implicit scheme an explicit Euler scheme is used we have

$$
(3.4) \qquad \begin{cases} \min \ \hat{J}(u) = \int_\Omega F(y)dx + \frac{\beta}{2}|u|^2_U \\ \text{over} \ \ u \in U, \quad \text{subject to} \\ \frac{1}{\delta t}y - \frac{1}{Re}\,\Delta y + \nabla p = R + Bu \ \text{ in } \ \Omega, \\ -\text{div} \ \ y = 0 \ \text{ in } \ \Omega, \\ y = 0 \ \text{ on } \ \delta\Omega, \end{cases}
$$

where $R = \frac{1}{\delta t}\,y_{i-1} - (y_{i-1} \cdot \nabla)y_{i-1} + f(t_i)$. In this case information from time level t_{i-1} is passed to t_i solely through the inhomogeneity R. Let u_i denote a solution to (3.4) and as before set $(y_i, p_i) = (y(u_i), p(u_i))$. Then the optimality system for (3.4) consists of the equality constraints in (3.4) together with

$$
(3.5) \qquad \begin{cases} \frac{1}{\delta t}\xi_i - \frac{1}{Re}\Delta\xi_i + \nabla p_i = -F'(y_i) \ \text{ in } \ \Omega, \\ -\text{div} \ \ \xi_i = 0 \ \text{ in } \ \Omega, \\ \xi_i = 0 \ \text{ on } \ \partial\Omega, \end{cases}
$$

and

$$
(3.6) \qquad \beta u_i = B^*\xi_i \ \text{ in } \ \Omega.
$$

In the optimality system for (3.4) the coupling between primal and adjoint equations occurs only through F.

The *instantaneous control strategy* considered as open loop control consits in iteratively solving (3.4) (or (3.1) or some variation thereof) for optimal controls u_i and associated states (y_i, p_i). - The feedback use and interpretation of instantaneous control consists in evaluating F in (3.5) on the basis of data available from the real system, solving (3.5) for (ξ_i, π_i), evaluating the optimal control at time level t_i via (3.6), applying it to the system to obtain the new observation and the continue on the next time level. The instantaneous control approach replaces (P) by a sequence of stationary optimal control problems. Clearly it cannot be claimed that a solution to (P) is obtained by this technique and its justification therefore needs to be addressed. - In the context of control of fluids the instantaneous control technique was probably first discussed in [CTMK] and utilized and refined in several papers thereafter, see e.g. [CHK] and the references given there.

One justification for the instantaneous control strategy is its success in achieving the control objective in numerical tests for diverse control problems in fluid mechanics, we refer to [B, CHK, CTMK] and the references given there. Further frameworks for the analysis of the instantaneous control strategy are discussed next.

(i) In [IK1] the connection between instantaneous control and receding horizon control was pointed out. Receding horizon control is a well-known technique in optimal control of nonlinear ordinary differential equations, see e.g. [CA] and the references given there. To briefly explain some concepts we follow [IK1] and consider the infinite–horizon optimal control problem in finite dimensions:

$$(3.7)\qquad \begin{cases} \inf \displaystyle\int_0^\infty f^\circ(x(t), u(t))dt \quad \text{over} \quad u \\ \text{subject to} \\ \frac{d}{dt}\, x(t) = f(x(t), u(t)),\ t > 0 \\ x(0) = x_0,\ x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m, \end{cases}$$

where $f^\circ\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Next (3.7) is replaced by a sequence of finite horizon problems. Let $T > 0$ denote the so–called prediction horizon, let G denote a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}$, and consider

$$(3.8)\qquad \begin{cases} \inf \displaystyle\int_{kT}^{(k+1)T} f^\circ(x(t), u(t))dt + G(x((k+1)T)), \quad \text{over} \quad u \\ \text{subject to} \\ \frac{d}{dt}\, x(t) = f(x(t), u(t)),\ t \in (kT, (k+1)T], \\ x(kT) = \bar{x}_k(kT), \end{cases}$$

where $\bar{x}_k$ denotes the solution on $[(k-1)T, kT]$ which is assumed to exist. Let $\bar{x}$ denote the function defined on $[0,\infty)$ which arises from concatenation of the solutions $\bar{x}_k$, $k = 2, \cdots$, to (3.8) which are assumed to exist. So far the replacement of (3.7) by the sequence of problems (3.8) has almost exclusively been justified by means of the asymptotic stabilization property for (3.7), which can be guarenteed under appropriate assumptions on f and G and/or additional explicit constraints on the state x in (3.7). The framework that best fits the application to the discretized Navier–Stokes equations uses the concept of closed loop dissipativity.

Definition 3.1 *Problem (3.7) is called closed loop dissipative, if there exist a feedback law* $u = -K(x)$ *and* $\alpha > 0$ *such that*

$$f(x, -K(x)) \cdot \alpha\, x + f^\circ(x, -K(x)) \leq 0 \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

Examples for closed loop dissipative systems are given in [IK1]. Assume henceforth that (3.7) is closed loop dissipative, and set $G(x) = \frac{\alpha}{2}|x|^2_{\mathbb{R}^n}$. We define for $T > 0$

$$V_T(x) = \inf \left\{ \int_0^T f^\circ(x(t), u(t))dt \colon \tfrac{d}{dt}x(t) = f(x(t), u(t)), x(0) = x \right\},$$

and analogously $V\colon \mathbb{R}^n \to \mathbb{R}$ denotes the minimal value functional for (3.7). Then for $0 \leq \hat{T} \leq T$ it can be shown that

$$V(x) \leq V_T(x) \leq V_{\hat{T}}(x) \leq G(x) = \tfrac{\alpha}{2}\, x^2_{\mathbb{R}^n} \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{3.9}$$

Therefore, a longer prediction horizon results in a better estimate of $V(x)$. Moreover, if

$$V_T(x) \leq \rho_T G(x) = \tfrac{\alpha \rho_T}{2}\, |x|^2_2 \quad \text{with} \quad \rho_T < 1, \tag{3.10}$$

then

$$|\bar{x}(kT)|^2_{\mathbb{R}^n} \leq \rho_T^k |x(0)|^2_{\mathbb{R}^n}, \quad \text{for all} \quad k \geq 1, \tag{3.11}$$

and hence $\bar{x}(kT) \to 0$ for $k \to \infty$. Estimate (3.10) holds, for example for $f^\circ(x, u) = \beta_1|x|^2 + \beta_2|u|^2$. - If (3.7) is not necessarily closed loop dissipative, then similar results can be obtained if G is chosen as a control–Liapunov function. Related results are also available for discrete–time systems. The instantaneous control strategy is an extreme case with only one discrete time predication horizon step.

(ii) Another analysis for the receding horizon optimal control concept applied to the Navier–Stokes equations was proposed in [HY]. To briefly explain the approach we consider

$$\text{(3.12)} \qquad \min \ \widetilde{J}(y,u) = \tfrac{\alpha}{2}\int_0^\infty \int_\Omega |y-Y|^2 dx\, dt \ + \ \tfrac{\beta}{2}\int_0^\infty \int_\Omega |u-U|^2 dx\, dt,$$

subject to (2.1) with $B = I$ and Y, U given. The infinite horizon problem (3.12) is replaced by a sequence of finite horizon problems

$$\text{(3.13)} \qquad \min \ \widetilde{J}_k(y,u) = \tfrac{\alpha}{2}\int_{kT}^{(k+1)T} |y-Y|^2 dx\, dt \ + \ \tfrac{\beta}{2}\int_{kT}^{(k+1)T} |u-U|^2 dx\, dt,$$

subject to (2.1) with initial condition $y(kT,\cdot) = \bar{y}_k(kT,\cdot)$, where $\bar{y}_k$ is the solution to (3.13) on $[(k-1)T, kT]$. Let $\bar{y}$ denote the function constructed from $\{\bar{y}_k\}_{k=1}^\infty$ by concatenation. The main assumption for the analysis in [HY] are

(C1) $\qquad U = \frac{d}{dt}Y - \frac{1}{Re}\Delta Y + (Y\cdot\nabla)Y$

(C2) $\qquad T, \frac{\alpha Re^2}{\beta}, Re$ (relative to $\|Y\|_{L^\infty(0,\infty;L^4(\Omega))}$) are sufficiently small.

According to (C1) the optimal control body forces u in (3.12) should be "close" to the body forces U corresponding to the desired flow field Y. With (C1), (C2) and appropriate technical assumptions holding, it can be shown that there exist constants $M > 0$ and $\kappa > 0$ such that

$$\|\bar{y}(t) - Y(t)\|_{H^1(\Omega)^2} \le M\, e^{-\kappa t}\|y(0) - Y(0)\|^2_{L^2(\Omega)^2}, \quad \text{for } \ t > 0.$$

(iii) In [HV] the authors analyze an instantaneous control strategy based on an implicit time stepping scheme. They consider optimal control of Burgers equation

$$\text{(3.14)} \qquad \min \ J(y,u) = \tfrac{1}{2}\int_0^\infty \int_\Omega |y-z|^2 dx\, dt \ + \ \tfrac{\beta}{2}\int_0^T |u|^2 dx\, dt,$$

subject to $u \in L^2(0,T;L^2(\Omega))$ and

$$\begin{cases} \frac{d}{dt}y(t) - \nu\Delta y(t) + y(t)y_x(t) = Bu(t) \ \text{ in } \ Q, \\ y(t,0) = y(t,1) = 0, \ t > 0, \\ y(0,\cdot) = y_0 \ \text{ in } \ \Omega, \end{cases}$$

where $\beta > 0$, $\nu > 0$ and z are given, and $\Omega = (0,1)$. Let A denote the negative Laplacian, $b(y) = yy_x$ and let $h > 0$ stand for the step size. Given $\{u_k^\circ\}_{k=1}^\infty$ and setting $z_k = z(t_k)$ we consider the algorithm

1. $k = 0$, $t_o = 0$

2. Solve for (y, λ):

$$(I + hA)y = y_k + h\, b(y_k) + Bu_k^\circ$$
$$(I + hA)\lambda = -(y - z_k).$$

3. Set $\nabla J(y(u_k^\circ), u_k^\circ) = \beta u_k^\circ - B^*\lambda$.

4. Given $\rho > 0$ set $u_{k+1} = u_k^\circ - \rho\, \nabla \hat{J}(u_k^\circ)$.

5. Solve $(I + hA)y_{k+1} = y_k + h\, b(y_k) + Bu_{k+1}$.

6. Set $t_{k+1} = t_k + h$, $k = k + 1$ and return to 2.

The two equations in 2. are readily seen as primal and adjoint system of a discrete–time linear–quadratic optimal control problem. Accordingly an optimal step length for ρ in 4. can easily be computed. Under the assumption that $B = I$ and $u_k^\circ = 0$, for all k, the determination of y_{k+1} in the above algorithm is equivalent to

$$(3.15) \qquad (I + hA)y_{k+1} = y_k + h\, b(y_k) - \rho\, S_h^2(y_k - z_k) - h\, \rho\, S_h^2(b(y_k) - A\, z_k,\ y_k = \phi,$$

where S_h is the solution operator to $v - \nu\, h\, v'' = f$, with homogenous Dirichlet boundary conditions. If h is sufficiently small and appropriate technical assumptions are satisfied, then there exists $\kappa \in (0, 1)$ such that

$$|y_k - z_k|_{L^2(\Omega)} \leq \kappa^k |y_0 - z_0|_{L^2(\Omega)},$$

and hence $|y_k - z_k|_{L^2(\Omega)}$ for $k \to \infty$.

(iv) In a recent paper [H2] the close connection between instantaneous control, the multiple shooting approach, well–known as numerical method in optimal control for ordinary differential equations, and the Gauss-Seidel method applied to a discrete–time reformulation of continuous time optimal control problems is pointed out. It is shown that certain instantaneous control techniques coincide with the first step of a forward Gauss–Seidel iteration applied to the discrete time problems. The analysis in [H2], which is carried out for linear quadratic problems, can be extended to certain nonlinear problems and will be instrumental to improve numerical aspects of instantaneous control and receding horizon strategies.

4 Reduced order methods

A powerful and structurally completely different possibility to solve optimal control problems for complex systems is the use of reduced order methods. The underlying idea consists in projecting the partial differential equation onto some lower–dimensional state–space, to project the cost–functional accordingly and to solve the resulting lower–dimensional problem. A popular method for obtaining reduced order methods is based on proper orthogonal decomposition (POD). An alternative is given by reduced basis methods. We shall explain these two techniques and turn to POD first.

Let X denote a Hilbert space with inner product $\langle\cdot,\cdot\rangle_X$. In the case of (2.1) it could be the closure of $\{v \in C_0^\infty(\Omega)^2\colon \text{ div } v = 0\}$ in $L^2(\Omega)^2$ or $H^1(\Omega)^2$, so that elements of X are divergence–free. For given $n \in \mathbb{N}$ let

$$0 = t_0 < t_1 < \cdots < t_n \le T,$$

denote a grid in the interval $[0,T]$. Let $\{y_j\}_{j=0}^n$ denote the velocity–components of the solution (y_j, p_j) to (2.1) at the grid points $\{t_j\}$ corresponding to some fixed reference control. POD does not address the question how these solutions, which are referred to as snap shots, are obtained. They must be available from an independent numerical technique or from experimental data. Let us set $\mathcal{V} =$ span $\{y_j\}_{j=0}^n$ and $d = \dim \mathcal{V}$. If $\{\psi_i\}_{i=1}^d$ denotes an orthonormal basis for $\mathcal{V}$ then each member of $\mathcal{V}$ can be expressed as

$$y_j = \sum_{i=1}^d \langle y_j, \psi_i\rangle_X\ \psi_i,\ j = 0, \cdots, n. \tag{4.1}$$

The method of POD consists in choosing an orthonormal basis such that for every $\ell \in \{1,\cdots,d\}$ the mean square error between $y_j, j = 0,\cdots,n$ and the corresponding ℓ–th partial sum in (4.1) is minimized on average:

$$\begin{cases} \min\limits_{\{\psi_i\}_{i=1}^\ell} \displaystyle\sum_{j=0}^n |y_j - \sum_{i=1}^\ell \langle y_j, \psi_i\rangle_X \psi_i|_X^2 \\ \\ \text{subject to } \ \langle \psi_i, \psi_j\rangle = \delta_{ij} \ \text{ for } \ 1 \le i, j \le \ell. \end{cases} \tag{4.2}$$

The solution $\{\psi_i\}_{i=1}^\ell$ to (4.2) is called POD basis of rank ℓ. It is characterized by a necessary optimality condition. We introduce the bounded linear operator $\mathcal{Y}\colon \mathbb{R}^{n+1} \to X$ by

$$\mathcal{Y}\, v = \sum_{j=0}^n v_j y_j.$$

Its adjoint $\mathcal{Y}^*\colon X \to \mathbb{R}^{n+1}$ is given by

$$\mathcal{Y}^* z = \left(\langle z, y_0\rangle_X, \cdots,\ \langle z, y_n\rangle_X\right)^T, \ \text{ for } \ z \in X.$$

It follows that $\mathcal{R} = \mathcal{Y}\mathcal{Y}^*$ and $\mathcal{K} = \mathcal{Y}^*\mathcal{Y}$ are given by

$$\mathcal{R} = \sum_{j=0}^n \langle \cdot, y_j\rangle_X y_j \ \text{ and } \ (\mathcal{K})_{ij} = \langle y_j, y_i\rangle_X,$$

respectively. Using a Lagrangian framework the optimality condition for (4.2) is given by

$$\mathcal{R}\,\psi = \lambda\,\psi.$$

Note that $\mathcal{R}$ is bounded, selfadjoint, non–negative, and since it has a finite dimensional range it is also compact. By Hilbert–Schmidt theory there exists an orthonormal basis $\{\psi_i\}_{i\in\mathbb{N}}$ for X and a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of non–negative real numbers so that

$$\mathcal{R}\psi_i = \lambda_i\psi_i,\ \lambda_1 \geq \cdots \geq \lambda_d > 0,\ \lambda_i = 0 \ \text{ for } \ i > d,$$

and $\mathcal{V} =$ span $\{\psi_i\}_{i=1}^d$. Setting

$$v_i = \tfrac{1}{\sqrt{\lambda_i}}\,\mathcal{Y}^*\,\psi_i,\ i = 1, \cdots d,$$

we find $\mathcal{R}\ v_i = \lambda_i\,\psi_i$ and $(v_i, v_j)_{\mathbb{R}^{n+1}} = \delta_{ij}$. Thus $\{v_i\}_{i=1}^d$ is an orthonormal basis of eigenvectors of $\mathcal{R}$ for the image of $\mathcal{R}$. Conversely, $\{\psi_i\}_{i=1}^d$ can be obtained from $\{v_i\}_{i=1}^d$ by means of $\psi_i = \frac{1}{\sqrt{\lambda_i}}\,\mathcal{Y}\,v_i,\ i = 1, \cdots, d$. The sequence $\{\psi_i\}_{i=1}^{\ell}$ solves (4.2). Note also that due to orthonormality of $\{\psi_i\}_{j=1}^{\ell}$ the "min–expression" in (4.2) can be replaced by $\sum_{j=0}^n \sum_{i=1}^{\ell} |\langle y_j, \psi_i\rangle_X|^2$.

If $\{\psi_i\}_{i=1}^{\ell}$ is the POD basis of rank $\ell \leq d$ then we have the following error formula:

$$\sum_{j=0}^{n} |y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i\rangle_X \psi_i|_X^2 = \sum_{i=\ell+1}^{d} \lambda_i. \tag{4.3}$$

For computations the spatial variable must be discretized as well. Then both $\mathcal{R}$ and $\mathcal{K}$ are matrices and the computation of the POD basis will be carried out by whichever matrix has smaller dimension. In finite dimensions, moreover, the close connection between POD and singular value analysis becomes quite obvious.

The question about the choice of ℓ is certainly a critical one. It is commonly resolved by defining the relative information content

$$I(\ell) = \sum_{k=1}^{\ell} \lambda_k / \sum_{k=1}^{n} \lambda_k.$$

If a basis is required that contains $\delta\%$ of the total information then ℓ is determined according to

$$\ell = \text{ argmin } \{I(m)\colon I(m) \geq \delta\}. \tag{4.4}$$

The reduced dynamical system is obtained by a Galerkin approximation applied to (2.1); i.e. one makes an Ansatz

$$y^{\ell}(t) = \sum_{i=1}^{\ell} \alpha_i(t)\,\psi_i,$$

and the coefficients $\alpha_i(t)$ are determined from

$$\begin{cases} (y_t^{\ell}(t), \psi_i) + \frac{1}{Re}(\nabla y^{\ell}(t), \nabla\psi_i) + ((y^{\ell}\cdot\nabla)y^{\ell}, \psi_i) = (Bu(t), \psi_i),\ 1 \leq i \leq \ell, \\ y^{\ell}(0) = P^{\ell} y_o, \end{cases} \tag{4.5}$$

where P^ℓ denotes the projection onto span $\{\psi_i\}_{i=1}^\ell$. In (4.5) the inner products are in $L^2(\Omega)$ and for simplicity we assumed $g = 0$ in (2.1). Note that the divergence–free condition is incorporated in the basis elements and hence it does not explicitly enter into (4.4). Making a further Ansatz for the controls:

$$u^\ell(t) = \sum_{i=1}^{\ell} \gamma_i(t)\psi_i,$$

(4.5) can be expressed as

$$\text{(4.6)} \qquad \begin{cases} \dot\alpha(t) + A\,\alpha(t) + n(\alpha(t)) = \mathcal{B}\,\gamma(t), \\ \alpha(0) = \alpha_0, \end{cases}$$

with A and $\mathcal{B}$ matrices and n a nonlinear mapping. Inserting the expressions for y^ℓ and u^ℓ into $J(y,u)$, the model problem (P) can be expressed as finite dimensional control problem of the form

$$(\mathrm{P}_\ell) \qquad \begin{cases} \min \displaystyle\int_0^T \mathcal{F}(\alpha(t))dt + \tfrac{\beta}{2}\int_0^T \gamma^T(t)\,Q\,\gamma(t)dt \\ \text{over } \ \gamma \in L^2(0,T;\mathbb{R}^\ell), \ \ \text{subject (4.5).} \end{cases}$$

Differently from a generic approximation the POD–based system reduction leading to (P_ℓ) has the property that its basis elements are related to the structure of the dynamical system (2.1). The basis elements, however, are computed for a reference control, which does not represent the optimal control for (P_ℓ). Hence the problem of unmodelled system dynamics occurs. It can partially be compensated by repeatedly adapting the POD basis leading to the following

Algorithm

1. Initialize the snapshot set $\{y_j^0\}_{j=0}^n$, and set $i = 0$.
2. Compute ℓ according to (4.4) with n replaced by $n(i+1)$.
3. Compute the POD basis and solve (P_ℓ) for γ^i.
4. Compute the state y^i according to $u^i(t) = \Sigma\ \gamma_j^i(t)\psi_j$, add resulting snapshots $\{y_j^i\}_{j=0}^n$ to existing snapshots.
5. Check stopping criterion, set $i = i + 1$, goto 2.

The above algorithm was suggested in [AH]. An alternative adaptive POD–based strategy combined with a trust–region approach was proposed in [AFS]. - For a general treatment of POD application to dynamical systems we refer to [BHL] and the references there. - The use of POD–based system reduction in optimal control recently attracted a significant amount of attention, we refer to [AK, HK3, KTB, KV1, LT, LT1], for example. In some of these references the complexity of the system is such that without system reduction (or some alternative suboptimal technique) the problems could not be solved within reasonable computing and/or manpower time. - Closing our discussion on POD–approximation we mention a recent result in [Z] on the relation between POD and balanced truncation for linear systems, and an error estimate for the POD approximation in the $\mathcal{H}_\infty$–norm. Convergence rate estimates for Galerkin–POD approximation of nonlinear dynamical systems are given in [KV2, KV3].

Let us now turn to the *reduced basis method*, which was first proposed for system reduction in structural problems and which was utilized for optimal control of fluids in [IR, IS]. Consider a stationary parameter–dependent equation formally expressed as

$$\mathcal{E}(x,\lambda) = 0 \quad \text{for} \quad x \in X \quad \text{and} \quad \lambda \in \Lambda, \tag{4.7}$$

where X stands for the state–space of the differential equation and Λ for the parameter space. In applications of the reduced basis method to control problems, λ denotes the control variable. Order reduction by means of the reduced basis method proceeds in two steps. In the first one the reduced basis subspace $X_R \subset X$ is determined. In the second step linear combinations of X_R called reduced basis function are determined which properly accomodate the boundary conditions of the differential equation.

1. Taylor Subspace: In this case the reduced basis functions are linear combinations of Taylor basis functions generated by computing the Taylor expansion of $x(\lambda)$ at a reference value λ^*. The reduced basis subspace is

   $$X_R = \text{ span } \{x_j = \frac{\partial^j x}{\partial \lambda^j}|_{\lambda=\lambda^*} : j = 0, \cdots, M\},$$

   where $M \in \mathbb{N}$. Equations for $\frac{\partial^j x}{\partial \lambda^j}$, $j = 1, \cdots, M$ are obtained from the implicit function theorem applied to (4.7), e.g.

   $$\mathcal{E}_x(x(\lambda^*), \lambda^*) x_1 = -\mathcal{E}_\lambda(x(\lambda^*), \lambda^*).$$

2. Lagrange Subspace: Here the reduced basis functions are linear combinations of basis functions generated by solving the nonlinear system(4.7) at various parameters λ_j. The reduced basis subspace is

   $$X_R = \text{ span } \{x_j = x(\lambda_j) : j = 1, \cdots, M\}.$$

3. Hermite Subspace: Here X_R is a combination of the Taylor and Lagrange subspaces.

Let us illustrate one possibility of obtaining Lagrange subspace reduced basis functions by a procedure which suggests itself for boundary velocity control on $\Gamma_c \subset \partial\Omega$. We define the reduced subspace as

$$V_R = \text{ span } \{y_i \colon y_i|_{\partial\Omega} = v_i\tau,\ i = 1, \cdots, M\}$$

where y_i satisfies

$$(4.8) \qquad \begin{cases} -\frac{1}{Re}\Delta y_i + (y_i \cdot \nabla)y_i + \nabla p_i = f \text{ in } \Omega \\ -\text{div } p_i = 0 \text{ in } \Omega \\ y_i = b \text{ on } \partial\Omega \backslash \Gamma_c,\ y_i = v_i\tau \text{ on } \Gamma_c, \end{cases}$$

with $\{v_i\}_{i=1}^M$ given boundary velocities, b fixed and τ the unit tangent vector to $\partial\Omega$. Let y_0 denote the solution to (4.8) with $v_i = 0$. Then reduced basis functions $\{\varphi_i\}_{i=1}^m$ are defined as

$$\varphi_i = y_M + a_i y_i + c_i y_0,\ i = 1, \cdots, M,$$

where the constants a_i and c_i are chosen such that homogenous boundary conditions are enforced on $\partial\Omega$. A reduced order solution

$$y^R(t) = \varphi_0 + \sum_{i=1}^{M} \alpha_i(t)\varphi_i$$

to the time–dependent version of (4.8) can be obtained by means of a Galerkin procedure with test functions $\{\varphi_i\}_{i=1}^M$. In an optimal control problem with $u = \sum_{i=1}^{\ell} \beta_i(t) v_i \tau$, $\ell < M$, as boundary control, and $\{\beta_i(t)\}_{i=1}^{\ell}$ as control parameters, an appropriate Ansatz would be

$$y^R(t) = \varphi_0 + \sum_{i=\ell+1}^{M} \alpha_i(t)\varphi_i + \sum_{i=1}^{\ell} \beta_i(t) y_i,$$

with test functions $\{\varphi_i\}_{i=\ell+1}^M$ for the Galerkin scheme. The details to obtain the finite dimensional optimal control problems are given in [IS].

5 Comments on suboptimal closed loop methods

So far we focused on open loop methods. In this final section we briefly survey some of the concepts developed for suboptimal feedback computations and partially follow [BTB].

To explain the ideas it will be convenient to consider a variant of (3.7):

$$(5.1) \qquad \begin{cases} \min \displaystyle\int_0^\infty (x^T Q\, x + u^T R\, u) dt \\ \\ \text{subject to} \\ \\ \frac{d}{dt} x(t) = g(x(t)) + B\, u(x(t)) \\ \\ x(0) = x_0, \end{cases}$$

where $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function and B is an $n \times m$ matrix. The optimal feedback control for (5.1) is known to be of the form

$$(5.2) \qquad u^*(x) = -\tfrac{1}{2}\, R^{-1}\, B^T\, V_x(x),$$

where V is the solution to the Hamilton–Jacobi–Bellman equation

$$(5.3) \qquad V_x^T(x) f(x) - \tfrac{1}{4}\, V_x^T(x) B\, R^{-1}\, B^T\, V_x(x) + x^T\, Q\, x = 0.$$

In case g is linear and $g(x) = A\, x$ with A an $n \times n$–matrix, $V_x(x)$ in (5.2) is replaced by $\Pi\, x$ with Π the positive definite solution to

$$(5.4) \qquad \Pi\, A + A\, \Pi - \Pi\, B\, R^{-1} B^T \Pi + Q = 0.$$

The necessity of resorting to suboptimal strategies stems from the fact that it is rather difficult to numerically realize the Hamilton–Jacobi–Bellman equation unless n is small.

One possibility of obtaining a suboptimal feedback solution is to *linearize* g at a nominal solution $\bar{x}$ and to utilize a Riccati feedback controllers based on $A = g_x(\bar{x})$. An alternative is to use a *power series expansion* for the value function, i.e. $V(x) = \sum_{n=0}^\infty V_n(x)$, where $V_n(x) = O(x^{n+2})$ and the associated expansion

$$f(x) = A_0\, x + \sum_{n=2}^\infty f_n(x) \;\text{ with }\; f_n(x) = O(x^n) \;\text{ in (5.3).}$$

The resulting equation for $n = 0$ is the Riccati–equation (5.4). A third possibility is to use "*state–dependent* Riccati–equations". The idea is to write the nonlinear term in (5.1) as $g(x) = A(x)x$ and to consider

$$\Pi(x)\, A(x) + A^T(x)\Pi(x) - \Pi(x)\, B\, R^{-1} B^T\, \Pi(x) + Q = 0,$$

and to procede by using a power series expansion for $\Pi(x)$.

We mention three further possibilities of obtaining suboptimal feedback solutions which, unlike the previous ones, do not utilize the Riccati equation. As allready described in Section 3, the *instantaneous control* method with explicit time stepping has a natural interpretation as feedback control method. We now describe, how *interpolation of two-point boundary value solutions* provides a feedback mechanism. For this purpose we recall that the optimality condition for (5.1) is given by

$$\tag{5.5} \begin{cases} \frac{d}{dt}x(t) = g(x(t)) - \frac{1}{2}\, B\, R^{-1}B^T p(t), \quad x(0) = x_0 \\ \\ \frac{d}{dt}p(t) = -g_x^T(x(t))p(t) - 2\, Q\, x(t), \quad \lim_{t\to\infty} p(t) = 0. \end{cases}$$

The relation to the open–loop control is given by

$$\tag{5.6} u(t) = -\tfrac{1}{2}\, R^{-1}B^T p(t).$$

For numerical realization the terminal condition for the adjoint equation in (5.5) is replaced by $p(T) = 0$ for large T. For the following arguments it is convenient to indicate the dependence of p on x, so that $p(t) = p(t, x(t))$. Assuming that T is large with respect to t, (5.6) is approximated by

$$\tag{5.7} u(t) \cong -\tfrac{1}{2}\, R^{-1}B^T p(0, x(t)).$$

The practical interpretation of (5.7) is the following: When the system has reached the state $\bar{x}(t)$ the corresponding feedback control is set $-\frac{1}{2}R^{-1}B^T p(0, \bar{x}(t))$ where $p(0, \bar{x}(t))$ is the second component of the solution to (5.5) evaluate at $t = 0$, and the initial condition in (5.5) is set to $x(0) = \bar{x}(t)$. This feedback strategy can be realized by precomputing the solutions to (5.5) with initial conditions $x(0) = x_j$, with $\{x_j\}_{j=1}^L$ chosen in a neighborhood of the expected optimal trajectory. The feedback solution at state $\bar{x}(t)$ is then obtained by proper interpolation of the values $\{p(0, x_j)\}_{j=1}^L$.

The last suboptimal strategy that we describe here is especially well suited for control synthesis of systems arising in fluid mechanics. We consider the controlled system

$$\tag{5.8} \begin{cases} \frac{d}{dt}\, x(t) + A\, x(t) + F(x(t)) = B\, u(t) \\ \\ x(0) = x_0, \end{cases}$$

where A is a nonnegative selfadjoint operator in a Hilbert space, F is a locally Lipschitz nonlinear operator satisfying

$$\tag{5.9} \langle F(x) - F(x_e), x - x_e \rangle_H = 0, \quad \text{for all } \ x \in \text{dom } A,$$

where x_e is an equilibrium solution to (5.8), and B is of the form

$$B\, u = \sum_{i=1}^{m}\, u_i\, b_i.$$

Note that (5.9) is satisfied e.g. for the convection term arising in the Navier–Stokes equation. For Q a nonnegative selfadjoint operator we consider

$$\tag{5.10} \begin{cases} \min\ \frac{1}{2} \int_0^\infty (\langle x(t) - x_e, (A + Q)(x(t) - x_e) \rangle_H + |u(t)|^2) dt \\ \\ \text{subject to (5.8)}, \end{cases}$$

and seek feedback solutions of the form

$$u_i = -\gamma_i(t)\langle b_i, x(t) - x_e\rangle_H,\ \gamma_i(t) \geq 0. \tag{5.11}$$

Under some technical assumptions it can be shown that the closed loop system (5.8), (5.11) is closed loop dissipative. The optimal feedback control (i.e. the optimal choices for γ_i) can again be characterized by a Hamilton–Jacobi–Bellman equation which, however, differs somewhat from (5.3) due to the constraints $\gamma_i \geq 0$. The structure of this equation suggests that $c(x)(x-x_e)$, with c a real–valued function, is an appropriate Ansatz for V_x. The value of $c(x)$ can be obtained from the Hamilton–Jacobi–Bellman equation. The details of this approach and numerical examples are given in [IK].

6 Conclusions

We addressed selected suboptimal stategies for optimal control of partial differential equations with emphasis on examples in fluid dynamics. The impact of such methods, we hope, will be a significant one, since they provide a means of solving practical problems which may otherwise be quite untractable. The reader will have noticed that many interesting questions for the methods we presented still need to be answered. Also many additional systems–theoretical aspects may become numerically feasible for large scale problems by suboptimal techniques. We mention robust control, the theory of dynamical observers, estimators and compensators. While we focused on suboptimal strategies here, this is not to indicate that exact methods would not be of equal importance and require further research. Second order methods, [H1, HK2, L], and numerical methods for constrained problems, for example, present interesting challanges.

References

[AH] K. Afanasiev and M. Hinze: *Adaptive control of a wake flow using proper orthogonal decomposition*,

[AK] J. A. Atwell and B. King: *Proper orthogonal decomposition for reduced basis feedback controllers for parabolic systems*, Math. and Comput. Modeling, to appear.

[AT] F. Abergel and R. Temam: *On some control problems in fluid mechanics*, Theoret. Comput. Fluid Dynamics. **1** (1990), 303–325.

[AFS] E. Arian, M. Fahl and E. W. Sachs: *Trust region proper orthogonal decomposition for flow control*, ICASE–Report 2000-25.

[B] M. Berggren: *Numerical solution of a flow–control problem: vorticity reduction by dynamic boundary action*, SIAM Journal on Sci. Comput. **19** (1998), 829–860.

[BHL] G. Berkooz, P. Holmes and J. L. Lumley: *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*, Cambridge University Press, Cambridge, 1996.

[BTB] S. C. Beeler, H. T. Tran and H. T. Banks: *Feedback control methodologies for nonlinear systems*, Preprint, North Carolina State University.

[BTZ] T. R. Bewley, R. Temam and M. Ziane: *A general framework for robust control in fluid mechanics*, to appear in Physica D.

[CA] H. Chen and F. Allgöwer: *A quasi–infinite horizon nonlinear model predictive control scheme with guaranteed stability*, Automatica, **34** (1998), 1205–1217.

[CF] P. Constantin and C. Foias: *Navier–Stokes Equations*, The University of Chicago Press, Chicago, 1988.

[CHK] H. Choi, M. Hinze and K. Kunisch: *Instantaneous control of backward facing step flow*, Applied Numerical Mathematics, **31** (1999), 133–158.

[CTMK] H. Choi, R. Temam, P. Moin and J. Kim: *Feedback control for unsteady flow and its application to the stochastic Burgers equation*, J. Fluid Mech. **253** (1993), 509–543.

[GM1] M. D. Gunzburger and S. Mansversisi: *Analysis and approximation of the velocity tracking problem for Navier–Stokes flows with distributed control*, SIAM J. Control and Optimization,

[GM2] M. D. Gunzburger and S. Mansversisi: *The velocity tracking problem for Navier–Stokes flows with boundary controls*, SIAM J. Control and Optimization **37** (1999),

[GR] V. Girault and P. A. Raviart: *Finite Element Methods for Navier–Stokes Equations*, Springer Verlag, Berlin, 1986.

[H1] M. Heinkenschloss: *Formulation and analysis of a sequential quadratic programming method for the Dirichlet boundary control of Navier–Stokes flow*, preprint.

[H2] M. Heinkenschloss: *Time–domain decomposition iterative methods for the solution of distributed linear quadratic optimal control problems*, preprint, Rice University, 2000.

[HGMNP] J. W. He, R. Glowinski, R. Metcalfe, A. Nordlander and J. Periaux: *Active control and drag optimization for flow past a circular cylinder*, J. Comp. Physics, **163** (2000), 83–117.

[HK1] M. Hinze and K. Kunisch: *On suboptimal control strategies for the Navier–Stokes equations*, ESAIM: Proceedings, **4** (1998), 181–198.

[HK2] M. Hinze and K. Kunisch: *Second order methods for optimal control of time–dependent fluid flow*, preprint.

[HK3] M. Hinze and K. Kunisch: *Three control methods for time–dependent fluid flow*, to appear in Journal of Flow, Control and Combustion.

[HR] L. S. Hou and S. S. Ravindran: *A penalized Neumann control approach for solving an optimal Dirichlet control problem for the Navier–Stokes equations*, SIAM J. Control Optimization. **36** (1998), 1795–1814.

[HV] M. Hinze and S. Volkwein: *Instantaneous control for the instationary Burgers equation–convergence analysis and numerical implementation*, preprint, University of Graz, 1999.

[HY] L. S. Hou and Y. Yan: *Dynamics and approximation of a velocity tracking problem for the Navier Stokes flows with piecewise distributed controls*, SIAM J. Control Optimization. **35** (1998), 1847–1885.

[IK] K. Ito and S. Kang: *A dissipative feedback control synthesis for systems arising in fluid dynamics*, SIAM J. Control and Optimization. **32** (1994), 831–854.

[IK1] K. Ito and K. Kunisch: *On asymptotic properties of receding horizon optimal control*, preprint.

[IR] K. Ito and S. S. Ravindran: *Reduced basis method for control problems governed by partial differential equations*, in Desch W. et al. (eds) Control and Estimation of Distributed Parameter Systems, Internat. Conference in Vorau, Austria, Internat. Series on Numerical Mathematics, Vol.**126**, Birkhäuser, Basel, 1998, 153–168.

[IS] K. Ito and J. D. Schroeter: *Reduced order feedback synthesis for viscous incompressible flows*, preprint, North Carolina State University.

[KTB] G. M. Kepler, H. T. Tran and H. T. Banks: *Compensator control for chemical vapor deposition film growth using reduced order design models*, preprint, North Carolina State University, 1999.

[KV1] K. Kunisch and S. Volkwein: *Control of Burger's equation by reduced order approach using proper orthogonal decomposition*, Journal of Optimization Theory and Applications, 102(1999), 345–371.

[KV2] K. Kunisch and S. Volkwein: *Galerkin proper orthogonal decomposition methods for parabolic systems*,to appear in Numeerische Mathematik.

[KV3] K. Kunisch and S. Volkwein: *Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics*, submitted.

[L] M. Laumen: *Newton's mesh independence principle for a class of optimal shape design problems*, SIAM J. Control and Optimization **37** (1999), 1088.

[LT] H. V. Ly and H. T. Tran: *Proper orthogonal decomposition for flow calculations and optimal control in a horizontal CVD reactor*, Quart. Appl. Math. (2000),

[LT1] C. H. Lee and H. T. Tran: *Reduced order feedback control for liquid film growth*, preprint, North Carolina State University.

[T] R. Temam: *Navier–Stokes equations*, North–Holland, Amsterdam, 1979.

[V] S. Volkwein: *Optimal control of a phase–field model using the proper orthogonal decomposition*, Zeitschrift für Angewandte Mathematik und Mechanik, **81** (2001), 83–97.

[Z] H. Zwart: *P.O.D. for linear systems*, preprint.