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Problem of crack perturbation based on level sets and velocities

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We describe cracks with the help of a given velocity as zero-level sets of a non-negative function satisfying a transport equation. For smooth velocities this description is equivalent to the coordinate transformation of a domain containing the crack inside. Analytical examples of cracks described by smooth as well as discontinuous velocities are presented in 2D and 3D domains. Based on a level-set formulation we consider the crack perturbation problem subject to a non-penetration condition and derive the formula for the shape derivative.

1 Introduction

For the description of invertible movements of geometric objects with regular boundaries, classical techniques employing perturbations of the identity operator or (equivalently) the velocity method can be used, and sensitivities of geometry dependent functionals can be calculated [5, 14, 27]. Alternatively, level sets methods can be employed for the description of the movement of shape variables [2, 11, 21–23, 26]. These methods can also be used in a natural way to define shape perturbations and to obtain sensitivity results, as developed for instance in [3, 10]. The level set approach is based on equations of Hamilton-Jacobi type (the "level-set equation"), which under appropriate conditions have a unique, global in time, viscosity solution [4, 20]. This is in contrast to classical methods based on transformation of the shape which are generally local in time. The major advantage of the level set method, however, lies in its flexibility, which allows to treat topology changes for the geometric variable in a unified way.

In this work, we are especially interested in investigating geometric objects with cracks, i.e. objects with irregular boundaries like domains containing a crack in the interior. Problems of crack perturbation arise in the framework of fracture mechanics for determination of the propagation of cracks in solids [9, 25, 28, 29]. It is generally accepted that a local fracture criterion involves the energy release rate at the crack tip, which can be expressed as a shape derivative of a potential energy functional. Using shape sensitivity analysis for models in non-smooth domains with rectilinear cracks subject to inequality constraints, representations of the shape derivative were obtained in [12, 13]. In crack models without constraints the first and high-order derivatives for cracks with curvilinear shapes were derived in [15–17]. These considerations were based on perturbations of the identity operator and the corresponding coordinate transformation of domains with cracks.

In the present paper we use a level-set formulation for the description of cracks and their propagation. Standard level-set methods are not applicable here because, from a geometric viewpoint, the crack is a general object of zero measure with co-dimension one a.e. (e.g. an open curve, a branching surface, a T-junction, etc) but not necessarily the boundary of an open set (which always separates 'inside' from 'outside'). Henceforth, we define a (moving) crack as the zero-level set of a *non-negative* (time-dependent) function ρ. Such an approach was used in [1] to describe a mean-curvature flow of manifolds. If the crack moves along a given vector field V, and all data are sufficiently smooth, then $\rho(t, y)$ satisfies an equation of transport type [6]:

$$\rho_t + V^\top \nabla\rho = 0. \tag{T}$$

We shall utilize the implicit formulation (T) in two different ways. Suppose first that we are given a moving family of cracks Γ_t, $t \geq 0$. It is quite natural to define $\rho^d(t, y) = \operatorname{dist}(y, \Gamma_t)$ as the corresponding level set function for which

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$\Gamma_t = \{y : \rho^d(t,y) = 0\}$. Solving the algebraic equation

$$\rho_t^d + V^\top \nabla \rho^d = 0 \tag{A}$$

for V, a natural extension of the velocity vector field (which a-priori is defined only on Γ) onto a neighborhood of the moving crack is obtained. Such an extension is a necessary ingredient for any kind of shape sensitivity analysis either in the classical as well as in the level set context. Construction of extension velocities (A) for certain families of moving cracks, and the equivalence between the implicit formulation (T) and the propagation of the crack via coordinate transformation methods are proved in Sect. 2. The corresponding transformation function and its inverse are constructed as the solutions of non-linear ODEs and transport equations, respectively, where the method of characteristics is used to establish the connection. Next a localization procedure for the construction of the velocity field for cracks located inside bounded domains is carried out with the help of cut-off functions. We illustrate this construction in examples describing analytically curvilinear cracks and cracks with curvilinear fronts.

We treat discontinuous velocities on the basis of ℓ^p-monotonicity property and the existence theorem of [24]. In this case we give the analytical example of a discontinuous velocity V providing splitting of a crack at the bifurcation point into two branches, which are also with a kink. The analytical construction is justified numerically with a level-set based algorithm solving the transport equation (T). For more numerical results, see [19].

In Sect. 3 a crack perturbation problem for a constrained elastic model subject to a non-penetration condition is considered. Based on the above results we argue a formula for the shape derivative of the quadratic cost functional associated with the potential energy of a solid with a crack. In contrast to [13] we use a primal-dual variational formulation of the problem accounting curvilinear cracks, see [18]. This procedure allows a description of the shape derivative with reduced smoothness requirements for the velocity. The result of calculation of the derivatives of the energy functional for cracks represented by typical parametric curves and surfaces is presented in Sect. 4.

2 Cracks and their propagation based on level sets

In this section we give a level-set description for moving cracks with a given velocity. We show that the implicit level-set formulation is equivalent to the classical formulation of crack movement using invertible transformations if the velocity vector field is sufficiently smooth. The implicit formulation, however, is more general in the sense that it also makes sense for discontinuous velocity vector fields. In this situation, the cracks for different times need not be homeomorphic. Next, we present some illustrative examples for 1- and 2-dimensional cracks. Starting with families of moving cracks, we construct from the geometrical objects (the cracks) a time dependent level set function $\rho(t,y)$ which has — for each time instance — the prescribed crack as its zero level set. The velocity vector field is found by inserting the level set function ρ into the transport equation (T) and solving it for V. A localization procedure allows to consider the movement of bounded cracks. The section concludes with description of discontinuous velocities providing the splitting of a crack at the bifurcation point.

2.1 Cracks in $\mathbb{R}^N$

We start with the representation of a family of moving cracks in $\mathbb{R}^N (N = 2,3)$ as zero sets of a time-dependent *non-negative* level-set function $\rho(t,y)$. We shall construct ρ as the generalized solution to the Cauchy problem for a linear transport equation with a given velocity vector field V. The solution of the transport equation can be explicitly constructed using the method of characteristics. Moreover, the flow map $R(t,\cdot)$ for the characteristic equation provides a family of invertible transformations which is used to define the moving cracks as the transformed initial crack.

Let the crack Υ_0 be a compact subset of $\mathbb{R}^N$ with finite $(N-1)$-dimensional Hausdorff measure. Let us first choose a function $\rho_0 \in C_u^{0,1}(\mathbb{R}^N)$, where, with the subscript "u", we indicate global uniform Lipschitz continuity of the elements in the respective function space. We suppose that $\rho_0 \geq 0$ and

$$\Upsilon_0 = \{x \in \mathbb{R}^N :\ \rho_0(x) = 0\}, \quad \mathbb{R}^N \setminus \overline{\Upsilon}_0 = \{x \in \mathbb{R}^N :\ \rho_0(x) > 0\}. \tag{1}$$

For instance, we can take the distance $\rho_0(x) = \inf_{y \in \Upsilon_0} |x - y|$, which is Lipschitz continuous in $\mathbb{R}^N$ (see [5, Theorem 2.1, p. 154]). Next, we choose a velocity vector field

$$V = (V_1, ..., V_N)^\top \in C([0,\infty); C_u^{0,1}(\mathbb{R}^N))^N, \tag{2}$$

which describes the movement of the crack.

Let us consider the Cauchy problem for the transport equation:

$$\begin{cases} \frac{\partial \rho}{\partial t}(t,y) + V(t,y)^\top \nabla \rho(t,y) = 0, \\ \rho(0,y) = \rho_0(y). \end{cases} \tag{3}$$

Our aim is to use $\rho(t)$ as a level-set function to define co-dimension one sets similar to (1) but for $t > 0$. We first recall the method of characteristics thus getting a constructive representation of ρ.

Lemma 1. *For arbitrary* $T > 0$ *the Cauchy problem* (3) *has a unique solution* $\rho \in C_u^{0,1}((0,T)\times\mathbb{R}^N)^N$ *which satisfies* (3) *point-wise almost everywhere and in the distributional sense. Moreover, the solution has the form*

$$\rho(t,y) = \rho_0(R^{-1}(t,y)) \tag{4}$$

with an invertible mapping $R \in C^1([0,T];C_u^{0,1}(\mathbb{R}^N))^N$ *such that* $R^{-1} \in C_u^{0,1}((0,T)\times\mathbb{R}^N)^N$. *If* $\rho_0 \geq 0$ *we also obtain* $\rho \geq 0$.

Proof. We first recall proof of the existence of the solution ρ using the classical method of characteristics. In a second step, we prove the asserted smoothness of the transformation R^{-1}. We introduce the characteristic equations

$$\begin{cases} \frac{dR}{dt}(t) = V(t,R(t)), \\ R(0) = x. \end{cases} \tag{5}$$

Due to assumption (2) system (5) has the unique classical solution. Let $R:(0,T)\times\mathbb{R}^N \to \mathbb{R}^N$ denote the flow map with respect to (5). It can be proved (see [5, Theorem 4.1]) that $R(t,\cdot)$ is invertible for any t and

$$R \in C^1([0,T];C_u^{0,1}(\mathbb{R}^N))^N \text{ and } R^{-1} \in C([0,T];C_u^{0,1}(\mathbb{R}^N))^N. \tag{6}$$

We now define (according to (4))

$$\rho(t,y) = \rho_0(R^{-1}(t,y)), \tag{7}$$

which implies that

$$\rho(t,R(t,x)) = \rho(s,R(s,x)) = \rho_0(x), \tag{8}$$

i.e. ρ is constant along characteristics and

$$\rho \in C_u^{0,1}((0,T)\times\mathbb{R}^N))^N. \tag{9}$$

Differentiating (8) with respect to t, we find that ρ satisfies (3) almost everywhere on $(0,T)\times\mathbb{R}^N$ and in the distributional sense, too (see [7, Theorem 5, p. 131]).

Let us now consider the system of transport equations

$$\begin{cases} \frac{\partial S_i}{\partial t}(t,y) + V(t,y)^\top \nabla S_i(t,y) = 0 & , \quad i = 1,...,N, \\ S_i(0,y) = y_i & , \quad i = 1,...,N. \end{cases} \tag{10}$$

Applying the arguments above to the system (10) we obtain that there exists a solution S_i which is Lipschitz continuous with respect to both its variables. Since (by construction) S_i is constant along characteristics, we have

$$S_i(t,R(t,y)) = S_i(0,R(0,y)) = y_i,$$

or, if we set $S = (S_1,\ldots,S_n)^\top$, we obtain

$$S(t,R(t,y)) = y \text{ for all } y \in \mathbb{R}^N.$$

Thus, $S(t,\cdot) = R^{-1}(t,\cdot)$ and R^{-1} is Lipschitz continuous also in t and satisfies (10). □

The inverse function R^{-1} admits an increase of smoothness in t at the expense of a decrease of the corresponding smoothness in y. Indeed, let us consider the relation

$$y = R(t,R^{-1}(t,y)), \quad t \in (0,T). \tag{11}$$

By differentiating Eq. (11) with respect to y (note that this is admissible due to (6) and (9)) we obtain

$$I = \frac{\partial R}{\partial x}(t,R^{-1}(t,y))\frac{\partial R^{-1}}{\partial y}(t,y) \quad \text{a.e. } \mathbb{R}^{N\times N}. \tag{12}$$

On the other hand, let us differentiate formally Eq. (11) with respect to t:

$$0 = \frac{\partial R}{\partial t}(t, R^{-1}(t,y)) + \frac{\partial R}{\partial x}(t, R^{-1}(t,y)) \frac{\partial R^{-1}}{\partial t}(t,y). \tag{13}$$

Multiplying (12) by $\partial R^{-1}/\partial t$ and (13) by $\partial R^{-1}/\partial y$ and subtracting the two expressions yields

$$\frac{\partial R^{-1}}{\partial t}(t,y) = -\frac{\partial R^{-1}}{\partial y}(t,y)\frac{\partial R}{\partial t}(t, R^{-1}(t,y)) \in C([0,T]; L^\infty(\mathbb{R}^N))^N.$$

We therefore find:

$$R^{-1} \in C^1([0,T]; L^\infty(\mathbb{R}^N))^N.$$

Thus, we obtain an additional degree of regularity with respect to time at the expense of a decrease in spatial regularity.

Due to Lemma 1 we can take the solution $\rho(t) \geq 0$ of (3) as a level-set function to define the sets:

$$\Upsilon_t = \{y \in \mathbb{R}^N : \ \rho(t,y) = 0\}, \ \mathbb{R}^N \setminus \overline{\Upsilon}_t = \{y \in \mathbb{R}^N : \ \rho(t,y) > 0\}. \tag{14}$$

Theorem 2. *The zero level-sets* (14) *define a family of cracks* Υ_t *in* $\mathbb{R}^N$*, depending on the parameter* $t \in [0,T)$*, where* $R(t)$ *establishes a one-to-one correspondence between* Υ_t *and* Υ_0.

Proof. In view of (8) the solution R of problem (5) yields the direct mapping:

$$y = R(t,x): \quad \Upsilon_0 \ \to \ \Upsilon_t, \quad \mathbb{R}^N \setminus \overline{\Upsilon}_0 \ \to \ \mathbb{R}^N \setminus \overline{\Upsilon}_t. \tag{15}$$

Due to (7) the inverse function R^{-1} from (10) fulfills the inverse mapping:

$$x = R^{-1}(t,y): \quad \Upsilon_t \ \to \ \Upsilon_0, \quad \mathbb{R}^N \setminus \overline{\Upsilon}_t \ \to \ \mathbb{R}^N \setminus \overline{\Upsilon}_0. \tag{16}$$

Due to the one-to-one correspondence, Υ_t is an $(N-1)$-dimensional manifold in $\mathbb{R}^N$ with the same degree of smoothness as Υ_0. □

Thus, for a given velocity V, solutions of (5), (10), and (3) describe cracks in (1), (14) by the corresponding level sets and establish the coordinate transformation (15), (16) between them.

Conversely, given a coordinate transformation between a-priori known cracks one can determine the velocity to find a level-set function expressing these cracks as zero-level sets. Indeed, let the one-to-one coordinate transformation (15), (16) be given by functions R with $R(0,x) = x$ and R^{-1} such that the properties (6) and (9) hold true. For the velocity V defined as

$$V(t,y) = \frac{\partial R}{\partial t}(t, R^{-1}(t,y)), \tag{17}$$

the function R is then a solution to problem (5), and R^{-1} to (10). When $\rho_0 \geq 0$ describes the crack Υ_0 in (1), then by the arguments of Lemma 1 the level-set function $\rho(t) \geq 0$ obtained from (7) describes Υ_t by (14). This procedure is carried out for several instructive test examples in the following sections.

Let us stress the underdetermination of the velocity V corresponding to the fixed crack family Υ_t. Utilizing problems (5) and (10) for a function $\widetilde{V}$ different from V results in transformations $\widetilde{R}$ and $\widetilde{R}^{-1}$ different from R and R^{-1}. Thus given Υ_t the velocity is only determined on the crack and the specific extension away from the crack is not unique. This issue will be discussed in examples later.

Now we relate the above results to the propagation of cracks. In fact, for the initial crack Υ_0 fixed at $t = 0$ in (1), the velocity $V(t)$ known a-priori determines via (3) the crack Υ_t in (14) moving with respect to the time parameter t. At each t, the restriction of $V(t)$ from $\mathbb{R}^N$ onto the crack Υ_t determines its local direction of propagation, which can be decomposed into normal and tangential components. The former implies changes of the crack shape, and the latter describes its prolongation. In our consideration we make no assumptions on the velocity direction. Nevertheless, analysis of cracks in fracture mechanics deals with the propagation of cracks preserving the previous shape. Accounting this interest we will get examples of the crack propagation in the tangential direction, but general results remain true for arbitrary velocities.

2.2 Curvilinear cracks in $\mathbb{R}^2$

To illustrate the results of the previous section we present, as a first example, a level-set description of a family of curvilinear cracks in the plane (i.e. $N = 2$). Starting with the given family we construct analytically the solutions to (3), (5), (10), and the corresponding velocity.

Let a smooth function $\psi \in C_u^{1,1}(\mathbb{R})$ be given. Consider the family of left-side unbounded cracks Υ_t

$$\Upsilon_t = \{y = (y_1, y_2)^\top \in \mathbb{R}^2 : \quad y_1 \le t, \quad y_2 = \psi(y_1)\}, \quad t \ge 0.$$

We describe them as level sets

$$\Upsilon_t = \{y \in \mathbb{R}^2 : \quad \rho(t, y) = 0\}$$

with the level-set function

$$\rho(t, y) = [y_1 - t]^+ + |y_2 - \psi(y_1)|. \tag{18}$$

Note that ρ is Lipschitz continuous. Here the superscript $+$ means the positive part. By substituting (18) into Eq. (3) we deduce the relation a.e. in $(0, T) \times \mathbb{R}^2$:

$$-\mathcal{H}(y_1 - t) + \Big(\mathcal{H}(y_1 - t) - \operatorname{sign}(y_2 - \psi)\psi'\Big)V_1 + \operatorname{sign}(y_2 - \psi)V_2 = 0,$$

where we use the notation

$$\mathcal{H}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}, \quad \operatorname{sign}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}.$$

It can be satisfied with the time-independent velocity

$$V(y) = (1, \psi'(y_1))^\top \in C_u^{0,1}(\mathbb{R}^2)^2. \tag{19}$$

With the vector field V in (19) the one-to-one transformation between Υ_0 and Υ_t can be constructed by solving the ODE system (5). The solution is given as

$$R(t, x) = (x_1 + t, x_2 + \psi(x_1 + t) - \psi(x_1))^\top$$

with inverse transformation

$$R^{-1}(t, y) = (y_1 - t, y_2 - \psi(y_1) + \psi(y_1 - t))^\top.$$

It is easily checked that R^{-1} satisfies (10). If we substitute the last two representations into (17) we regain the velocity V from (19) as the equivalence result predicts. Moreover, ρ satisfies (7) with $\rho_0(x) = x_1^+ + |x_2 - \psi(x_1)|$. Such a coordinate transformation was used in [17]. Note that for the tangential vector to Υ_t

$$\tau(y_1) = (1, \psi'(y_1))^\top (1 + (\psi'(y_1))^2)^{-1/2}$$

we have $V = \tau|V|$, i.e the velocity in (19) is tangential to Υ_t.

These constructions are not unique. One can derive another level-set function which describes the same family of cracks. For instance, the function

$$\rho(t, y) = [y_1 - t]^+ + |y_2 - \psi(y_1 - [y_1 - t]^+)|$$

solves problem (3) for the time-dependent velocity

$$V(t, y) = (1, \psi'(y_1 - [y_1 - t]^+))^\top$$

with the initial function $\rho_0(x) = x_1^+ + |x_2 - \psi(-x_1^-)|$.

Using a discontinuous velocity vector field we can consider the phenomenon of kinking of the crack. For instance, taking $\psi(y_1) = y_1^+$ in our example implies the velocity

$$V(y) = (1, \mathcal{H}(y_1))^\top,$$

which is discontinuous at $y_1 = 0$. The solution to the transport Eq. (3) in this case is given by the construction (18).

2.3 Cracks with a curvilinear front in $\mathbb{R}^3$

We present the next example in 3 space dimensions starting with plane cracks with a curvilinear front. Such cracks are often considered in the fracture modeling. Let us consider the layer $Q = \{y = (y_1, y_2, y_3)^\top \in \mathbb{R}^2 \times (0, Y)\}$ with a plane crack Υ_t contained inside Q. The crack is assumed to be located in the plane $y_2 = 0$ and bounded by the lines $y_3 = 0$, $y_3 = Y$, and by the crack front γ_t. We describe γ_t by the function $\phi(t, y_3) \in C^1([0,\infty); C_u^{0,1}(\mathbb{R}))$. Then

$$\Upsilon_t = \{y \in Q : \quad y_1 \le \phi(t, y_3),\ y_2 = 0,\ 0 \le y_3 \le Y\}, \quad t \ge 0.$$

With the Lipschitz continuous level-set function

$$\rho(t, y) = [y_1 - \phi(t, y_3)]^+ + |y_2| \tag{20}$$

the crack can be defined equivalently as

$$\Upsilon_t = \{y \in \mathbb{R}^3 : \quad \rho(t, y) = 0\} \cap Q.$$

First, we find a velocity vector field corresponding to the movement of the given crack Υ_t. Substituting ρ from (20) into problem (3) yields

$$-\mathcal{H}(y_1 - \phi)\frac{\partial \phi}{\partial t} + \mathcal{H}(y_1 - \phi)V_1 + \operatorname{sign}(y_2)V_2 - \mathcal{H}(y_1 - \phi)\frac{\partial \phi}{\partial y_3}V_3 = 0 \quad \text{a.e. } (0, T) \times \mathbb{R}^3.$$

To stay in the layer Q during the transformation $y = R(t, x)$ obtained from (5), we suppose that $y_3 = x_3$, or $\frac{dR_3}{dt} = 0$ is fulfilled. This implies $V_3 = 0$. We then take the velocity V in the form

$$V(t, y) = \Big(\frac{\partial \phi}{\partial t}(t, y_3), 0, 0\Big)^\top.$$

By solving (5) we derive the corresponding transformations

$$R(t, x) = (x_1 + \phi(t, x_3) - \phi(0, x_3), x_2, x_3)^\top, \quad R^{-1}(t, y) = (y_1 - \phi(t, y_3) + \phi(0, y_3), y_2, y_3)^\top.$$

Lemma 1 ensures that ρ as defined in (20) solves (3) for $\rho_0(x) = [x_1 - \phi(0, x_3)]^+ + |x_2|$. Note that V is tangential to the crack plane $y_2 = 0$.

One can also determine the crack front from a given velocity. Indeed, let the first component $V_1 \in C([0,\infty); C_u^{0,1}(\mathbb{R}))$ of the velocity $V = (V_1, 0, 0)^\top$ be given. This implies a family of cracks Υ_t with the curvilinear crack front γ_t described by the function

$$\phi(t, y_3) = \phi(0, y_3) + \int_0^t V_1(s, y_3)\, ds.$$

The regularity of velocity fields V assumed in (2) admits the presence of singularities like finite corners at the crack front. For instance, taking $V_1(y_3) = [y_3 - c]^+$ with a constant c fixed on the interval $(0, Y)$, we have that the time-independent velocity $V = (V_1, 0, 0)^\top \in C_u^{0,1}(\mathbb{R}^3)^3$ is admissible here. Then the corresponding function

$$\phi(t, y_3) = \phi(0, y_3) + t[y_3 - c]^+$$

generates a corner at $y_3 = c$ on the crack front γ_t for $t > 0$. This is in spite of the fact that the initial crack front at $t = 0$ may be smooth.

Now we combine this example with results of Sect. 2.2 to describe non-planar cracks (i.e. cracks which are not contained in a plane). Let the crack be given as

$$\Upsilon_t = \{y \in Q : \quad y_1 \le \phi(t, y_3),\ y_2 = \psi(y_1, y_3),\ 0 \le y_3 \le Y\}, \quad t \ge 0,$$

with given functions $\phi \in C^1([0,\infty); C_u^{0,1}(\mathbb{R}))$ and ψ, $\frac{\partial \psi}{\partial y_1} \in C_u^{0,1}(\mathbb{R}^2)$. The crack can equivalently be described as

$$\Upsilon_t = \{y \in \mathbb{R}^3 : \quad \rho(t, y) = 0\} \cap Q$$

with the level-set function

$$\rho(t,y) = [y_1 - \phi(t,y_3)]^+ + |y_2 - \psi(y_1,y_3)|.$$

To fulfill (3) the unknown velocity field V has to satisfy the equation

$$-\mathcal{H}(y_1-\phi)\frac{\partial\phi}{\partial t} + \Big(\mathcal{H}(y_1-\phi) - \mathrm{sign}(y_2-\psi)\frac{\partial\psi}{\partial y_1}\Big)V_1$$
$$+\mathrm{sign}(y_2-\psi)V_2 - \Big(\mathcal{H}(y_1-\phi)\frac{\partial\phi}{\partial y_3} + \mathrm{sign}(y_2-\psi)\frac{\partial\psi}{\partial y_3}\Big)V_3 = 0$$

almost everywhere in $(0,T)\times\mathbb{R}^3$. The velocity chosen as

$$V(y) = \Big(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial t}\frac{\partial\psi}{\partial y_1}, 0\Big)^\top$$

fulfills the desired relation and provides the one-to-one correspondence by

$$R(t,x) = \begin{pmatrix} x_1 + \phi(t,x_3) - \phi(0,x_3) \\ x_2 + \psi(x_1 + \phi(t,x_3) - \phi(0,x_3), x_3) - \psi(x_1,x_3) \\ x_3 \end{pmatrix},$$

$$R^{-1}(t,y) = \begin{pmatrix} y_1 - \phi(t,y_3) + \phi(0,y_3) \\ y_2 - \psi(y_1,y_3) + \psi(y_1 - \phi(t,y_3) + \phi(0,y_3), y_3) \\ y_3 \end{pmatrix}.$$

2.4 The crack in a bounded domain

In practice one has to handle problems in bounded domains. To get a constructive description, we use the considerations of Sect. 2.1 and suggest further the cut-off technique localizing a velocity on the support of a properly chosen cut-off function.

We suppose that the previous considerations from Sect. 2.1 are valid and Υ_0, V are given. Let the initial crack Γ_0 be a bounded part of the crack Υ_0 defined in $\mathbb{R}^N$ by (1). It may be the whole Υ_0 if Υ_0 is bounded. We assume that Γ_0 is located inside some bounded domain $\Omega\subset\mathbb{R}^N$ with Lipschitz boundary Γ, and that $\Gamma_0 = \Upsilon_0$ for those points in Ω where Γ_0 is defined.

Let us choose a cut-off function $\chi(t,y)\in C([0,\infty);C^{0,1}_u(\mathbb{R}^N))$ such that $\chi(t)=0$ outside some set $B_0(t)\subset\mathbb{R}^N$, $\chi(t)=1$ inside some set $B_1(t)\subset\mathbb{R}^N$, and $B_1(t)\subset B_0(t)\subset\Omega$. For the reference velocity V from (2) we define the velocity

$$\Lambda(t,y) = \chi(t,y)V(t,y), \tag{21}$$

which is cut off outside $B_1(t)$. Next we find the transformation

$$\Phi\in C^1([0,T];C^{0,1}_u(\mathbb{R}^N))^N \tag{22}$$

as the unique solution to the ODE-problem

$$\begin{cases} \frac{d\Phi}{dt}(t) = \Lambda(t,\Phi(t)), \\ \Phi(0) = x. \end{cases} \tag{23}$$

The inverse transformation

$$\Phi^{-1}\in C^{0,1}_u((0,T)\times\mathbb{R}^N)^N \tag{24}$$

is obtained as solution to the transport equations

$$\begin{cases} \frac{\partial\Phi^{-1}}{\partial t}(t,y) + \frac{\partial\Phi^{-1}}{\partial y}(t,y)\Lambda(t,y) = 0, \\ \Phi^{-1}(0,y) = y. \end{cases} \tag{25}$$

The one-to-one coordinate transformation $y = \Phi(t,x)$ with Φ from (22), (23) and its inverse $x = \Phi^{-1}(t,y)$ from (24), (25) together define the family of cracks Γ_t via the mapping

$$\Phi(t):\ \Gamma_0\ \to\ \Gamma_t,\quad \Phi^{-1}(t):\ \Gamma_t\ \to\ \Gamma_0. \tag{26}$$

Since $\Lambda = 0$ in $\mathbb{R}^N \setminus B_0(t)$, and by (23) it follows that

$$\Phi(t) = \Phi^{-1}(t) = I:\ \mathbb{R}^N \setminus B_0(t)\ \to\ \mathbb{R}^N \setminus B_0(t). \tag{27}$$

Due to $B_0(t) \subset \Omega$ the crack Γ_t remains located inside Ω, and we can define a domain with crack as $\Omega_t = \Omega \setminus \overline{\Gamma}_t$ for $t \in [0,T)$. Then relations (26) and (27) provide also the one-to-one mapping of the domains with cracks

$$\Phi(t):\ \Omega_0\ \to\ \Omega_t,\quad \Phi^{-1}(t):\ \Omega_t\ \to\ \Omega_0. \tag{28}$$

In the examples from Sections 2.2 and 2.3 we constructed analytical expressions of the velocity, level-set function, and coordinate transformation corresponding to the reference cracks in $\mathbb{R}^2$ and $\mathbb{R}^3$. Now we have to multiply the constructed reference velocities with appropriate cut-off functions to bring these examples into a bounded domain. This localization procedure will give us an analytical representation of the cut-off velocity Λ in (21). However, it is difficult to construct analytically solutions to problems (23) and (25) in view of the presence of the cut-off function χ. In the same manner we can describe propagation of three or more tips of branching and merging cracks, combine velocities in different directions, and so on.

Next we provide a sufficient condition guaranteeing that cracks obtained from the reference velocity on the one hand and from its cut-off version on the other hand are the same. The condition is deduced and explained for the example on curvilinear cracks.

Let Γ_t be a bounded crack in Ω defined by (26) with solution Φ to (23). The velocity Λ is constructed in (21) as a cut-off of V. Solving (3) with reference velocity V we find another family of cracks Υ_t in $\mathbb{R}^N$ defined by (14). Now we provide a sufficient condition for $\Gamma_t = \Upsilon_t$ in Ω. Comparing problems (5) and (23) for $x = \Phi^{-1}(t,y)$ with $y \in B_1(t)$ where $\chi(t) = 1$, we conclude that

$$\Gamma_t = \Upsilon_t \quad \text{in } B_1(t). \tag{29}$$

From (27) it follows that

$$\Gamma_t = \Gamma_0 = \Upsilon_0 \quad \text{in } \Omega \setminus B_0(t), \tag{30}$$

where Γ_0 was defined. It remains to consider a neighborhood $\mathcal{O} \subset \mathbb{R}^N$ of the part $\Gamma_t \cap (B_0(t) \setminus B_1(t))$ of the crack.

Lemma 3. *If the condition*

$$\frac{\partial \rho}{\partial t}(t,y) = 0 \quad \textit{a.e. } \mathcal{O} \tag{31}$$

for the level-set function ρ from (3) *is satisfied, then*

$$\Gamma_t = \begin{cases} \Upsilon_t & \textit{in } B_0(t), \\ \Upsilon_0 & \textit{in } (\Omega \setminus B_0(t)) \cap \Gamma_0. \end{cases}$$

P r o o f. From (31) it follows that

$$\Gamma_t = \Upsilon_t \quad \text{in } B_0(t) \setminus B_1(t). \tag{32}$$

In fact, multiplying (3) with χ, we deduce from (31) that

$$\frac{\partial \rho}{\partial t} + \Lambda^\top \nabla \rho = 0 \quad \text{a.e. } \mathcal{O}.$$

The same arguments as for identity (8) provide us with the formula

$$\rho(t, \Phi(t,x)) = \rho_0(x), \quad \Phi(t,x) \in \mathcal{O},$$

or, equivalently,

$$\rho(t,y) = \rho_0(\Phi^{-1}(t,y)), \quad y \in \mathcal{O}. \tag{33}$$

Due to (26) the points $x = \Phi^{-1}(t,y)$ with $y \in \Gamma_t$ belong to Γ_0, and $\rho_0(x) = 0$ since $\Gamma_0 = \Upsilon_0$ here. From (33) we find $\rho(t,y) = 0$, i.e. $y \in \Upsilon_t$. This proves formula (32). Summarizing (29), (30), and (32) ends the proof. □

If condition (31) is not satisfied, then the crack Γ_t may differ from Υ_t on the set $B_0(t) \setminus B_1(t)$.

As an example illustrating Lemma 3 we consider the bounded curvilinear cracks

$$\Gamma_t = \{y \in \mathbb{R}^2 : \quad y_2 = \psi(y_1),\ a(t) < y_1 < b(t)\}, \quad a(t) < b(t), \quad t \geq 0,$$

with given $\psi \in C^{1,1}_u(\mathbb{R})$ and $a, b \in C^1([0,\infty))$. Following the arguments of Sect. 2.2 we define the left-unbounded cracks in $\mathbb{R}^2$:

$$\Upsilon^1_t = \{y \in \mathbb{R}^2 : \quad y_2 = \psi(y_1), \quad y_1 < b(t)\},$$

which can be described by the level-set function

$$\rho^1(t,y) = [y_1 - b(t)]^+ + |y_2 - \psi(y_1)|$$

with the corresponding velocity

$$V^1(t,y) = b'(t)(1, \psi'(y_1))^\top \in C([0,\infty); C^{0,1}_u(\mathbb{R}^2))^2.$$

Similarly, for the right-unbounded cracks in $\mathbb{R}^2$:

$$\Upsilon^2_t = \{y \in \mathbb{R}^2 : \quad y_2 = \psi(y_1), \quad y_1 > a(t)\}$$

we find that

$$\rho^2(t,y) = [a(t) - y_1]^+ + |y_2 - \psi(y_1)|, \quad V^2(t,y) = a'(t)(1, -\psi'(y_1))^\top.$$

Then Γ_t can be expressed as the intersection of Υ^1_t and Υ^2_t.

Take cut-off functions χ^1 and χ^2 with disjoint supports such that $\chi^1 = 1$ and $\chi^2 = 1$ in neighborhoods of the crack tips $(b(t), \psi(b(t)))^\top$ and $(a(t), \psi(a(t)))^\top$, respectively. For the velocity

$$\Lambda = \chi^1 V^1 + \chi^2 V^2$$

the solutions of (23) and (25) determine a one-to-one coordinate transformation between the initial crack

$$\Gamma_0 = \{y \in \mathbb{R}^2 : \quad y_2 = \psi(y_1),\ a(0) < y_1 < b(0)\}$$

and the transformed crack

$$\hat{\Gamma}_t = \{y \in \mathbb{R}^2 : \quad y = \Phi(x) \quad \text{for all } x \in \Gamma_0\},$$

for any fixed t. A simple calculation shows that

$$\frac{\partial \rho^1}{\partial t} = -b'\,\mathcal{H}(y_1 - b) = 0 \quad \text{for } y_1 < b,$$
$$\frac{\partial \rho^2}{\partial t} = a'\,\mathcal{H}(a - y_1) = 0 \quad \text{for } y_1 > a.$$

Hence, for an appropriate choice of the cut-off functions χ^1 and χ^2 condition (31) is satisfied. Due to Lemma 3 it follows that $\hat{\Gamma}_t = \Upsilon^1_t \cap \Upsilon^2_t$, i.e. $\hat{\Gamma}_t = \Gamma_t$. This fact provides the one-to-one correspondence (26) for the reference crack Γ_t moving with the velocity Λ in "time" $t \geq 0$, and (28) for its complement in Ω.

For a generalization of formula (21) the velocity Λ can also be chosen as a sum of velocities multiplied by the corresponding cut-off functions: $\Lambda = \chi^1 V^1 + ... + \chi^M V^M$, $M \geq 1$. For each of them, the previous arguments remain valid.

2.5 Discontinuous velocities

For the treatment of discontinuous velocities we introduce the condition of ℓ^p-monotonicity ($1 \leq p < \infty$) of a velocity Λ:

$$\begin{aligned} &\|x_1 - y_1\|^{p-1}_{\ell^{p-1}} \operatorname{sign}(x_1 - y_1)(\Lambda_1(t,x) - \Lambda_1(t,y)) + \|x_2 - y_2\|^{p-1}_{\ell^{p-1}} \operatorname{sign}(x_2 - y_2)(\Lambda_2(t,x) - \Lambda_2(t,y)) \\ &\geq -K(t)\|x - y\|^p_{\ell^p}, \quad \text{with } K \in L^1([0,T]), \quad \text{for allmost all } x, y \in \mathbb{R}^N \text{ and } t \in (0,T). \end{aligned} \tag{34}$$

For $p = 2$ this coincides with the Filippov monotonicity condition [8].

Proposition 4. *(see [24]) If Λ is a measurable function fulfilling* (34)*, then for every locally Lipschitz continuous ρ_0, there exists a locally Lipschitz continuous function ρ satisfying*

$$\begin{cases} \frac{\partial\rho}{\partial t}(t,y) + \Lambda(t,y)^\top \nabla\rho(t,y) = 0 \quad \textit{a.e. } (0,T)\times\mathbb{R}^N, \\ \rho(0,y) = \rho_0(y). \end{cases} \tag{35}$$

The foundation of Proposition 4 lies in generalized characteristics for the nonlinear ODE (23), which enjoy a backward uniqueness property.

Based on this result we investigate the following example of a nonsmooth (discontinuous) velocity providing the splitting of a crack. Consider the family of cracks:

$$\Upsilon_t = \big\{y\in\mathbb{R}^2: \quad \{y_1 \le \min(0, t-t^*),\ y_2 = 0 \text{ for } t\ge 0\} \cup \{y_1 = 0,\ |y_2| \le t - t^* \text{ for } t \ge t^*\}\big\}, \tag{36}$$

with fixed $t^*\ge 0$. As $t = t^*$ they split (also with a kinking) into two branches with the angle of $\pm\pi/2$ with respect to the y_1-direction at the bifurcation point $y = 0$.

To construct the velocity field for (36) we separate $\mathbb{R}^2$ into two regions:

$$D^1 = \{y\in\mathbb{R}^2: \quad y_1 + a|y_2| \le 0\},$$
$$D^2 = \{y\in\mathbb{R}^2: \quad y_1 + a|y_2| \ge 0\},$$

with arbitrary constant $0\le a<\infty$. First for $0\le t<t^*$ the crack $\Upsilon_t = \{y\in\mathbb{R}^2:\ y_1\le t-t^*, y_2 = 0\}$ admits the implicit description

$$\Upsilon_t = \{y\in D^1: \quad \rho^1(t,y) = 0\}$$

with a non-negative function ρ^1 of the anisotropic distance

$$\rho^1 = [y_1 - t + t^*]^+ + |y_2| \quad \text{for } 0\le t<t^*. \tag{37}$$

By substituting (37) into the transport equation we deduce the relation

$$-\mathcal{H}(y_1 - t + t^*) + \mathcal{H}(y_1 - t + t^*)V_1 + \operatorname{sign}(y_2)V_2 = 0 \quad \text{a.e. } (0,t^*)\times D^1.$$

Evidently, this relation can be satisfied with the constant velocity $V^1 = (1,0)^\top$ in $(0,t^*)\times D^1$. According to Sect. 2.4 we extend V^1 onto $\mathbb{R}^2$ in the following way. For small $\delta>0$ we denote

$$\mathcal{H}^\delta(x) = \begin{cases} 1 & \text{for } x\ge\delta \\ \frac{x}{\delta} & \text{for } 0<x<\delta \\ 0 & \text{for } x\le 0 \end{cases}$$

and construct the piecewise-linear velocity

$$\Lambda^1(y) = (\Lambda^1_1, \Lambda^1_2)^\top = \big(\mathcal{H}^\delta(\delta - y_1 - a|y_2|), 0\big)^\top \quad \text{in } (0,t^*)\times\mathbb{R}^2. \tag{38}$$

Note that $\Lambda^1 = V^1$ on D^1. Calculating the expression on the left-hand side of (34) we deduce the following estimate

$$\begin{aligned} &\|x_1 - y_1\|^{p-1}_{\ell^{p-1}} \operatorname{sign}(x_1 - y_1)(\Lambda^1_1(t,x) - \Lambda^1_1(t,y)) \\ &\ge -\|x_1 - y_1\|^{p-1}_{\ell^{p-1}} \frac{1}{\delta}\max(1,a)\|x-y\|_{\ell^1} \ge -\frac{1}{\delta}\max(1,a)\|x-y\|^p_{\ell^p} \quad \text{a.e. } \mathbb{R}^N. \end{aligned} \tag{39}$$

Secondly, for $t\ge t^*$ the crack Υ_t can be described in D^2 by the implicit surface

$$\Upsilon_t = \{y\in D^2: \quad \rho^2(t,y) = 0\}$$

with the non-negative function

$$\rho^2 = |y_1| + [|y_2| - t + t^*]^+. \tag{40}$$

By substituting (40) into the transport equation we have

$$-\mathcal{H}(|y_2| - t + t^*) + \mathrm{sign}(y_1)V_1 + \mathcal{H}(|y_2| - t + t^*)\mathrm{sign}(y_2)V_2 = 0 \quad \text{a.e. } (t^*, \infty) \times D^2.$$

The velocity chosen as $V^2(y) = (0, \mathrm{sign}(y_2))^\top$ in $(t^*, \infty) \times D^2$ fulfills the above relation. Note that its second component V_2^2 is discontinuous in the space variables across the line $\{y_1 > 0, y_2 = 0\}$. We take the piecewise-linear extension onto $\mathbb{R}^2$ as

$$\Lambda^2(y) = (\Lambda_1^2, \Lambda_2^2)^\top = \big(0, \mathrm{sign}(y_2)(1 - \mathcal{H}^\delta(-y_1 - a|y_2|))\big)^\top \quad \text{in } (0, t^*) \times \mathbb{R}^2 \tag{41}$$

with the property that $\Lambda^2 = V^2$ on D^2. Similar to (39) calculations get

$$\begin{aligned} &\|x_2 - y_2\|_{\ell^{p-1}}^{p-1} \mathrm{sign}(x_2 - y_2)(\Lambda_2^2(t,x) - \Lambda_2^2(t,y)) \\ &\geq -\|x_2 - y_2\|_{\ell^{p-1}}^{p-1} \frac{1}{\delta} \max(1,a)\|x-y\|_{\ell^1} \geq -\frac{1}{\delta}\max(1,a)\|x-y\|_{\ell^p}^p \quad \text{a.e. } \mathbb{R}^N. \end{aligned} \tag{42}$$

Combining (38) with (41)

$$\Lambda(t,y) = \begin{cases} \Lambda^1(y) & \text{for } 0 \leq t < t^* \\ \Lambda^2(y) & \text{for } t \geq t^* \end{cases} \tag{43}$$

results in a velocity field which is continuous for $0 \leq t < t^*$, discontinuous at time $t = t^*$, and it is discontinuous in the space variables across the line $\{y_1 > -\delta, y_2 = 0\}$ for $t \geq t^*$. Nevertheless, due to the estimates (39) and (42) the velocity Λ in (43) enjoys the ℓ^p-monotonicity property (34) with $K(t) = \delta^{-1}\max(1,a)$. Hence, by Proposition 4, problem (35) has a solution ρ, which is locally Lipschitz continuous in $(0,T) \times \mathbb{R}^2$. We will calculate this solution numerically to justify that it satisfies the implicit description

$$\Upsilon_t = \{y \in \mathbb{R}^2 : \quad \rho(t,y) = 0\} \tag{44}$$

of the non-smooth crack Υ_t in (36).

For solution of problem (35) the following algorithm is proposed.

Algorithm 1. *Define the initial crack Υ_0, velocity $\Lambda(t,x)$, time discretization $t_k = k\Delta t$.*

(0) Set $k = 0$; start with the non-negative function of isotropic distance:

$$\rho(t_k, x) = \min_{z \in \Upsilon_0} \|x - z\|_{\ell^2}.$$

(1) Set $t_{k+1} = t_k + \Delta t$, compute:

$$\rho(t_{k+1}, x) = \rho(t_k, x) - \int \Lambda(t_k, x)^\top \nabla\rho(t_k, x)\,\Delta t. \tag{45}$$

(2) Find an ε-neighborhood of the crack:

$$\Upsilon_{t_{k+1}}(\varepsilon) = \{x : \quad \rho(t_{k+1}, x) \leq \varepsilon\}.$$

(5) Update $t_k = t_{k+1}$, go to step 1.

To solve (45) in Step 1 numerically we apply: a finite-difference approximation based on a Lax-Friedrichs flux, the WENO-approximation for the spatial derivatives, and the 3rd-order Runge–Kutta method for time integration. Eq. (45) is solved in a bounded domain endowed with an uniform grid of mesh-size h.

The initialization in Step 0 is taken for the initial crack

$$\Upsilon_0 = \{x \in \mathbb{R}^2 : \quad x_1 \leq -t^*,\ x_2 = 0\} \quad \text{with } t^* = 0.1.$$

As $t = t^* = 0.1$ the reference crack Υ_t in (36) attains the bifurcation point $x = 0$. Choosing $h = 0.01$ and $\Delta t = 0.001$ (to guarantee the CFL-condition) we realize Algorithm 1 for the velocity Λ from (43) with parameters $\delta = h$ and $a = 0.5$. The numerical result is plotted in Fig. 1 in the fixed square domain as $t = \{0, 0.1, 0.3, 0.6\}$ after respectively $\{0, 100, 300, 600\}$ time-iterations. The computed level-set function $\rho(t)$ and its 10 equal-sized contours projected onto the (x_1, x_2)-plane are

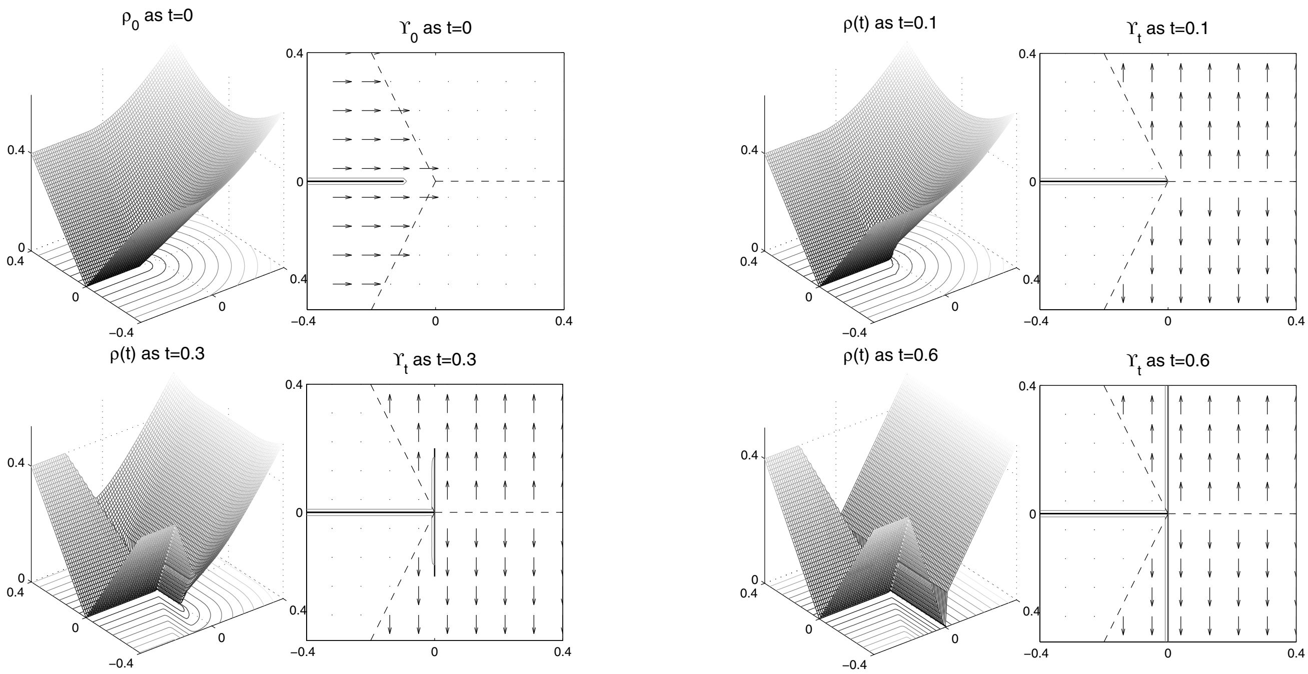

Fig. 1 The splitting crack obtained by Algorithm 1.

depicted in the left plots of Fig. 1. In the right plots we show the velocity fields $\Lambda(t)$ which are zero in the white regions, the reference crack Υ_t marked with a thick solid-line, and its neighborhood $\Upsilon_t(\varepsilon)$ found in Step 2 with $\varepsilon = h$ (the contour depicted with a thin solid-line). It can be seen in Fig. 1 that the contours surround Υ_t during the propagation steps. The function ρ obtained from Eq. (45) differs for various values of the parameters δ and a of the velocity Λ. In the numerical calculations we observe that the choice of $a \leq 1$ has visually better performance than $a > 1$. This fact can be related to the lower bound in estimates (39) and (42). Nevertheless, appropriate ($\varepsilon \geq h$) ε-neighborhoods $\Upsilon_t(\varepsilon)$ found in Step 2 are always close to the reference crack Υ_t.

Note that, when the bifurcation occurs, the tip $(0,0)^\top$ of the crack $\Upsilon_{t}*$ possesses the velocity $\Lambda(t^*, 0) = 0$. Therefore, a velocity vector known a-priori only at the tip of a propagating crack is not sufficient to describe the crack splitting. From our numerical experiments we can conclude that the full velocity field distributed in a neighborhood of the crack tip affects its bifurcation phenomena.

The other combination of (38) with (41) as

$$\Lambda(y) = \Lambda^1(y) + \Lambda^2(y) \quad (\text{with } a > 0) \tag{46}$$

yields a time-independent velocity field, which also satisfies condition (34), hence Proposition 4. The numerical results of Algorithm 1 with Λ from (46) are comparable with those shown in Fig. 1.

3 The problem of crack perturbation

In this section we treat a crack perturbation problem as a propagation of the crack with given velocity at a fixed state of the system. Here the coordinate transformations of bounded domains with cracks are utilized following the level-set formulation and equivalence results of Sect. 2. We consider a crack model subject to a non-penetration condition and argue a formula for its shape derivative. This formula is useful, for example, in fracture mechanics since it describes the energy release rate for the propagation of the crack. The formula includes a smooth velocity vector field Λ, which can be constructed for a given family of cracks by following the lines of Sect. 2.

3.1 The constrained crack model with non-penetration condition

The classic theory of cracks assumes boundary conditions of equality type describing stress-free crack faces. This may result in their overlapping. To prevent this physically inconsistent behavior, constrained models with cracks were suggested, which are subject to inequality constraints implying non-penetration between opposite crack faces. We start with the model formulation.

Let a solid occupy the domain Ω_t with the crack Γ_t as introduced in Sect. 2.4. We look for a vector $u = (u_1, ..., u_N)^\top$ in Ω_t describing admissible displacements satisfying the non-penetration condition at the crack Γ_t (see [12]):

$$H_t = \{u \in H^1(\Omega_t)^N : \quad u = 0 \quad \text{on } \Gamma\},$$
$$K_t = \{u \in H_t : \quad [\![u]\!]^\top \nu^t \geq 0 \quad \text{on } \Gamma_t\}.$$

Here ν^t denotes a normal vector chosen at Γ_t, and $[\![u]\!]$ represents the jump of u across the crack. We consider the quadratic cost functional associated with the potential energy of the solid with the crack according to

$$P_t(u; \Omega_t) = \frac{1}{2} \int_{\Omega_t} A(t, y)\Big\langle \frac{\partial u}{\partial y}, \frac{\partial u}{\partial y} \Big\rangle \, dy - \int_{\Omega_t} F(t, y)\langle u \rangle \, dy, \tag{47}$$

for $u \in H^1(\Omega_t)^N$, with the coefficients

$$A\langle \cdot\,, \cdot \rangle, F\langle \cdot \rangle \in C^1([0, \infty); C(\mathbb{R}^N)), \quad \frac{\partial A}{\partial y}\langle \cdot\,, \cdot \rangle, \frac{\partial F}{\partial y}\langle \cdot \rangle \in C([0, \infty) \times \mathbb{R}^N)^N. \tag{48}$$

The quadratic operator A is assumed to be bilinear, symmetric, uniformly positive. It represents the density of the elastic energy. Further F expresses given forces. The equilibrium problem for the solid with the crack under the non-penetration condition is given by

$$u^t \in K_t : \quad P_t(u^t; \Omega_t) \leq P_t(u; \Omega_t) \quad \text{for all } u \in K_t, \tag{49}$$

or, equivalently, as the variational inequality

$$\int_{\Omega_t} A(t, y)\Big\langle \frac{\partial u^t}{\partial y}, \frac{\partial (u - u^t)}{\partial y} \Big\rangle \, dy \geq \int_{\Omega_t} F(t, y)\langle u - u^t \rangle \, dy \quad \text{for all } u \in K_t.$$

Uniform positiveness of A implies that (49) is well defined. At the crack we have $[\![u^t]\!]^\top \nu^t \in H^{1/2}_{00}(\Gamma_t)$, where $H^{1/2}_{00}(\Gamma_t)$ is the space of functions in $H^{1/2}(\Gamma_t)$ which admit a continuation by zero on an extension of Γ_t into Ω_t, see [12]. Since the trace of H_t onto $H^{1/2}_{00}(\Gamma_t)^N$ is surjective there exists a Lagrange multiplier $\lambda^t \in M_t$ from the dual cone

$$M_t = \{\lambda \in H^{1/2}_{00}(\Gamma_t)^\star : \quad \langle \lambda, \xi \rangle_{\Gamma_t} \geq 0 \quad \text{for all } 0 \leq \xi \in H^{1/2}_{00}(\Gamma_t)\},$$

where $\langle \cdot\,, \cdot \rangle_{\Gamma_C}$ stands for the duality pairing between the spaces $H^{1/2}_{00}(\Gamma_t)$ and $H^{1/2}_{00}(\Gamma_t)^\star$, such that

$$\int_{\Omega_t} A(t, y)\Big\langle \frac{\partial u^t}{\partial y}, \frac{\partial u}{\partial y} \Big\rangle \, dy - \langle \lambda^t, [\![u]\!]^\top \nu^t \rangle_{\Gamma_t} = \int_{\Omega_t} F(t, y)\langle u \rangle \, dy \quad \text{for all } u \in H_t, \tag{50}$$

$$\langle \lambda - \lambda^t, [\![u^t]\!]^\top \nu^t \rangle_{\Gamma_t} \geq 0 \quad \text{for all } \lambda \in M_t. \tag{51}$$

Relations (50) and (51) yield the primal-dual variational formulation of the equilibrium problem (49), and they express an optimality condition for the minimax point

$$(u^t, \lambda^t) \in H_t \times M_t : \quad L_t(u^t, \lambda; \Omega_t) \leq L_t(u^t, \lambda^t; \Omega_t) \leq L_t(u, \lambda^t; \Omega_t) \quad \text{for all } (u, \lambda) \in H_t \times M_t \tag{52}$$

of the Lagrangian

$$L_t(u, \lambda; \Omega_t) = \Pi_t(u; \Omega_t) - \langle \lambda, [\![u]\!]^\top \nu^t \rangle_{\Gamma_t}.$$

Note that due to $\langle \lambda^t, [\![u^t]\!]^\top \nu^t \rangle_{\Gamma_t} = 0$ in (51) we conclude with identity

$$L_t(u^t, \lambda^t; \Omega_t) = \Pi_t(u^t; \Omega_t). \tag{53}$$

3.2 Shape derivative of the cost functional

We consider the reference crack Γ_t moving in "time" t with given velocity Λ in the bounded domain Ω, as described in Sect. 2.4 and the perturbed crack Γ_{t+s}. A one-to-one coordinate transformation of the domain with crack is obtained by solving problems (23) and (25) for Λ. It provides us with an asymptotic expansion with respect to s of the perturbed model specified in Sect. 3.1. We obtain the shape derivative of the cost functional (47) in dependence of the velocity Λ argued by the minimax formulation (52).

For $t \in [0, T)$ and the increment s we define in accordance to (28) the one-to-one coordinate transformation

$$z = \Psi(s, y) : \ \Omega_t \ \to \ \Omega_{t+s}, \quad y = \Psi^{-1}(s, z) : \ \Omega_{t+s} \ \to \ \Omega_t \tag{54}$$

by setting

$$\Psi(s, y) = \Phi(t + s, \Phi^{-1}(t, y)).$$

Due to (22) and (23) we can decompose these functions in s as follows:

$$\begin{aligned}\Psi(s, y) = \Psi(0, y) + s\frac{\partial \Psi}{\partial s}(0, y) + r(s) &= \Phi(t, \Phi^{-1}(t, y)) + s\frac{\partial \Phi}{\partial t}(t, \Phi^{-1}(t, y)) + r(s)\\ &= y + s\Lambda(t, y) + r(s), \quad \|r(s)\|_{C^{0,1}_u(\mathbb{R}^N)^N} = o(s).\end{aligned} \tag{55}$$

In view of the decomposition (55) the following expansion of the Jacobian and the functional matrices of the transformation (54) hold:

$$\begin{aligned}\Big|\frac{\partial \Psi}{\partial y}\Big| &= 1 + s(\operatorname{div} \Lambda) + r_1(s), \quad \|r_1(s)\|_{L^\infty(\mathbb{R}^N)} = o(s),\\ \frac{\partial \Psi}{\partial y} &= I + s\frac{\partial \Lambda}{\partial y} + r_2(s), \quad \|r_2(s)\|_{L^\infty(\mathbb{R}^N)^{N\times N}} = o(s),\\ \Big(\frac{\partial \Psi}{\partial y}\Big)^{-1} &= I - s\frac{\partial \Lambda}{\partial y} + r_3(s), \quad \|r_3(s)\|_{L^\infty(\mathbb{R}^N)^{N\times N}} = o(s).\end{aligned} \tag{56}$$

Similar to (47) we define the perturbed functional:

$$P_{t+s}(v; \Omega_{t+s}) = \frac{1}{2}\int\limits_{\Omega_{t+s}} A(t + s, z)\Big\langle\frac{\partial v}{\partial z}, \frac{\partial v}{\partial z}\Big\rangle\, dz - \int\limits_{\Omega_{t+s}} F(t + s, z)\langle v\rangle\, dz \tag{57}$$

for $v \in H^1(\Omega_{t+s})^N$, and the perturbed constrained minimization problem for $u^{t+s} \in K_{t+s}$:

$$P_{t+s}(u^{t+s}; \Omega_{t+s}) \le P_{t+s}(v; \Omega_{t+s}) \quad \text{for all } v \in K_{t+s}. \tag{58}$$

It describes a minimax point $(u^{t+s}, \lambda^{t+s}) \in H_{t+s} \times M_{t+s}$:

$$L_{t+s}(u^{t+s}, \mu; \Omega_{t+s}) \le L_{t+s}(u^{t+s}, \lambda^{t+s}; \Omega_{t+s}) \le L_{t+s}(v, \lambda^{t+s}; \Omega_{t+s}) \quad \text{for all } (v, \mu) \in H_{t+s} \times M_{t+s} \tag{59}$$

of the Lagrangian $L_{t+s}(v, \mu; \Omega_{t+s}) = \Pi_{t+s}(v; \Omega_{t+s}) - \langle \mu, [\![v]\!]^\top \nu^{t+s}\rangle_{\Gamma_{t+s}}$. Note that similarly to (53) we have the identity

$$L_{t+s}(u^{t+s}, \lambda^{t+s}; \Omega_{t+s}) = \Pi_{t+s}(u^{t+s}; \Omega_{t+s}). \tag{60}$$

Now our aim is to adopt the coordinate transformation (54) to the perturbed minimax problem (59). We apply (54) to the integrals in (57) and obtain

$$P_{t+s}(v; \Omega_{t+s}) = \bar{P}_{t+s}(v \circ \Psi; \Omega_t) \quad \text{for } v \in H_{t+s}, \tag{61}$$

$$\begin{aligned}\bar{P}_{t+s}(u; \Omega_t) =& \frac{1}{2}\int\limits_{\Omega_t}\Big|\frac{\partial \Psi}{\partial y}\Big| A(t + s, \Psi(s, y))\Big\langle\frac{\partial u}{\partial y}\Big(\frac{\partial \Psi}{\partial y}\Big)^{-1}, \frac{\partial u}{\partial y}\Big(\frac{\partial \Psi}{\partial y}\Big)^{-1}\Big\rangle\, dy\\ &- \int\limits_{\Omega_t}\Big|\frac{\partial \Psi}{\partial y}\Big| F(t + s, \Psi(s, y))\langle u\rangle\, dy \quad \text{for } u \in H_t,\end{aligned} \tag{62}$$

due to $\frac{\partial v}{\partial z} = \frac{\partial(v\circ\Psi)}{\partial y}(\frac{\partial\Psi}{\partial y})^{-1}$. Transformation of the duality pairing can be defined in the dual sense:

$$\langle \mu\circ\Psi, \xi\rangle_{\Gamma_t} = \langle \mu, (\omega^{-1}\xi)\circ\Psi^{-1}\rangle_{\Gamma_{t+s}} \quad \text{for } \xi\in H^{1/2}_{00}(\Gamma_t),$$
$$\langle \lambda\circ\Psi^{-1}, \eta\rangle_{\Gamma_{t+s}} = \langle \lambda, \omega(\eta\circ\Psi)\rangle_{\Gamma_t} \quad \text{for } \eta\in H^{1/2}_{00}(\Gamma_{t+s}), \tag{63}$$

with the Jacobian at the crack

$$\omega = \Big|\frac{\partial\Psi}{\partial y}\Big|\,\Big|\Big(\Big(\frac{\partial\Psi}{\partial y}\Big)^{-1}\Big)^{\top}\nu^t\Big|. \tag{64}$$

From (61) and (63) we obtain transformation of the Lagrangian in the form

$$L_{t+s}(v,\mu;\Omega_{t+s}) = \bar L_{t+s}(v\circ\Psi,\mu\circ\Psi;\Omega_t) = \bar P_{t+s}(v\circ\Psi;\Omega_t) - \langle \mu\circ\Psi, \omega[\![v\circ\Psi]\!]^{\top}(\nu^{t+s}\circ\Psi)\rangle_{\Gamma_t}. \tag{65}$$

It can be verified that one-to-one correspondence holds between the sets:

$$(v,\mu)\in H_{t+s}\times M_{t+s} \Rightarrow (v\circ\Psi,\mu\circ\Psi)\in H_t\times M_t,$$
$$(u,\lambda)\in H_t\times M_t \Rightarrow (u\circ\Psi^{-1},\lambda\circ\Psi^{-1})\in H_{t+s}\times M_{t+s}. \tag{66}$$

Using (65) and (66) we deduce from (59) that $(u^{t+s}\circ\Psi, \lambda^{t+s}\circ\Psi)\in H_t\times M_t$ denotes the minimax point of the transformed Lagrangian:

$$\bar L_{t+s}(u^{t+s}\circ\Psi,\lambda;\Omega_t) \le \bar L_{t+s}(u^{t+s}\circ\Psi,\lambda^{t+s}\circ\Psi;\Omega_t) \le \bar L_{t+s}(u,\lambda^{t+s}\circ\Psi;\Omega_t) \quad \text{for all } (u,\lambda)\in H_t\times M_t, \tag{67}$$

and an optimality condition is expressed by the relations:

$$\int_{\Omega_t}\Big|\frac{\partial\Psi}{\partial y}\Big|A(t+s,\Psi(s,y))\Big\langle \frac{\partial(u^{t+s}\circ\Psi)}{\partial y}\Big(\frac{\partial\Psi}{\partial y}\Big)^{-1}, \frac{\partial u}{\partial y}\Big(\frac{\partial\Psi}{\partial y}\Big)^{-1}\Big\rangle\,dy$$
$$-\langle \lambda^{t+s}\circ\Psi, \omega[\![u]\!]^{\top}(\nu^{t+s}\circ\Psi)\rangle_{\Gamma_t} = \int_{\Omega_t}\Big|\frac{\partial\Psi}{\partial y}\Big|F(t+s,\Psi(s,y))\langle u\rangle\,dy \quad \text{for all } u\in H_t, \tag{68}$$

$$\langle \lambda-\lambda^{t+s}\circ\Psi, \omega[\![u^{t+s}\circ\Psi]\!]^{\top}(\nu^{t+s}\circ\Psi)\rangle_{\Gamma_t} \ge 0 \quad \text{for all } \lambda\in M_t. \tag{69}$$

From (65) and (69) it follows the identity

$$\bar L_{t+s}(u^{t+s}\circ\Psi,\lambda^{t+s}\circ\Psi;\Omega_t) = \bar P_{t+s}(u^{t+s}\circ\Psi;\Omega_t). \tag{70}$$

Our next aim is to get an asymptotic expansion with respect to s in (67). Due to the regularity assumptions (48) and the expansions (56) we get the asymptotic representation of the functional $\bar P_{t+s}$ from (62) with respect to s as follows:

$$\bar P_{t+s}(u;\Omega_t) = P_t(u;\Omega_t) + s\,P'_t(u;\Omega_t) + o(s) \quad \text{for } u\in H_t, \tag{71}$$

with the first asymptotic term given by

$$P'_t(u;\Omega_t) = \frac12\int_{\Omega_t}\Big\{(\operatorname{div}\Lambda)A(t,y) + \frac{\partial A}{\partial t}(t,y) + \Lambda^{\top}\frac{\partial A}{\partial y}(t,y)\Big\}\Big\langle \frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\Big\rangle\,dy$$
$$-\int_{\Omega_t}A(t,y)\Big\langle \frac{\partial u}{\partial y}\frac{\partial\Lambda}{\partial y}, \frac{\partial u}{\partial y}\Big\rangle\,dy - \int_{\Omega_t}\Big\{(\operatorname{div}\Lambda)F(t,y) + \frac{\partial F}{\partial t}(t,y) + \Lambda^{\top}\frac{\partial F}{\partial y}(t,y)\Big\}\langle u\rangle\,dy. \tag{72}$$

Due to (56) the decomposition of (64) with respect to s yields

$$\omega = 1 + s\,\operatorname{div}_{\Gamma_t}\Lambda + o(s), \quad \operatorname{div}_{\Gamma_t}\Lambda = \operatorname{div}\Lambda - (\nu^t)^{\top}\frac{\partial\Lambda}{\partial y}\nu^t. \tag{73}$$

From representation of the normal vector at the crack (see [27])

$$\nu^{t+s} \circ \Psi = \Big(\Big(\frac{\partial \Psi}{\partial y} \Big)^{-1} \Big)^{\top} \nu^t \Big| \Big(\Big(\frac{\partial \Psi}{\partial y} \Big)^{-1} \Big)^{\top} \nu^t \Big|^{-1}$$

we obtain the following decomposition with respect to s

$$\nu^{t+s} \circ \Psi = \nu^t + s \Big(\big((\nu^t)^{\top} \frac{\partial \Lambda}{\partial y} \nu^t \big) \nu^t - \Big(\frac{\partial \Lambda}{\partial y} \Big)^{\top} \nu^t \Big) + o(s). \tag{74}$$

Using (71), (73), and (74) results in the asymptotic representation of the Lagrangian as

$$\bar{L}_{t+s}(u, \lambda; \Omega_t) = L(u, \lambda; \Omega_t) + s\, L'_t(u, \lambda; \Omega_t) + o(s), \tag{75}$$

$$L'_t(u, \lambda; \Omega_t) = P'_t(u; \Omega_t) - \langle \lambda, [\![u]\!]^{\top} \big(\mathrm{div}\Lambda \nu^t - \Big(\frac{\partial \Lambda}{\partial y} \Big)^{\top} \nu^t \big) \rangle_{\Gamma_t}. \tag{76}$$

In view of (70) and expansion (71) the substitution of $u = 0$ into the second inequality in (67) implies

$$0 \geq P_t(u^{t+s} \circ \Psi; \Omega_t) + s\, P'_t(u^{t+s} \circ \Psi; \Omega_t) + o(s),$$

and the uniform estimate of the norm $\|u^{t+s} \circ \Psi\|_{H^1(\Omega_t)^N}$ follows for small s. From (68) we can estimate $\lambda^{t+s} \circ \Psi \in M_t$ in the $H^{1/2}_{00}(\Gamma_t)^{\star}$ norm, and obtain the estimate

$$\|u^{t+s} \circ \Psi\|_{H^1(\Omega_t)^N} + \|\lambda^{t+s} \circ \Psi\|_{H^{1/2}_{00}(\Gamma_t)^{\star}} \leq \mathrm{const}. \tag{77}$$

Therefore, there exists a subsequence of the solutions such that

$$u^{t+s} \circ \Psi \to u^t \text{ weakly in } H^1(\Omega_t)^N \text{ as } s \to 0, \quad u^t \in H_t,$$

$$\lambda^{t+s} \circ \Psi \to \lambda^t \text{ *-weakly in } H^{1/2}_{00}(\Gamma_t)^{\star} \text{ as } s \to 0, \quad \lambda^t \in M_t. \tag{78}$$

Since the quadratic functional P_t is weakly lower semi-continuous, we have in view of relations (67), (75), (77), and (78) for all $(u, \lambda) \in H_t \times M_t$:

$$\begin{aligned}
&L_t(u^t, \lambda; \Omega_t) \leq \limsup_{s \to 0} L_t(u^{t+s} \circ \Psi, \lambda; \Omega_t) \leq \limsup_{s \to 0} \bar{L}_{t+s}(u^{t+s} \circ \Psi, \lambda; \Omega_t) \\
&\leq \limsup_{s \to 0} \bar{L}_{t+s}(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) \leq \limsup_{s \to 0} \bar{L}_{t+s}(u^t, \lambda^{t+s} \circ \Psi; \Omega_t) = L_t(u^t, \lambda^t; \Omega_t) \\
&\leq \liminf_{s \to 0} L_t(u^{t+s} \circ \Psi, \lambda^t; \Omega_t) \leq \liminf_{s \to 0} \bar{L}_{t+s}(u^{t+s} \circ \Psi, \lambda^t; \Omega_t) \\
&\leq \liminf_{s \to 0} \bar{L}_{t+s}(u^{t+s} \circ \Phi, \lambda^{t+s} \circ \Psi; \Omega_t) \leq \liminf_{s \to 0} \bar{L}_{t+s}(u, \lambda^{t+s} \circ \Psi; \Omega_t) \leq L_t(u, \lambda^t; \Omega_t).
\end{aligned}$$

Thus (u^t, λ^t) solves (52). Substituting $u = u^t$ into (67), using again the expansion (75), estimate (77), and the weak convergence (78), the estimate

$$\begin{aligned}
I(s) &:= \frac{1}{2} \int_{\Omega_t} A(t,y) \Big\langle \frac{\partial}{\partial y}(u^{t+s} \circ \Psi - u^t), \frac{\partial}{\partial y}(u^{t+s} \circ \Psi - u^t) \Big\rangle \, dy \\
&= - \int_{\Omega_t} A(t,y) \Big\langle \frac{\partial u^t}{\partial y}, \frac{\partial}{\partial y}(u^{t+s} \circ \Psi - u^t) \Big\rangle \, dy \\
&\quad + \frac{1}{2} \int_{\Omega_t} A(t,y) \Big\langle \frac{\partial (u^{t+s} \circ \Psi)}{\partial y}, \frac{\partial (u^{t+s} \circ \Psi)}{\partial y} \Big\rangle \, dy - \frac{1}{2} \int_{\Omega_t} A(t,y) \Big\langle \frac{\partial u^t}{\partial y}, \frac{\partial u^t}{\partial y} \Big\rangle \, dy \\
&= - \int_{\Omega_t} A(t,y) \Big\langle \frac{\partial u^t}{\partial y}, \frac{\partial}{\partial y}(u^{t+s} \circ \Psi - u^t) \Big\rangle \, dy + \int_{\Omega_t} F(t,y) \langle u^{t+s} \circ \Psi - u^t \rangle \, dy \\
&\quad + \bar{L}_{t+s}(u^{t+s} \circ \Phi, \lambda^{t+s} \circ \Psi; \Omega_t) - \bar{L}_{t+s}(u^t, \lambda^{t+s} \circ \Psi; \Omega_t) + O(s) \\
&\leq - \int_{\Omega_t} A(t,y) \Big\langle \frac{\partial u^t}{\partial y}, \frac{\partial}{\partial y}(u^{t+s} \circ \Psi - u^t) \Big\rangle \, dy + \int_{\Omega_t} F(t,y) \langle u^{t+s} \circ \Psi - u^t \rangle \, dy + O(s)
\end{aligned}$$

leads to $I(s) \to 0$ as $s \to 0$. Uniform positive definiteness of A implies that $u^{t+s} \circ \Psi \to u^t$ strongly in $H^1(\Omega_t)^N$ as $s \to 0$. The subtraction of (50) from (68) yields

$$\langle \lambda^{t+s} \circ \Psi - \lambda^t, [\![u]\!]^\top \nu^t \rangle_{\Gamma_t} = -\int_{\Omega_t} A(t,y) \Big\langle \frac{\partial}{\partial y}(u^{t+s} \circ \Psi - u^t), \frac{\partial u}{\partial y} \Big\rangle \, dy + \langle \lambda^{t+s} \circ \Psi, [\![u]\!]^\top (\nu^{t+s} \circ \Psi - \nu^t) \rangle_{\Gamma_t} + O(s),$$

which provides convergence of $\lambda^{t+s} \circ \Psi \to \lambda^t$ in the $H^{1/2}_{00}(\Gamma_t)^\star$ norm. As a consequence we obtain the strong convergence in (78):

$$(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi) \to (u^t, \lambda^t) \quad \text{strongly in } H^1(\Omega_t)^N \times H^{1/2}_{00}(\Gamma_t)^\star \text{ as } s \to 0. \tag{79}$$

Finally we find the derivative of the cost functional from (47) with respect to the perturbation parameter s. Let us substitute $u = u^t$ in the second inequality (67) and $\lambda = \lambda^{t+s} \circ \Psi$ in the first inequality (52), use representation (65) and expansion (75) to obtain the upper bound:

$$\begin{aligned} &L_{t+s}(u^{t+s}, \lambda^{t+s}; \Omega_{t+s}) - L_t(u^t, \lambda^t; \Omega_t) = \bar{L}_{t+s}(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) - L_t(u^t, \lambda^t; \Omega_t) \\ &\le \bar{L}_{t+s}(u^t, \lambda^{t+s} \circ \Psi; \Omega_t) - L_t(u^t, \lambda^t; \Omega_t) \le s\, L'_t(u^t, \lambda^{t+s} \circ \Psi; \Omega_t) + o(s). \end{aligned}$$

Similarly, the substitution of $u = u^{t+s} \circ \Psi$ in (52) and $\lambda = \lambda^t$ in (67), and the use of the uniform estimate (77) provide the following lower bound:

$$\begin{aligned} &L_{t+s}(u^{t+s}, \lambda^{t+s}; \Omega_{t+s}) - L(u^t, \lambda^t; \Omega_t) = \bar{L}_{t+s}(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) - L_t(u^t, \lambda^t; \Omega_t) \\ &\ge \bar{L}_{t+s}(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) - L_t(u^{t+s} \circ \Psi, \lambda^t; \Omega_t) \ge s\, L'_t(u^{t+s} \circ \Psi, \lambda^t; \Omega_t) + o(s). \end{aligned}$$

By dividing the last two inequalities by s and passing to the limit $s \to 0$ we obtain, using (79):

$$\lim_{s \to 0} \frac{L_{t+s}(u^{t+s}, \lambda^{t+s}; \Omega_{t+s}) - L_t(u^t, \lambda^t; \Omega_t)}{s} = L'_t(u^t, \lambda^t; \Omega_t). \tag{80}$$

Recalling the identities (53) and (60), from (80) we arrive at the following theorem.

Theorem 5. *Assuming a velocity* Λ *of the form* (21)*, there exists for every* $t \in (0, T)$ *a derivative of the cost functional* (47) *in the sense that*

$$\lim_{s \to 0} \frac{P_{t+s}(u^{t+s}; \Omega_{t+s}) - P_t(u^t; \Omega_t)}{s} = L'_t(u^t, \lambda^t; \Omega_t),$$

represented by (76) *and* (72) *with* $(u, \lambda) = (u^t, \lambda^t)$ *being a solution to the minimax problem* (52).

If the crack is regular enough such that its normal vector ν^t is differentiable, then we can rewrite more precisely the boundary term in expression (76). In fact, using decomposition of the vectors into the normal and the tangential components as

$$\begin{aligned} &[\![u^t]\!] = \big\{ [\![u^t]\!]^\top \nu^t \big\} \nu^t + [\![(u^t)_{\Gamma_t}]\!], \\ &\mathrm{div}\Lambda \nu^t - \Big(\frac{\partial \Lambda}{\partial y}\Big)^\top \nu^t = \big\{ \mathrm{div}_{\Gamma_t}\Lambda - \Lambda^\top \frac{\partial \nu^t}{\partial y} \nu^t + (\nu^t)^\top \Big(\frac{\partial \nu^t}{\partial y}\Big)^\top \Lambda \big\} \nu^t - \nabla_{\Gamma_t}(\Lambda^\top \nu^t) + \Big(\Big(\frac{\partial \nu^t}{\partial y}\Big)^\top \Lambda \Big)_{\Gamma_t}, \end{aligned}$$

with the notation of $\nabla_{\Gamma_t}\xi = \nabla\xi - (\nabla\xi^\top \nu^t)\nu^t$, we arrive from (76) at the equivalent formulation

$$L'_t(u^t, \lambda^t; \Omega_t) = P'_t(u^t; \Omega_t) - \langle \lambda^t, [\![(u^t)_{\Gamma_t}]\!]^\top \big\{ \Big(\Big(\frac{\partial \nu^t}{\partial y}\Big)^\top \Lambda \Big)_{\Gamma_t} - \nabla_{\Gamma_t}(\Lambda^\top \nu^t) \big\} \rangle_{\Gamma_t}, \tag{81}$$

using the complementarity conditions fulfilled between λ^t and $[\![u^t]\!]^\top \nu^t$ at the crack Γ_t. We stress the fact that for rectilinear (planar) cracks with $\nu^t = \mathrm{const}$ and for the tangential velocities Λ such that $\Lambda^\top \nu^t = 0$, it follows from (81) that the boundary term is equal to zero and consequently

$$L'_t(u^t, \lambda^t; \Omega_t) = P'_t(u^t; \Omega_t). \tag{82}$$

Thus considering curvilinear (non-planar) cracks results in the presence of the boundary term in (81) when compared to the equality (82). The expression for $P'_t(u^t;\Omega_t)$ was obtained for the primal variational formulation of the crack problem in [13]. In the latter reference, for the final representation of the shape derivative by formula similar to (72), it is assumed that

$$\Lambda \in C^1(0,T;W^{2,\infty}_{loc}(\mathbb{R}^N))^N.$$

In our work, this representation involves only

$$\Lambda \in C([0,T];C^{0,1}_u(\mathbb{R}^N))^N = C([0,T];W^{1,\infty}(\mathbb{R}^N))^N,$$

thus relaxing the smoothness requirement with respect to both spatial and temporal variables.

4 Derivatives of the energy functional with respect to crack shapes

In this section we combine the result of Sect. 2 with the result of Sect. 3 to calculate derivatives of the energy functional with respect to perturbation of the crack shape. In fact, representation of the derivative in Sect. 3 involves a (smooth) velocity Λ which is assumed to be known a-priori. We construct the velocities for families of propagating cracks by following the implicit representation of cracks by zero-level sets of Sect. 2. For this purpose we suggest that a crack propagates along the (smooth) path, which can be represented by a parametric curve or surface. In the following consideration the shape functions describing cracks and their paths are assumed to be smooth enough.

Using a decomposition of the vectors into their tangential and normal components at the crack Γ_t, we utilize the complementarity conditions fulfilled between λ^t and $[\![u^t]\!]^\top \nu^t$ at Γ_t and derive from (76) the equivalent expression of the derivative of the energy functional P:

$$L'_t(u^t,\lambda^t;\Omega_t) = P'_t(u^t;\Omega_t) + I_t, \quad I_t = \langle \lambda^t, [\![(u^t)_{\Gamma_t}]\!]^\top \{ \Big(\Big(\frac{\partial \Lambda}{\partial y}\Big)^\top \nu^t \Big)_{\Gamma_t} \rangle_{\Gamma_t}, \tag{83}$$

$$\begin{aligned} P'_t(u^t;\Omega_t) = \frac{1}{2} \int_{\Omega_t} \Big\{ (\operatorname{div}\Lambda) A(t,y) + \frac{\partial A}{\partial t}(t,y) + \Lambda^\top \frac{\partial A}{\partial y}(t,y) \Big\} \Big\langle \frac{\partial u^t}{\partial y}, \frac{\partial u^t}{\partial y} \Big\rangle \, dy \\ - \int_{\Omega_t} A(t,y) \Big\langle \frac{\partial u^t}{\partial y} \frac{\partial \Lambda}{\partial y}, \frac{\partial u^t}{\partial y} \Big\rangle \, dy \\ - \int_{\Omega_t} \Big\{ (\operatorname{div}\Lambda) F(t,y) + \frac{\partial F}{\partial t}(t,y) + \Lambda^\top \frac{\partial F}{\partial y}(t,y) \Big\} \langle u^t \rangle \, dy. \end{aligned} \tag{84}$$

The expression of P'_t in (84) repeats (72). In the following consideration we pay a special attention to the expression of the boundary term I_t in (83).

We stress the fact that the value of $-L'_t$ implies the energy release rate, which is commonly used for the Griffith fracture criterion in fracture mechanics. Note also that, if $\frac{\partial A}{\partial t} = \frac{\partial F}{\partial t} = 0$, then relation (83) determines a linear continuous functional with respect to the velocity Λ and its first derivatives $\frac{\partial \Lambda}{\partial y}$. The respective coefficients can be related to the energy momentum tensor of Eshelby.

Cracks propagating along a parametric curve represented in Cartesian coordinates. Recalling the example of Sect. 2.2 we start with the family of cracks given implicitly in a bounded domain $\Omega \subset \mathbb{R}^2$ by

$$\Gamma_t = \{ y \in \Omega : \quad [y_1 - \phi(t)]^+ + |y_2 - \psi(y_1)| = 0 \}.$$

The crack tip $(\phi(t), \psi(\phi(t)))^\top$ is assumed to be located inside the support of some cut-off function $\chi(y)$ in Ω within a time interval $(0,T)$. Due to the result of Sect. 2.4, specifically from (19) and (21) we have

$$\begin{aligned} &\Lambda = \chi \phi' (1,\psi')^\top, \quad \frac{\partial \Lambda}{\partial y} = \phi' \begin{pmatrix} \frac{\partial \chi}{\partial y_1} & \frac{\partial \chi}{\partial y_2} \\ \frac{\partial \chi}{\partial y_1}\psi' + \chi \psi'' & \frac{\partial \chi}{\partial y_2}\psi' \end{pmatrix}, \quad \operatorname{div}\Lambda = \phi' \Big(\frac{\partial \chi}{\partial y_1} + \frac{\partial \chi}{\partial y_2}\psi' \Big), \\ &\nu^t = Z^{-1/2}(-\psi',1)^\top, \quad Z = 1 + (\psi')^2. \end{aligned} \tag{85}$$

Substituting expressions (85) in (83) and (84) provides a formula for calculation of the derivative $L'_t = P'_t + I_t$ of the energy functional P with respect to the crack Γ_t propagating along the parametric curve $y_2 - \psi(y_1) = 0$. In the case that A is the linear elasticity operator, the form of P'_t is presented in [17]. The boundary term I_t in (83) takes the particular form

$$I_t = \phi' \langle \lambda^t, \chi \psi'' Z^{-3/2} \big([\![u_1^t]\!] + \psi' [\![u_2^t]\!] \big) \rangle_{\Gamma_t}.$$

Note that $\phi'(t)$ implies the physical velocity of propagation of the crack tip along this curve, and it appears as a multiplier in the expression of L'_t.

Cracks propagating along a parametric curve represented in polar coordinates. Let a family of cracks Γ_t in $\Omega \subset \mathbb{R}^2$ be given on the curve $\theta - \psi(r) = 0$ in polar coordinates $\{y_1 = r\cos\theta, y_2 = r\sin\theta\}$ as

$$\Gamma_t = \{(r,\theta)^\top \in \mathbb{R}^2_+ : \quad [r - \phi(t)]^+ + |\theta - \psi(r)| = 0\} \cap \Omega$$

with $\phi(0) = r_0 > 0$. It can be verified that the function $\rho(t,r,\theta) = [r - \phi(t)]^+ + |\theta - \psi(r)|$ satisfies the transport equation in polar coordinates

$$\frac{\partial \rho}{\partial t} + V_r \frac{\partial \rho}{\partial r} + V_\theta \frac{1}{r} \frac{\partial \rho}{\partial \theta} = 0 \quad \text{a.e. } (0,\infty) \times \mathbb{R}^2_+$$

with $V_r = \phi'$, $V_\theta = \phi' r \psi'$. The velocity in Cartesian coordinates has the form

$$(V_1, V_2)^\top = \phi' (\cos\theta - \sin\theta r\psi', \sin\theta + \cos\theta r\psi')^\top.$$

We assume that a cut-off function $\chi(r)$ in Ω can be chosen such that the crack tip $\phi(t)(\cos\psi(\phi(t)), \sin\psi(\phi(t)))^\top$ lies inside the support of χ for $t \in (0,T)$. Hence our construction results in the following expression of the velocity $\Lambda = \chi V$ and its derivatives:

$$\Lambda = \chi\phi' (\tau_1, \tau_2)^\top, \quad \tau = \begin{pmatrix} \cos\theta - \sin\theta r\psi' \\ \sin\theta + \cos\theta r\psi' \end{pmatrix}, \quad \operatorname{div}\Lambda = \phi' \Big(\chi' + \frac{\chi}{r} \Big),$$

$$\frac{\partial \Lambda}{\partial y} = \phi' \begin{pmatrix} \chi' \cos\theta \tau_1 + \chi \sin\theta \big(-\cos\theta r\psi'' + \frac{\sin\theta}{r} \big) & \chi' \sin\theta \tau_1 - \chi \sin\theta \big(\sin\theta r\psi'' + \frac{\cos\theta}{r} \big) - \chi\psi' \\ \chi' \cos\theta \tau_2 + \chi \cos\theta \big(\cos\theta r\psi'' - \frac{\sin\theta}{r} \big) + \chi\psi' & \chi' \sin\theta \tau_2 + \chi \cos\theta \big(\sin\theta r\psi'' + \frac{\cos\theta}{r} \big) \end{pmatrix}.$$

Substituting it in (84) we find P'_t. Using the normal vector at Γ_t

$$\nu^t = Z^{-1/2} (-\sin\theta - \cos\theta r\psi', \cos\theta - \sin\theta r\psi')^\top, \quad Z = 1 + (r\psi')^2,$$

we calculate I_t in (83) as

$$I_t = \phi' \langle \lambda^t, \chi Z^{-3/2} \big(2\psi' + r\psi'' + r^2 (\psi')^3 \big) \big(\tau_1 [\![u_1^t]\!] + \tau_2 [\![u_2^t]\!] \big) \rangle_{\Gamma_t}.$$

Cracks propagating along a parametric surface represented in Cartesian coordinates. Recalling the example of Sect. 2.3 we consider the family of cracks given in the layer $\Omega = \Omega_2 \times (0,Y)$, where $\Omega_2 \in \mathbb{R}^2$ is bounded, as

$$\Gamma_t = \{ y \in \Omega : \quad [y_1 - \phi(t,y_3)]^+ + |y_2 - \psi(y_1,y_3)| = 0 \}.$$

This formulation implies that the crack front γ_t is described by a non-closed curve $\{y_1 = \phi(t,y_3), y_2 = \psi(\phi(t,y_3),y_3)\}$ with respect to $y_3 \in (0,Y)$. With the help of a cut-off function $\chi(y_1,y_2)$ in Ω_2 such that $\gamma_t \subset \operatorname{supp}\chi \times (0,Y)$ for $t \in (0,T)$ we construct the velocity vector field

$$\Lambda = \chi(y_1,y_2) \frac{\partial \phi}{\partial t}(t,y_3) \Big(1, \frac{\partial \psi}{\partial y_1}(y_1,y_3), 0 \Big)^\top.$$

Calculating its derivatives

$$\frac{\partial \Lambda}{\partial y} = \begin{pmatrix} \frac{\partial \phi}{\partial t} \frac{\partial \chi}{\partial y_1} & \frac{\partial \phi}{\partial t} \frac{\partial \chi}{\partial y_2} & \chi \frac{\partial^2 \phi}{\partial t \partial y_3} \\ \frac{\partial \phi}{\partial t} \big(\frac{\partial \chi}{\partial y_1} \frac{\partial \psi}{\partial y_1} + \chi \frac{\partial^2 \psi}{\partial y_1^2} \big) & \frac{\partial \phi}{\partial t} \frac{\partial \chi}{\partial y_2} \frac{\partial \psi}{\partial y_1} & \chi \big(\frac{\partial \phi}{\partial t} \frac{\partial^2 \psi}{\partial y_1 \partial y_3} + \frac{\partial^2 \phi}{\partial t \partial y_3} \frac{\partial \psi}{\partial y_1} \big) \\ 0 & 0 & 0 \end{pmatrix}$$

we substitute them into (84) to determine P_t'. With the normal vector at the crack path $y_2 - \psi(y_1, y_3) = 0$:

$$\nu^t = Z^{-1/2}\Big(-\frac{\partial\psi}{\partial y_1}, 1, -\frac{\partial\psi}{\partial y_3}\Big)^\top, \quad Z = 1 + \Big(\frac{\partial\psi}{\partial y_1}\Big)^2 + \Big(\frac{\partial\psi}{\partial y_3}\Big)^2,$$

we derive the expression of the boundary term I_t in (83)

$$I_t = \langle \lambda^t, \chi\frac{\partial\phi}{\partial t} Z^{-3/2}\big(\tau_1[\![u_1^t]\!] + \tau_2[\![u_2^t]\!] + \tau_3[\![u_3^t]\!]\big)\rangle_{\Gamma_t},$$

where the tangential vector $\tau = (\tau_1, \tau_2, \tau_3)^\top$ is

$$\tau_1 = \Big(1 + \Big(\frac{\partial\psi}{\partial y_3}\Big)^2\Big)\frac{\partial^2\psi}{\partial y_1^2} - \frac{\partial\psi}{\partial y_1}\frac{\partial\psi}{\partial y_3}\frac{\partial^2\psi}{\partial y_1\partial y_3}, \quad \tau_2 = \frac{\partial\psi}{\partial y_1}\frac{\partial^2\psi}{\partial y_1^2} + \frac{\partial\psi}{\partial y_3}\frac{\partial^2\psi}{\partial y_1\partial y_3},$$

$$\tau_3 = \frac{\partial^2\psi}{\partial y_1\partial y_3}\Big(1 + \Big(\frac{\partial\psi}{\partial y_1}\Big)^2 - \frac{\partial\psi}{\partial y_1}\frac{\partial\psi}{\partial y_3}\Big).$$

Cracks propagating along a parametric surface represented in cylindric coordinates. Let a family of cracks Γ_t in a bounded domain $\Omega \subset \mathbb{R}^3$ be given on the surface $y_2 - \psi(r, \eta) = 0$ in cylindric coordinates $\{y_1 = r\sin\eta, y_2, y_3 = r\cos\eta\}$ by the implicit representation

$$\Gamma_t = \{(r, y_2, \eta)^\top \in \mathbb{R}_+ \times \mathbb{R} \times [0, 2\pi] : \quad [r - \phi(t, \eta)]^+ + |y_2 - \psi(r, \eta)| = 0\} \cap \Omega$$

with a periodic function $\phi(t, 0) = \phi(t, 2\pi)$, and let $\phi(0, \eta) > 0$. This formulation describes the crack front which is represented by a closed curve $\{r = \phi(t, \eta), y_2 = \psi(\phi(t, \eta), \eta)\}$ with respect to the polar angle η. To satisfy the transport equation in cylindric coordinates for $\rho(t, r, y_2, \eta) = [r - \phi(t, \eta)]^+ + |y_2 - \psi(r, \eta)|$ almost everywhere

$$-\frac{\partial\phi}{\partial t}\mathcal{H}(r-\phi) + V_r\Big(\mathcal{H}(r-\phi) - \frac{\partial\psi}{\partial r}\mathrm{sign}(y_2-\psi)\Big) + V_2\mathrm{sign}(y_2-\psi) - \frac{1}{r}V_\eta\Big(\frac{\partial\phi}{\partial\eta}\mathcal{H}(r-\phi) + \frac{\partial\psi}{\partial\eta}\mathrm{sign}(y_2-\psi)\Big) = 0,$$

we get $V_r = \frac{\partial\phi}{\partial t}$, $V_2 = \frac{\partial\phi}{\partial t}\frac{\partial\psi}{\partial r}$, $V_\eta = 0$. The velocity vector in Cartesian coordinates takes the form $V = \frac{\partial\phi}{\partial t}(\sin\eta, \frac{\partial\psi}{\partial r}, \cos\eta)^\top$. Multiplying it with a cut-off function $\chi(r, y_2)$ in Ω such that $\Gamma_t \subset \mathrm{supp}\,\chi$ for $t \in (0, T)$ we derive the velocity Λ in the bounded domain Ω:

$$\Lambda = \chi\frac{\partial\phi}{\partial t}(\sin\eta, \frac{\partial\psi}{\partial r}, \cos\eta)^\top, \quad \mathrm{div}\,\Lambda = \frac{\partial\phi}{\partial t}\Big(\frac{\partial\chi}{\partial r} + \frac{\chi}{r} + \frac{\partial\chi}{\partial y_2}\frac{\partial\psi}{\partial r}\Big),$$

$$\frac{\partial\Lambda}{\partial y_1} = \begin{pmatrix} \sin^2\eta\frac{\partial\phi}{\partial t}\frac{\partial\chi}{\partial r} + \chi\frac{\cos\eta}{r}\big(\sin\eta\frac{\partial^2\phi}{\partial t\partial\eta} + \cos\eta\frac{\partial\phi}{\partial t}\big) \\ \sin\eta\frac{\partial\phi}{\partial t}\big(\frac{\partial\chi}{\partial r}\frac{\partial\psi}{\partial r} + \chi\frac{\partial^2\psi}{\partial r^2}\big) + \chi\frac{\cos\eta}{r}\big(\frac{\partial\psi}{\partial r}\frac{\partial^2\phi}{\partial t\partial\eta} + \frac{\partial^2\psi}{\partial r\partial\eta}\frac{\partial\phi}{\partial t}\big) \\ \sin\eta\cos\eta\frac{\partial\phi}{\partial t}\frac{\partial\chi}{\partial r} + \chi\frac{\cos\eta}{r}\big(\cos\eta\frac{\partial^2\phi}{\partial t\partial\eta} - \sin\eta\frac{\partial\phi}{\partial t}\big) \end{pmatrix},$$

$$\frac{\partial\Lambda}{\partial y_2} = \frac{\partial\phi}{\partial t}\frac{\partial\chi}{\partial y_2}\Big(\sin\eta, \frac{\partial\psi}{\partial r}, \cos\eta\Big)^\top,$$

$$\frac{\partial\Lambda}{\partial y_3} = \begin{pmatrix} \sin\eta\cos\eta\frac{\partial\phi}{\partial t}\frac{\partial\chi}{\partial r} - \chi\frac{\sin\eta}{r}\big(\sin\eta\frac{\partial^2\phi}{\partial t\partial\eta} + \cos\eta\frac{\partial\phi}{\partial t}\big) \\ \cos\eta\frac{\partial\phi}{\partial t}\big(\frac{\partial\chi}{\partial r}\frac{\partial\psi}{\partial r} + \chi\frac{\partial^2\psi}{\partial r^2}\big) - \chi\frac{\sin\eta}{r}\big(\frac{\partial\psi}{\partial r}\frac{\partial^2\phi}{\partial t\partial\eta} + \frac{\partial^2\psi}{\partial r\partial\eta}\frac{\partial\phi}{\partial t}\big) \\ \cos^2\eta\frac{\partial\phi}{\partial t}\frac{\partial\chi}{\partial r} - \chi\frac{\sin\eta}{r}\big(\cos\eta\frac{\partial^2\phi}{\partial t\partial\eta} - \sin\eta\frac{\partial\phi}{\partial t}\big) \end{pmatrix},$$

which provides a formula for P_t' in in (84). Its particular form for the operator A of linear elasticity is presented in [15]. With the normal vector at the surface $y_2 - \psi(r, \eta) = 0$:

$$\nu^t = Z^{-1/2}\Big(-\cos\eta\frac{\partial\psi}{\partial\eta} - \sin\eta r\frac{\partial\psi}{\partial r}, r, \sin\eta\frac{\partial\psi}{\partial\eta} - \cos\eta r\frac{\partial\psi}{\partial r}\Big)^\top, \quad Z = r^2 + \Big(\frac{\partial\psi}{\partial\eta}\Big)^2 + \Big(r\frac{\partial\psi}{\partial r}\Big)^2,$$

calculation of the boundary term I_t in (83) implies

$$I_t = \langle \lambda^t, \chi\frac{\partial\phi}{\partial t} Z^{-3/2}\big(-(\cos\eta ra + \sin\eta b)[\![u_1^t]\!] + c[\![u_2^t]\!] + (\sin\eta ra - \cos\eta b)[\![u_3^t]\!]\big)\rangle_{\Gamma_t},$$

where

$$a = \Big(1 + \Big(\frac{\partial\psi}{\partial r}\Big)^2\Big)\Big(\frac{\partial\psi}{\partial\eta} - r\frac{\partial^2\psi}{\partial r\partial\eta}\Big) + r\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial\eta}\frac{\partial^2\psi}{\partial r^2},$$

$$b = -\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial\eta}\Big(\frac{\partial\psi}{\partial\eta} - r\frac{\partial^2\psi}{\partial r\partial\eta}\Big) - \Big(r^2 + \Big(\frac{\partial\psi}{\partial\eta}\Big)^2\Big)r\frac{\partial^2\psi}{\partial r^2}, \quad c = -\frac{\partial\psi}{\partial\eta}\Big(\frac{\partial\psi}{\partial\eta} - r\frac{\partial^2\psi}{\partial r\partial\eta}\Big) + r^3\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial r^2}.$$

Cracks propagating along a parametric surface represented in spherical coordinates. Let a family of cracks in $\Omega \subset \mathbb{R}^3$ be given on the surface $\theta - \psi(r,\eta) = 0$ in spherical coordinates $\{y_1 = r\sin\theta\sin\eta, y_2 = r\cos\theta, r_3 = r\sin\theta\cos\eta\}$ as

$$\Gamma_t = \{(r,\theta,\eta)^\top \in \mathbb{R}^3_+ : \quad [r - \phi(t,\eta)]^+ + |\theta - \psi(r,\eta)| = 0\} \cap \Omega$$

with the crack front $\gamma_t = \{(r,\theta,\eta)^\top \in \mathbb{R}^3_+ : r = \phi(t,\eta), \theta = \psi(\phi(t,\eta),\eta)\}$. We assume that a cut-off function $\chi(r)$ in Ω exists such that $\Gamma_t \subset \operatorname{supp}\chi$ for $t \in (0,T)$ with some $T > 0$. The transport equation in spherical coordinates takes the form

$$\frac{\partial\rho}{\partial t} + V_r\frac{\partial\rho}{\partial r} + V_\theta\frac{1}{r}\frac{\partial\rho}{\partial\theta} + V_\eta\frac{1}{r\sin\theta}\frac{\partial\rho}{\partial\eta} = 0 \quad \text{a.e. } (0,T)\times\mathbb{R}^3_+.$$

It is satisfied by $\rho(t,r,\theta,\eta) = [r - \phi(t,\eta)]^+ + |\theta - \psi(r,\eta)|$ and $V_r = \frac{\partial\phi}{\partial t}$, $V_\theta = r\frac{\partial\phi}{\partial t}\frac{\partial\psi}{\partial r}$, $V_\eta = 0$. Hence we can express the velocity vector in Cartesian coordinates as

$$\Lambda = \chi\frac{\partial\phi}{\partial t}\Big(\sin\eta\Big(\sin\theta - \cos\theta r\frac{\partial\psi}{\partial r}\Big), \cos\theta - \sin\theta r\frac{\partial\psi}{\partial r}, \cos\eta\Big(\sin\theta + \cos\theta r\frac{\partial\psi}{\partial r}\Big)\Big)^\top.$$

The normal vector to the surface $\theta - \psi(r,\eta) = 0$ is determined as

$$\nu^t = Z^{-1/2}\begin{pmatrix} \frac{\cos\theta\sin\eta}{r} - \sin\theta\sin\eta\frac{\partial\psi}{\partial r} - \frac{\cos\eta}{r\sin\theta}\frac{\partial\psi}{\partial\eta} \\ -\frac{\sin\theta}{r} - \cos\theta\frac{\partial\psi}{\partial r} \\ \frac{\cos\theta\cos\eta}{r} - \sin\theta\cos\eta\frac{\partial\psi}{\partial r} + \frac{\sin\eta}{r\sin\theta}\frac{\partial\psi}{\partial\eta} \end{pmatrix}, \quad Z = \frac{1}{r^2} + \Big(\frac{\partial\psi}{\partial r}\Big)^2 + \Big(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\eta}\Big)^2.$$

The substitution of last two expressions in (83) and (84) will provide us with a formula for calculation of the derivative L'_t.

In conclusion, using the transport equation, first we constructed the smooth velocity vector fields for the families of propagating cracks represented by parametric curves and surfaces. Secondly, we obtained the formulas for the derivative of the energy functional with respect to the shape of these curves and surfaces. In this construction the level-set formulation of cracks given in Sect. 2 is used as a tool for the perturbation problem of Sect. 3.

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References

[1] L. Ambrosio and H. M. Soner, Level set approach to mean curvature flow in arbitrary codimension, J. Differential Geom. **43**, 693–737 (1996).
[2] M. Burger, A level set method for inverse problems, Inverse Problems **47** (5), 1327–1355 (2001).
[3] M. Burger, B. Hackl, and W. Ring, Incorporating topological derivatives into level set methods, J. Comput. Phys. **194**, 344–362 (2004).
[4] M. G. Crandall and P. L. Lions, On existence and uniqueness of solutions of Hamilton-Jacobi equations, Nonlinear Anal. T M A **10**, 353–370 (1986).
[5] M. C. Delfour and J.-P. Zolesio, Shapes and Geometries (SIAM, Philadelphia, 2001).
[6] P. J. DiPerna and P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. **98**, 511–547 (1989).
[7] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions (CRC Press, Boca Raton, 1992).
[8] A. F. Filippov, Differential equations with discontinuous right-hand side, Amer. Math. Soc. Transl. Ser. 2, Vol. 42, pp. 199–231 (AMS, Providence, 1964).

[9] A. Friedman and Y. Liu, Propagation of cracks in elastic media, Arch. Rat. Mech. Anal. **136** (3), 235–290 (1996).
[10] M. Hintermüller and W. Ring, A second order shape optimization approach for image segmentation, SIAM J. Appl. Math. **64** (2), 442–467 (2003/04).
[11] K. Ito, K. Kunisch, and Z. Li, Level-set function approach to an inverse interface problem, Inverse Problems **17** (5), 1225–1242 (2001).
[12] A. M. Khludnev and V. A. Kovtunenko, Analysis of Cracks in Solids (WIT-Press, Southampton, Boston, 2000).
[13] A. M. Khludnev, K. Ohtsuka, and J. Sokolowski, On derivative of energy functional for elastic bodies with cracks and unilateral conditions, Quart. Appl. Math. **60** (1), 99–109 (2002).
[14] A. M. Khludnev and J. Sokolowski, Modelling and Control in Solid Mechanics (Birkhäuser, Basel, Boston, Berlin, 1997).
[15] V. A. Kovtunenko, Sensitivity of interfacial cracks to non-linear crack front perturbations, J. Appl. Math. Mech. (ZAMM) **82** (6), 387–398 (2002).
[16] V. A. Kovtunenko, Shape sensitivity of a plane crack front, Math. Meth. Appl. Sci. **26**, 359–374 (2003).
[17] V. A. Kovtunenko, Shape sensitivity of curvilinear cracks on interface to non-linear perturbations, J. Appl. Math. Phys. (ZAMP) **54** (3), 410–423 (2003).
[18] V. A. Kovtunenko, Primal-dual methods of shape sensitivity analysis for curvilinear cracks with non-penetration, IMA J. Appl. Math. **71** (5), 635–657 (2006).
[19] V. A. Kovtunenko, K. Kunisch, and W. Ring, Propagation and bifurcation of cracks based on implicit surfaces and discontinuous velocities, Comput. Visual. Sci., to appear.
[20] P. L. Lions, Generalized Solutions of Hamilton-Jacobi Equations, Research Notes Math. 69 (Pitman, Boston, 1982).
[21] A. Litman, D. Lesselier, and F. Santosa, Reconstruction of a two-dimensional binary obstacle by controlled evolution of a level-set, Inverse Problems **14**, 685–706 (1998).
[22] S. Osher and R. Fedkiw, Level Set Methods and Dynamic Implicit Surfaces, Appl. Math. Sci. **153** (Springer, NY, 2003).
[23] S. Osher and J. Sethian, Fronts propagating with curvature dependent speed: Algorithms based on Hamilton-Jacobi formulations, J. Comput. Phys. **79**, 12–49 (1988).
[24] G. Petrova and B. Popov, Linear transport equations with μ-monotone coefficients, J. Math. Anal. Appl. **260**, 307–324 (2001).
[25] J. R. Rice, First-order variation in elastic fields due to variation in location of a planar crack front, J. Appl. Mech. **52**, 571–579 (1985).
[26] J. A. Sethian, Level Set Methods: Evolving interfaces in geometry, fluid mechanics, computer vision, and material science (Cambridge University Press, 1996).
[27] J. Sokolowski and J.-P. Zolesio, Introduction to Shape Optimization. Shape Sensitivity Analysis (Springer-Verlag, Berlin, 1992).
[28] M. Stolarska and D. L. Chopp, Modelling thermal fatigue cracking in integrated circuits by level sets and the extended finite element method, Int. J. Engng. Sci. **41**, 2381–2410 (2003).
[29] N. Sukumar, N. Moës, B. Moran, and T. Belytschko, Extended finite element method for three-dimensional crack modelling, Int. J. Numer. Meth. Engng. **48**, 1549–1570 (2000).

