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Nonconvex problem for crack with nonpenetration

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The problem with crack under nonlinear boundary conditions is considered as a minimization of the total potential energy functional. The functional is nonconvex by assuming the surface energy at a crack presented in a general form. The correctness properties of the nonconvex minimization problem with constraints are investigated. Applying the shape sensitivity analysis, the problem of shape perturbation is formulated, and the derivative of the total potential energy functional with respect to the perturbation parameter is calculated. Examples on the rectilinear and the planar cracks are presented.

1 Motivation and modeling

The classical formulation of problems with cracks implies the linear boundary condition of the stress-free faces of a crack. Analysis of the linear crack problems using the method of matched expansions is founded in [16]. To prevent overlapping of the crack faces possible in the linear case, in [9] the nonpenetration condition was suggested. For collection of such nonlinear crack models with nonpenetration and their mathematical foundations, see [10]. The problems with conditions of nonpenetration together with the given friction at a crack was studied in [1], with the Coulomb friction in [12].

On the other hand, the quasistatic process of crack propagation can be treated as an energy minimization problem, see the related works [5,6,17]. This question is closely connected with the derivative of potential energy with respect to parameters of the crack shape, the so-called energy release rate [4,8,18,19]. Following [20], the shape sensitivity technique was recently extended for the linear and nonlinear crack problems to calculate the first derivative of energy [11] and corresponding local characteristics of the crack growth [13,14].

Within an energetic approach the fracture criterion is usually based on the Griffith hypothesis [7]. This implies the constant density of the surface energy distributed at a crack, that is presented in Fig. 1 as the constant function $g(t) = g_0$ in dependence on the crack opening t, $t \geq 0$. For refining the Griffith hypothesis, the cohesive forces at the crack should be taken into consideration, see [2,3,15]. This refinement leads to nonlinear functions $g(t)$ of the density of surface energy with the principal condition $g(0) = 0$, $g'(0) < \infty$, which allows the crack faces to close up in the vicinity of the crack. In Fig. 1 we present schematically such curves in dependence of the crack opening and corresponding to the cohesive, cohesive with softening, and elastoplasticity conditions. Excepting the linear case, the function describing the surface energy density is

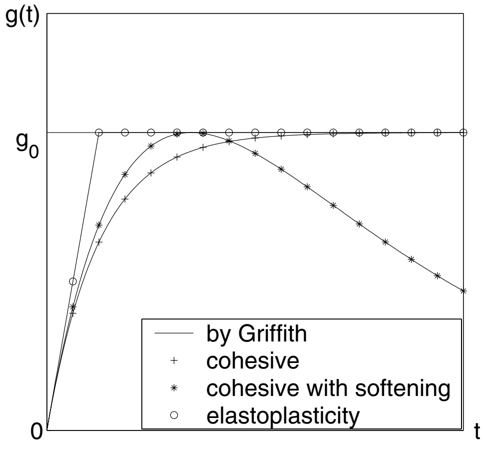

Fig. 1 Functions g of the density of surface energy.

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in general nonconvex, or may be concave. Mathematically this leads to the nonconvex constrained minimization problem, which is under our consideration. To manage this problem, in Sect. 2 we prove its solvability basing on the weakly lower semicontinuity and coercivity properties of the energy functional. For regular functions g we formulate the necessary condition for the minimizer in the form of variational inequality. The unique solution of this inequality provides us the necessary and sufficient conditions for the minimizer. The last two sections of the paper deals with the crack perturbation problem connected with stability of the crack and its advance.

2 Nonconvex constrained minimization problem with crack

The nonconvex constrained minimization problem for cracks is formulated and analyzed with respect to the well-posedness property in the first subsection, and the optimality conditions are argued in the second subsection.

2.1 Formulation and well-posedness of the optimization problem

Let $\Omega \subset \mathbf{R}^N$, $N = 2, 3$, be a bounded domain and its boundary Γ consist of two parts $\Gamma_\mathcal{D}$ and $\Gamma_\mathcal{N}$. Let us suppose the crack Γ_0 located inside Ω as an open curve or a surface as $N = 2$ or $N = 3$, respectively. We assume that the domain Ω can be split into two subdomains with Lipschitz-continuous boundaries and common interface Σ, which intersects $\Gamma_\mathcal{D}$, such that $\Gamma_0 \subset \Sigma$. By choosing the unit normal vector $\nu^0 = (\nu_1^0, \ldots, \nu_N^0)$ to Γ_0, two opposite crack faces $\Gamma_0^\pm$ can be distinguished, which correspond to $\pm\nu^0$, respectively. We denote $\Omega_0 = \Omega \setminus \overline{\Gamma}_0$. The geometric assumptions formulated above provide us a correctness of the variational formulation of elasticity problem in the domain Ω_0 with the crack Γ_0.

Let us consider the linear elasticity model of nonhomogeneous anisotropic solid. For the displacement vector $u = (u_1, \ldots, u_N)$ we introduce the symmetric tensors of strains and stresses

$$\sigma_{ij}(u) = c_{ijkl}\epsilon_{kl}(u)\,, \quad \epsilon_{ij}(u) = 0.5(u_{i,j} + u_{j,i})\,, \quad i,j = 1, \ldots, N\,,$$

with the elasticity coefficients $c_{ijkl} \in L^\infty\left(\mathbf{R}^N\right)$, $i,j,k,l = 1, \ldots, N$,

$$c_{ijkl} = c_{jikl} = c_{klij}\,, \quad c_0\xi_{ij}\xi_{ij} \le c_{ijkl}\xi_{kl}\xi_{ij} \le C_0\xi_{ij}\xi_{ij}\,, \quad c_0, C_0 > 0\,.$$

Let $f = (f_1, \ldots, f_N) \in L^2(\mathbf{R}^N)^N$ be a given volume load. We define the total potential energy of the solid in domain Ω_0 as the sum of potential and surface energies, namely

$$\begin{aligned} T\left(u;\Omega_0\right) &= P\left(u;\Omega_0\right) + S\left(u;\Omega_0\right)\,, \\ P\left(u;\Omega_0\right) &= 0.5\int_{\Omega_0} \sigma_{ij}\left(u\right)\epsilon_{ij}\left(u\right)\,dx - \int_{\Omega_0} f_i u_i\,dx\,, \\ S\left(u;\Omega_0\right) &= \int_{\Gamma_0} g\left(\llbracket u \rrbracket\,\nu^0\right)\,ds. \end{aligned} \tag{1}$$

Here g means a density of the surface energy distributed at the crack. It depends on the crack opening $\llbracket u \rrbracket\,\nu^0$, where $\llbracket u \rrbracket$ denotes the jump $u|_{\Gamma_0^+} - u|_{\Gamma_0^-}$. From the physical point of view, the function g can obey the following principal cases as presented in Fig. 1. In the general case, we assume $g \in C^{0,1}(\overline{\mathbf{R}})$ and

$$g \ge 0 \quad \text{on } \mathbf{R}^+\,. \tag{2}$$

For displacements we introduce the Sobolev space

$$\widetilde{H}^1\left(\Omega_0\right) = \{u = (u_1, \ldots, u_N) \in H^1\left(\Omega_0\right)^N\,, \quad u = 0 \quad \text{on } \Gamma_\mathcal{D}\}\,,$$

which includes the condition of clamping at the part $\Gamma_\mathcal{D}$ of external boundary Γ. By physical reasons, the overlapping of crack faces is not acceptable. This fact can be described with the help of the nonpenetration condition as follows:

$$\llbracket u \rrbracket\,\nu^0 \ge 0 \quad \text{on } \Gamma_0\,. \tag{3}$$

This gets the convex closed set of admissible displacements

$$K_0 = \{u \in \widetilde{H}^1\left(\Omega_0\right)\,, \quad u \quad \text{satisfies(3)}\}\,.$$

We formulate the problem of equilibrium of the solid with crack under given load as a minimization of the total potential energy functional,

$$T\left(u^0;\Omega_0\right) \le T\left(v;\Omega_0\right) \quad \forall v \in K_0\,, \tag{4}$$

where the function T is presented by relations (1). Note that, for the Griffith case $g = \mathrm{const}$ problem (4) coincides with the classical problem of minimization of the potential energy

$$P\left(u^0;\Omega_0\right) \le P\left(v;\Omega_0\right) \quad \forall v \in K_0\,.$$

Theorem 1. *There exists the solution $u^0 \in K_0$ of the minimization problem* (4).

Proof. Due to the properties (2) and (3) we have $S(u;\Omega_0) \ge 0$ for all $u \in K_0$. Therefore, T is coercive on K_0. The functional P is quadratic and then weakly lower semicontinuous. Let $u^n \to u$ weakly in $\widetilde{H}^1(\Omega_0)$ as $n \to \infty$. By the compactness property we have $[\![u^n]\!]\nu^0 \to [\![u]\!]\nu^0$ strongly in $L^2(\Gamma_0)$ as $n \to \infty$. From the Lipschitz-continuity property of g, it follows that

$$|S\left(u^n;\Omega_0\right) - S\left(u;\Omega_0\right)| \le c_1 \|\, [\![u^n]\!]\,\nu^0 - [\![u]\!]\,\nu^0\|_{0,\Gamma_0}$$

and S is a weakly continuous functional. Therefore, the functional T is weakly lower semicontinuous.

Now take the minimizing sequence $u^n \in K_0$ such that $T(u^n;\Omega_0) \to T_0 = \inf\limits_{v\in K_0} T(v;\Omega_0)$. The coercivity of T implies the boundedness of u^n. Then $u^n \to u^0$ weakly in $\widetilde{H}^1(\Omega_0)$ as $n \to \infty$ and $u^0 \in K_0$ because of the weak closedness of K_0. From the weakly lower semicontinuity property of T the estimate follows

$$T_0 \le T\left(u^0;\Omega_0\right) \le \liminf T\left(u^n;\Omega_0\right) = T_0\,.$$

This completes the proof. □

2.2 Optimality conditions

We establish the necessary and sufficient optimality conditions for the optimization problem (4) provided that g satisfies additional regularity properties.

Theorem 2. *If $g \in C^1(\overline{\mathbf{R}})$, then the following variational inequality*

$$\int\limits_{\Omega_0} \sigma_{ij}\left(u^0\right)\epsilon_{ij}\left(v-u^0\right)\,dx + \int\limits_{\Gamma_0} g'\left([\![u^0]\!]\,\nu^0\right)\left([\![v]\!]\,\nu^0 - [\![u^0]\!]\,\nu^0\right)\,ds \ge \int\limits_{\Omega_0} f_i\left(v-u^0\right)_i\,dx \quad \forall v \in K_0 \tag{5}$$

yields the necessary condition of solvability for the minimization problem (4).

Proof. To prove Theorem 2, we take the element $w = tv + (1-t)u^0$ with $v \in K_0$, $0 < t < 1$, substitute it into the inequality $T(u^0;\Omega_0) \le T(w;\Omega_0)$ divided with t, and then pass to the lower limit at $t \to 0$ due to the differentiability and weakly lower semicontinuity properties of T. □

Assuming that the solution $u^0 \in K_0$ of (5) is smooth enough, we can apply the Green formula to deduce formally from the variational inequality (5) the following relations

$$\begin{cases} -\sigma_{ij,j}\left(u^0\right) = f_i\,, & i = 1,\dots,N\,,\quad \text{in } \Omega_0\,;\\ u^0 = 0\,, & \text{on } \Gamma_{\mathcal{D}}\,;\\ \sigma_{ij}\left(u^0\right)\nu_j = 0\,, & i = 1,\dots,N\,,\quad \text{on } \Gamma_{\mathcal{N}}\,;\\ \left[\!\left[\sigma_{ij}\left(u^0\right)\nu_j^0\right]\!\right] = 0\,,\quad \sigma_{ij}\left(u^0\right)\nu_j^0 - \sigma_{kj}\left(u^0\right)\nu_j^0\nu_k^0\nu_i^0 = 0\,, & i = 1,\dots,N\,,\\ [\![u^0]\!]\,\nu^0 \ge 0\,,\quad \sigma_{kj}\left(u^0\right)\nu_j^0\nu_k^0 - g'\left([\![u^0]\!]\,\nu^0\right) \le 0, & \text{on } \Gamma_0\,.\\ \left(\sigma_{kj}\left(u^0\right)\nu_j^0\nu_k^0 - g'\left([\![u^0]\!]\,\nu^0\right)\right)[\![u^0]\!]\,\nu^0 = 0, & \end{cases} \tag{6}$$

When the crack Γ_0 is of the $C^{1,1}$-class, the crack opening and stresses at the crack in (6) can be defined as functions from the space $H^{1/2}_{00}(\Gamma_0)$ and its dual space $H^{1/2}_{00}(\Gamma_0)^\star$, respectively. For details see [10]. From the mechanical point of view, the term $g'([\![u^0]\!]\nu^0)$ describes the cohesive force between crack faces.

Theorem 3. *If* $g \in C^{1,1}\left(\overline{\mathbf{R}}\right)$ *with* $|g'(t) - g'(s)| \leq c_2|t - s|\ \forall t, s$, *and the uniform estimate holds*

$$(c_0 - \delta)\, \|u\|^2_{1,\Omega_0} - c_2 \|\, [\![u]\!]\, \nu^0\|^2_{0,\Gamma_0} \geq 0 \quad \forall u \in \widetilde{H}^1(\Omega_0)\ , \quad \delta > 0\,, \tag{7}$$

then problems (4) *and* (5) *are equivalent and their solution* $u^0 \in K_0$ *is unique.*

Proof. Firstly, in view of Theorem 1 and Theorem 2, for such g there exists a solution to problem (5). It can be not unique. Secondly, consider two variational inequalities (5) for two different solutions u^1 and u^2, respectively. Substituting $v = u^2$ in the first inequality, $v = u^1$ in the second one, and summing them, in a standard way one gets

$$\int\limits_{\Omega_0} \sigma_{ij}\left(u^1 - u^2\right) \epsilon_{ij}\left(u^1 - u^2\right)\, dx + \int\limits_{\Gamma_0} \left(g'\left([\![u^1]\!]\, \nu^0\right) - g'\left([\![u^2]\!]\, \nu^0\right)\right)\left([\![u^1]\!]\, \nu^0 - [\![u^2]\!]\, \nu^0\right)\, ds \leq 0\,.$$

Therefore, due to the Lipschitz-continuity property of g', the fulfillment of assumption (7) leads to the conclusion that $\delta \|u^1 - u^2\|^2_{1,\Omega_0} \leq 0$ and the solution of problem (5) is unique. Because any solution of the minimization problem (4) fulfills (5), it is also unique and coincides with the solution $u^0 \in K_0$ of the variational inequality (5). The proof is completed. □

The required estimate (7) can be provided by that g, which is small in the C^2-norm. The examples are $g = \mathrm{const}$, $g' = \mathrm{const}$, $g = g_0 p$ with constant $g_0 << \|p\|_{C^2}$.

3 Crack perturbation problem

In three subsections we perturb the crack shape with the help of a coordinate transformation, establish the shape differentiability of the total potential energy functional, and give the example of planar cracks, respectively.

3.1 Perturbed problem with the crack

For small parameter ε we introduce the perturbation of shape given by the function $\Phi \in C^1(\mathbf{R}; W^{1,\infty}(\mathbf{R}^N))^N$. Let $\Phi(0)(x) = x$, $x \in \mathbf{R}^N$, i.e. $\Phi(\varepsilon)$ is a small perturbation of the identity operator. We fix ε. The coordinate transformation $y = \Phi(\varepsilon)(x)$ transforms the initial domain Ω_0 onto the perturbed domain $\Omega_\varepsilon = \Phi(\varepsilon)(\Omega) \setminus \overline{\Gamma}_\varepsilon$ with the perturbed crack $\Gamma_\varepsilon = \Phi(\varepsilon)(\Gamma_0)$. We suppose that $\Phi(\varepsilon)(\Omega)$, Γ_ε, and $\Phi(\varepsilon)(\Gamma_\mathcal{D})$ fulfill the geometric assumptions formulated for Ω, Γ_0, and $\Gamma_\mathcal{D}$, too. For ε small enough, the Jacobian of transformation admits the asymptotic expansion

$$J(\varepsilon) = \left|\partial\Phi/\partial x\right|(\varepsilon) = 1 + \varepsilon \operatorname{div} V + o(\varepsilon) \quad \text{a.e. } \mathbf{R}^N\,,$$

which provides its positiveness. Here $V \in W^{1,\infty}(\mathbf{R}^N)^N$ denotes the velocity vector $\partial\Phi/\partial\varepsilon|_{\varepsilon=0}$. Therefore, the correspondence $\Phi(\varepsilon) : \Omega_0 \to \Omega_\varepsilon$ is one-to-one, i.e. there exists the inverse transformation $\Phi^{-1}(\varepsilon) : \Omega_\varepsilon \to \Omega_0$ with $\Phi^{-1}(\varepsilon) \in W^{1,\infty}(\mathbf{R}^N)^N$.

Let ν^ε be a unit normal vector to Γ_ε. We introduce the Sobolev space

$$\widetilde{H}^1(\Omega_\varepsilon) = \{u \in H^1(\Omega_\varepsilon)^N\ , \quad u = 0 \quad \text{on } \Phi(\varepsilon)(\Gamma_\mathcal{D})\}$$

and the convex closed set

$$K_\varepsilon = \{u \in \widetilde{H}^1(\Omega_\varepsilon)\ , \quad [\![u]\!]\, \nu^\varepsilon \geq 0 \quad \text{on } \Gamma_\varepsilon\}\,,$$

where $[\![u]\!]$ denotes here the jump of u on Γ_ε. Due to the regularity properties marked above, the transformation $\Phi(\varepsilon)$ yields the one-to-one correspondence between $\widetilde{H}^1(\Omega_0)$ and $\widetilde{H}^1(\Omega_\varepsilon)$. This means that $u \circ \Phi(\varepsilon) \in \widetilde{H}^1(\Omega_0)$ for all $u \in \widetilde{H}^1(\Omega_\varepsilon)$, and $u \circ \Phi^{-1}(\varepsilon) \in \widetilde{H}^1(\Omega_\varepsilon)$ for all $u \in \widetilde{H}^1(\Omega_0)$. We assume the one-to-one correspondence between the sets K_0 and K_ε, too. The sufficient condition for this assumption can be provided by the geometric constraint

$$\nu^\varepsilon \circ \Phi(\varepsilon) = \nu^0\,. \tag{8}$$

To fulfill condition (8), in the example of Sect. 3.3 we consider planar cracks with $\nu^\varepsilon = \nu^0 = \mathrm{const}$.

Similar to (1), we introduce the functional of the total potential energy in the perturbed domain with crack by

$$T(u; \Omega_\varepsilon) = 0.5 \int\limits_{\Omega_\varepsilon} \sigma_{ij}(u)\epsilon_{ij}(u)\, dy - \int\limits_{\Omega_\varepsilon} f_i u_i\, dy + \int\limits_{\Gamma_\varepsilon} g\left([\![u]\!]\, \nu^\varepsilon\right)\, ds\,, \quad u \in K_\varepsilon\,, \tag{9}$$

and consider the minimization problem

$$T\left(u^{\varepsilon};\Omega_{\varepsilon}\right)\leq T\left(v;\Omega_{\varepsilon}\right)\quad\forall v\in K_{\varepsilon}\,. \tag{10}$$

Following Theorem 1, there exists the solution $u^{\varepsilon}\in K_{\varepsilon}$ of problem (10).

Let us now apply the coordinate transformation $\Phi(\varepsilon)$ to the integrals in formula (9). Due to the assumption (8), this gets the representation $T(u;\Omega_{\varepsilon})=T(\varepsilon)(u\circ\Phi(\varepsilon);\Omega_0)$ with the following functional

$$\begin{aligned}T(\varepsilon)\,(u;\Omega_0)&=0.5\int\limits_{\Omega_0}J(\varepsilon)(c_{ijkl}\circ\Phi(\varepsilon))E_{kl}(\Psi(\varepsilon);u)E_{ij}(\Psi(\varepsilon);u)\,dx\\&\qquad-\int\limits_{\Omega_0}J(\varepsilon)(f_i\circ\Phi(\varepsilon))u_i\,dx+\int\limits_{\Gamma_0}J(\varepsilon)|\Psi(\varepsilon)\nu^0|g\left(\llbracket u\rrbracket\,\nu^0\right)\,ds\,,\quad u\in K_0\,,\end{aligned} \tag{11}$$

where $E_{ij}(\Psi(\varepsilon);u)=0.5(u_{i,k}\Psi_{kj}(\varepsilon)+u_{j,k}\Psi_{ki}(\varepsilon))$ and $\Psi(\varepsilon)=(\partial\Phi/\partial x)^{-1}(\varepsilon)$. According to [20], the boundary term in (11) possesses an expansion in ε as

$$\begin{aligned}J(\varepsilon)|\Psi(\varepsilon)\nu^0|&=1+\varepsilon\,\mathrm{div}_{\Gamma_0}(V)+o(\varepsilon)\,,\\\mathrm{div}_{\Gamma_0}(V)&=\mathrm{div}(V)+(\partial V/\partial x)_{ij}\nu_j^0\nu_i^0.\end{aligned}$$

In view of the one-to-one correspondence between K_0 and K_{ε}, it follows from (10) that

$$T(\varepsilon)\left(u^{\varepsilon}\circ\Phi(\varepsilon);\Omega_0\right)=T\left(u^{\varepsilon};\Omega_{\varepsilon}\right)\leq T\left(v\circ\Phi^{-1}(\varepsilon);\Omega_{\varepsilon}\right)=T(\varepsilon)(v;\Omega_0)\,,\quad v\in K_0\,.$$

Therefore, the following theorem is true.

Theorem 4. *Under assumption* (8)*, the transformed function* $u^{\varepsilon}\circ\Phi(\varepsilon)\in K_0$ *solves the minimization problem*

$$T(\varepsilon)\left(u^{\varepsilon}\circ\Phi(\varepsilon);\Omega_0\right)\leq T(\varepsilon)\left(v;\Omega_0\right)\quad\forall v\in K_0\,. \tag{12}$$

3.2 Shape differentiablity of the energy

For the perturbation problem we assume the enough smoothness of data, namely $f\in C^1(\mathbf{R}^N)^N$ and $c_{ijkl}\in C^1(\mathbf{R}^N)$, $i,j,k,l=1,\ldots,N$. Applying then the Taylor expansion, it follows from formula (11) the asymptotic representations

$$\begin{aligned}T(\varepsilon)\,(u;\Omega_0)&=T\left(u;\Omega_0\right)+R_1\left(\varepsilon,u;\Omega_0\right)\,,\\|R_1\left(\varepsilon,u;\Omega_0\right)|&\leq O\left(\varepsilon\right)r_1\left(\|u\|_{1,\Omega_0}\right),\end{aligned} \tag{13}$$

and

$$\begin{aligned}T(\varepsilon)\,(u;\Omega_0)&=T\left(u;\Omega_0\right)+\varepsilon\,T'\left(V,u;\Omega_0\right)+R_2\left(\varepsilon,u;\Omega_0\right)\,,\\|R_2\left(\varepsilon,u;\Omega_0\right)|&\leq o(\varepsilon)r_2\left(\|u\|_{1,\Omega_0}\right),\end{aligned} \tag{14}$$

with continuous functionals R_1, R_2 and positive quadratic functions r_1, r_2, respectively. The functional T' in expansion (14) has the form:

$$\begin{aligned}T'\left(V,u;\Omega_0\right)&=\int\limits_{\Omega_0}\left[0.5\,\mathrm{div}(Vc_{ijkl})\epsilon_{kl}(u)\epsilon_{ij}(u)-\sigma_{ij}(u)E_{ij}(\partial V/\partial x;u)\right]dx\\&\qquad-\int\limits_{\Omega_0}\mathrm{div}(Vf_i)u_i\,dx+\int\limits_{\Gamma_0}\mathrm{div}_{\Gamma_0}(V)g\left(\llbracket u\rrbracket\,\nu^0\right)\,ds\,.\end{aligned} \tag{15}$$

Using Theorem 4, we substitute $v=0$ in inequality (12) and can evaluate

$$\begin{aligned}&\int\limits_{\Gamma_0}J(\varepsilon)|\Psi(\varepsilon)\nu^0|g(0)\,ds\geq T(\varepsilon)\left(u^{\varepsilon}\circ\Phi(\varepsilon);\Omega_0\right)\\&\qquad\geq0.5\int\limits_{\Omega_0}\sigma_{ij}\left(u^{\varepsilon}\circ\Phi(\varepsilon)\right)\epsilon_{ij}\left(u^{\varepsilon}\circ\Phi(\varepsilon)\right)\,dx-\int\limits_{\Omega_0}f_i\left(u^{\varepsilon}\circ\Phi(\varepsilon)\right)_i\,dx-O(\varepsilon)r_1\left(\|u^{\varepsilon}\circ\Phi(\varepsilon)\|_{1,\Omega_0}\right)\end{aligned}$$

in view of expansion (13). Therefore, for ε small enough this gets the uniform estimate

$$\|u^\varepsilon \circ \Phi(\varepsilon)\|_{1,\Omega_0} \le U_0 \,. \tag{16}$$

Due to (16), there exists a subsequence of solutions, still marked by ε, such that

$$u^\varepsilon \circ \Phi(\varepsilon) \;\to\; u^0 \quad \text{weakly in } \widetilde{H}^1(\Omega_0) \quad \text{as } \varepsilon \to\; 0 \tag{17}$$

with the limit function $u^0 \in K_0$. Using the weakly lower semicontinuity property T, weak convergence (17), and expansion (13), we can pass to the limit in (12) in the sense

$$\begin{aligned} T\left(v;\Omega_0\right) = \lim T(\varepsilon)\left(v;\Omega_0\right) &\ge \liminf T(\varepsilon)\left(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) \\ &\ge \liminf T\left(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) - \lim O(\varepsilon) r_1\left(U_0\right) \ge T\left(u^0;\Omega_0\right) \end{aligned}$$

for any $v \in K_0$. Therefore, the limit function $u^0 \in K_0$ is a solution of the minimization problem (4). Substituting $v = u^0$ in inequality (12) and applying expansion (13) again, we deduce that

$$\begin{aligned} &0.5 \int_{\Omega_0} \sigma_{ij}\left(u^\varepsilon \circ \Phi(\varepsilon) - u^0\right) \epsilon_{ij}\left(u^\varepsilon \circ \Phi(\varepsilon) - u^0\right)\, dx \\ &\le - \int_{\Omega_0} \sigma_{ij}\left(u^0\right) \epsilon_{ij}\left(u^\varepsilon \circ \Phi(\varepsilon) - u^0\right)\, dx + \int_{\Omega_0} f_i \left(u^\varepsilon \circ \Phi(\varepsilon) - u^0\right)_i\, dx \\ &\quad - \int_{\Gamma_0} \left[g\left(\left[\!\left[u^\varepsilon \circ \Phi(\varepsilon)\right]\!\right] \nu^0\right) - g\left(\left[\!\left[u^0\right]\!\right] \nu^0\right)\right] ds + 2O(\varepsilon) r_1\left(U_0\right) \;\to\; 0 \end{aligned}$$

as $\varepsilon \to 0$ due to the weak convergence (17). Therefore,

$$u^\varepsilon \circ \Phi(\varepsilon) \;\to\; u^0 \quad \text{strongly in} \quad \widetilde{H}^1\left(\Omega_0\right) \quad \text{as} \quad \varepsilon \to\; 0 \,. \tag{18}$$

Finally, we find the derivative of the total potential energy, given by (9), with respect to the perturbation parameter ε. Let us substitute $v = u^0$ in inequality (12) and use expansion (14) to evaluate the following difference from above:

$$\begin{aligned} T(\varepsilon)\left(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) - T\left(u^0;\Omega_0\right) &\le T(\varepsilon)\left(u^0;\Omega_0\right) - T\left(u^0;\Omega_0\right) \\ &= \varepsilon T'\left(V, u^0;\Omega_0\right) + R_2\left(\varepsilon, u^0;\Omega_0\right) . \end{aligned} \tag{19}$$

Similarly, we substitute $v = u^\varepsilon \circ \Phi(\varepsilon)$ in (4) and use (14) for the estimation of the same difference from below:

$$\begin{aligned} T(\varepsilon)\left(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) - T\left(u^0;\Omega_0\right) &\ge T(\varepsilon)\left(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) - T\left(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) \\ &= \varepsilon T'\left(V, u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) + R_2\left(\varepsilon, u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) . \end{aligned} \tag{20}$$

Dividing last two inequalities with ε and passing to the limit as $\varepsilon \to 0$, due to the uniform estimate (16), expansion (14), and strong convergence (18) we conclude that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left[T(\varepsilon)\left(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) - T\left(u^0;\Omega_0\right)\right] = T'\left(V, u^0;\Omega_0\right) \;.$$

Because $T(\varepsilon)(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0) = T(u^\varepsilon;\Omega_\varepsilon)$, the following theorem holds.

Theorem 5. *Under assumptions of Theorem* 4*, the following limit exists*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left[T\left(u^\varepsilon;\Omega_\varepsilon\right) - T\left(u^0;\Omega_0\right)\right] = T'\left(V, u^0;\Omega_0\right) \;. \tag{21}$$

From Theorem 5 we can obtain the following consequence. Inequalities (19) and (20) together with formula (21) imply that

$$\begin{aligned} \lim_{\varepsilon\to 0} \varepsilon^{-1}\left[T(\varepsilon)\left(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) - T\left(u^0;\Omega_0\right)\right] &= \lim_{\varepsilon\to 0} \varepsilon^{-1}\left[T(\varepsilon)\left(u^0;\Omega_0\right) - T\left(u^0;\Omega_0\right)\right] \\ &= \lim_{\varepsilon\to 0} \varepsilon^{-1}\left[T(\varepsilon)\left(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right) - T\left(u^\varepsilon \circ \Phi(\varepsilon);\Omega_0\right)\right]. \end{aligned}$$

From the following evident decompositions

$$T(\varepsilon)\left(u^\varepsilon\circ\Phi(\varepsilon);\Omega_0\right)-T\left(u^0;\Omega_0\right)=T(\varepsilon)\left(u^\varepsilon\circ\Phi(\varepsilon);\Omega_0\right)-T(\varepsilon)\left(u^0;\Omega_0\right)+T(\varepsilon)\left(u^0;\Omega_0\right)-T\left(u^0;\Omega_0\right)$$
$$=T(\varepsilon)\left(u^\varepsilon\circ\Phi(\varepsilon);\Omega_0\right)-T\left(u^\varepsilon\circ\Phi(\varepsilon);\Omega_0\right)+T\left(u^\varepsilon\circ\Phi(\varepsilon);\Omega_0\right)-T\left(u^0;\Omega_0\right)$$

it follows two additional relations

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\big[T(\varepsilon)\left(u^\varepsilon\circ\Phi(\varepsilon);\Omega_0\right)-T(\varepsilon)\left(u^0;\Omega_0\right)\big]=0\,,$$
$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\big[T\left(u^\varepsilon\circ\Phi(\varepsilon);\Omega_0\right)-T\left(u^0;\Omega_0\right)\big]=0.$$

3.3 Example of planar crack

For sufficiently regular g let us consider the planar crack Γ_0 with the normal $\nu^0=(0,0,1)$, which lies on the plane $x_3=0$ inside the domain $\Omega\subset\mathbf{R}^3$. In the polar coordinates (r,ϕ) with $x_1=r\cos\phi$, $x_2=r\sin\phi$, we describe the crack by the function $R\in C^{0,1}([0,2\pi])$, $R>0$, $R(0)=R(2\pi)$, as

$$\Gamma_0=\{r<R(\phi)\,,\quad 0\le\phi\le 2\pi\,,\quad x_3=0\}\,.$$

Let us choose the perturbation function $h\in W^{1,\infty}(0,2\pi)$, $h(0)=h(2\pi)$. The perturbed crack Γ_ε on the same plane $x_3=0$ with $\nu^\varepsilon=\nu^0$ is considered as

$$\Gamma_\varepsilon=\{r<R(\phi)+\varepsilon h(\phi)\,,\quad 0\le\phi\le 2\pi\,,\quad x_3=0\}\subset\Omega\,.$$

Let $\eta\in W^{1,\infty}(\mathbf{R}^3)$ be a cut-off function with $\eta=1$ near the crack front $\{r=R(\phi),\ 0\le\phi\le 2\pi,\ x_3=0\}$, and $\mathrm{supp}(\eta)\subset\Omega$, $0\notin\mathrm{supp}(\eta)$. Such η can be constructed with the support in a tor surrounding the crack front.

The perturbation $\Phi(\varepsilon)$ constructed in the following way

$$\Phi(\varepsilon)=(x_1+\varepsilon h\theta_1,x_2+\varepsilon h\theta_2,x_3)\,,\quad \theta_\alpha=x_\alpha\theta,\ \alpha=1,2\,,\quad \theta=\eta/r\,,$$

transforms Γ_0 onto Γ_ε, and $\Omega_0=\Omega\setminus\overline{\Gamma}_0$ onto $\Omega_\varepsilon=\Omega\setminus\overline{\Gamma}_\varepsilon$. For the velocity vector V, in this case $V=(h\theta_1,h\theta_2,0)$, due to $\phi_{,\alpha}x_\alpha=0$ we can calculate

$$\mathrm{div}_{\Gamma_0}(V)=\mathrm{div}(V)=h\theta_{\alpha,\alpha}\,,$$
$$\mathrm{div}(Vf)=h(\theta_\alpha f)_{,\alpha}\,,$$
$$E_{ij}(\partial V/\partial x;u)=0.5(u_{i,\alpha}(h\theta_\alpha)_{,j}+u_{j,\alpha}(h\theta_\alpha)_{,i})\,,\quad i,j=1,2,3.$$

Therefore, formula (15) and Theorem 5 give us the derivative of the total potential energy functional T in the form:

$$T'(V,u^0;\Omega_0)=0.5\int_{\Omega_0}\big[h(\theta_\alpha c_{ijkl})_{,\alpha}\epsilon_{kl}\left(u^0\right)\epsilon_{ij}\left(u^0\right)$$
$$-\sigma_{ij}\left(u^0\right)(u^0_{i,\alpha}(h\theta_\alpha)_{,j}+u^0_{j,\alpha}(h\theta_\alpha)_{,i})\big]\,dx-\int_{\Omega_0}h(\theta_\alpha f_i)_{,\alpha}u^0_i\,dx+\int_{\Gamma_0}h\theta_{\alpha,\alpha}g\left(\llbracket u^0_3\rrbracket\right)\,ds$$

for any admissible function h of the crack front perturbation.

4 Advance of rectilinear crack

In this part we connect the results of the previous section to the classical J-integral for rectilinear cracks, which is related to the crack advance in fracture mechanics.

We assume the sufficient regularity for g. Let $\Omega\subset\mathbf{R}^2$, the crack be rectilinear with the normal $\nu^0=(0,1)$, namely,

$$\Gamma_0=\{0<x_1<l\,,\quad x_2=0\}\subset\Omega\,,\quad l>0\,,$$

and the unperturbed domain be $\Omega_0=\Omega\setminus\overline{\Gamma}_0$. Here parameter l means the crack length. Let $\chi\in W^{1,\infty}(\mathbf{R}^2)$ be a cut-off function such that $\chi=1$ in a small neighborhood B of the crack tip $(l,0)$ and $B\subset\mathrm{supp}(\chi)\subset B_1\subset\Omega$ with $0\notin B_1$.

We denote by $n = (n_1, n_2)$ the unit outward normal vector to ∂B, and $D = B_1 \setminus B$. Let us apply the perturbation $\Phi(\varepsilon)(x) = (x_1 + \varepsilon\chi(x), x_2)$, $x \in \mathbf{R}^2$, with the velocity vector $V = (\chi, 0)$. It transforms Ω_0 onto the domain $\Omega_\varepsilon = \Omega \setminus \overline{\Gamma}_\varepsilon$ with the crack

$$\Gamma_\varepsilon = \{0 < x_1 < l + \varepsilon\,, \quad x_2 = 0\} \subset \Omega$$

of the perturbed length $l + \varepsilon$. The chosen normal $\nu^\varepsilon = (0, 1)$ coincides with ν^0, then condition (8) is fulfilled.

By Theorem 5, in this case from formula (15) we obtain the derivative of the total potential energy in the form

$$T'\left(V, u^0; \Omega_0\right) = 0.5 \int\limits_{\Omega_0} \left[(\chi c_{ijkl})_{,1} \epsilon_{kl}\left(u^0\right) \epsilon_{ij}\left(u^0\right) - \sigma_{ij}\left(u^0\right) \left(u^0_{i,1}\chi_{,j} + u^0_{j,1}\chi_{,i}\right) \right] dx - \int\limits_{\Omega_0} (\chi f_i)_{,1} u^0_i \, dx + \int\limits_{\Gamma_0} \chi_{,1} g\left(\left[\!\left[u^0_2\right]\!\right]\right) \, ds \,. \tag{22}$$

The solution $u^0 \in K_0$ of problem (5) possesses the additional H^2-regularity outside of neighborhoods of the crack tips, i.e. $u^0 \in H^2(\overline{D})$. Therefore, in $\Omega_0 \setminus \overline{B}$ we can integrate by parts the corresponding integrals in expression (22). It gets

$$T'\left(V, u^0; \Omega_0 \setminus \overline{B}\right) = \int\limits_{\Omega_0 \setminus B} \chi u^0_{i,1} \left(\sigma_{ij,j}\left(u^0\right) + f_i\right) \, dx + \int\limits_{\partial B} f_i u^0_i n_1 \, ds - \int\limits_{\partial B} \sigma_{ij}\left(u^0\right) \left[0.5 \epsilon_{ij}\left(u^0\right) n_1 - u^0_{i,1} n_j\right] \, ds + \int\limits_{\Gamma_0 \cap D} \chi_{,1} g\left(\left[\!\left[u^0_2\right]\!\right]\right) \, ds + \int\limits_{\Gamma_0 \cap D} \chi \left[\!\left[\sigma_{i2}\left(u^0\right) u^0_{i,1}\right]\!\right] \, ds \,. \tag{23}$$

Now let us use the relations (6). In our case they take the form:

$$-\sigma_{ij,j}\left(u^0\right) = f_i \,, \quad i = 1, 2 \,, \tag{24}$$

inside the domain Ω_0, and

$$\left[\!\left[\sigma_{i2}\left(u^0\right)\right]\!\right] = 0 \,, \quad i = 1, 2 \,, \quad \sigma_{12}\left(u^0\right) = 0 \,, \tag{25}$$

$$\left[\!\left[u^0_2\right]\!\right] \geq 0 \,, \quad \sigma_{22}\left(u^0\right) - g'\left(\left[\!\left[u^0_2\right]\!\right]\right) \leq 0 \,, \quad \left(\sigma_{22}\left(u^0\right) - g'\left(\left[\!\left[u^0_2\right]\!\right]\right)\right) \left[\!\left[u^0_2\right]\!\right] = 0 \tag{26}$$

at the crack Γ_0. By the equilibrium eqs. (24), the first integral in formula (23) is zero. By relations (25) for the stresses at the crack, the last two integrals in (23) are equal to

$$I = \int\limits_{\Gamma_0 \cap D} \left(\chi_{,1} g\left(\left[\!\left[u^0_2\right]\!\right]\right) + \chi \sigma_{22}\left(u^0\right) \left[\!\left[u^0_{2,1}\right]\!\right]\right) \, ds \,.$$

We can integrate I by parts along $\Gamma_0 \cap D$, it gets

$$I = \int\limits_{\Gamma_0 \cap D} \chi \left[\!\left[u^0_{2,1}\right]\!\right] \left(\sigma_{22}\left(u^0\right) - g'\left(\left[\!\left[u^0_2\right]\!\right]\right)\right) ds + g\big(\left[\!\left[u^0_2(\partial B \cap \Gamma_0)\right]\!\right]\big) \,.$$

In view of relations (26) held at $\Gamma_0 \cap D$, we have either $\left[\!\left[u^0_2\right]\!\right] = 0$ and then $\left[\!\left[u^0_{2,1}\right]\!\right] = 0$, or $\left[\!\left[u^0_2\right]\!\right] > 0$ and then $\sigma_{22}(u^0) - g'(\left[\!\left[u^0_2\right]\!\right]) = 0$. Therefore, $I = g\big(\left[\!\left[u^0_2(\partial B \cap \Gamma_0)\right]\!\right]\big)$ and

$$T'\left(V, u^0; \Omega_0 \setminus \overline{B}\right) = -\int\limits_{\partial B} \sigma_{ij}\left(u^0\right) \left[0.5 \epsilon_{ij}\left(u^0\right) n_1 - u^0_{i,1} n_j\right] \, ds + \int\limits_{\partial B} f_i u^0_i n_1 \, ds + g\big(\left[\!\left[u^0_2(\partial B \cap \Gamma_0)\right]\!\right]\big) \,.$$

On the other hand, in B we have $\chi = 1$ and

$$T'\left(V, u^0; B \setminus \overline{\Gamma}_0\right) = 0.5 \int\limits_{B \setminus \Gamma_0} \left(c_{ijkl,1} \epsilon_{kl}\left(u^0\right) \epsilon_{ij}\left(u^0\right) + f_i u^0_{i,1}\right) \, dx - \int\limits_{\partial B} f_i u^0_i n_1 \, ds \,.$$

Last two formulas together give us the final representation

$$T'\left(V,u^0;\Omega_0\right) = 0.5 \int\limits_{B\setminus\Gamma_0} \left(c_{ijkl,1}\epsilon_{kl}\left(u^0\right)\epsilon_{ij}\left(u^0\right) + f_i u^0_{i,1}\right) dx - \int\limits_{\partial B} \sigma_{ij}\left(u^0\right)\left[0.5\epsilon_{ij}\left(u^0\right)n_1 - u^0_{i,1}n_j\right] ds + g\big(\llbracket u^0_2(\partial B\cap\Gamma_0)\rrbracket\big)\,. \tag{27}$$

The integral over $B\setminus\overline{\Gamma}_0$ can be omitted in (27) without loss of generality by assuming that $\{c_{ijkl,1}\} = 0$ and $f = 0$ in a neighborhood of the crack tip. Since the neighborhood B is arbitrary, the unique derivative expressed as

$$T'\left(V,u^0;\Omega_0\right) = -\int\limits_{\partial B} \sigma_{ij}\left(u^0\right)\left[0.5\epsilon_{ij}\left(u^0\right)n_1 - u^0_{i,1}n_j\right] ds + g\big(\llbracket u^0_2(\partial B\cap\Gamma_0)\rrbracket\big) \tag{28}$$

is independent of the closed path ∂B surrounding the crack tip. The integral over ∂B here coincides with the well-known in classical fracture mechanics Cherepanov-Rice integral:

$$J = \int\limits_{\partial B} \sigma_{ij}\left(u^0\right)\left[0.5\epsilon_{ij}\left(u^0\right)n_1 - u^0_{i,1}n_j\right] ds\,. \tag{29}$$

The term $g\big(\llbracket u^0_2(\partial B\cap\Gamma_0)\rrbracket\big)$ expresses some δ-function at a crack. For comparison, the Griffith hypothesis $g = g_0$ leads to the path-independence property of the Cherepanov-Rice integral J given in (29). In this relation the surface energy density occurs as an additional term to J in (28).

The derivative T' is useful for the criterions of crack advance adopted in mechanics. Thus, for $g = g_0$ the necessary condition of extrema of T with respect to the crack length parameter l implies $T' = 0$ and coincides with the Griffith fracture criterion $-J + g_0 = 0$. In the general case it should be modified according to (28).

Let us make a remark on the sign of derivative in this case. For $\varepsilon \geq 0$ we have $\Gamma_0 \subseteq \Gamma_\varepsilon$, $\Omega_\varepsilon \subseteq \Omega_0$, and $\widetilde{H}^1(\Omega_0) \subseteq \widetilde{H}^1(\Omega_\varepsilon)$ in view of $\Phi(\varepsilon)(\Gamma) = \Gamma$, because of $\chi = 0$ near Γ. Then $K_0 \subseteq K_\varepsilon$, and we can substitute $u^0 \in K_0$ as an element of the set K_ε into inequality (10) as a test function. It gets $T(u^\varepsilon;\Omega_\varepsilon) \leq T(u^0;\Omega_\varepsilon)$. Due to $\llbracket u^0\rrbracket = 0$ at $\Gamma_\varepsilon\setminus\Gamma_0$ we obtain $T(u^0;\Omega_\varepsilon) = T(u^0;\Omega_0) + \int\limits_{\Gamma_\varepsilon\setminus\Gamma_0} g(0)\,ds$. Together with the previous inequality this implies that

$$T(u^\varepsilon;\Omega_\varepsilon) - T\left(u^0;\Omega_0\right) \leq g(0)\,\mathrm{meas}(\Gamma_\varepsilon\setminus\Gamma_0)\,.$$

If $g(0) = 0$, then $T(u^\varepsilon;\Omega_\varepsilon)$ is a nonincreasing function of parameter ε, and $T'(V,u^0;\Omega_0) \leq 0$ by Theorem 5. If $g = g_0$, by the same arguments we deduce that $T'(V,u^0;\Omega_0) \leq g_0$, then the derivative can be negative as well as positive. In the latter case a minima of T may be attained at the extrema point where $T' = 0$, and the former case $g(0) = 0$ implies an unstable extrema.

5 Conclusion

The account of surface energies depending on the crack opening results in nonconvex constrained minimization of the total potential energy subject to nonpenetration conditions between the crack surfaces. The optimization problem is well posed, and its optimality conditions are satisfied for suitably regular functions describing the surface energy density. Formula for the shape derivative of the energy with respect to crack perturbations is derived and applied for planar and rectilinear cracks. For the rectilinear crack it involves the surface energy density as an additional term to the J-integral, thus influencing to the crack advance.

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