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Shape sensitivity of curvilinear cracks on interface to non-linear perturbations

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Abstract. The 2D-model of an anisotropic, non-homogeneous, bonded elastic solid with a crack on the interface is considered. We state the linear problem with the stress-free boundary condition at the crack faces in addition to the transmission condition at the connected part of the interface. The sensitivity of the model to non-linear perturbations of the curvilinear crack along the interface is investigated. We obtain the asymptotic expansion and the corresponding derivatives of the potential energy functional with respect to the crack length via the material derivatives of the solution. This allows us to describe the growth or stationarity, and the local optimality conditions by the Griffith rupture criterion. The integral expression of the energy release rate for the considered problems is obtained, and the Cherepanov-Rice integral is discussed.

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1. Introduction

The first peculiarity of crack problems is connected with the non-regular character of the boundaries. Basic analysis of the singularities arising in non-smooth domains was carried out in [17], [19], [22], [12]. Another principle question concerns the crack propagation, which depends on the rupture criterion used. By the Griffith approach, growth of a crack is defined by the variations of an energy. The first derivative of the potential energy (the energy release rate) was obtained and utilized in [23], [5], [13], [7–10] and other works. Knowledge of the higher-order derivatives allows to investigate the stability of the crack and to derive the quasi-static models of its local propagation as shown in [22].

In general, the crack perturbation deals with the singular perturbations by shape. The shape sensitivity of smooth boundaries was analyzed in [26], [27]. Following [14], analysis of the smooth crack shape can be reduced to the regular perturbations by transforming the domain. This approach was applied to non-linear crack problems with the Signorini or friction conditions in [16], [15], [2], [11]. By this, the first derivatives only are available in comparison with linear crack problems under the stress-free boundary condition as in [18], [3].

When a crack has the curved or kinked path as in [20], [7], [1], some distinguishes can appear in adopted rupture criterions. It is shown here that the energy release rate for curvilinear cracks in general can not be expressed as the path-independent integral of the Cherepanov-Rice type due to the curvature near the crack tip.

The interface cracks in bonded materials were investigated in [24], [4], [28], [21], [6]. We generalize here the crack sensitivity approach to the bonded solids, which have the same mathematical description in the weak formulation as the homogeneous bodies. Thus, the presented work combines the curvilinear form of a crack under the stress-free condition with the transmission condition at the interface. In our consideration we use the general variational principles.

2. Non-linear perturbation of a crack

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with the boundary Γ of the class $C^{0,1}$, and $\overline{\Omega} = \Omega \cup \Gamma$. Suppose that Ω is separated by the curve Σ into two sub-domains Ω^1 and Ω^2, i.e. $\overline{\Omega}^1 \cap \overline{\Omega}^2 = \overline{\Sigma}$ and $\overline{\Omega}^1 \cup \overline{\Omega}^2 = \overline{\Omega}$. We assume that Σ in $\mathbf{R}^2$ is given by a smooth function ψ such that

$$\Sigma = \{x_2 = \psi(x_1), \quad -l_0 < x_1 < l_1\}, \quad l_0, l_1 > 0.$$

The chosen normal vector $\nu = (\nu_1, \nu_2)$ to Σ fits its positive Σ^+ and negative Σ^- faces. Let us fix the internal part Γ_l of this curve, namely

$$\Gamma_l = \{x_2 = \psi(x_1), \quad 0 < x_1 < l\}, \quad 0 < l < l_1,$$

with the parameter l defining the length of Γ_l in projection onto the x_1-axis. Denote by $\Omega_l = \Omega \setminus \overline{\Gamma}_l$ the domain bounded by Γ and $\overline{\Gamma}_l^+$, $\overline{\Gamma}_l^-$. We will consider a bonded elastic body occupying the domain Ω_l with the interface crack Γ_l. By this we mean that two, may be different materials fill the sub-domains Ω^1 and Ω^2 of Ω_l, respectively, detached by the interface Σ, and they are connected together at the part $\overline{\Sigma} \setminus \Gamma_l$.

We look for the displacement vector $u = (u_1, u_2)$ in the solid possessing the linear Hooke law

$$\sigma_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2,$$

with the coefficients c_{ijkl} composed from the elasticity coefficients c^1_{ijkl} and c^2_{ijkl} of each solid Ω^1 and Ω^2, respectively:

$$c_{ijkl} = \begin{cases} c^1_{ijkl} & \text{in } \overline{\Omega}^1 \\ c^2_{ijkl} & \text{in } \overline{\Omega}^2 \end{cases}, \quad i, j, k, l, = 1, 2.$$

We assume that $c^m_{ijkl} \in C^2(\overline{\Omega}^m)$, $m = 1, 2$, are elliptic and symmetric as usual. Let $f = (f_1, f_2) \in [C^2(\overline{\Omega})]^2$ be a given force. The potential energy of the solid is expressed by the functional

$$\Pi(u;\Omega_l) = \frac{1}{2}\int_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(u) - \int_{\Omega_l} f_i u_i. \tag{1}$$

For a small parameter ε, let us consider the perturbed along the interface Σ crack $\Gamma_{l+\varepsilon}$ of the length-parameter $l+\varepsilon$:

$$\Gamma_{l+\varepsilon} = \{x_2 = \psi(x_1), \quad 0 < x_1 < l+\varepsilon\}, \quad 0 < l+\varepsilon < l_1.$$

In the perturbed domain $\Omega_{l+\varepsilon} = \Omega \setminus \overline{\Gamma}_{l+\varepsilon}$ the potential energy functional has the same form:

$$\Pi(u;\Omega_{l+\varepsilon}) = \frac{1}{2}\int_{\Omega_{l+\varepsilon}} \sigma_{ij}(u)\varepsilon_{ij}(u) - \int_{\Omega_{l+\varepsilon}} f_i u_i. \tag{2}$$

We will transform the perturbed domain onto the initial one by a local coordinate transformation following the idea of ([16]). Let $B_\delta \subset \mathbf{R}^2$ be a circle of the radius $\delta > 0$ centered at the crack tip $(l, \psi(l))$. We suppose δ being small enough so that $\overline{B}_\delta \subset \Omega$ and the second crack tip $(0, \psi(0))$ is separated from B_δ. Choose the smooth cut-off function χ such that $supp\,\chi \subset B_\delta$ and $\chi \equiv 1$ in $B_{\delta/2}$. For small ε with $|\varepsilon| < \delta/2$ we construct the non-linear in ε, one-to-one coordinate transformation

$$\left\{ \begin{array}{l} y_1 = x_1 + \varepsilon\chi(x) \\ y_2 = x_2 + \psi(x_1 + \varepsilon\chi(x)) - \psi(x_1) \end{array} \right. , \quad (y_1, y_2) \in \Omega_{l+\varepsilon}, \quad (x_1, x_2) \in \Omega_l, \tag{3}$$

which maps Ω_l to $\Omega_{l+\varepsilon}$. The functional matrix of the transformation

$$\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)} = \begin{pmatrix} 1+\varepsilon\chi_{,1} & , \ \varepsilon\chi_{,2} \\ (1+\varepsilon\chi_{,1})\psi'(x_1+\varepsilon\chi) - \psi'(x_1) & , \ 1+\varepsilon\chi_{,2}\psi'(x_1+\varepsilon\chi) \end{pmatrix}$$

has the Jacobian

$$J = 1 + \varepsilon\frac{\partial\chi}{\partial s}, \qquad \frac{\partial}{\partial s} \equiv \frac{\partial}{\partial x_1} + \psi'(x_1)\frac{\partial}{\partial x_2},$$

where $\frac{\partial}{\partial s}$ means the differentiation along the interface, and $J > 0$ for ε small. Its inverse matrix is of the form:

$$\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} = \begin{pmatrix} 1 - \dfrac{\varepsilon\chi_{,1}}{J} + \dfrac{\varepsilon\chi_{,2}}{J}\Delta_\varepsilon\psi' & , \ -\dfrac{\varepsilon\chi_{,2}}{J} \\ -\dfrac{\varepsilon\psi'(x_1)\chi_{,1}}{J} - \dfrac{1+\varepsilon\chi_{,1}}{J}\Delta_\varepsilon\psi' & , \ 1 - \dfrac{\varepsilon\psi'(x_1)\chi_{,2}}{J} \end{pmatrix} \tag{4}$$

for $\Delta_\varepsilon\psi' \equiv \psi'(x_1+\varepsilon\chi) - \psi'(x_1)$. Denote by $\widehat{u}(x)$, $x \in \Omega_l$, the transformed function $u(y)$, $y \in \Omega_{l+\varepsilon}$, namely

$$u(y) = u(x_1 + \varepsilon\chi(x), x_2 + \psi(x_1+\varepsilon\chi(x)) - \psi(x_1)) \equiv \widehat{u}(x).$$

Using (4), we can rewrite the derivatives as

$$\frac{\partial u}{\partial y_i} = \frac{\partial \widehat{u}}{\partial x_i} - \frac{\varepsilon}{J}\chi_{,i}\frac{\partial \widehat{u}}{\partial s} + \frac{\Delta_\varepsilon\psi'}{J}\delta_{i1}\left(\frac{\partial \widehat{u}}{\partial x_1}(x_1+\varepsilon\chi)_{,2} - \frac{\partial \widehat{u}}{\partial x_2}(x_1+\varepsilon\chi)_{,1}\right)$$

for $i = 1, 2$, and, therefore, the strain and the stress tensors as

$$\begin{aligned} \varepsilon_{ij}(u) &= \varepsilon_{ij}(\widehat{u}) - \frac{\varepsilon}{J} E^s_{ij}(\chi; \widehat{u}) + \frac{\Delta_\varepsilon \psi'}{J} M_{ij}(x_1 + \varepsilon\chi; \widehat{u}), \quad i, j = 1, 2, \\ \sigma_{ij}(u) &= \widehat{c}_{ijkl} \left(\varepsilon_{kl}(\widehat{u}) - \frac{\varepsilon}{J} E^s_{kl}(\chi; \widehat{u}) + \frac{\Delta_\varepsilon \psi'}{J} M_{kl}(x_1 + \varepsilon\chi; \widehat{u}) \right), \end{aligned} \tag{5}$$

where

$$\widehat{c}_{ijkl}(x) = \begin{cases} c^1_{ijkl}(x_1 + \varepsilon\chi, x_2 + \psi(x_1 + \varepsilon\chi) - \psi(x_1)) & \text{in } \overline{\Omega}^1 \\ c^2_{ijkl}(x_1 + \varepsilon\chi, x_2 + \psi(x_1 + \varepsilon\chi) - \psi(x_1)) & \text{in } \overline{\Omega}^2 \end{cases}, \ i, j, k, l, = 1, 2,$$

with the notation used from

$$E^s_{ij}(\chi; w) = \frac{1}{2} \left(\chi_{,i} \frac{\partial w_j}{\partial s} + \chi_{,j} \frac{\partial w_i}{\partial s} \right), \quad \Sigma^s_{ij}(\chi; w) = c_{ijkl} E^s_{kl}(\chi; w),$$

$$M_{ij}(\xi; w) = \frac{1}{2} \Big(\delta_{i1}(w_{j,1}\xi_{,2} - w_{j,2}\xi_{,1}) + \delta_{j1}(w_{i,1}\xi_{,2} - w_{i,2}\xi_{,1}) \Big),$$

$$N_{ij}(\xi; w) = c_{ijkl} M_{kl}(\xi; w), \quad i, j = 1, 2, \quad w = (w_1, w_2).$$

Tensor $\{E^s_{ij}\}$ describes the tangential to Σ deformation, and $\{M_{ij}\}$ – the rotation. Note that for rectilinear cracks we have $\psi'' = 0$, i.e. $\psi' = const$, therefore $\Delta_\varepsilon \psi' = 0$ and only the tangential deformations E^s_{ij} arise. Applying the transformation (3) to the integrals in (2), we substitute (5) and deduce the formula $\Pi(u; \Omega_{l+\varepsilon}) = \Pi_\varepsilon(\widehat{u}; \Omega_l)$ with

$$\begin{aligned} \Pi_\varepsilon(w; \Omega_l) \equiv \frac{1}{2} \int\limits_{\Omega_l} J \widehat{c}_{ijkl} \Big(\varepsilon_{kl}(w) - \frac{\varepsilon}{J} E^s_{kl}(\chi; w) + \frac{\Delta_\varepsilon \psi'}{J} M_{kl}(x_1 + \varepsilon\chi; w) \Big) \\ \times \Big(\varepsilon_{ij}(w) - \frac{\varepsilon}{J} E^s_{ij}(\chi; w) + \frac{\Delta_\varepsilon \psi'}{J} M_{ij}(x_1 + \varepsilon\chi; w) \Big) - \int\limits_{\Omega_l} J \widehat{f}_i w_i. \end{aligned} \tag{6}$$

Thus, we have proved the following lemma.

Lemma 1. *The identity* $\Pi(u; \Omega_{l+\varepsilon}) = \Pi_\varepsilon(\widehat{u}; \Omega_l)$ *holds for the function* $u(y)$, $y \in \Omega_{l+\varepsilon}$, *smooth enough.*

3. Shape sensitivity of the curvilinear crack

In the previous section we have obtained the preliminary results. Now let us formulate the linear problem implying that the crack faces are free of stresses.

While we have the bonded solid, one needs to fulfill the transmission condition of the coupling at the connected part of the interface (see for example ([25]):

$$[u_i] = 0, \quad [\sigma_{ij}(u)\nu_j] = 0, \quad i = 1, 2, \quad \text{on } \overline{\Sigma} \setminus \Gamma_l,$$

where $[\xi]$ denotes the jump $\xi|_{\Sigma^+} - \xi|_{\Sigma^-}$ at the corresponding part of Σ. Introduce the space of admissible displacements

$$\widetilde{H}^1(\Omega_l) = \{u = (u_1, u_2) \in [H^1(\Omega_l)]^2, \quad u = 0 \quad \text{on } \Gamma\},$$

which includes the zero displacement condition at the external boundary Γ and the zero jump condition for displacements at the part $\overline{\Sigma} \setminus \Gamma_l$ of the interface. The equilibrium of a solid with a crack is described by the minimization problem of the potential energy:

$$\Pi(u; \Omega_l) = \inf_{v \in \widetilde{H}^1(\Omega_l)} \Pi(v; \Omega_l), \tag{7}$$

which is equivalent to the variational equation

$$\int_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(v) = \int_{\Omega_l} f_i v_i \quad \forall v \in \widetilde{H}^1(\Omega_l). \tag{8}$$

By the well known properties of the functional Π, due to the Korn inequality provided $u = 0$ on Γ, there exists the unique solution $u \in \widetilde{H}^1(\Omega_l)$ of the problems (7) or (8). The corresponding boundary value problem for (8) is of the form:

$$-\sigma_{ij,j}(u) = f_i, \quad i = 1, 2, \quad \text{in } \Omega_l;$$

$$[u_i] = 0, \quad [\sigma_{ij}(u)\nu_j] = 0, \quad i = 1, 2, \quad \text{on } \overline{\Sigma} \setminus \Gamma_l;$$

$$\sigma_{ij}(u)\nu_j = 0, \quad i = 1, 2, \quad \text{on } \Gamma_l^{\pm}; \quad u = 0 \quad \text{on } \Gamma.$$

The first string of boundary conditions here implies the transmission condition fulfilled.

For the perturbed domain $\Omega_{l+\varepsilon}$, the minimization problem

$$\Pi(u^\varepsilon; \Omega_{l+\varepsilon}) = \inf_{v \in \widetilde{H}^1(\Omega_{l+\varepsilon})} \Pi(v; \Omega_{l+\varepsilon})$$

or, equivalently, the variational equation

$$\int_{\Omega_{l+\varepsilon}} \sigma_{ij}(u^\varepsilon)\varepsilon_{ij}(v) = \int_{\Omega_{l+\varepsilon}} f_i v_i \quad \forall v \in \widetilde{H}^1(\Omega_{l+\varepsilon}) \tag{9}$$

has the unique solution $u^\varepsilon \in \widetilde{H}^1(\Omega_{l+\varepsilon})$ by the same reason as for problem (8). Due to Lemma 1, the transformed function $\widehat{u}^\varepsilon \in \widetilde{H}^1(\Omega_l)$ is a unique solution of the problem

$$\Pi_\varepsilon(\widehat{u}^\varepsilon; \Omega_l) = \inf_{v \in \widetilde{H}^1(\Omega_l)} \Pi_\varepsilon(v; \Omega_l). \tag{10}$$

From (6) and (10) we obtain then the equation

$$\int_{\Omega_l} J\widehat{c}_{ijkl}\Big(\varepsilon_{kl}(\widehat{u}^\varepsilon) - \frac{\varepsilon}{J}E^s_{kl}(\chi; \widehat{u}^\varepsilon) + \frac{\Delta_\varepsilon \psi'}{J} M_{kl}(x_1 + \varepsilon\chi; \widehat{u}^\varepsilon)\Big) \tag{11}$$

$$\times\Big(\varepsilon_{ij}(v)-\frac{\varepsilon}{J}E^s_{ij}(\chi;v)+\frac{\Delta_\varepsilon\psi'}{J}M_{ij}(x_1+\varepsilon\chi;v)\Big)=\int\limits_{\Omega_l}J\widehat{f}_iv_i\quad\forall v\in\widetilde{H}^1(\Omega_l).$$

Note that the substitution of $v=\widehat{u}^\varepsilon$ in (11) possesses for ε small enough the uniform in ε estimate

$$\|\widehat{u}^\varepsilon\|_{\widetilde{H}^1(\Omega_l)}\le c. \tag{12}$$

We seek for the representation of the solution $\widehat{u}^\varepsilon$ of (11) by the series in ε of the form

$$\widehat{u}^\varepsilon=u+\varepsilon\dot{u}+\frac{\varepsilon^2}{2}\ddot{u}+o(\varepsilon^2). \tag{13}$$

The functions $\dot{u}$ and $\ddot{u}$ are called the strong material derivatives of the solution of the first and the second order, respectively.

In a standard way, one can obtain the next series in ε :

$$\Delta_\varepsilon\psi'=\varepsilon\chi\psi''+\frac{\varepsilon^2}{2}\chi^2\psi'''+o(\varepsilon^2), \tag{14}$$

$$\widehat{f}_i=f_i+\varepsilon\chi\frac{\partial f_i}{\partial s}+\frac{\varepsilon^2}{2}\chi^2\frac{\partial^2 f_i}{\partial s^2}+o(\varepsilon^2),\quad i=1,2, \tag{15}$$

where

$$\frac{\partial^2}{\partial s^2}=\frac{\partial^2}{\partial x_1^2}+\psi''\frac{\partial}{\partial x_2}+2\psi'\frac{\partial^2}{\partial x_1x_2}+(\psi')^2\frac{\partial^2}{\partial x_2^2}.$$

It follows then from (15) together with definition of J that

$$J\widehat{f}_i=f_i+\varepsilon\frac{\partial}{\partial s}\big(\chi f_i\big)+\frac{\varepsilon^2}{2}\frac{\partial}{\partial s}\Big(\chi^2\frac{\partial f_i}{\partial s}\Big)+o(\varepsilon^2),\quad i=1,2. \tag{16}$$

Moreover, in analogy with (15) we have

$$\widehat{c}_{ijkl}=c_{ijkl}+\varepsilon\chi\frac{\partial c_{ijkl}}{\partial s}+\frac{\varepsilon^2}{2}\chi^2\frac{\partial^2 c_{ijkl}}{\partial s^2}+o(\varepsilon^2),\quad i,j,k,l=1,2, \tag{17}$$

and the derivatives

$$\frac{\partial c_{ijkl}}{\partial s}=\begin{cases}\dfrac{\partial c^1_{ijkl}}{\partial s} & \text{in } \overline{\Omega}^1\\[2ex] \dfrac{\partial c^2_{ijkl}}{\partial s} & \text{in } \overline{\Omega}^2\end{cases},\qquad \frac{\partial^2 c_{ijkl}}{\partial s^2}=\begin{cases}\dfrac{\partial^2 c^1_{ijkl}}{\partial s^2} & \text{in } \overline{\Omega}^1\\[2ex] \dfrac{\partial^2 c^2_{ijkl}}{\partial s^2} & \text{in } \overline{\Omega}^2\end{cases}$$

are correctly defined at least on $supp\,\chi$ because the differentiation $\frac{\partial}{\partial s}$ along the interface Σ keeps the elasticity coefficients $\{c^m_{ijkl}\}$ in the corresponding set $\overline{\Omega}^m$, $m=1,2$. Therefore, it follows from (14), (17) and definition of J that

$$J\widehat{c}_{ijkl}=c_{ijkl}+\varepsilon\frac{\partial}{\partial s}\big(\chi c_{ijkl}\big)+\frac{\varepsilon^2}{2}\frac{\partial}{\partial s}\Big(\chi^2\frac{\partial c_{ijkl}}{\partial s}\Big)+o(\varepsilon^2),$$

$$\Delta_\varepsilon\psi'\widehat{c}_{ijkl}=\varepsilon\chi\psi''c_{ijkl}+\frac{\varepsilon^2}{2}\chi^2\Big(\psi'''c_{ijkl}+2\psi''\frac{\partial c_{ijkl}}{\partial s}\Big)+o(\varepsilon^2),\quad i,j,k,l=1,2.$$

Using (16) we can decompose the right-hand side of (11) in the series by ε as

$$\int_{\Omega_l} J\widehat{f}_i v_i = \int_{\Omega_l} \Big(f_i + \varepsilon \frac{\partial}{\partial s}(\chi f_i) + \frac{\varepsilon^2}{2}\frac{\partial}{\partial s}\Big(\chi^2 \frac{\partial f_i}{\partial s}\Big)\Big) v_i + o(\varepsilon^2).$$

By the previous formulas, the left-hand side of (11) gets

$$\int_{\Omega_l} J\widehat{c}_{ijkl}\Big(\varepsilon_{kl}(\widehat{u}^\varepsilon) - \frac{\varepsilon}{J}E^s_{kl}(\chi;\widehat{u}^\varepsilon) + \frac{\Delta_\varepsilon \psi'}{J} M_{kl}(x_1+\varepsilon\chi;\widehat{u}^\varepsilon)\Big)$$

$$\times\Big(\varepsilon_{ij}(v) - \frac{\varepsilon}{J}E^s_{ij}(\chi;v) + \frac{\Delta_\varepsilon \psi'}{J} M_{ij}(x_1+\varepsilon\chi;v)\Big)$$

$$= \int_{\Omega_l} \sigma_{ij}(\widehat{u}^\varepsilon)\varepsilon_{ij}(v) + \varepsilon A^1(\widehat{u}^\varepsilon, v) + \varepsilon^2 A^2(\widehat{u}^\varepsilon, v) + o(\varepsilon^2),$$

with the corresponding bilinear forms A^1 and A^2 as follows:

$$A^1(u,v) = \int_{\Omega_l} \Big(\frac{\partial}{\partial s}(\chi c_{ijkl})\varepsilon_{kl}(u)\varepsilon_{ij}(v) - \sigma_{ij}(u)E^s_{ij}(\chi;v) - E^s_{ij}(\chi;u)\sigma_{ij}(v)$$

$$+\chi\psi''\Big(\sigma_{ij}(u)M_{ij}(x_1;v) + M_{ij}(x_1;u)\sigma_{ij}(v)\Big)\Big),$$

$$A^2(u,v) = \int_{\Omega_l} \Big(\frac{1}{2}\frac{\partial}{\partial s}\Big(\chi^2\frac{\partial c_{ijkl}}{\partial s}\Big)\varepsilon_{kl}(u)\varepsilon_{ij}(v)$$

$$-\chi\frac{\partial c_{ijkl}}{\partial s}\Big(\varepsilon_{kl}(u)E^s_{ij}(\chi;v) + E^s_{kl}(\chi;u)\varepsilon_{ij}(v)\Big) + \Sigma^s_{ij}(\chi;u)E^s_{ij}(\chi;v)$$

$$+\chi^2\Big(\frac{1}{2}\psi'''c_{ijkl} + \psi''\frac{\partial c_{ijkl}}{\partial s}\Big)\Big(\varepsilon_{kl}(u)M_{ij}(x_1;v) + M_{kl}(x_1;u)\varepsilon_{ij}(v)\Big)$$

$$+(\chi\psi'')^2 N_{ij}(x_1;u)M_{ij}(x_1;v) + \chi\psi''\Big(\sigma_{ij}(u)M_{ij}(\chi;v) + M_{ij}(\chi;u)\sigma_{ij}(v)$$

$$-E^s_{ij}(\chi;u)N_{ij}(x_1;v) - N_{ij}(x_1;u)E^s_{ij}(\chi;v)\Big)\Big).$$

Note that already the first term A^1 in the above expansion of the operator of the problem (11) depends on the crack curvature by the term with ψ''. For rectilinear cracks we have $\psi'' \equiv 0$, therefore the bilinear forms A^1 and A^2 for rectilinear cracks possess more simple form:

$$A^1(u,v) = \int_{\Omega_l} \Big(\frac{\partial}{\partial s}(\chi c_{ijkl})\varepsilon_{kl}(u)\varepsilon_{ij}(v) - \sigma_{ij}(u)E^s_{ij}(\chi;v) - E^s_{ij}(\chi;u)\sigma_{ij}(v)\Big),$$

$$A^2(u,v) = \int_{\Omega_l} \Big(\frac{1}{2}\frac{\partial}{\partial s}\Big(\chi^2\frac{\partial c_{ijkl}}{\partial s}\Big)\varepsilon_{kl}(u)\varepsilon_{ij}(v)$$

$$-\chi \frac{\partial c_{ijkl}}{\partial s}\Big(\varepsilon_{kl}(u)E^s_{ij}(\chi;v) + E^s_{kl}(\chi;u)\varepsilon_{ij}(v)\Big) + \Sigma^s_{ij}(\chi;u)E^s_{ij}(\chi;v)\Big).$$

After the substitution of (13) in (11), we can conclude formally from (11) that

$$0 = \Big\{\int_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(v) - \int_{\Omega_l} f_i v_i\Big\} + \varepsilon\Big\{\int_{\Omega_l} \sigma_{ij}(\dot u)\varepsilon_{ij}(v) + A^1(u,v) - \int_{\Omega_l} \frac{\partial}{\partial s}(\chi f_i)v_i\Big\}$$

$$+\frac{\varepsilon^2}{2}\Big\{\int_{\Omega_l} \sigma_{ij}(\ddot u)\varepsilon_{ij}(v) + 2A^1(\dot u,v) + 2A^2(u,v) - \int_{\Omega_l} \frac{\partial}{\partial s}\Big(\chi^2 \frac{\partial f_i}{\partial s}\Big)v_i\Big\} + o(\varepsilon^2).$$

Let us now define the functions $\dot u, \ddot u \in \widetilde H^1(\Omega_l)$ as unique solutions of the corresponding problems

$$\int_{\Omega_l} \sigma_{ij}(\dot u)\varepsilon_{ij}(v) = \int_{\Omega_l} \frac{\partial}{\partial s}(\chi f_i)v_i - A^1(u,v) \quad \forall v \in \widetilde H^1(\Omega_l), \tag{18}$$

$$\int_{\Omega_l} \sigma_{ij}(\ddot u)\varepsilon_{ij}(v) = \int_{\Omega_l} \frac{\partial}{\partial s}\Big(\chi^2 \frac{\partial f_i}{\partial s}\Big)v_i - 2A^1(\dot u,v) - 2A^2(u,v) \quad \forall v \in \widetilde H^1(\Omega_l). \tag{19}$$

Subtracting (8) from (11), we have due to the above decomposition that

$$\int_{\Omega_l} \sigma_{ij}(\widehat u^\varepsilon - u)\varepsilon_{ij}(v) = \varepsilon\Big(\int_{\Omega_l} \frac{\partial}{\partial s}(\chi f_i)v_i - A^1(\widehat u^\varepsilon, v)\Big) + o(\varepsilon).$$

Taking $v = \widehat u^\varepsilon - u$ and applying the Korn and Hölder inequalities, we finally obtain the estimate

$$\|\widehat u^\varepsilon - u\|_{\widetilde H^1(\Omega_l)} \le c\varepsilon \tag{20}$$

because of (12). The subtraction of (8) and (18) multiplied with ε from (11) leads to the relation

$$\int_{\Omega_l} \sigma_{ij}(\widehat u^\varepsilon - u - \varepsilon \dot u)\varepsilon_{ij}(v) = \frac{\varepsilon^2}{2}\int_{\Omega_l} \frac{\partial}{\partial s}\Big(\chi^2 \frac{\partial f_i}{\partial s}\Big)v_i$$

$$-\varepsilon A^1(\widehat u^\varepsilon - u, v) - \varepsilon^2 A^2(\widehat u^\varepsilon, v) + o(\varepsilon^2),$$

and therefore, in view of (12) and (20), gives the next estimate

$$\|\widehat u^\varepsilon - u - \varepsilon \dot u\|_{\widetilde H^1(\Omega_l)} \le c\varepsilon^2. \tag{21}$$

Extending the expansion of integrals in (11) up to the ε^3-terms, by the same arguments we have

$$\int_{\Omega_l} \sigma_{ij}(\widehat u^\varepsilon - u - \varepsilon \dot u - \frac{\varepsilon^2}{2}\ddot u)\varepsilon_{ij}(v) = \frac{\varepsilon^3}{6}\int_{\Omega_l} \frac{\partial}{\partial s}\Big(\chi^3 \frac{\partial^2 f_i}{\partial s^2}\Big)v_i$$

$$-\varepsilon A^1(\widehat{u}^\varepsilon - u - \varepsilon\dot{u}, v) - \varepsilon^2 A^2(\widehat{u}^\varepsilon - u, v) - \varepsilon^3 A^3(\widehat{u}^\varepsilon, v) + o(\varepsilon^3)$$

and the estimate

$$\|\widehat{u}^\varepsilon - u - \varepsilon\dot{u} - \frac{\varepsilon^2}{2}\ddot{u}\|_{\widetilde{H}^1(\Omega_l)} \le c\varepsilon^3 \tag{22}$$

because of (12), (20), (21). Thus, we have proved the following result.

Theorem 1. *For the crack problem considered, there exist the strong material derivatives* $\dot{u}$, $\ddot{u}$ *given by problems* (18), (19), *and the expansion* (13) *holds with estimates* (20)–(22).

If substitute the solution u of (8) in the functional (1), then we can define the potential energy of a body with a crack as a function $\mathcal{P} : (0, l_1) \to \mathbf{R}$ of the parameter l expressing the crack length in projection onto the x_1-axis. Precisely, substituting $v = u$ in (8), it follows from (1) that

$$\mathcal{P}(l) \equiv \Pi(u; \Omega_l) = -\frac{1}{2}\int\limits_{\Omega_l} f_i u_i.$$

For the solution u^ε of (9), applying the transformation (3), we get analogously

$$\mathcal{P}(l+\varepsilon) \equiv \Pi(u^\varepsilon; \Omega_{l+\varepsilon}) = -\frac{1}{2}\int\limits_{\Omega_{l+\varepsilon}} f_i u_i^\varepsilon = -\frac{1}{2}\int\limits_{\Omega_l} J\widehat{f}_i \widehat{u}_i^\varepsilon.$$

Taking the expansions (13) and (16) for the above expression due to Theorem 1, one can deduce a series in ε of the function $\mathcal{P}$,

$$\mathcal{P}(l+\varepsilon) = \mathcal{P}(l) + \varepsilon\mathcal{P}'(l) + \frac{\varepsilon^2}{2}\mathcal{P}''(l) + o(\varepsilon^2), \tag{23}$$

with the corresponding first and second derivatives of $\mathcal{P}$,

$$\begin{aligned} \mathcal{P}'(l) &= -\frac{1}{2}\int\limits_{\Omega_l}\Big(\frac{\partial}{\partial s}(\chi f_i)u_i + f_i\dot{u}_i\Big), \\ \mathcal{P}''(l) &= -\frac{1}{2}\int\limits_{\Omega_l}\bigg(\frac{\partial}{\partial s}\Big(\chi^2\frac{\partial f_i}{\partial s}\Big)u_i + 2\frac{\partial}{\partial s}(\chi f_i)\dot{u}_i + f_i\ddot{u}_i\bigg). \end{aligned} \tag{24}$$

Let us note that integrals in $\mathcal{P}(l) = -\frac{1}{2}\int\limits_{\Omega_l} f_i u_i$ and in $\mathcal{P}(l+\varepsilon) = -\frac{1}{2}\int\limits_{\Omega_{l+\varepsilon}} f_i u_i^\varepsilon$ do not depend on the cut-off function χ, therefore the derivatives $\mathcal{P}'(l)$ and $\mathcal{P}''(l)$ are also independent of the cut-off function in spite the expressions (24) include χ. This means that if one takes two different functions χ^1 and χ^2 in (24), these integrals must have the same value.

We can reduce the order of the material derivatives included in formulas (24). Indeed, taking $v = \dot{u}$ in (8) and $v = u$ in (18), we have

$$\int_{\Omega_l} f_i \dot{u}_i = \int_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(\dot{u}) = \int_{\Omega_l} \sigma_{ij}(\dot{u})\varepsilon_{ij}(u) = \int_{\Omega_l} \frac{\partial}{\partial s}(\chi f_i) u_i - A^1(u,u),$$

and therefore

$$\mathcal{P}'(l) = -\int_{\Omega_l} \frac{\partial}{\partial s}(\chi f_i) u_i + \frac{1}{2} A^1(u,u). \tag{25}$$

Analogously, the substitution of $v = \ddot{u}$ in (8) and $v = u$ in (19) yields

$$\int_{\Omega_l} f_i \ddot{u}_i = \int_{\Omega_l} \sigma_{ij}(\ddot{u})\varepsilon_{ij}(u) = \int_{\Omega_l} \frac{\partial}{\partial s}\Big(\chi^2 \frac{\partial f_i}{\partial s}\Big) u_i - 2A^1(\dot{u},u) - 2A^2(u,u).$$

Moreover, (18) with $v = \dot{u}$,

$$\int_{\Omega_l} \sigma_{ij}(\dot{u})\varepsilon_{ij}(\dot{u}) = \int_{\Omega_l} \frac{\partial}{\partial s}(\chi f_i)\dot{u}_i - A^1(u,\dot{u}),$$

leads to the relation

$$\mathcal{P}''(l) = -\int_{\Omega_l} \frac{\partial}{\partial s}\Big(\chi^2 \frac{\partial f_i}{\partial s}\Big) u_i + A^2(u,u) - \int_{\Omega_l} \sigma_{ij}(\dot{u})\varepsilon_{ij}(\dot{u}). \tag{26}$$

Theorem 2. *For the crack problem considered, there exist the derivatives $\mathcal{P}'(l)$, $\mathcal{P}''(l)$ of the potential energy with respect to the crack length given by formulas* (24) *or* (25), (26)*, and the refined Griffith formula* (23) *holds.*

Knowing these derivatives, we can seek the locally optimal state of the crack. For fixed l we introduce the total potential energy

$$\mathcal{T}(0) = \mathcal{P}(l) + \gamma \operatorname{meas}\Gamma_l, \quad \gamma > 0, \quad \operatorname{meas}\Gamma_l = \int_0^l \sqrt{1+\psi'(t)^2}\, dt, \tag{27}$$

where the last term denotes the surface energy of the crack with the constant density γ by the Griffith hypothesis. Using representation (23), we get the local quadratic approximation of $\mathcal{T}$ as follows:

$$\mathcal{T}(\varepsilon) \equiv \mathcal{P}(l+\varepsilon) + \gamma \int_0^{l+\varepsilon} \sqrt{1+\psi'(t)^2}\, dt = \mathcal{T}(0) + \varepsilon\big(\mathcal{P}'(l) + \gamma\sqrt{1+\psi'(l)^2}\big)$$
$$+ \frac{\varepsilon^2}{2}\Big(\mathcal{P}''(l) + \gamma \frac{\psi'(l)\psi''(l)}{\sqrt{1+\psi'(l)^2}}\Big) + o(\varepsilon^2).$$

By the Griffith criterion of the crack propagation, the total potential energy turns out to be minimal. Its local extremality condition is expressed by

$$0=\frac{d\mathcal{T}}{d\varepsilon}(\varepsilon)=\mathcal{P}'(l)+\gamma\sqrt{1+\psi'(l)^2}+\varepsilon\Big(\mathcal{P}''(l)+\gamma\frac{\psi'(l)\psi''(l)}{\sqrt{1+\psi'(l)^2}}\Big)+o(\varepsilon).$$

The optimal value of ε minimizing the quadratic approximation of $\mathcal{T}$ depends on the sign of the corresponding coefficients

$$a=\mathcal{P}'(l)+\gamma\sqrt{1+\psi'(l)^2},\quad b=\mathcal{P}''(l)+\gamma\frac{\psi'(l)\psi''(l)}{\sqrt{1+\psi'(l)^2}}.$$

Thus, when $b>0$ and $a<0$, we can find $\varepsilon^\star=-\frac{a}{b}>0$ and say that the interface crack tending the length-parameter $l+\varepsilon^\star$ is locally more preferable for the given load f from the energetic point of view. When $b\le 0$ and $a<0$, then the crack is locally unstable. The another case $a\ge 0$ yields the local stationarity of the initial crack with the length-parameter l.

Note that, applying this approach, one can find all the higher-order derivatives of the potential energy and use them also.

4. Energy release rate

We deduce now from the first derivative of the potential energy given by (25), namely

$$\mathcal{P}'(l)=\int\limits_{\Omega_l}\Big(-\frac{\partial}{\partial s}(\chi f_i)u_i+\frac{1}{2}\frac{\partial(\chi c_{ijkl})}{\partial s}\varepsilon_{kl}(u)\varepsilon_{ij}(u) -\sigma_{ij}(u)E^s_{ij}(\chi;u)+\chi\psi''\sigma_{ij}(u)M_{ij}(x_1;u)\Big), \tag{28}$$

a representation like the Cherepanov-Rice integral.

Let $B\subset\mathbf{R}^2$ be a neighborhood of the crack tip $(l,\psi(l))$ bounded by the smooth contour ∂B with the outward normal vector $n=(n_1,n_2)$, and let the cut-off function χ equal 1 identically in $\overline{B}=B\cup\partial B$. For example, one can choose B by the ball $B_{\delta/2}$ as before. Therefore, the integral in (28) over $B\setminus\overline{\Gamma}_l$ takes the form

$$\int\limits_{B\setminus\Gamma_l}\Big(f_i\frac{\partial u_i}{\partial s}+\frac{1}{2}\frac{\partial c_{ijkl}}{\partial s}\varepsilon_{kl}(u)\varepsilon_{ij}(u)+\psi''\sigma_{ij}(u)M_{ij}(x_1;u)\Big) -\int\limits_{\partial B}f_iu_i(n_1+\psi'n_2) \tag{29}$$

due to $\chi=1$ and the Green formula

$$-\int\limits_{B\setminus\Gamma_l}\frac{\partial f_i}{\partial s}u_i=\int\limits_{B\setminus\Gamma_l}f_i\frac{\partial u_i}{\partial s}-\int\limits_{\partial B}f_iu_i(n_1+\psi'n_2).$$

In the domain $\Omega_l \setminus \overline{B}$ excluding the crack tip, in view of the local smoothness of the solution u of (8) we can integrate by parts to deduce that

$$\int\limits_{\Omega_l \setminus B} \Big(-\frac{\partial}{\partial s}(\chi f_i)u_i + \frac{1}{2}\frac{\partial(\chi c_{ijkl})}{\partial s}\varepsilon_{kl}(u)\varepsilon_{ij}(u) - \sigma_{ij}(u)E^s_{ij}(\chi; u)$$

$$+\chi\psi''\sigma_{ij}(u)M_{ij}(x_1; u)\Big) \tag{30}$$

$$= \int\limits_{\partial B} \Big(\big(f_i u_i - \frac{1}{2}\sigma_{ij}(u)\varepsilon_{ij}(u)\big)(n_1 + \psi' n_2) + \sigma_{ij}(u) n_j \frac{\partial u_i}{\partial s}\Big)$$

$$+ \int\limits_{\Omega_l \setminus B} \chi\big(f_i + \sigma_{ij,j}(u)\big)\frac{\partial u_i}{\partial s} + \int\limits_{\Gamma_l \setminus B} [\chi \sigma_{ij}(u)\nu_j \frac{\partial u_i}{\partial s}].$$

The integral over $\Omega_l \setminus \overline{B}$ in the right-hand side of (30) is equal to zero because the equilibrium equations are fulfilled, and by the stress-free boundary condition $\sigma_{ij}(u)\nu_j = 0$, $i = 1, 2$, at the crack faces, the boundary integral over $\Gamma_l \setminus \overline{B}$ in (30) is also zeroth. Consequently, we obtain the following representation of (28) as a sum of (29) and (30):

$$\mathcal{P}'(l) = \int\limits_{\partial B} \sigma_{ij}(u)\Big(n_j \frac{\partial u_i}{\partial s} - \frac{1}{2}(n_1 + \psi' n_2)\varepsilon_{ij}(u)\Big)$$

$$+ \int\limits_{B \setminus \Gamma_l} \Big(f_i \frac{\partial u_i}{\partial s} + \frac{1}{2}\frac{\partial c_{ijkl}}{\partial s}\varepsilon_{kl}(u)\varepsilon_{ij}(u) + \psi''\sigma_{ij}(u)M_{ij}(x_1; u)\Big). \tag{31}$$

Theorem 3. *The energy release rate* $-\mathcal{P}'(l)$ *for the problem* (8) *yields the representation* (31).

Firstly, expression (31) does not contain the cut-off function in comparison with (28). Second, the integral over the neighborhood $B \setminus \overline{\Gamma}_l$ of the crack tip in general can not be excluded from (31) because of the presence of the curvature ψ''. The only case $\psi'' = 0$ consists of the rectilinear cracks, that implies, in our case for the load f and elasticity coefficients c_{ijkl} with $f = 0$ and $\frac{\partial c_{ijkl}}{\partial s} = 0$ in the neighborhood of the crack tip, the Cherepanov-Rice integral

$$-\mathcal{P}'(l) = \int\limits_{\partial B} \sigma_{ij}(u)\Big(\frac{1}{2}(n_1 + \psi' n_2)\varepsilon_{ij}(u) - n_j \frac{\partial u_i}{\partial s}\Big)$$

to be invariable for every closed contour ∂B surrounding the crack tip.

5. Conclusion

For the considered linear crack problem under the stress-free boundary condition at the crack faces, there exist all the derivatives of the potential energy functional with respect to the crack perturbation. The energy release rate gets the Cherepanov-Rice integral independent of a path for zero curvature near the crack tip. The curvilinear crack given on interface of the bonded solid is well-defined thanks to differentiation along the crack. Note that the same reason allows to take the load also piecewise-smooth through the interface.

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