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Sensitivity of cracks in 2D-Lamé problem via material derivatives

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Abstract. The Lamé model of a two-dimensional solid with a crack under the stress-free boundary condition of the Neumann type at the crack faces is considered. We investigate the sensitivity of the problem to the crack perturbation. By constructing the material derivatives of the solution as iterative solutions of the same elasticity problem with specified right-hand sides, derivatives of the energy functional and of the stress intensity factors with respect to the crack length of an arbitrary order are obtained providing the corresponding asymptotic expansions. In particular, this implies the local optimality condition for finding of the crack length and the quasi-static model of the local crack propagation by the Griffith rupture criterion.

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1. Introduction

Analysis of the singularities arising near the crack tips in linear models was carried out in [12], [14], [15], [7]. Non-linear models with unilateral constraints at the crack were considered in [10], [13]. Description of the quasi-static crack propagation requires involving of a suitable fracture criterion in terms of the energy, such as the energy release rate or J-integral (see [8], [9], [6]), or in terms of the stress intensity factors as in [15], [5], [1].

Following the Griffith hypothesis, question of the crack propagation depends on the derivatives of the energy functional with respect to the crack perturbation. The first-order variations to the linear crack problems were outlined in [4], [16], [2] and other works, the high-order variations – in [15] by using the singular perturbations method. The presented smooth transformation of the cracked domain allows to adopt the regular perturbations theory to the considered problem. Thus, we apply the technique of the shape sensitivity analysis developed in [18], [11], [17], [3] to describe all the derivatives of the energy functional via material derivatives of the solution. To construct the material derivatives or their stress intensity factors, the iterations of the same elasticity problem are required. In comparison with the asymptotic methods, we use the energetic solutions in the variational sense of the H^1-class.

2. Variation of cracks

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with a boundary Γ of the class $C^{2,1}$, and $\overline{\Omega} = \Omega \cup \Gamma$. Assume that the segment $[0, l_0]$ of the x_1-axis lies inside Ω. We define the set $\Gamma_l = (0, l) \times \{0\}$ in $\mathbf{R}^2$, where $0 < l < l_0$. The normal vector $(0, 1)$ to Γ_l fits its positive and negative faces $\Gamma_l^{\pm} = \Gamma_l \cap \{x \in \mathbf{R}^2,\ \pm x_2 \geq 0\}$, respectively. Denote $\Omega_l = \Omega \setminus \overline{\Gamma}_l$, where $\overline{\Gamma}_l = \Gamma_l \cup \partial\Gamma_l$ and $\partial\Gamma_l$ consists of the points $(0, 0)$, $(l, 0)$. Then the boundary of Ω_l is the union of Γ, $\Gamma_l^{\pm}$, $\partial\Gamma_l$. We consider the two-dimensional elastic body occupying the domain Ω_l with the crack Γ_l.

Let $f = (f_1, f_2) \in [C^\infty(\overline{\Omega})]^2$ be a given force. We look for the displacement vector $u = (u_1, u_2)$ and use the notation for the linear strains $\varepsilon_{ij}(u)$ and stresses $\sigma_{ij}(u)$, $i, j = 1, 2$, given by the Lamé law

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda\delta_{ij}\big(\varepsilon_{11}(u) + \varepsilon_{22}(u)\big).$$

Introduce the space

$$H^{1,0}(\Omega_l) = \{u = (u_1, u_2) \in [H^1(\Omega_l)]^2, \quad u = 0 \quad \text{on } \Gamma\},$$

which includes the jam condition at Γ. At the crack faces $\Gamma_l^{\pm}$ we assume the usual stress-free condition of the Neumann type $\sigma_{12}(u) = \sigma_{22}(u) = 0$. The equilibrium state of the solid corresponds to the minimum of the potential energy

$$\Pi(u; \Omega_l) = \frac{1}{2}\int\limits_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(u) - \int\limits_{\Omega_l} f_i u_i \tag{1}$$

and can be described by the variational equality

$$\int\limits_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(v) = \int\limits_{\Omega_l} f_i v_i \quad \forall v \in H^{1,0}(\Omega_l). \tag{2}$$

By the Korn inequality provided that $u = 0$ on Γ, there exists the unique solution $u \in H^{1,0}(\Omega_l)$ to the problem (2). The corresponding boundary value problem is of the form:

$$\begin{gathered} -\sigma_{ij,j}(u) = f_i, \quad i = 1, 2, \quad \text{in } \Omega_l, \\ \sigma_{i2}(u) = 0, \quad i = 1, 2, \quad \text{on } \Gamma_l^{\pm}, \quad u = 0 \quad \text{on } \Gamma. \end{gathered} \tag{3}$$

One can obtain, in a standard way, the additional H^2-smoothness of the solution u inside Ω_l up to the boundary Γ and $\Gamma_l^{\pm}$ excepting neighborhoods of $\partial\Gamma_l$.

For a small parameter $\varepsilon > 0$, let us consider the perturbed crack $\Gamma_{l+\varepsilon} = (0, l+\varepsilon) \times \{0\}$. In the perturbed domain $\Omega_{l+\varepsilon} = \Omega \setminus \overline{\Gamma}_{l+\varepsilon}$, we also have the unique solution $u^\varepsilon \in H^{1,0}(\Omega_{l+\varepsilon})$ to the equilibrium problem

$$\int_{\Omega_{l+\varepsilon}} \sigma_{ij}(u^\varepsilon)\varepsilon_{ij}(v) = \int_{\Omega_{l+\varepsilon}} f_i v_i \quad \forall v \in H^{1,0}(\Omega_{l+\varepsilon}). \tag{4}$$

We vary the shape of the crack by a local coordinate transformation of the domain like in [10]. Let B_δ be a circle of the radius $\delta > 0$ centered in the crack tip $(l, 0)$, with $\delta < \min\{l; l_0 - l\}$ and $\overline{B}_\delta \subset \Omega$. Choose the smooth cut-off function χ, $0 \le \chi \le 1$, such that $\operatorname{supp} \chi \subset B_\delta$ and $\chi \equiv 1$ in $B_{\delta/2}$. For small $\varepsilon < \delta/2$, we construct the one-to-one coordinate transformation

$$y_1 = x_1 + \varepsilon\chi(x_1, x_2), \quad y_2 = x_2, \quad (y_1, y_2) \in \Omega_{l+\varepsilon}, \quad (x_1, x_2) \in \Omega_l, \tag{5}$$

with the Jacobian $J = 1 + \varepsilon\chi_{,1}$, which transforms Ω_l to $\Omega_{l+\varepsilon}$. Denote by $\widehat{u}(x)$, $x \in \Omega_l$, the transformed function $u(y)$, $y \in \Omega_{l+\varepsilon}$, namely

$$u(y) = u(x_1 + \varepsilon\chi(x), x_2) \equiv \widehat{u}(x).$$

Using (5), rewrite the derivatives

$$u_{i,j} = \widehat{u}_{i,j} - \frac{\varepsilon}{J}\chi_{,j}\widehat{u}_{i,1}, \quad i, j = 1, 2,$$

and therefore,

$$\begin{gathered} \varepsilon_{ij}(u) = \varepsilon_{ij}(\widehat{u}) - \frac{\varepsilon}{J}E_{ij}(\chi; \widehat{u}), \quad E_{ij}(\chi; \widehat{u}) = \frac{1}{2}\Big(\chi_{,i}\widehat{u}_{j,1} + \chi_{,j}\widehat{u}_{i,1}\Big), \\ \sigma_{ij}(u) = \sigma_{ij}(\widehat{u}) - \frac{\varepsilon}{J}\Sigma_{ij}(\chi; \widehat{u}), \\ \Sigma_{ij}(\chi; \widehat{u}) = 2\mu E_{ij}(\chi; \widehat{u}) + \lambda\delta_{ij}\big(E_{11}(\chi; \widehat{u}) + E_{22}(\chi; \widehat{u})\big). \end{gathered} \tag{6}$$

Applying the transformation (5) to the integrals in (4), we substitute (6) and obtain the equation

$$\int_{\Omega_l} \sigma_{ij}(\widehat{u^\varepsilon})\varepsilon_{ij}(v) + \varepsilon A(\widehat{u^\varepsilon}, v) + \varepsilon^2 B\big[\frac{1}{J}\big](\widehat{u^\varepsilon}, v) = \int_{\Omega_l} J\widehat{f_i}v_i \quad \forall v \in H^{1,0}(\Omega_l), \tag{7}$$

where the bilinear forms A and $B[\,\cdot\,]$ are as follows:

$$A(u, v) = \int_{\Omega_l} \Big(\chi_{,1}\sigma_{ij}(u)\varepsilon_{ij}(v) - \sigma_{ij}(u)E_{ij}(\chi; v) - \Sigma_{ij}(\chi; u)\varepsilon_{ij}(v)\Big),$$

$$B[w](u,v) = \int\limits_{\Omega_l} w \cdot \Sigma_{ij}(\chi;u) E_{ij}(\chi;v).$$

Thus, $\widehat{u}^\varepsilon \in H^{1,0}(\Omega_l)$ is the unique solution to the problem (7).

Note that, for small ε, the uniform in ε estimate follows from (7):

$$\|\widehat{u}^\varepsilon\|_{H^1(\Omega_l)} \leq \text{const.} \tag{8}$$

Substituting $v = u$ in (2), one gets the evident relation

$$\int\limits_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(u) = \int\limits_{\Omega_l} f_i u_i. \tag{9}$$

3. Material derivatives

We seek for the global expansion of the solution $\widehat{u}^\varepsilon$ of (7) in the form

$$\widehat{u}^\varepsilon = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \overset{(n)}{u}, \quad \overset{(0)}{u} = u. \tag{10}$$

Following [18], the functions $\overset{(n)}{u}$, $n \geq 1$, are called the material derivatives of the order n of the solution. First, we write the series in ε of $1/J$,

$$\frac{1}{J} = \frac{1}{1+\varepsilon\chi_{,1}} = \sum_{n=0}^{\infty} \bigl(-\varepsilon\chi_{,1}\bigr)^n,$$

and multiply it with (10):

$$\frac{1}{J}\widehat{u}^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \left(\sum_{k=0}^{n} \frac{1}{k!} \bigl(-\chi_{,1}\bigr)^{n-k} \overset{(k)}{u} \right). \tag{11}$$

Second, in accordance with (5), by the infinite differentiability of f, one can deduce the representation

$$\widehat{f} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \chi^n \frac{\partial^n f}{\partial x_1^n}, \tag{12}$$

and therefore,

$$J\widehat{f} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left(\chi^n \frac{\partial^n f}{\partial x_1^n} + n\chi^{n-1}\chi_{,1} \frac{\partial^{n-1} f}{\partial x_1^{n-1}} \right) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left(\chi^n \frac{\partial^{n-1} f}{\partial x_1^{n-1}} \right)_{,1}. \tag{13}$$

We mean here the relation $(\partial^{-1}f/\partial x_1^{-1})_{,1} = f$ as $n = 0$. Let us now substitute (10), (11), (13) in (7) to obtain formally that

$$\sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int\limits_{\Omega_l} \sigma_{ij}(\overset{(n)}{u})\varepsilon_{ij}(v) + \sum_{n=0}^{\infty} \frac{\varepsilon^{n+1}}{n!} A(\overset{(n)}{u}, v)$$

$$+ \sum_{n=0}^{\infty} \frac{\varepsilon^{n+2}}{n!} \sum_{k=0}^{n} \frac{n!}{k!} B\Big[\big(-\chi_{,1}\big)^{n-k}\Big](\overset{(k)}{u}, v) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int\limits_{\Omega_l} \left(\chi^n \frac{\partial^{n-1} f_i}{\partial x_1^{n-1}}\right)_{,1} v_i,$$

or, the same

$$\sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \Bigg(\int\limits_{\Omega_l} \sigma_{ij}(\overset{(n)}{u})\varepsilon_{ij}(v) + nA(\overset{(n-1)}{u}, v) \tag{14}$$

$$+n(n-1) \sum_{k=0}^{n-2} \frac{(n-2)!}{k!} B\Big[\big(-\chi_{,1}\big)^{n-2-k}\Big](\overset{(k)}{u}, v) - \int\limits_{\Omega_l} \left(\chi^n \frac{\partial^{n-1} f_i}{\partial x_1^{n-1}}\right)_{,1} v_i \Bigg) = 0.$$

We should define the functions $\overset{(n)}{u} \in H^{1,0}(\Omega_l)$ as the unique solutions of the following iterative problems

$$\int\limits_{\Omega_l} \sigma_{ij}(\overset{(n)}{u})\varepsilon_{ij}(v) = \int\limits_{\Omega_l} \left(\chi^n \frac{\partial^{n-1} f_i}{\partial x_1^{n-1}}\right)_{,1} v_i - nA(\overset{(n-1)}{u}, v) \tag{15}$$

$$-n(n-1) \sum_{k=0}^{n-2} \frac{(n-2)!}{k!} B\Big[\big(-\chi_{,1}\big)^{n-2-k}\Big](\overset{(k)}{u}, v) \quad \forall v \in H^{1,0}(\Omega_l),$$

with the initial value $\overset{(0)}{u} = u$, where u is the solution of (2). For example, for $n = 1$:

$$\int\limits_{\Omega_l} \sigma_{ij}(\dot{u})\varepsilon_{ij}(v) = \int\limits_{\Omega_l} (\chi f_i)_{,1} v_i - A(u, v); \tag{16}$$

for $n = 2$:

$$\int\limits_{\Omega_l} \sigma_{ij}(\ddot{u})\varepsilon_{ij}(v) = \int\limits_{\Omega_l} \big(\chi^2 f_{i,1}\big)_{,1} v_i - 2A(\dot{u}, v) - 2B[1](u, v); \tag{17}$$

and so on.

Subtracting (2) from (7), we write

$$\int\limits_{\Omega_l} \sigma_{ij}(\widehat{u}^\varepsilon - u)\varepsilon_{ij}(v) = \int\limits_{\Omega_l} \big(\widehat{f}_i - f_i + \varepsilon\chi_{,1}\widehat{f}_i\big) v_i - \varepsilon A(\widehat{u}^\varepsilon, v) - \varepsilon^2 B\big[\frac{1}{J}\big](\widehat{u}^\varepsilon, v).$$

One can substitute $v = \widehat{u}^\varepsilon - u$ here and easily deduce, due to (8), (12) and Korn's inequality, that

$$\|\widehat{u}^\varepsilon - u\|_{H^1(\Omega_l)} \le c\varepsilon. \tag{18}$$

Analogously, subtracting (2) and the corresponding equations in (15) from (7), we can write in view of the decomposition (14) the next relation

$$\int\limits_{\Omega_l} \sigma_{ij}\Big(\widehat{u}^\varepsilon - \sum_{k=0}^{n} \frac{\varepsilon^k}{k!} \overset{(k)}{u}\Big)\varepsilon_{ij}(v) = \int\limits_{\Omega_l} \Big(\Big(\widehat{f}_i - \sum_{k=0}^{n} \frac{\varepsilon^k}{k!}\chi^k \frac{\partial^k f_i}{\partial x_1^k}\Big)$$

$$+\varepsilon\chi_{,1}\Big(\widehat{f}_i - \sum_{k=0}^{n-1} \frac{\varepsilon^k}{k!}\chi^k \frac{\partial^k f_i}{\partial x_1^k}\Big)\Big)v_i - \varepsilon A\big(\widehat{u}^\varepsilon - \sum_{k=0}^{n-1} \frac{\varepsilon^k}{k!} \overset{(k)}{u}, v\big)$$

$$-\varepsilon^2 \sum_{m=0}^{n-2} B\Big[\big(-\varepsilon\chi_{,1}\big)^{n-2-m}\Big]\big(\widehat{u}^\varepsilon - \sum_{k=0}^{m} \frac{\varepsilon^k}{k!} \overset{(k)}{u}, v\big) - \varepsilon^{n+1} B\Big[\frac{1}{J}\big(-\chi_{,1}\big)^{n-1}\Big](\widehat{u}^\varepsilon, v)$$

and substitute $v = \widehat{u}^\varepsilon - \sum\limits_{k=0}^{n} \frac{\varepsilon^k}{k!} \overset{(k)}{u}$ as a test function to obtain, due to (8) and (18), the estimates for $n > 0$:

$$\|\widehat{u}^\varepsilon - \sum_{k=0}^{n} \frac{\varepsilon^k}{k!} \overset{(k)}{u}\|_{H^1(\Omega_l)} \le c\varepsilon^{n+1}. \tag{19}$$

Thus, the following theorem is proved.

Theorem 1. *There exist the material derivatives $\dot{u}$, $\ddot{u}$,..., $\overset{(n)}{u}$,... given as the solutions of the problems* (15)*, such that the expansion* (10) *holds with the estimate* (19) *for any $n \ge 0$.*

The integration by parts in (15)–(17) is meaningful because of the local H^2-smoothness of the solutions in $B_\delta \setminus B_{\delta/2}$, and $\chi \equiv 1$ in the neighborhood $B_{\delta/2}$ of the considered crack tip, $\chi \equiv 0$ outside B_δ. Therefore it implies with (3) the following boundary value problems for the material derivatives. For $n = 1$, using (3):

$$-\sigma_{ij,j}(\dot{u}) = \chi f_{i,1} - \chi_{,j}\sigma_{ij,1}(u) - \Sigma_{ij,j}(\chi; u) = -\sigma_{ij,j}(\chi u_{,1}), \quad i = 1,2, \quad \text{in } \Omega_l,$$

$$\sigma_{12}(\dot{u}) = \mu\big(\chi_{,2}u_{1,1} - \chi_{,1}u_{1,2}\big), \quad \sigma_{22}(\dot{u}) = (\lambda + 2\mu)\big(\chi_{,2}u_{2,1} - \chi_{,1}u_{2,2}\big) \text{ on } \Gamma_l^\pm,$$

$$\dot{u} = 0 \quad \text{on } \Gamma;$$

for $n = 2$:

$$-\sigma_{ij,j}(\ddot{u}) = \big(\chi^2 f_{i,1}\big)_{,1} + 2\big(\chi_{,1}\sigma_{ij,j}(\dot{u}) - \chi_{,j}\sigma_{ij,1}(\dot{u}) - \Sigma_{ij,j}(\chi; \dot{u})\big)$$

$$+2\Big(\chi_{,j}\Sigma_{ij}(\chi;u)\Big)_{,1}, \quad i=1,2, \quad \text{in } \Omega_l,$$

$$\sigma_{12}(\ddot{u}) = 2\mu\big(\chi_{,2}\dot{u}_{1,1} - \chi_{,1}\dot{u}_{1,2}\big), \quad \sigma_{22}(\ddot{u}) = 2(\lambda+2\mu)\big(\chi_{,2}\dot{u}_{2,1} - \chi_{,1}\dot{u}_{2,2}\big) \text{ on } \Gamma_l^{\pm},$$

$$\ddot{u} = 0 \quad \text{on } \Gamma;$$

and for the arbitrary n:

$$-\sigma_{ij,j}(\overset{(n)}{u}) = \Big(\chi^n \frac{\partial^{n-1} f_i}{\partial x_1^{n-1}}\Big)_{,1} + n\big(\chi_{,1}\sigma_{ij,j}(\overset{(n-1)}{u}) - \chi_{,j}\sigma_{ij,1}(\overset{(n-1)}{u}) - \Sigma_{ij,j}(\chi;\overset{(n-1)}{u})\big)$$

$$+n(n-1)\sum_{k=0}^{n-2}\frac{(n-2)!}{k!}\Big[\big(-\chi_{,1}\big)^{n-2-k}\chi_{,j}\Sigma_{ij}(\chi;\overset{(k)}{u})\Big]_{,1}, \quad i=1,2, \quad \text{in } \Omega_l,$$

$$\sigma_{12}(\overset{(n)}{u}) = n\mu\big(\chi_{,2}\overset{(n-1)}{u}_{1,1} - \chi_{,1}\overset{(n-1)}{u}_{1,2}\big), \tag{20}$$

$$\sigma_{22}(\overset{(n)}{u}) = n(\lambda+2\mu)\big(\chi_{,2}\overset{(n-1)}{u}_{2,1} - \chi_{,1}\overset{(n-1)}{u}_{2,2}\big) \quad \text{on } \Gamma_l^{\pm},$$

$$\overset{(n)}{u} = 0 \quad \text{on } \Gamma.$$

One can see from the above that, if $u \in [H^2(\Omega_l)]^2$ then $\dot{u} = \chi u_{,1}$, if $u \in [H^{n+1}(\Omega_l)]^2$ then $\overset{(n)}{u} = \chi^n \partial^n u/\partial x_1^n$, $n \in \mathbf{N}$, that corresponds to the representation (12).

4. Derivatives of the energy functional

Let us substitute the solution u of (2) in (1) and define the potential energy as the function $\mathcal{P}:\,(0,l_0) \to \mathbf{R}$ depending on the crack length l. In view of (9), this function has the form

$$\mathcal{P}(l) = -\frac{1}{2}\int\limits_{\Omega_l} f_i u_i. \tag{21}$$

For the solution u^ε of the equation (4), applying the transformation (5), we also get

$$\mathcal{P}(l+\varepsilon) = -\frac{1}{2}\int\limits_{\Omega_{l+\varepsilon}} f_i u_i^\varepsilon = -\frac{1}{2}\int\limits_{\Omega_l} J\widehat{f}_i\widehat{u}_i^\varepsilon. \tag{22}$$

One can substitute the representations (10), (13) in (22) to obtain the formula

$$\mathcal{P}(l+\varepsilon) = -\frac{1}{2}\sum_{n=0}^{\infty}\varepsilon^n\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\int\limits_{\Omega_l}\left(\chi^{n-k}\frac{\partial^{n-k-1} f_i}{\partial x_1^{n-k-1}}\right)_{,1}\overset{(k)}{u}_i.$$

Therefore, thanks to Theorem 1, using (21) we get the expansion

$$\mathcal{P}(l+\varepsilon)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{P}^{(n)}(l),\quad \mathcal{P}^{(0)}(l)=\mathcal{P}(l),\tag{23}$$

where the derivatives of $\mathcal{P}$ at the point l are as follows:

$$\mathcal{P}^{(n)}(l)=-\frac{1}{2}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\int\limits_{\Omega_l}\left(\chi^{n-k}\frac{\partial^{n-k-1}f_i}{\partial x_1^{n-k-1}}\right)_{,1}\overset{(k)}{u}_i.\tag{24}$$

In particular, we have

$$\mathcal{P}'(l)=-\frac{1}{2}\int\limits_{\Omega_l}\big((\chi f_i)_{,1}u_i+f_i\dot{u}_i\big),\tag{25}$$

$$\mathcal{P}''(l)=-\frac{1}{2}\int\limits_{\Omega_l}\Big(\big(\chi^2 f_{i,1}\big)_{,1}u_i+2(\chi f_i)_{,1}\dot{u}_i+f_i\ddot{u}_i\Big).$$

Using equations (15)–(17), we can reduce the order n of the material derivatives in (24) to $n-1$. Indeed, substitute $v=\dot{u}$ in (2) and $v=u$ in (16), then

$$\int\limits_{\Omega_l}f_i\dot{u}_i=\int\limits_{\Omega_l}\sigma_{ij}(\dot{u})\varepsilon_{ij}(u)=\int\limits_{\Omega_l}(\chi f_i)_{,1}u_i-A(u,u),$$

and therefore (25) takes the form

$$\mathcal{P}'(l)=-\int\limits_{\Omega_l}(\chi f_i)_{,1}u_i+\frac{1}{2}A(u,u)\tag{26}$$

$$=\int\limits_{\Omega_l}\Big[-(\chi f_i)_{,1}u_i+\Big(\frac{1}{2}\chi_{,1}\sigma_{ij}(u)-\Sigma_{ij}(\chi;u)\Big)\varepsilon_{ij}(u)\Big].$$

Analogously, the substitution of $v=\overset{(n)}{u}$ in (2) and of $v=u$ in (15) gives

$$\int\limits_{\Omega_l}f_i\overset{(n)}{u}_i=\int\limits_{\Omega_l}\sigma_{ij}(\overset{(n)}{u})\varepsilon_{ij}(u)=\int\limits_{\Omega_l}\left(\chi^n\frac{\partial^{n-1}f_i}{\partial x_1^{n-1}}\right)_{,1}u_i-nA(\overset{(n-1)}{u},u)\tag{27}$$

$$-n(n-1)\sum_{k=0}^{n-2}\frac{(n-2)!}{k!}B\Big[\big(-\chi_{,1}\big)^{n-2-k}\Big](\overset{(k)}{u},u).$$

Moreover, let us take $v = \overset{(n-1)}{u}$ in (16), then

$$n \int_{\Omega_l} \sigma_{ij}(\overset{(n-1)}{u})\varepsilon_{ij}(\dot{u}) = n \int_{\Omega_l} (\chi f_i)_{,1} \overset{(n-1)}{u}{}_i - nA(u, \overset{(n-1)}{u}). \tag{28}$$

Combining (24), (27), (28) together, we deduce the following formula

$$\begin{aligned} \mathcal{P}^{(n)}(l) = &- \int_{\Omega_l} \left(\chi^n \frac{\partial^{n-1} f_i}{\partial x_1^{n-1}} \right)_{,1} u_i \\ &- \frac{1}{2} \sum_{k=1}^{n-2} \frac{n!}{(n-k)!k!} \int_{\Omega_l} \left(\chi^{n-k} \frac{\partial^{n-k-1} f_i}{\partial x_1^{n-k-1}} \right)_{,1} \overset{(k)}{u}{}_i \\ &- \frac{n}{2} \int_{\Omega_l} \sigma_{ij}(\overset{(n-1)}{u})\varepsilon_{ij}(\dot{u}) + \frac{n(n-1)}{2} \sum_{k=0}^{n-2} \frac{(n-2)!}{k!} B\Big[\big(-\chi_{,1}\big)^{n-2-k} \Big](\overset{(k)}{u}, u). \end{aligned} \tag{29}$$

In particular, for $n = 2$:

$$\begin{aligned} \mathcal{P}''(l) &= \int_{\Omega_l} \Big(-\big(\chi^2 f_{i,1}\big)_{,1} u_i - \sigma_{ij}(\dot{u})\varepsilon_{ij}(\dot{u}) \Big) + B[1](u,u) \\ &= \int_{\Omega_l} \Big(-\big(\chi^2 f_{i,1}\big)_{,1} u_i - \sigma_{ij}(\dot{u})\varepsilon_{ij}(\dot{u}) + \Sigma_{ij}(\chi; u) E_{ij}(\chi; u) \Big). \end{aligned}$$

Theorem 2. *There exist the derivatives of the potential energy functional with respect to the crack length in expansion* (23) *given by formulas* (24) *or* (29).

Note that integrals in (21) and (22) do not depend on χ, therefore all values of the derivatives in expansion (23) are also independent on the cut-off function.

5. Stress intensity factors

In fracture mechanics the crack propagation is usually interpreted with the help of the stress intensity factors (SIF). We remind its definition in connection with the material derivatives considered.

Introduce the local polar coordinates in a neighborhood of the crack tip $(l, 0)$ as

$$x_1 - l = r\cos\phi, \quad x_2 = r\sin\phi, \quad r \geq 0, \quad |\phi| \leq \pi.$$

The crack faces $\Gamma_l^\pm$ correspond to the values $\phi = \pm\pi$, respectively. The normal $n = (n_1, n_2)$ to the boundary of the circle B_δ centered in $(l, 0)$ is $(\cos\phi, \sin\phi)$. Let us define the smooth vector-functions Φ^1, Φ^2, which in the Descartes basis (x_1, x_2) are given as follows:

$$\Phi^1(\phi) = \big((2\kappa - 1)\cos\frac{\phi}{2} - \cos\frac{3\phi}{2}, (2\kappa + 1)\sin\frac{\phi}{2} - \sin\frac{3\phi}{2}\big),$$

$$\Phi^2(\phi) = \big((2\kappa + 3)\sin\frac{\phi}{2} + \sin\frac{3\phi}{2}, -(2\kappa - 3)\cos\frac{\phi}{2} - \cos\frac{3\phi}{2}\big), \quad \kappa = \frac{\lambda + 3\mu}{\lambda + \mu}.$$

The functions Φ^m, $m = 1, 2$, possess the properties $\sigma_{ij,j}(\sqrt{r}\Phi^m) = 0$ as $|\phi| < \pi$ and $\sigma_{i2}(\sqrt{r}\Phi^m) = 0$ as $\phi = \pm\pi$, $i = 1, 2$. Assume also that the cut-off function χ depends on r only. It is well known that the solution u of the Lamé system (2) admits the unique representation in the form (see [7]):

$$u = \chi(r)\sqrt{r}K_m\Phi^m(\phi) + w, \quad w \in [H^2(\Omega_l)]^2. \tag{30}$$

The coefficients K_1, K_2 here are called SIF. To obtain them, construct the functions following the idea of [15]:

$$\zeta^m = \chi(\sqrt{r}\Phi^m)_{,1} + V^m, \quad m = 1, 2, \tag{31}$$

where

$$(\sqrt{r}\Phi^1)_{,1} = \frac{1}{2\sqrt{r}}\big((2\kappa - 3)\cos\frac{\phi}{2} + \cos\frac{5\phi}{2}, -(2\kappa + 3)\sin\frac{\phi}{2} + \sin\frac{5\phi}{2}\big),$$

$$(\sqrt{r}\Phi^2)_{,1} = \frac{1}{2\sqrt{r}}\big(-(2\kappa + 1)\sin\frac{\phi}{2} - \sin\frac{5\phi}{2}, -(2\kappa - 1)\cos\frac{\phi}{2} + \cos\frac{5\phi}{2}\big),$$

and $V^m \in H^{1,0}(\Omega_l)$ are the unique solutions of the problems

$$\int\limits_{\Omega_l} \sigma_{ij}(V^m)\varepsilon_{ij}(v) = A(\sqrt{r}\Phi^m, v) \quad \forall v \in H^{1,0}(\Omega_l), \quad m = 1, 2. \tag{32}$$

In view of the properties $\sigma_{ij,j}\big((\sqrt{r}\Phi^m)_{,1}\big) = 0$ as $|\phi| < \pi$ and $\sigma_{i2}\big((\sqrt{r}\Phi^m)_{,1}\big) = 0$ as $\phi = \pm\pi$, $i = 1, 2$, by the local regularity of the constructed functions outside B_δ, one can see from (31), (32) that ζ^m, $m = 1, 2$, are the nontrivial functions satisfying the homogeneous relations

$$\begin{gathered} -\sigma_{ij,j}(\zeta^m) = 0, \quad i = 1, 2, \quad \text{in } \Omega_l, \\ \sigma_{i2}(\zeta^m) = 0, \quad i = 1, 2, \quad \text{on } \Gamma_l^\pm, \quad \zeta^m = 0 \quad \text{on } \Gamma. \end{gathered} \tag{33}$$

Again, because ζ^m are of the H^1-class outside a neighborhood of the crack tip $(l,0)$, one can apply the Green formula in the domain $\Omega_l \setminus \overline{B}_t$, $0<t<\delta/2$. Using (3) and (33), we get then

$$\int\limits_{\Omega_l\setminus B_t} f_i\zeta_i^m = -\int\limits_{\Omega_l\setminus B_t} \sigma_{ij,j}(u)\zeta_i^m = \int\limits_{\partial B_t} \big(\sigma_{ij}(u)n_j\zeta_i^m - \sigma_{ij}(\zeta^m)n_j u_i\big). \tag{34}$$

The substitution of (30), (31) in the right hand-side of (34) reduces it to the relation

$$\int\limits_{\Omega_l\setminus B_t} f_i\zeta_i^m = -\alpha K_m + I_m(t), \quad \alpha = \frac{16\pi\mu(\lambda+2\mu)}{\lambda+\mu},$$

where the the last term denotes the integral

$$I_m(t) = t\int\limits_{|\phi|<\pi} \Big(\sigma_{ij}\big(\sqrt{r}(K_1\Phi^1+K_2\Phi^2)\big)n_jV_i^m + \sigma_{ij}(w)n_j\big((\sqrt{r}\Phi_i^m)_{,1}+V_i^m\big)$$
$$-\sigma_{ij}\big((\sqrt{r}\Phi^m)_{,1}\big)n_jw_i - \sigma_{ij}(V^m)n_j\big(\sqrt{r}(K_1\Phi_i^1+K_2\Phi_i^2)+w_i\big)\Big)\Big|_{r=t}\,d\phi.$$

One can see that $I_m(t)\to 0$ as $t\to 0$, $m=1,2$. Therefore, the passing to the limit in (34) as $t\to 0$ leads finally to the relation

$$K_m = -\frac{1}{\alpha}\int\limits_{\Omega_l} f_i\zeta_i^m, \quad m=1,2.$$

For the material derivatives $\overset{(n)}{u}$ as the solutions of the Lamé problem (15), the same as (30) representation is valid for $n\ge 1$,

$$\overset{(n)}{u} = \chi(r)\sqrt{r}\,\overset{(n)}{K}_m\Phi^m(\phi) + \overset{(n)}{w}, \quad \overset{(n)}{w}\in[H^2(\Omega_l)]^2. \tag{35}$$

To find the corresponding SIF $\overset{(n)}{K}_1$, $\overset{(n)}{K}_2$, we use the same arguments as for (34) with the next Green formula for non-homogeneous Neumann boundary conditions at $\Gamma_l^\pm$,

$$-\int\limits_{\Omega_l\setminus B_t} \sigma_{ij,j}(\overset{(n)}{u})\zeta_i^m = \int\limits_{\partial B_t}\big(\sigma_{ij}(\overset{(n)}{u})n_j\zeta_i^m - \sigma_{ij}(\zeta^m)n_j\overset{(n)}{u}_i\big) + \int\limits_{\Gamma_l\setminus B_t} [\![\sigma_{i2}(\overset{(n)}{u})\zeta_i^m]\!],$$

where the thick brackets denote the jump at Γ_l, i.e. $[\![u]\!] = u|_{\Gamma_l^+} - u|_{\Gamma_l^-}$. In view of (20), the analogous formula hold for finding $\overset{(n)}{K}_m$:

$$\overset{(n)}{K}_m = -\frac{1}{\alpha}\int\limits_{\Omega_l}\bigg(\Big(\chi^n\frac{\partial^{n-1}f_i}{\partial x_1^{n-1}}\Big)_{,1} + n\big(\chi_{,1}\sigma_{ij,j}(\overset{(n-1)}{u}) - \chi_{,j}\sigma_{ij,1}(\overset{(n-1)}{u}) - \Sigma_{ij,j}(\chi;\overset{(n-1)}{u})$$

$$+n(n-1)\sum_{k=0}^{n-2}\frac{(n-2)!}{k!}\Big[\big(-\chi_{,1}\big)^{n-2-k}\chi_{,j}\Sigma_{ij}(\chi;\overset{(k)}{u})\Big]_{,1}\Big)\zeta_i^m \tag{36}$$

$$+\frac{n}{\alpha}\int_{\Gamma_l}[\![\mu\big(\chi_{,2}\overset{(n-1)}{u}_{1,1}-\chi_{,1}\overset{(n-1)}{u}_{1,2}\big)\zeta_1^m+(\lambda+2\mu)\big(\chi_{,2}\overset{(n-1)}{u}_{2,1}-\chi_{,1}\overset{(n-1)}{u}_{2,2}\big)\zeta_2^m]\!].$$

In particular, (36) as $n=1$ is of the form:

$$\dot{K}_m=\frac{1}{\alpha}\int_{\Omega_l}\sigma_{ij,j}\big(\chi u_{,1}\big)\zeta_i^m$$

$$+\frac{1}{\alpha}\int_{\Gamma_l}[\![\mu(\chi_{,2}u_{1,1}-\chi_{,1}u_{1,2})\zeta_1^m+(\lambda+2\mu)(\chi_{,2}u_{2,1}-\chi_{,1}u_{2,2})\zeta_2^m]\!].$$

Summing together (30) and (35) with the corresponding multipliers, thanks to Theorem 1, we deduce the following representation to the solution $\widehat{u}^\varepsilon$ of (7),

$$\widehat{u}^\varepsilon=\chi(r)\sqrt{r}K_m^\varepsilon\Phi^m(\phi)+W,\quad W\in[H^2(\Omega_l)]^2, \tag{37}$$

with

$$K_m^\varepsilon=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\overset{(n)}{K}_m,\quad \overset{(0)}{K}_m=K_m,\quad m=1,2, \tag{38}$$

where $\overset{(n)}{K}_m$ are given by (36). Because the transformation (5) does not change the constant SIF, the same formula is also true for the solution u^ε of (4),

$$u^\varepsilon=\chi\sqrt{r_\varepsilon}K_m^\varepsilon\Phi^m(\phi_\varepsilon)+w^\varepsilon,\quad w^\varepsilon\in[H^2(\Omega_{l+\varepsilon})]^2, \tag{39}$$

with the coefficients K_1^ε, K_2^ε from (38) and the polar coordinates $(r_\varepsilon,\phi_\varepsilon)$ at the crack tip $(0,l+\varepsilon)$.

6. Derivatives of SIF

Let us define the vector-function $\mathcal{K}:\ (0,l_0)\to\mathbf{R}^2$ by the equality $\mathcal{K}(l)=(K_1,K_2)$ with K_m, $m=1,2$, from (30). It follows from (39) that $\mathcal{K}(l+\varepsilon)=(K_1^\varepsilon,K_2^\varepsilon)$, and, by (38), we get the asymptotic formula

$$\mathcal{K}(l+\varepsilon)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{K}^{(n)}(l),\quad \mathcal{K}^{(0)}(l)=\mathcal{K}(l),\quad \mathcal{K}^{(n)}(l)=(\overset{(n)}{K}_1,\overset{(n)}{K}_2), \tag{40}$$

with the derivatives $\mathcal{K}^{(n)}(l)$ of the order n at the point l of SIF. We formulate this result as the following theorem.

Theorem 3. *There exist the derivatives of SIF with respect to the crack length in expansion* (40) *given by the formula* (36).

Let us now rewrite the derivatives of the potential energy using the derivatives of SIF in the following way. First, by the smoothness of the solution u of (2) outside a neighborhood of the crack tip $(l, 0)$, we can integrate (26) by parts in $\Omega_l \setminus \overline{B}_{\delta/2}$ to obtain that

$$\mathcal{P}'(l) = \int\limits_{\Omega_l \setminus B_{\delta/2}} \chi(\sigma_{ij,j}(u) + f_i)u_{i,1} + \int\limits_{\Gamma_l \setminus B_{\delta/2}} \chi[\![\sigma_{i2}(u)u_{i,1}]\!]$$

$$+ \int\limits_{\partial B_{\delta/2}} \Big(-\frac{1}{2}\sigma_{ij}(u)\varepsilon_{ij}(u)n_1 + \sigma_{ij}(u)n_j u_{i,1} + f_i u_i n_1\Big) + \int\limits_{B_{\delta/2}\setminus\Gamma_l} f_i u_{i,1} - \int\limits_{\partial B_{\delta/2}} f_i u_i n_1$$

$$= \int\limits_{B_{\delta/2}\setminus\Gamma_l} f_i u_{i,1} + \int\limits_{\partial B_{\delta/2}} \big(\sigma_{ij}(u)n_j u_{i,1} - \frac{1}{2}\sigma_{ij}(u)\varepsilon_{ij}(u)n_1\big)$$

due to the relations (3). Again, substituting the representation (30) in the last formula, we deduce

$$\mathcal{P}'(l) = -\frac{8\mu(\lambda+2\mu)}{\lambda+\mu} \int\limits_{|\phi|<\pi} \big(K_1^2 \sin^2\phi + K_2^2 \cos^2\phi\big)\, d\phi + I(\delta), \tag{41}$$

where

$$I(\delta) = \frac{\delta}{2} \int\limits_{|r|<\frac{\delta}{2}, |\phi|<\pi} f_i u_{i,1}\, dr\, d\phi + \frac{\delta}{2} \int\limits_{|\phi|<\pi} \Big(\sigma_{ij}\big(\sqrt{r}(K_1\Phi^1 + K_2\Phi^2)\big)n_j w_{i,1}$$

$$+\sigma_{ij}(w)n_j\big(\sqrt{r}(K_1\Phi_i^1 + K_2\Phi_i^2) + w_i\big)_{,1} - \frac{1}{2}\sigma_{ij}\big(\sqrt{r}(K_1\Phi^1 + K_2\Phi^2)\big)\varepsilon_{ij}(w)n_1$$

$$-\frac{1}{2}\sigma_{ij}(w)\varepsilon_{ij}\big(\sqrt{r}(K_1\Phi^1 + K_2\Phi^2) + w\big)n_1\Big)\Big|_{r=\frac{\delta}{2}}\, d\phi.$$

One can easily see that $I(\delta) \to 0$ as $\delta \to 0$. Because the derivative $\mathcal{P}'(l)$ does not depend on δ, we can pass to the limit in (41) as $\delta \to 0$ and obtain

$$\mathcal{P}'(l) = -\frac{\alpha}{2}\mathcal{K}^2(l), \quad \mathcal{K}^2(l) = K_1^2 + K_2^2. \tag{42}$$

By Theorems 2 and 3, using the representations (23) and (40), it follows from (42) that

$$\mathcal{P}'(l+\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\mathcal{P}^{(n+1)}(l) = -\frac{\alpha}{2}\mathcal{K}^2(l+\varepsilon) = -\frac{\alpha}{2}\Big(\mathcal{K}_1^2(l+\varepsilon) + \mathcal{K}_2^2(l+\varepsilon)\Big)$$

$$= -\frac{\alpha}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{\varepsilon^n}{k!(n-k)!}\Big(\mathcal{K}_1^{(k)}(l)\mathcal{K}_1^{(n-k)}(l)+\mathcal{K}_2^{(k)}(l)\mathcal{K}_2^{(n-k)}(l)\Big)\right).$$

The comparison of the terms with the similar power of ε leads to

$$\mathcal{P}^{(n+1)}(l) = -\frac{\alpha}{2}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\Big(\mathcal{K}_1^{(k)}(l)\mathcal{K}_1^{(n-k)}(l)+\mathcal{K}_2^{(k)}(l)\mathcal{K}_2^{(n-k)}(l)\Big). \tag{43}$$

In particular, for $n=1$, formulas (43) together with (38) give

$$\mathcal{P}''(l) = -\alpha\big(K_1\dot{K}_1 + K_2\dot{K}_2\big)$$

and so on. Thus, the next theorem follows.

Theorem 4. *The derivatives of the potential energy with respect to the crack length in expansion* (23) *can be calculated with the help of the formulas* (42), (43), *where SIF are taken from* (36).

7. Locally optimal cracks

By adding the surface energy to (1), let us introduce the function $\mathcal{T}:\,(0,l_0)\to\mathbf{R}$ of the total potential energy as

$$\mathcal{T}(l) = \mathcal{P}(l) + \gamma\,\mathrm{meas}\,\Gamma_l, \quad \gamma>0, \quad \mathrm{meas}\,\Gamma_l = l.$$

Then we get from Theorem 2 and (23) that

$$\mathcal{T}(l+\varepsilon) = \mathcal{T}(l) + \varepsilon\big(\gamma+\mathcal{P}'(l)\big) + \sum_{n=2}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{P}^{(n)}(l). \tag{44}$$

By the Griffith criterion of the crack propagation, the total potential energy turns out to be minimal. The extremality condition of (44) in ε implies

$$0 = \frac{d\mathcal{T}}{d\varepsilon} = \gamma + \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{P}^{(n+1)}(l).$$

In particular, we can use the linear in ε condition

$$\gamma + \mathcal{P}'(l) + \varepsilon\,\mathcal{P}''(l) = 0 \tag{45}$$

to seek the locally optimal crack length $l+\varepsilon$ minimizing the quadratic approximation of the total potential energy:

$$\mathcal{T}(l+\varepsilon) \approx \mathcal{T}(l) + \varepsilon\big(\gamma+\mathcal{P}'(l)\big) + \frac{\varepsilon^2}{2}\mathcal{P}''(l) \equiv T(\varepsilon). \tag{46}$$

Minimum in (46) is provided by the positiveness of the second derivative,

$$\mathcal{P}''(l) > 0. \tag{47}$$

We formulate the quasi-static model of the local crack propagation under the assumption (47). Let $t \geq 0$ be a loading parameter. We consider the load $f(t) = tf$, that leads to the fact that $u(t) = tu$, $\overset{(n)}{u}(t) = t\overset{(n)}{u}$, $n = 1, 2, ...$, are the solutions of equilibrium problems (2), (15), respectively. Therefore, the multiplier t^2 appears in derivatives of the potential energy given by (26), (29), and the functional (46) reduces to

$$T_t(\varepsilon) \equiv \mathcal{T}(l) + \varepsilon\big(\gamma + t^2\mathcal{P}'(l)\big) + \frac{\varepsilon^2}{2}t^2\mathcal{P}''(l).$$

We look for the propagating crack, i.e. for the value $\varepsilon(t) \geq 0$ of the crack length $l + \varepsilon(t)$ minimizing T_t for each $t \geq 0$. Due to the positiveness requirement constrained $\varepsilon(t)$, instead of (45) we obtain the algebraic variational inequality

$$\varepsilon(t) \geq 0, \quad \big(\gamma + t^2\mathcal{P}'(l) + t^2\mathcal{P}''(l)\varepsilon(t)\big)(\bar{\varepsilon} - \varepsilon(t)) \geq 0 \quad \forall \bar{\varepsilon} \geq 0. \tag{48}$$

If $\mathcal{P}'(l) \geq 0$, then $\varepsilon(t) \equiv 0$ is a solution of (48) because $\gamma > 0$, that means stationarity of the crack. For $\mathcal{P}'(l) < 0$, by the same reason we have

$$\varepsilon(t) = 0, \quad 0 \leq t \leq t_{\text{critical}}, \quad t_{\text{critical}} = \sqrt{\frac{\gamma}{-\mathcal{P}'(l)}}. \tag{49}$$

The crack growth starts only after reaching this critical value, and the solution of (48) is given then by

$$\varepsilon(t) = \frac{-\mathcal{P}'(l)}{\mathcal{P}''(l)} - \frac{\gamma}{\mathcal{P}''(l)} \cdot \frac{1}{t^2}, \quad t > t_{\text{critical}}, \tag{50}$$

which has the finite asymptotic $-\mathcal{P}'(l)/\mathcal{P}''(l)$ for t big. Thus, the continuous function $\varepsilon(t)$ written in (49), (50) shows us the quasi-static, local crack propagation.

8. Conclusion

For the 2D-problem describing solids with cracks, we have constructed the global expansion (10) of the solution with the material derivatives of an arbitrary order given by formulas (15) or (20), and their stress intensity factors – by (36). This allows us to find all the derivatives of the potential energy in respect to variation of the crack length with the help of formulas (24), or (29), or (43), and to describe the local crack propagation by the Griffith rupture criterion.

Note, for iterative calculations of the material derivatives or their SIF, in view of the above consideration one needs only to be able to find the solutions of the linear elasticity problem in the week sense

$$\int\limits_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(v) = F(v)$$

with the corresponding right-hand sides F.

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