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Sensitivity of Interfacial Cracks to Non-linear Crack Front Perturbations

The 3D-elasticity model of an anisotropic, non-homogeneous, bonded solid is considered. The interface is thought of as being a smooth surface comprising the connected part under the transmission condition and the crack under the stress-free boundary condition. We investigate the sensitivity of the model to the non-linear perturbation of the crack front along the interface. Expansions of the energy functionals at least up to the second-order terms are obtained by global derivatives of the solution with respect to the shape of the crack front. These derivatives are constructed over the whole non-smooth domain as iterative solutions of the same elasticity problem with specified fictitious forces. We consider only energetic solutions of the H^1-class using the week formulation of the elasticity problem. Properties of the constructed derivatives of the energy functionals are discussed.

Key words: crack, fracture, shape sensitivity, shape differentiation

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1. Introduction

On the Griffith rupture hypothesis the crack propagation depends on the derivatives of energy with respect to the crack perturbation. The first derivative deals with the initiation of the crack growth, and the knowledge of the high-order derivatives allows one to investigate the stability of the crack. The treatment of these questions suggests the shape sensitivity analysis.

Variations of the crack are usually utilized by the local asymptotic methods developed in the singular perturbation theory. When constructing the smooth transformation of cracked domains after KHLUDNEV, SOKOLOWSKI [6], KHLUDNEV, KOVTUNENKO [5], BACH, KHLUDNEV, KOVTUNENKO [1], the shape sensitivity analysis can be reduced to regular perturbations. Thus, for linear crack problems concerning the stress-free boundary condition of the Neumann type imposed at the crack faces, it is possible to construct the global asymptotic expansion of the perturbed solution over the whole domain in spite of the non-smooth character of boundaries. We define the corresponding derivatives of the solution by iterating the initial elasticity problem. This procedure has a generalized differentiation meaning by the crack shape. The shape sensitivity analysis for problems in domains with smooth boundaries is treated by SIMON [11], SOKOLOWSKI, ZOLESIO [12].

Another peculiarity of the considered model deals with the fact that the crack front is described by some arbitrary function. Therefore, we have to define the shape variations by distributed parameters. Shape variations of curvilinear cracks in 2D-domains are studied by KOVTUNENKO [7], perturbations of the function describing a plane crack front for the 3D-case are investigated by KOVTUNENKO [8]. In the present work we investigate the general 3D-elasticity model of a solid with a crack given at a smooth interfacial surface of the body.

The interfacial cracks in fracture mechanics are studied respectively by WILLIS [13], COMMINOU [4], RICE [10], NAZAROV [9], DUDUCHAVA, SÄNDIG, WENDLAND [3]. The asymptotic analysis shows the possibility of oscillatory solutions appearing near the crack vicinity. Using only the weak formulation of the crack problem stated, we extend here the shape sensitivity technique for homogeneous solids to the bonded solids.

2. Formulation of the elasticity problem

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with the boundary Γ of the class $C^{0,1}$, and denote $\bar{\Omega} = \Omega \cup \Gamma$. Let Ω consist of two sub-domains Ω^1 and Ω^2 detached by the surface Σ, i.e. $\bar{\Omega} = \bar{\Omega}^1 \cup \bar{\Omega}^2$, $\bar{\Omega}^1 \cap \bar{\Omega}^2 = \bar{\Sigma}$. We assume that Σ in $\mathbf{R}^3$ is given by the smooth function ψ as follows:

$$x_3 = \psi(x_1, x_2), \qquad (x_1, x_2) \in \sigma, \qquad \sigma \subset \mathbf{R}^2,$$

and $\{0\}$ lies within the domain σ. The chosen normal vector $\nu = (\nu_1, \nu_2, \nu_3)$ to Σ fits its positive and negative faces $\Sigma^{\pm}$, respectively. Suppose that surface Σ consists of two parts Γ_0 and $\Sigma \setminus \bar{\Gamma}_0$ separated by the closed curve γ_0, such that $\bar{\Gamma}_0 \subset \Sigma$ with $\bar{\Gamma}_0 = \Gamma_0 \cup \gamma_0$. We assume that the boundary γ_0 of Γ_0 in $\mathbf{R}^3$ can be described by the Lipschitz continuous function $R(\phi)$, $R > 0$, $R(0) = R(2\pi)$, in the cylindric coordinates (r, ϕ, x_3) as follows:

$$\gamma_0 = \{r = R(\phi),\, 0 \le \phi \le 2\pi,\, x_3 = \psi(R(\phi)\cos\phi,\, R(\phi)\sin\phi)\}.$$

Let us denote $\Omega_0 = \Omega \setminus \bar{\Gamma}_0$. We can consider a bonded solid as occupying the domain Ω_0 with the crack Γ_0 inside on the interface Σ. This solid consists of two, may be different materials occupying the sub-domains Ω^1 and Ω^2 with the interface Σ, which are connected at the part $\bar{\Sigma} \setminus \Gamma_0$ of this interface.

Let the elasticity coefficients $c^m_{ijkl} \in C^\infty(\bar{\Omega}^m)$, $m = 1, 2$, of each solid Ω^1 and Ω^2 are elliptic and symmetric, as it usually is. Composing them, we define the elasticity coefficients in the whole body as follows:

$$c_{ijkl}(x) = \begin{cases} c^1_{ijkl}(x), & x \in \bar{\Omega}^1 \\ c^2_{ijkl}(x), & x \in \bar{\Omega}^2 \end{cases}, \qquad i, j, k, l, = 1, 2, 3\,,$$

with the same symmetry and ellipticity properties. We look for the displacement vector $u = (u_1, u_2, u_3)$ in the bonded solid following the linear Hooke law

$$\sigma_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u)\,, \qquad \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i})\,, \qquad i, j = 1, 2, 3\,.$$

We suppose the stress-free boundary condition

$$\sigma_{ij}(u)\,\nu_j = 0\,, \qquad i = 1, 2, 3\,,$$

fulfilled at the crack Γ_0, and the transmission condition

$$[\![u_i]\!] = 0\,, \qquad [\![\sigma_{ij}(u)\,\nu_j]\!] = 0\,, \qquad i = 1, 2, 3\,, \quad \text{on} \quad \bar{\Sigma} \setminus \Gamma_0\,. \tag{1}$$

Here $[\![u]\!] = u|_{\Sigma^+} - u|_{\Sigma^-}$ denotes the jump on the proper part of the interface Σ. Introduce the space of admissible displacements

$$\tilde{H}^1(\Omega_0) = \{u \in [H^1(\Omega_0)]^3,\ u = 0 \text{ on } \Gamma\}\,,$$

which includes the zero displacement condition at the external boundary Γ and the zero jump condition for the displacements at the connected part of the interface, i.e. $[\![u]\!] = 0$ at $\bar{\Sigma} \setminus \Gamma_0$.

Let $f = (f_1, f_2, f_3)$, $f_i \in C^\infty(\bar{\Omega})$, $i = 1, 2, 3$, be the given force. Consider the potential energy functional depending on the domain by the following equation:

$$\Pi(u; \Omega_0) = \tfrac{1}{2} \int_{\Omega_0} \sigma_{ij}(u)\,\varepsilon_{ij}(u) - \int_{\Omega_0} f_i u_i\,, \tag{2}$$

which is coercive when $u = 0$ at Γ due to the well-known Korn inequality. The minimization of (2) over $u \in \tilde{H}^1(\Omega_0)$ yields the existence of a unique solution $u^0 \in \tilde{H}^1(\Omega_0)$ of the equilibrium problem as follows:

$$\int_{\Omega_0} \sigma_{ij}(u^0)\,\varepsilon_{ij}(v) = \int_{\Omega_0} f_i v_i \quad \forall v \in \tilde{H}^1(\Omega_0)\,. \tag{3}$$

The variational equation (3) implies the following equilibrium equations:

$$-\sigma_{ij,j}(u^0) = f_i\,, \qquad i = 1, 2, 3\,, \quad \text{a.e. in } \Omega_0$$

and the corresponding boundary conditions:

$$[\![\sigma_{ij}(u^0)\,\nu_j]\!] = 0\,, \qquad i = 1, 2, 3\,, \quad \text{on } \Sigma\,;$$

$$\sigma_{ij}(u^0)\,\nu_j = 0\,, \qquad i = 1, 2, 3\,, \quad \text{on } \Gamma_0\,;$$

$$[\![u^0]\!] = 0 \quad \text{a.e. on } \bar{\Sigma} \setminus \Gamma_0\,; \qquad u^0 = 0 \quad \text{a.e. on } \Gamma\,.$$

The stresses $\sigma_{ij}(u^0)\,\nu_j$, $i = 1, 2, 3$, are herein defined at the interface in terms of dual spaces. In particular, the above boundary relations include the transmission condition (1) fulfilled in the weak formulation.

3. Non-linear perturbation of a surface crack

We perturb the fixed crack front γ_0 to the close contour γ_h by the function $h \in C^1([0, 2\pi])$ with $h(0) = h(2\pi)$, $h'(0) = h'(2\pi)$, which defines the projection of the surface crack onto $\mathbf{R}^2$, as follows:

$$\gamma_h = \{r = R(\phi) + h(\phi),\ 0 \le \phi \le 2\pi,\ x_3 = \psi((R + h)\cos\phi,\ (R + h)\sin\phi)\}\,.$$

Assume that γ_0 and γ_h both are located inside a tor $\omega \subset \mathbf{R}^3$ such that $\bar{\omega} \subset \Omega$ and $\{0\} \notin \omega$. The perturbed crack front γ_h bounds a perturbed surface crack Γ_h posed at the interface Σ. In the perturbed domain $\Omega_h = \Omega \setminus \bar{\Gamma}_h$, by the same reason as before, there exists a unique solution $u^h \in \tilde{H}^1(\Omega_h)$,

$$\tilde{H}^1(\Omega_h) = \{u \in [H^1(\Omega_h)]^3,\ u = 0 \text{ on } \Gamma\}\,,$$

of the equilibrium problem $\Pi(u;\Omega_h) \to \inf_{u\in\tilde{H}^1(\Omega_h)}$ with

$$\Pi(u;\Omega_h) = \tfrac{1}{2}\int\limits_{\Omega_h} \sigma_{ij}(u)\,\varepsilon_{ij}(u) - \int\limits_{\Omega_h} f_i u_i\,, \tag{4}$$

which is equivalent to the variational equation

$$\int\limits_{\Omega_h} \sigma_{ij}(u^h)\,\varepsilon_{ij}(v) = \int\limits_{\Omega_h} f_i v_i \quad \forall v \in \tilde{H}^1(\Omega_h)\,. \tag{5}$$

We next transform the perturbed domain Ω_h into the fixed one Ω_0. Let η be a non-negative, smooth cut-off function such that $\mathrm{supp}(\eta) \subset (\Omega\setminus\{0\})$ and $\eta\equiv 1$ in ω. By choosing ω and η we assume that every point $x=(x_1,x_2,x_3)\in\mathrm{supp}(\eta)$ defines uniquely the polar radius $r=r(x_1,x_2)$ and the polar angle $\phi=\phi(x_1,x_2)$ for the projection x onto $\mathbf{R}^2$ satisfying the standard relations

$$x_1 = r\cos\phi\,, \qquad x_2 = r\sin\phi\,.$$

For $x\in\Omega_0$ and $y\in\Omega_h$ we construct the coordinate transformation yielding the extension along Σ,

$$\begin{aligned} y_1 &= x_1 + h(\phi(x_1,x_2))\cdot\cos\phi(x_1,x_2)\cdot\eta(x)\,,\\ y_2 &= x_2 + h(\phi(x_1,x_2))\cdot\sin\phi(x_1,x_2)\cdot\eta(x)\,,\\ y_3 &= x_3 + \psi(y_1,y_2) - \psi(x_1,x_2)\,, \end{aligned} \tag{6}$$

which is nonlinear in h. Here the cut-off function η localizes the transformation near by the crack front. For simplicity sake we introduce the notation

$$\theta = \frac{\eta}{r}\,, \qquad \theta_\alpha = x_\alpha\theta\,, \qquad \alpha = 1,2\,, \qquad \Delta_h\psi = \psi(x_1+h\theta_1, x_2+h\theta_2) - \psi(x_1,x_2)\,.$$

Under its transformation eq. (6) takes the form

$$\{y_1 = x_1 + h\theta_1,\ y_2 = x_2 + h\theta_2,\ y_3 = x_3 + \Delta_h\psi\}$$

and has the functional matrix

$$\frac{\partial(y_1,y_2,y_3)}{\partial x_1} = (1+(h\theta_1)_{,1},\ (h\theta_2)_{,1},\ (x_\alpha+h\theta_\alpha)_{,1}\,\Delta_h\psi_{,\alpha} + (h\theta_\alpha)_{,1}\,\psi_{,\alpha})\,,$$

$$\frac{\partial(y_1,y_2,y_3)}{\partial x_2} = ((h\theta_1)_{,2},\ 1+(h\theta_2)_{,2},\ (x_\alpha+h\theta_\alpha)_{,2}\,\Delta_h\psi_{,\alpha} + (h\theta_\alpha)_{,2}\,\psi_{,\alpha})\,,$$

$$\frac{\partial(y_1,y_2,y_3)}{\partial x_3} = (h\theta_{1,3},\ h\theta_{2,3},\ 1+h\theta_{\alpha,3}(\Delta_h\psi_{,\alpha}+\psi_{,\alpha}))\,.$$

In view of the property $x_\alpha\phi_{,\alpha}=0$, its Jacobian is equal to

$$J = 1 + h\theta_{\alpha,\alpha} + h\theta_{\alpha,3}\psi_{,\alpha} + \frac{h^2}{2}\,(\theta\theta_\alpha)_{,\alpha} + \frac{h^2}{2}\,(\theta\theta_\alpha)_{,3}\,\psi_{,\alpha}\,.$$

Introduce the differentiation along the principal directions of Σ,

$$\frac{\partial}{\partial s_\alpha} = \frac{\partial}{\partial x_\alpha} + \psi_{,\alpha}\,\frac{\partial}{\partial x_3}\,, \qquad \alpha = 1,2\,,$$

which yields the next representation of the Jacobian

$$J = 1 + h\,\frac{\partial\theta_\alpha}{\partial s_\alpha} + \frac{h^2}{2}\,\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\,. \tag{7}$$

Formula (7) suggests that $J > 0$ for $\|h\|_{C([0,2\pi])}$ is rather small, and the correspondence (6) is then one-to-one. Therefore, its inverse functional matrix can be calculated as follows:

$$\frac{\partial x_1}{\partial y_1} = 1 - \frac{1}{J}\,(h\theta_1)_{,1} + \frac{h^2}{2J}\left[(\theta\theta_1)_{,3}\,\psi_{,1} - \frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\right] + \frac{1}{J}\left[h\theta_{1,3} + \frac{h^2}{2}\,(\theta\theta_1)_{,3}\right]\Delta_h\psi_{,1}\,,$$

$$\frac{\partial x_2}{\partial y_1} = -\frac{1}{J}\,(h\theta_2)_{,1} + \frac{h^2}{2J}\,(\theta\theta_2)_{,3}\,\psi_{,1} + \frac{1}{J}\left[h\theta_{2,3} + \frac{h^2}{2}\,(\theta\theta_2)_{,3}\right]\Delta_h\psi_{,1}\,,$$

$$\frac{\partial x_3}{\partial y_1} = -\frac{1}{J}\,(h\theta_\alpha)_{,1}\,\psi_{,\alpha} - \frac{h^2}{2J}\,(\theta\theta_\alpha)_{,\alpha}\,\psi_{,1} + \frac{1}{J}\left[h\theta_{\alpha,3}\psi_{,\alpha} + \frac{h^2}{2}\,(\theta\theta_\alpha)_{,3}\,\psi_{,\alpha} - J\right]\Delta_h\psi_{,1}\,,$$

$$\frac{\partial x_1}{\partial y_2} = -\frac{1}{J}\,(h\theta_1)_{,2} + \frac{h^2}{2J}\,(\theta\theta_1)_{,3}\,\psi_{,2} + \frac{1}{J}\left[h\theta_{1,3} + \frac{h^2}{2}\,(\theta\theta_1)_{,3}\right]\Delta_h\psi_{,2}\,,$$

$$\frac{\partial x_2}{\partial y_2} = 1 - \frac{1}{J}\,(h\theta_2)_{,2} + \frac{h^2}{2J}\left[(\theta\theta_2)_{,3}\,\psi_{,2} - \frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\right] + \frac{1}{J}\left[h\theta_{2,3} + \frac{h^2}{2}\,(\theta\theta_2)_{,3}\right]\Delta_h\psi_{,2}\,,$$

$$\frac{\partial x_3}{\partial y_2} = -\frac{1}{J}\,(h\theta_\alpha)_{,2}\,\psi_{,\alpha} - \frac{h^2}{2J}\,(\theta\theta_\alpha)_{,\alpha}\,\psi_{,2} + \frac{1}{J}\left[h\theta_{\alpha,3}\psi_{,\alpha} + \frac{h^2}{2}\,(\theta\theta_\alpha)_{,3}\,\psi_{,\alpha} - J\right]\Delta_h\psi_{,2}\,,$$

$$\frac{\partial x_1}{\partial y_3} = -\frac{1}{J}\,h\theta_{1,3} - \frac{h^2}{2J}\,(\theta\theta_1)_{,3}\,,\qquad \frac{\partial x_2}{\partial y_3} = -\frac{1}{J}\,h\theta_{2,3} - \frac{h^2}{2J}\,(\theta\theta_2)_{,3}\,,$$

$$\frac{\partial x_3}{\partial y_3} = 1 - \frac{1}{J}\,h\theta_{\alpha,3}\psi_{,\alpha} - \frac{h^2}{2J}\,(\theta\theta_\alpha)_{,3}\,\psi_{,\alpha}\,.$$

Let us introduce the notation 'hat' for the functions transformed by eq. (6):

$$u(y) = u(x_1 + h\theta_1, x_2 + h\theta_2, x_3 + \Delta_h\psi) \equiv \hat{u}(x)\,,\qquad x \in \Omega_0\,,\qquad y \in \Omega_h\,. \tag{8}$$

With the previous formulas for $\partial x/\partial y$, due to (8) we can rewrite the derivatives $\partial u/\partial y_i$ over $(\partial\hat{u}/\partial x_j)(\partial x_j/\partial y_i)$, $i = 1,2,3$, as follows:

$$u_{,1} = \hat{u}_{,1} - \frac{1}{J}\,(h\theta_\alpha)_{,1}\,\frac{\partial\hat{u}}{\partial s_\alpha} - \Delta_h\psi_{,1}\,\hat{u}_{,3} - \frac{h^2}{2J}\left[(\theta\theta_\alpha)_{,1}\,\frac{\partial\hat{u}}{\partial s_\alpha} + \frac{\partial(\theta\theta_2)}{\partial s_2}\,\frac{\partial\hat{u}}{\partial s_1} - \frac{\partial(\theta\theta_2)}{\partial s_1}\,\frac{\partial\hat{u}}{\partial s_2}\right]$$
$$+ \frac{\Delta_h\psi_{,1}}{J}\left[h\theta_{\alpha,3} + \frac{h^2}{2}\,(\theta\theta_\alpha)_{,3}\right]\frac{\partial\hat{u}}{\partial s_\alpha}\,,$$

$$u_{,2} = \hat{u}_{,2} - \frac{1}{J}\,(h\theta_\alpha)_{,2}\,\frac{\partial\hat{u}}{\partial s_\alpha} - \Delta_h\psi_{,2}\,\hat{u}_{,3} - \frac{h^2}{2J}\left[(\theta\theta_\alpha)_{,2}\,\frac{\partial\hat{u}}{\partial s_\alpha} + \frac{\partial(\theta\theta_1)}{\partial s_1}\,\frac{\partial\hat{u}}{\partial s_2} - \frac{\partial(\theta\theta_1)}{\partial s_2}\,\frac{\partial\hat{u}}{\partial s_1}\right]$$
$$+ \frac{\Delta_h\psi_{,2}}{J}\left[h\theta_{\alpha,3} + \frac{h^2}{2}\,(\theta\theta_\alpha)_{,3}\right]\frac{\partial\hat{u}}{\partial s_\alpha}\,,$$

$$u_{,3} = \hat{u}_{,3} - \frac{1}{J}\,h\theta_{\alpha,3}\,\frac{\partial\hat{u}}{\partial s_\alpha} - \frac{h^2}{2J}\,(\theta\theta_\alpha)_{,3}\,\frac{\partial\hat{u}}{\partial s_\alpha}\,,$$

or, in the general form

$$u_{,i} = \hat{u}_{,i} - \frac{1}{J}\,(h\theta_\alpha)_{,i}\,\frac{\partial\hat{u}}{\partial s_\alpha} - \Delta_h\psi_{,\alpha}x_{\alpha,i}\hat{u}_{,3} - \frac{h^2}{2J}\left[(\theta\theta_\alpha)_{,i}\,\frac{\partial\hat{u}}{\partial s_\alpha} + \left(\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\,\frac{\partial\hat{u}}{\partial s_\beta} - \frac{\partial(\theta\theta_\alpha)}{\partial s_\beta}\,\frac{\partial\hat{u}}{\partial s_\alpha}\right)x_{\beta,i}\right]$$
$$+ \frac{\Delta_h\psi_{,\beta}}{J}\left[h\theta_{\alpha,3} + \frac{h^2}{2}\,(\theta\theta_\alpha)_{,3}\right]\frac{\partial\hat{u}}{\partial s_\alpha}\,x_{\beta,i}\,. \tag{9}$$

Here and in what follows we use the Greek letters α, β, ξ, $\zeta = 1,2$.

Let us next rewrite the strain tensor on transformation (6). There arises an operator as evidenced by the following equation:

$$E_{ij}(w;u) = \tfrac{1}{2}(w_{,i}u_j + w_{,j}u_i)\,,\qquad i,j = 1,2,3\,,\qquad u = (u_1,u_2,u_3)\,. \tag{10}$$

For the sake of simplicity we introduce the auxiliary tensor for i, $j = 1,2,3$

$$M_{ij}(u) = E_{ij}\left(\theta\theta_\alpha;\,\frac{\partial u}{\partial s_\alpha}\right) + \frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\,E_{ij}\left(x_\beta;\,\frac{\partial u}{\partial s_\beta}\right) - \frac{\partial(\theta\theta_\alpha)}{\partial s_\beta}\,E_{ij}\left(x_\beta;\,\frac{\partial u}{\partial s_\alpha}\right). \tag{11}$$

With the notation (10) and (11), from eq. (9) it follows that

$$\varepsilon_{ij}(u) = \varepsilon_{ij}(\hat{u}) - \frac{1}{J}\, E_{ij}\left(h\theta_\alpha;\, \frac{\partial \hat{u}}{\partial s_\alpha}\right) - \Delta_h\psi_{,\alpha}\, E_{ij}(x_\alpha; \hat{u}_{,3}) - \frac{h^2}{2J}\, M_{ij}(\hat{u})$$

$$+ \frac{\Delta_h\psi_{,\alpha}}{J}\left[h\theta_{\beta,3} + \frac{h^2}{2}\,(\theta\theta_\beta)_{,3}\right] E_{ij}\left(x_\alpha;\, \frac{\partial \hat{u}}{\partial s_\beta}\right), \qquad i,j = 1,2,3\,. \tag{12}$$

By analogy with notation (10), (11), for the stress tensor we also define

$$\Sigma_{ij}(w;u) = c_{ijkl} E_{kl}(w;u)\,, \qquad N_{ij}(u) = c_{ijkl} M_{kl}(u)\,, \qquad i,j = 1,2,3\,. \tag{13}$$

While transformation (6) is constructed as a perturbation along the interface Σ, it keeps the elasticity coefficients c^m_{ijkl}, $i,j,k,l = 1,2,3$, in the corresponding domain Ω^m, $m = 1,2$. Consequently, relation (8) is also well-defined for them if one takes the form

$$c_{ijkl}(y) = \hat{c}_{ijkl}(x) \equiv \begin{cases} c^1_{ijkl}(x_1 + h\theta_1,\; x_2 + h\theta_2,\; x_3 + \Delta_h\psi)\,, & x \in \bar{\Omega}^1 \\ c^2_{ijkl}(x_1 + h\theta_1,\; x_2 + h\theta_2, x_3 + \Delta_h\psi)\,, & x \in \bar{\Omega}^2 \end{cases}.$$

Let us rewrite the problem (5) in Ω_h over the fixed domain Ω_0 by applying the transformation (6). Together with formulas (8) and (12) we reduce then eq. (5) to the following form:

$$\int\limits_{\Omega_0} J\hat{c}_{ijkl}\left(\varepsilon_{kl}(\hat{u}^h) - \frac{1}{J}\, E_{kl}\left(h\theta_\alpha;\, \frac{\partial \hat{u}^h}{\partial s_\alpha}\right) - \Delta_h\psi_{,\alpha}\, E_{kl}(x_\alpha; \hat{u}^h_{,3}) - \frac{h^2}{2J}\, M_{kl}(\hat{u}^h)\right.$$
$$\left. + \frac{\Delta_h\psi_{,\alpha}}{J}\left[h\theta_{\beta,3} + \frac{h^2}{2}\,(\theta\theta_\beta)_{,3}\right] E_{kl}\left(x_\alpha;\, \frac{\partial \hat{u}^h}{\partial s_\beta}\right)\right)$$
$$\times\left(\varepsilon_{ij}(v) - \frac{1}{J}\, E_{ij}\left(h\theta_\xi; \frac{\partial v}{\partial s_\xi}\right) - \Delta_h\psi_{,\xi}\, E_{ij}(x_\xi; v_{,3}) - \frac{h^2}{2J} M_{ij}(v)\right.$$
$$\left. + \frac{\Delta_h\psi_{,\xi}}{J}\left[h\theta_{\zeta,3} + \frac{h^2}{2}\,(\theta\theta_\zeta)_{,3}\right] E_{ij}\left(x_\xi;\, \frac{\partial v}{\partial s_\zeta}\right)\right)$$
$$= \int\limits_{\Omega_0} J\hat{f}_i v_i \quad \forall v \in \tilde{H}^1(\Omega_0) \tag{14}$$

due to the one-to-one correspondence property of the coordinate transformation (6). This yields the following result.

Theorem 1: *For small $\|h\|_{C([0,2\pi])}$, the function $\hat{u}^h \in \tilde{H}^1(\Omega_0)$, as a solution of the problem* (5) *transformed by eq.* (6)*, is a unique solution of the problem* (14).

Note that substituting $v = \hat{u}^h$ in eq. (14) and applying the Korn and Hölder inequalities, for small enough $\|h\|_{C^1([0,2\pi])}$ the uniform in the function h estimate results from (14):

$$\|\hat{u}^h\|_{\tilde{H}^1(\Omega_0)} \le \text{const}\,. \tag{15}$$

4. Expansion of the elasticity problem

We have rewritten the elasticity problem (5) in the perturbed domain over the auxiliary problem (14) in the fixed domain involving the perturbed elasticity operator. Let us next decompose this operator with respect to h.

Firstly, the supposed smoothness of ψ implies

$$\Delta_h\psi = h\theta_\alpha\psi_{,\alpha} + \frac{h^2}{2}\,\theta_\alpha\theta_\beta\psi_{,\alpha\beta} + \dots\,, \tag{16}$$

and therefore, it follows from representation (8) that

$$\hat{f} = f + h\theta_\alpha\,\frac{\partial f}{\partial s_\alpha} + \frac{h^2}{2}\,\theta_\alpha\theta_\beta\,\frac{\partial^2 f}{\partial s_\alpha\,\partial s_\beta} + \dots\,. \tag{17}$$

By multiplying eqs. (7) and (17), we have

$$J\hat{f} = f + h\,\frac{\partial(\theta_\alpha f)}{\partial s_\alpha} + \frac{h^2}{2}\left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\, f + \frac{\partial}{\partial s_\alpha}\left(\theta_\alpha\theta_\beta\,\frac{\partial f}{\partial s_\beta}\right)\right] + \dots\,. \tag{18}$$

Thus, relation (18) gets the expansion of the right member of the problem (14) in h:

$$\int\limits_{\Omega_0} J\hat{f}_i v_i = \int\limits_{\Omega_0} \left(f_i v_i + h \, \frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha} \, v_i + \frac{h^2}{2} \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha} \, f_i + \frac{\partial}{\partial s_\alpha} \left(\theta_\alpha \theta_\beta \, \frac{\partial f_i}{\partial s_\beta} \right) \right] v_i \right) + \dots \,. \tag{19}$$

By analogy with expansion (18) one obtains

$$J\hat{c}_{ijkl} = c_{ijkl} + h \, \frac{\partial(\theta_\alpha c_{ijkl})}{\partial s_\alpha} + \frac{h^2}{2} \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha} \, c_{ijkl} + \frac{\partial}{\partial s_\alpha} \left(\theta_\alpha \theta_\beta \, \frac{\partial c_{ijkl}}{\partial s_\beta} \right) \right] + \dots \,, \tag{20}$$

where

$$\frac{\partial c_{ijkl}}{\partial s_\alpha}(x) = \begin{cases} \dfrac{\partial c^1_{ijkl}}{\partial s_\alpha}\,(x), & x \in \bar{\Omega}^1 \\ \dfrac{\partial c^2_{ijkl}}{\partial s_\alpha}\,(x), & x \in \bar{\Omega}^2 \end{cases}, \qquad \alpha = 1, 2\,, \qquad i, j, k, l = 1, 2, 3\,.$$

Moreover, the following expression will be also required:

$$J\hat{c}_{ijkl}\, \Delta_h \psi_{,\xi} = h\theta_\alpha \psi_{,\alpha\xi} c_{ijkl} + \frac{h^2}{2} \, \theta_\alpha \left[\theta_\beta \psi_{,\alpha\beta\xi} c_{ijkl} + 2\psi_{,\alpha\xi} \, \frac{\partial(\theta_\beta c_{ijkl})}{\partial s_\beta} \right] + \dots \tag{21}$$

as is evident from eqs. (16) and (20). With formulas (16), (20), and (21) we decompose the left member of the problem (14) in h as follows:

$$\begin{aligned} &\int\limits_{\Omega_0} J\hat{c}_{ijkl} \left(\varepsilon_{kl}(\hat{u}^h) - \frac{1}{J} \, E_{kl} \left(h\theta_\alpha; \frac{\partial \hat{u}^h}{\partial s_\alpha} \right) - \Delta_h \psi_{,\alpha} \, E_{kl}(x_\alpha; \hat{u}^h_{,3}) - \frac{h^2}{2J} \, M_{kl}(\hat{u}^h) \right. \\ &\qquad\qquad \left. + \frac{\Delta_h \psi_{,\alpha}}{J} \left[h\theta_{\beta,3} + \frac{h^2}{2} \, (\theta\theta_\beta)_{,3} \right] E_{kl} \left(x_\alpha; \frac{\partial \hat{u}^h}{\partial s_\beta} \right) \right) \\ &\qquad \times \left(\varepsilon_{ij}(v) - \frac{1}{J} \, E_{ij} \left(h\theta_\xi; \frac{\partial v}{\partial s_\xi} \right) - \Delta_h \psi_{,\xi} \, E_{ij}(x_\xi; v_{,3}) - \frac{h^2}{2J} \, M_{ij}(v) \right. \\ &\qquad\qquad \left. + \frac{\Delta_h \psi_{,\xi}}{J} \left[h\theta_{\zeta,3} + \frac{h^2}{2} \, (\theta\theta_\zeta)_{,3} \right] E_{ij} \left(x_\xi; \frac{\partial v}{\partial s_\zeta} \right) \right) \\ &= \int\limits_{\Omega_0} \sigma_{ij}(\hat{u}^h) \, \varepsilon_{ij}(v) + A^1(h; \hat{u}^h, v) + \frac{1}{2} \, A^2(h^2; \hat{u}^h, v) + \dots \,, \end{aligned} \tag{22}$$

where the symmetric bilinear forms $A^1(h; \cdot\,, \cdot)$, $A^2(h^2; \cdot\,, \cdot)$, … can be expressed by the following equations:

$$\begin{aligned} A^1(h; u, v) \; &= \int\limits_{\Omega_0} \left(h \, \frac{\partial(\theta_\alpha c_{ijkl})}{\partial s_\alpha} \, \varepsilon_{kl}(u) \, \varepsilon_{ij}(v) - \sigma_{ij}(u) \, E_{ij} \left(h\theta_\alpha; \frac{\partial v}{\partial s_\alpha} \right) - E_{ij} \left(h\theta_\alpha; \frac{\partial u}{\partial s_\alpha} \right) \sigma_{ij}(v) \right. \\ &\qquad \left. - \, h\theta_\alpha \psi_{,\alpha\beta} [\sigma_{ij}(u) \, E_{ij}(x_\beta; v_{,3}) + E_{ij}(x_\beta; u_{,3}) \, \sigma_{ij}(v)] \right), \\ A^2(h^2; u, v) &= \int\limits_{\Omega_0} \left(h^2 \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha} \, c_{ijkl} + \frac{\partial}{\partial s_\alpha} \left(\theta_\alpha \theta_\beta \, \frac{\partial c_{ijkl}}{\partial s_\beta} \right) \right] \varepsilon_{kl}(u) \, \varepsilon_{ij}(v) \right. \\ &\qquad - \, 2h\theta_\alpha \, \frac{\partial c_{ijkl}}{\partial s_\alpha} \left[\varepsilon_{kl}(u) \, E_{ij} \left(h\theta_\beta; \frac{\partial v}{\partial s_\beta} \right) + E_{kl} \left(h\theta_\beta; \frac{\partial u}{\partial s_\beta} \right) \varepsilon_{ij}(v) \right] \\ &\qquad - \, h^2 \theta_\alpha \left(\theta_\beta \psi_{,\alpha\beta\xi} c_{ijkl} + 2\psi_{,\alpha\xi} \, \frac{\partial(\theta_\beta c_{ijkl})}{\partial s_\beta} \right) [\varepsilon_{kl}(u) \, E_{ij}(x_\xi; v_{,3}) + E_{kl}(x_\xi; u_{,3}) \, \varepsilon_{ij}(v)] \\ &\qquad - \, h^2 [\sigma_{ij}(u) \, M_{ij}(v) + M_{ij}(u) \, \sigma_{ij}(v)] + 2\boldsymbol{\Sigma}_{ij} \left(h\theta_\alpha; \frac{\partial u}{\partial s_\alpha} \right) E_{ij} \left(h\theta_\beta; \frac{\partial v}{\partial s_\beta} \right) \\ &\qquad - \, 2h^2 \theta_\alpha \theta_{\beta,3} \psi_{,\alpha\xi} \left[\sigma_{ij}(u) \, E_{ij} \left(x_\xi; \frac{\partial v}{\partial s_\beta} \right) + E_{ij} \left(x_\xi; \frac{\partial u}{\partial s_\beta} \right) \sigma_{ij}(v) \right] \\ &\qquad \left. + 2h^2 \theta_\alpha \theta_\beta \psi_{,\alpha\xi} \psi_{,\beta\zeta} \boldsymbol{\Sigma}_{ij}(x_\xi; u_{,3}) \, E_{ij}(x_\zeta; v_{,3}) \right), \end{aligned} \tag{23}$$

and so on, using the notation (13).

From the representation (23) it can be seen that already the first term A^1 in the expansion (22) depends on the curvature of Σ over the second derivatives of the function ψ. If the crack has a plane form, i.e. $\psi \equiv 0$, then $\partial/\partial s_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, 2$, and the formulas of eq. (23) admit then the simpler form

$$A^1(h; u, v) = \int_{\Omega_0} (h(\theta_\alpha c_{ijkl})_{,\alpha}\, \varepsilon_{kl}(u)\, \varepsilon_{ij}(v) - \sigma_{ij}(u)\, E_{ij}(h\theta_\alpha; v_{,\alpha}) - E_{ij}(h\theta_\alpha; u_{,\alpha})\, \sigma_{ij}(v)),$$

$$\begin{aligned} A^2(h^2; u, v) = \int_{\Omega_0} (& h^2[(\theta\theta_\alpha)_{,\alpha}\, c_{ijkl} + (\theta_\alpha\theta_\beta c_{ijkl,\beta})_{,\alpha}]\, \varepsilon_{kl}(u)\, \varepsilon_{ij}(v) \\ & - 2h\theta_\alpha c_{ijkl,\alpha}[\varepsilon_{kl}(u)\, E_{ij}(h\theta_\beta; v_{,\beta}) + E_{kl}(h\theta_\beta; u_{,\beta})\, \varepsilon_{ij}(v)] \\ & - h^2[\sigma_{ij}(u)\, M_{ij}(v) + M_{ij}(u)\, \sigma_{ij}(v)] + 2\Sigma_{ij}(h\theta_\alpha; u_{,\alpha})\, E_{ij}(h\theta_\beta; v_{,\beta})) \end{aligned}$$

with

$$M_{ij}(u) = (\theta\theta_\alpha)_{,\alpha}\, E_{ij}(x_\beta; u_{,\beta}) + (\theta\theta_\alpha)_{,3}\, E_{ij}(x_3; u_{,\alpha}).$$

5. Global expansion of the solution

After SIMON [11] we seek an expansion of the transformed solution $\hat{u}^h$ with respect to h in the form

$$\hat{u}^h = u^0 + \dot{u}(h) + \tfrac{1}{2}\ddot{u}(h^2) + \dots, \tag{24}$$

where $\dot{u}$ and $\ddot{u}$ are the first and the second global variations of the solution, respectively. Discussed here is a global variation because this expansion can be constructed over the whole domain Ω_0. For comparison, the local expansion of the untransformed solution

$$u^h = u^0 + u'(h) + \tfrac{1}{2}u''(h^2) + \dots$$

can be constructed for a compact subset of Ω_0, that is usually adopted by the local asymptotic analysis. Alternatively, after SOKOLOWSKI, ZOLESIO [12, p. 98, Def. 2.71] we have the next definition rewritten in our notation as follows.

Definition: The element $\dot{u}_h \in \tilde{H}^1(\Omega_0)$ is called the *weak or strong material derivative of* $u^0 \in \tilde{H}^1(\Omega_0)$ *in the direction of perturbation* h if there exists a limit

$$\dot{u}_h = \lim_{t \to 0} \frac{\hat{u}^{th} - u^0}{t}$$

by means the weak or strong convergence in $\tilde{H}^1(\Omega_0)$, respectively.

One can readily see that the existence of the first global variation $\dot{u}(h)$ possessing the property $\dot{u}(th) = t\dot{u}(h)$, $t = \text{const}$, implies the strong material derivatives $\dot{u}_h$ in all admissible directions h.

Let us substitute formally expansion (24) in the problem (14). Subsequent to decompositions (19) and (22), we have to define the derivatives $\dot{u}(h)$, $\ddot{u}(h^2), \dots$ from the following problems:

$$\int_{\Omega_0} \sigma_{ij}(\dot{u}(h))\, \varepsilon_{ij}(v) = \int_{\Omega_0} h\, \frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha}\, v_i - A^1(h; u^0, v) \quad \forall v \in \tilde{H}^1(\Omega_0), \tag{25}$$

$$\int_{\Omega_0} \sigma_{ij}(\ddot{u}(h^2))\, \varepsilon_{ij}(v) = \int_{\Omega_0} h^2 \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\, f_i + \frac{\partial}{\partial s_\alpha}\left(\theta_\alpha\theta_\beta\, \frac{\partial f_i}{\partial s_\beta}\right)\right] v_i - 2A^1(h; \dot{u}(h), v) - A^2(h^2; u^0, v) \quad \forall v \in \tilde{H}^1(\Omega_0), \tag{26}$$

and so on, which, together with the problem (3), is an iterative procedure. The right-hand sides of eqs. (25), (26) are at least linear continuous functionals on $\tilde{H}^1(\Omega_0)$, therefore, these problems are well defined, and there exist their unique solutions $\dot{u}(h)$, $\ddot{u}(h^2) \in \tilde{H}^1(\Omega_0)$. Eqs. (25) and (26) are identical to eq. (3) concerning the elasticity problems on some fictitious forces. This provides a procedure of the generalized differentiation with respect to the crack shape defined in the above weak sense.

Let us now obtain the estimates proving the correctness of the expansion (24). Owing to expansions (19) and (22) deduced, and subtracting eq. (3) from eq. (14), we have

$$\int_{\Omega_0} \sigma_{ij}(\hat{u}^h - u^0)\, \varepsilon_{ij}(v) = \int_{\Omega_0} h\, \frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha}\, v_i - A^1(h; \hat{u}^h, v) + O((h, h')^2).$$

Take here $v = \hat{u}^h - u^0$ and use the Korn and Hölder inequalities to evaluate the following norm due to the estimate (15)

$$\|\hat{u}^h - u^0\|_{\tilde{H}^1(\Omega_0)} \le c\|h\|_{C^1([0,2\pi])}. \tag{27}$$

On subtraction of eqs. (3) and (25) from eq. (14), expansions (19) and (22) yield

$$\int\limits_{\Omega_0} \sigma_{ij}(\hat{u}^h - u^0 - \dot{u}(h))\, \varepsilon_{ij}(v)$$

$$= \int\limits_{\Omega_0} \frac{h^2}{2} \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\, f_i + \frac{\partial}{\partial s_\alpha} \left(\theta_\alpha \theta_\beta\, \frac{\partial f_i}{\partial s_\beta}\right)\right] v_i - A^1(h; \hat{u}^h - u^0, v) - \frac{1}{2}\, A^2(h^2; \hat{u}^h, v) + O((h, h')^3)\,.$$

Consequently, once again from estimates (15) and (27) it follows that

$$\|\hat{u}^h - u^0 - \dot{u}(h)\|_{\tilde{H}^1(\Omega_0)} \le c\|h\|^2_{C^1([0,2\pi])}\,. \tag{28}$$

Analogously, using the next terms in expansions (19) and (22), for the same reason, the subtraction of eqs. (3), (25), and (26), multiplied by $\frac{1}{2}$, from eq. (14) provides, in view of estimates (15), (27), (28), the following result:

$$\|\hat{u}^h - u^0 - \dot{u}(h) - \tfrac{1}{2}\ddot{u}(h^2)\|_{\tilde{H}^1(\Omega_0)} \le c\|h\|^3_{C^1([0,2\pi])}\,. \tag{29}$$

We combine the above consideration into the following theorem.

Theorem 2: *For* $\|h\|_{C^1([0,2\pi])}$ *small enough, there exist global variations* $\dot{u}(h)$, $\ddot{u}(h^2)$ *of the solution, given as the iterative solutions of the elasticity problems* (25) *and* (26), *respectively, and the expansion* (24) *holds with the estimates* (27)–(29).

In the same manner one can continue this iterative procedure for all terms of the higher-order in the expansion (24).

6. Expansion of the potential energy

Let us substitute the solution u^h of the problem (5) for the potential energy form expressed by eq. (4) to define a functional $\mathcal{P}\colon C^1([0,2\pi]) \to \mathbf{R}$ by

$$\mathcal{P}(h) \equiv \Pi(u^h; \Omega_h) = -\tfrac{1}{2} \int\limits_{\Omega_h} f_i u_i^h \tag{30}$$

due to the evident equality

$$\int\limits_{\Omega_h} \sigma_{ij}(u^h)\, \varepsilon_{ij}(u^h) = \int\limits_{\Omega_h} f_i u_i^h\,.$$

Similarly, substituting the solution u^0 of the problem (3) for eq. (2), for the same reason, we have

$$\mathcal{P}(0) \equiv \Pi(u^0; \Omega_0) = -\tfrac{1}{2} \int\limits_{\Omega_0} f_i u_i^0\,. \tag{31}$$

If we next rewrite the integral in representation (30) on the transformation (6), this provides the following formula:

$$\mathcal{P}(h) = -\tfrac{1}{2} \int\limits_{\Omega_0} J \hat{f}_i \hat{u}_i^h\,. \tag{32}$$

By Theorem 2 we can substitute the expansion (24) in formula (32), providing together with eqs. (19) and (31) the expansion in series

$$\mathcal{P}(h) = \mathcal{P}(0) + \mathcal{P}'_0(h) + \tfrac{1}{2}\mathcal{P}''_0(h^2) + \ldots \tag{33}$$

with the corresponding derivatives

$$\mathcal{P}'_0(h) = -\frac{1}{2} \int\limits_{\Omega_0} \left(h\, \frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha}\, u_i^0 + f_i \dot{u}_i(h)\right), \tag{34}$$

$$\mathcal{P}''_0(h^2) = -\frac{1}{2} \int\limits_{\Omega_0} \left(h^2 \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\, f_i + \frac{\partial}{\partial s_\alpha} \left(\theta_\alpha \theta_\beta\, \frac{\partial f_i}{\partial s_\beta}\right)\right] u_i^0 + 2h\, \frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha}\, \dot{u}_i(h) + f_i \ddot{u}_i(h^2)\right). \tag{35}$$

Furthermore, we deduce the estimates proving the correctness of the expansion (33). Subtracting eq. (31) from eq. (32), it follows from relations (7), (15), (17), and (27) that

$$|\mathcal{P}(h) - \mathcal{P}(0)| = \frac{1}{2} \left|\int\limits_{\Omega_0} \left(f_i(\hat{u}_i^h - u_i^0) + (\hat{f}_i - f_i)\, \hat{u}_i^h + \left[h\, \frac{\partial\theta_\alpha}{\partial s_\alpha} + \frac{h^2}{2}\, \frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\right] \hat{f}_i \hat{u}_i^h\right)\right| \le c\|h\|_{C^1([0,2\pi])}\,. \tag{36}$$

Similarly, subtracting (31) and (34) from (32), we have

$$|\mathcal{P}(h) - \mathcal{P}(0) - \mathcal{P}_0'(h)| = \frac{1}{2} \Bigg| \int_{\Omega_0} \left(f_i(\hat{u}_i^h - u_i^0 - \dot{u}_i(h)) + \left(\hat{f}_i - f_i - h\theta_\alpha \frac{\partial f_i}{\partial s_\alpha}\right) \hat{u}_i^h + h\theta_\alpha \frac{\partial f_i}{\partial s_\alpha} (\hat{u}_i^h - u_i^0) + \frac{h^2}{2} \frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha} \hat{f}_i \hat{u}_i^h + h \frac{\partial \theta_\alpha}{\partial s_\alpha} [f_i(\hat{u}_i^h - u_i^0) + (\hat{f}_i - f_i)\, \hat{u}_i^h] \right) \Bigg| \le c\|h\|^2_{C^1([0,2\pi])} \qquad (37)$$

due to (7), (15), (17), (27), (28). The same consideration as for the second derivative (35) provides, in view of the estimate (29), the following result:

$$|\mathcal{P}(h) - \mathcal{P}(0) - \mathcal{P}_0'(h) - \tfrac{1}{2}\mathcal{P}_0''(h^2)| \le c\|h\|^3_{C^1([0,2\pi])} \,. \qquad (38)$$

We reduce the order of the global variations of the solution included in the formulas (34) and (35). Take $v = \dot{u}(h)$ in eq. (3) and $v = u^0$ in eq. (25) to deduce the identity

$$\int_{\Omega_0} f_i \dot{u}_i(h) = \int_{\Omega_0} \sigma_{ij}(u^0)\, \varepsilon_{ij}(\dot{u}(h)) = \int_{\Omega_0} \sigma_{ij}(\dot{u}(h))\, \varepsilon_{ij}(u^0) = \int_{\Omega_0} h \frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha} u_i^0 - A^1(h; u^0, u^0) \,.$$

Therefore, formula (34) gives

$$\mathcal{P}_0'(h) = - \int_{\Omega_0} h \frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha} u_i^0 + \frac{1}{2} A^1(h; u^0, u^0) \,. \qquad (39)$$

Similarly, taking $v = \ddot{u}(h^2)$ in eq. (3) and $v = u^0$ in eq. (26), we have

$$\int_{\Omega_0} f_i \ddot{u}_i(h^2) = \int_{\Omega_0} h^2 \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha} f_i + \frac{\partial}{\partial s_\alpha} \left(\theta_\alpha \theta_\beta \frac{\partial f_i}{\partial s_\beta} \right) \right] u_i^0 - 2A^1(h; \dot{u}(h), u^0) - A^2(h^2; u^0, u^0)$$

and get from formula (35)

$$\mathcal{P}_0''(h^2) = -\frac{1}{2} \int_{\Omega_0} \left(h^2 \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha} f_i + \frac{\partial}{\partial s_\alpha} \left(\theta_\alpha \theta_\beta \frac{\partial f_i}{\partial s_\beta} \right) \right] u_i^0 + h \frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha} \dot{u}_i(h) \right) + A^1(h; \dot{u}(h), u^0) + \frac{1}{2} A^2(h^2; u^0, u^0) \,.$$

Moreover, eq. (25) with $v = \dot{u}(h)$ implies

$$\int_{\Omega_0} \sigma_{ij}(\dot{u}(h))\, \varepsilon_{ij}(\dot{u}(h)) = \int_{\Omega_0} h \frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha} \dot{u}_i(h) - A^1(h; u^0, \dot{u}(h)) \,,$$

and, consequently, the final formula arises:

$$\mathcal{P}_0''(h^2) = - \int_{\Omega_0} \left(h^2 \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha} f_i + \frac{\partial}{\partial s_\alpha} \left(\theta_\alpha \theta_\beta \frac{\partial f_i}{\partial s_\beta} \right) \right] u_i^0 + \sigma_{ij}(\dot{u}(h))\, \varepsilon_{ij}(\dot{u}(h)) \right) + \frac{1}{2} A^2(h^2; u^0, u^0) \,. \qquad (40)$$

As a result, the next theorem is worth consideration.

Theorem 3: *For $\|h\|_{C^1([0,2\pi])}$ small, there exist derivatives $\mathcal{P}_0'(h)$ and $\mathcal{P}_0''(h^2)$ of the potential energy functional $\mathcal{P}$ with respect to the perturbation h of the initial front shape at $h = 0$ given by the integrals* (34), (35)*, or* (39), (40)*, respectively, which are independent of the cut-off function, and the expansion* (33) *holds with the estimates* (36)–(38).

Having the next terms in expansion (24) one can get the higher-order derivatives of the potential energy in expansion (33). Let us note that the first derivative $\mathcal{P}_0'(h)$ really depends only on the solution u^0 itself, the second derivative $\mathcal{P}_0''(h^2)$ can be defined already for u^0 together with its first variation $\dot{u}$, and so on.

To prove the independence of the cut-off function η for all the integral representations, which express the derivatives of the potential energy, let us consider two different functions η^1 and η^2. We denote the integrals in eqs. (34), (35) or in (39), (40) by $I_1(\eta) \equiv \mathcal{P}_0'(h)$, $I_2(\eta) \equiv \frac{1}{2}\mathcal{P}_0''(h^2)$, and take $h = th$ with a small positive parameter t. Then formula (33) holds for each η^m in the form

$$-\tfrac{1}{2} \int_{\Omega_{th}} f_i u_i^{th} = -\tfrac{1}{2} \int_{\Omega_0} f_i u_i^0 + tI_1(\eta^m) + t^2 I_2(\eta^m) + o(t^2) \,, \qquad m = 1, 2 \,.$$

Subtracting them, we have

$$t(I_1(\eta^1) - I_1(\eta^2)) + t^2(I_2(\eta^1) - I_2(\eta^2)) = o(t^2) \,. \qquad (41)$$

Divide eq. (41) by t and pass t to zero to obtain $I_1(\eta^1) = I_1(\eta^2)$, then divide the rest of eq. (41) by t^2 to see that $I_2(\eta^1) = I_2(\eta^2)$, too, for every admissible h. This proves the needed independence of η.

We now turn to the available properties of the derivatives of the potential energy with respect to the function h. Using the obvious formula

$$E_{ij}(hw; u) = hE_{ij}(w; u) + wh' E_{ij}(\phi; u)\,, \qquad i, j = 1, 2, 3\,,$$

provided by the definition (10), let us rewrite $\mathcal{P}_0'(h)$ from formula (39) by eq. (23) in the form

$$\mathcal{L}_1(h) = \int\limits_{\Omega_0} \left(h \left[-\frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha}\, u_i^0 + \frac{1}{2}\, \frac{\partial(\theta_\alpha c_{ijkl})}{\partial s_\alpha}\, \varepsilon_{kl}(u^0)\, \varepsilon_{ij}(u^0) - \sigma_{ij}(u^0)\, E_{ij}\left(\theta_\alpha; \frac{\partial u^0}{\partial s_\alpha}\right) \right.\right.$$
$$\left.\left. - \theta_\alpha \psi_{,\alpha\beta} \sigma_{ij}(u^0)\, E_{ij}(x_\beta; u^0_{,3}) \right] - h'\theta_\alpha \sigma_{ij}(u^0)\, E_{ij}\left(\phi; \frac{\partial u^0}{\partial s_\alpha}\right)\right),$$

which defines a linear continuous functional $\mathcal{L}_1 : C^1([0, 2\pi]) \to \mathbf{R}$. Similarly, from the problem (25) rewritten as

$$\int\limits_{\Omega_0} \sigma_{ij}(\dot u(h))\, \varepsilon_{ij}(v) = \int\limits_{\Omega_0} \left(h \left[\frac{\partial(\theta_\alpha f_i)}{\partial s_\alpha}\, v_i - \frac{\partial(\theta_\alpha c_{ijkl})}{\partial s_\alpha}\, \varepsilon_{kl}(u^0)\, \varepsilon_{ij}(v) + \sigma_{ij}(u^0)\, E_{ij}\left(\theta_\alpha; \frac{\partial v}{\partial s_\alpha}\right) + E_{ij}\left(\theta_\alpha; \frac{\partial u^0}{\partial s_\alpha}\right) \sigma_{ij}(v) \right.\right.$$
$$\left. + \theta_\alpha \psi_{,\alpha\beta} [\sigma_{ij}(u^0)\, E_{ij}(x_\beta; v_{,3}) + E_{ij}(x_\beta; u^0_{,3})\, \sigma_{ij}(v)] \right]$$
$$\left. - h'\theta_\alpha \left[\sigma_{ij}(u^0)\, E_{ij}\left(\phi; \frac{\partial v}{\partial s_\alpha}\right) + E_{ij}\left(\phi; \frac{\partial u^0}{\partial s_\alpha}\right) \sigma_{ij}(v) \right] \right) \qquad \forall v \in \tilde H^1(\Omega_0)$$

it follows that $\dot u$ possesses a linear continuous mapping from $C^1([0, 2\pi])$ to $\tilde H^1(\Omega_0)$. Therefore, we can define the functional $\mathcal{L}_2 : C^1([0, 2\pi]) \times C^1([0, 2\pi]) \to \mathbf{R}$, which relates to the second derivative $\mathcal{P}_0''(h^2)$ given in eq. (40), by

$$\mathcal{L}_2(h_1, h_2) = \int\limits_{\Omega_0} \left(h_1 h_2 \left[- \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\, f_i + \frac{\partial}{\partial s_\alpha} \left(\theta_\alpha \theta_\beta\, \frac{\partial f_i}{\partial s_\beta} \right) \right] u_i^0 \right.\right.$$
$$+ \frac{1}{2} \left[\frac{\partial(\theta\theta_\alpha)}{\partial s_\alpha}\, c_{ijkl} + \frac{\partial}{\partial s_\alpha} \left(\theta_\alpha \theta_\beta\, \frac{\partial c_{ijkl}}{\partial s_\beta} \right) \right] \varepsilon_{kl}(u^0)\, \varepsilon_{ij}(u^0) - 2\theta_\alpha\, \frac{\partial c_{ijkl}}{\partial s_\alpha}\, \varepsilon_{kl}(u^0)\, E_{ij}\left(\theta_\beta; \frac{\partial u^0}{\partial s_\beta}\right)$$
$$- \theta_\alpha \left[\theta_\beta \psi_{,\alpha\beta\xi}\, c_{ijkl} + 2\psi_{,\alpha\xi}\, \frac{\partial(\theta_\beta c_{ijkl})}{\partial s_\beta} \right] \varepsilon_{kl}(u^0)\, E_{ij}(x_\xi; u^0_{,3})$$
$$- \sigma_{ij}(u^0)\, M_{ij}(u^0) + \Sigma_{ij}\left(\theta_\alpha; \frac{\partial u^0}{\partial s_\alpha}\right) E_{ij}\left(\theta_\beta; \frac{\partial u^0}{\partial s_\beta}\right)$$
$$\left. - 2\theta_\alpha \theta_{\beta,3} \psi_{,\alpha\xi} \sigma_{ij}(u^0)\, E_{ij}\left(x_\xi; \frac{\partial u^0}{\partial s_\beta}\right) + \theta_\alpha \theta_\beta \psi_{,\alpha\xi} \psi_{,\beta\zeta} \Sigma_{ij}(x_\xi; u^0_{,3})\, E_{ij}(x_\zeta; u^0_{,3}) \right]$$
$$+ (h_1 h_2)' \left[-\theta_\alpha \theta_\beta\, \frac{\partial c_{ijkl}}{\partial s_\alpha}\, \varepsilon_{kl}(u^0)\, E_{ij}\left(\phi; \frac{\partial u^0}{\partial s_\beta}\right) + \Sigma_{ij}\left(\theta_\alpha; \frac{\partial u^0}{\partial s_\alpha}\right) E_{ij}\left(\phi; \frac{\partial u^0}{\partial s_\beta}\right) \right]$$
$$\left. + h_1' h_2' \theta_\alpha \theta_\beta \Sigma_{ij}\left(\phi; \frac{\partial u^0}{\partial s_\alpha}\right) E_{ij}\left(\phi; \frac{\partial u^0}{\partial s_\beta}\right) - \sigma_{ij}(\dot u(h_1))\, \varepsilon_{ij}(\dot u(h_2)) \right).$$

The form $\mathcal{L}_2$ is symmetric, bilinear, and continuous, but its positivity and convexity properties remain unknown.

To illustrate the situation with the positive definiteness property of $\mathcal{L}_2$, let us adduce the example from Bach, Kovtunenko, Sukhorukov [2]. In this example, we assume that the crack is a rectangle in the (x_1, x_2)-plane, the solid occupies a rectilinear parallelepiped in the (x_1, x_2, x_3)-coordinates, and there occurs an anti-plane stress distribution under the anti-symmetric traction force applied. The model can be reduced to the anti-plane problem for the displacement function $u_2(x_1, x_3)$ in the 2D-domain Ω_0 with a rectilinear crack, which, in this case, are defined by the parameter $R = \text{const}$ of the crack length. Then $h = \text{const}$ yields the perturbation of the crack length along x_1-axis, and the quadratic functional $\mathcal{L}_2$ takes the form $\mathcal{L}_2(h_1, h_2) = h_1 h_2 \mathcal{P}_0''$ with the second derivative $\mathcal{P}_0''$, which can be represented as

$$\mathcal{P}_0'' = \mu \int\limits_{\Omega_0} ((\chi')^2\, |u^0_{2,1}|^2 - |\nabla \dot u_2|^2)\,.$$

Obviously, this expression depends on the parameter R via the domain Ω_0. Realizing the numerical calculations of the model, we get the second derivative $\mathcal{P}_0''$, and therefore the form $\mathcal{L}_2$, can be positive as well as negative depending on R, under various loads chosen. Namely, it is negative when the crack is small or located far enough from the clamped boundary, and positive otherwise. These numerical experiments show that the positiveness of $\mathcal{L}_2$ can be realized in practice depending on the geometry of problem and, of course, physical data. However, in theory we cannot provide a priori some assertions on the subject, in particular, those due to the geometric nature of the obtained second derivative with respect to the shape.

7. Expansion of the total potential energy

Let us consider the surface energy $\mathcal{S}$ given by the Griffith hypothesis using the relations:

$$\mathcal{S}(h) = \int_{\Gamma_h} \gamma = \gamma \cdot \text{meas}\, \Gamma_h\,, \qquad \mathcal{S}(0) = \gamma \cdot \text{meas}\, \Gamma_0\,, \tag{42}$$

with the constant density $\gamma > 0$. In our notation we can express

$$\text{meas}\, \Gamma_0 = \int_0^{2\pi} \int_0^{R(\phi)} r \left[1 + \sum_\alpha \psi_{,\alpha}^2(r\cos\phi,\, r\sin\phi)\right]^{1/2} dr\, d\phi\,, \tag{43}$$

and analogously,

$$\text{meas}\, \Gamma_h = \int_0^{2\pi} \int_0^{(R+h)(\phi)} r \left[1 + \sum_\alpha \psi_{,\alpha}^2(r\cos\phi,\, r\sin\phi)\right]^{1/2} dr\, d\phi\,. \tag{44}$$

Expanding eq. (44) in series with respect to h, from eqs. (42) and (43) one gets

$$\mathcal{S}(h) = \mathcal{S}(0) + \mathcal{S}_0'(h) + \tfrac{1}{2}\mathcal{S}_0''(h^2) + \dots \tag{45}$$

with the corresponding derivatives

$$\mathcal{S}_0'(h) = \gamma \int_0^{2\pi} hR \left[1 + \sum_\alpha \psi_{,\alpha}^2(R\cos\phi,\, R\sin\phi)\right]^{1/2} d\phi\,,$$

$$\begin{aligned}\mathcal{S}_0''(h^2) = \gamma \int_0^{2\pi} h^2 \Big(1 &+ \sum_\alpha \psi_{,\alpha}^2(R\cos\phi,\, R\sin\phi) + R\psi_{,\alpha}(R\cos\phi,\, R\sin\phi)\\ &\times [\psi_{,\alpha 1}(R\cos\phi,\, R\sin\phi)\cdot\cos\phi + \psi_{,\alpha 2}(R\cos\phi,\, R\sin\phi)\cdot\sin\phi]\Big)\\ &\times \left[1 + \sum_\alpha \psi_{,\alpha}^2(R\cos\phi,\, R\sin\phi)\right]^{-1/2} d\phi\,.\end{aligned} \tag{46}$$

Again, $\mathcal{S}_0' \colon C([0,2\pi]) \to \mathbf{R}$ defines a linear continuous functional, and $\mathcal{S}_0''$ is associated by a symmetric, bilinear continuous form on $C([0,2\pi]) \times C([0,2\pi])$. Note that for the plane crack we have

$$\mathcal{S}_0'(h) = \gamma \int_0^{2\pi} hR\, d\phi\,, \qquad \mathcal{S}_0''(h^2) = \gamma \int_0^{2\pi} h^2\, d\phi$$

with the second derivative which is positive definite.

By Theorem 3 and formula (45), the total potential energy $\mathcal{T}$, as a sum of the potential energy $\mathcal{P}$ and the surface energy $\mathcal{S}$, admits the following expansion:

$$\mathcal{T}(h) = \mathcal{T}(0) + \mathcal{P}_0'(h) + \mathcal{S}_0'(h) + \tfrac{1}{2}(\mathcal{P}_0''(h^2) + \mathcal{S}_0''(h^2)) + o(\|h\|^2_{C^1([0,2\pi])})$$

with the derivatives from relations (39), (40), (46).

8. Conclusion

For the linear problem on surface cracks under the stress-free boundary condition at the crack faces, there exist all the derivatives of the potential energy with respect to the non-linear perturbation of the crack front along the given smooth surface. This surface can be considered as an interface among different materials connected over the region, which is complementary to the crack. Thanks to perturbations along the crack, the load can also be taken piecewise-smooth over this surface. The lack of the positive definiteness property of the second derivative of the potential energy does not allow one to state well the shape optimization problem for the crack front in terms of energy. In particular, this property is due to the geometric nature of the problem stated. The numerical examples show that the choice of data providing the positivity of the second derivative can be realized.

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