Siberian Mathematical Journal, Vol. 39, No. 5, 1998

VARIATIONAL AND BOUNDARY VALUE PROBLEMS IN THE PRESENCE OF FRICTION ON THE INNER BOUNDARY†)

V. A. Kovtunenko

UDC 517.97

We consider the problem of minimizing a nondifferentiable functional that is given in a two-dimensional domain whose inner boundary is a curve. The model describes a bend of a thin elastic plate with a vertical cut (crack) in the presence of a given friction force on the sides of the cut. The mathematical problem under study has two features connected firstly with irregularity of the boundary of the domain and secondly with nonsmoothness of the functions describing friction.

The theory of boundary value problems for elliptic equations in domains with piecewise smooth boundary was developed by V. A. Kondrat′ev, V. G. Maz′ya, B. A. Plamenevskiĭ, et al. [1, 2]. Here we give much attention to the asymptotic behavior of a solution near the angular points. Statements of mathematical models in irregular domains appear from physical problems with cracks and cuts [3–5]. Many mathematical questions in friction theory are still challenging. The problems with friction on the outer boundary were considered in [6÷8] in the framework of the theory of variational inequalities. Methods for regularization of variational inequalities can be found in [7, 9, 10]. The article [11] is devoted to various forms of the analytical approach to solving these inequalities in the one-dimensional case.

In the present article, we state the initial problem as a variational problem. With a suitable choice of spaces, we justify the Green's formula and find the boundary conditions on the inner boundary. Introducing the penalty function, we obtain an approximate problem with a small parameter which can be linearized by the iterative method. Using the apparatus of duality theory, we represent a solution to the problem as a saddle point of the Lagrangian. This enables us to construct an analytical method for finding a solution in the one-dimensional case. We give an example of an explicit solution to the one-dimensional problem.

1. Variational statement of the problem. Suppose that $\Omega_0 \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega_0$ and Ω_0 includes a nonclosed smooth curve Γ (which is considered as a cut in a plate). Denote $\Omega = \Omega_0 \setminus \Gamma$. Let n be the outward unit normal of $\partial\Omega_0$. Choose the direction of the unit normal $\nu = (\nu_1, \nu_2)$ of Γ which determines the positive and negative sides of the cut (Fig. 1). If s is a smooth function in Ω then we can speak of its jump on Γ: $[s] = s^+|_\Gamma - s^-|_\Gamma$. Moreover, it suffices to define the traces as follows: Extend Γ to a closed smooth curve Σ that divides Ω_0 into two domains: the inner domain Ω^- (for which ν is the outward normal) and the outer domain Ω^+ (for which $-\nu$ is the outward normal), as shown in Fig. 1. Then the traces on the boundaries $s^+|_\Gamma$ and $s^-|_\Gamma$ are defined with respect to Ω^+ and Ω^-.

Define the following bilinear form in second-order derivatives which is well known in elasticity:

$$b(u,v) = D\int_\Omega \{u_{xx}v_{xx} + u_{yy}v_{yy} + \kappa(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1-\kappa)u_{xy}v_{xy}\}\, d\Omega.$$

Here $D = Eh^3/3(1-\kappa^2)$: E is the Young modulus; $2h$ is the thickness of the plate; and $0 < \kappa < 1/2$ is the Poisson coefficient. We choose the basic function space

$$X = \{v \in H^2(\Omega),\ v = \partial v/\partial n = 0 \text{ on } \partial\Omega_0\}.$$

†) The research was financially supported by the Russian Foundation for Basic Research (Grant 97-01-00896).

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 39, No. 5, pp. 1060–1073, September–October, 1998. Original article submitted December 16, 1996. Revision submitted June 6, 1997.

The bilinear form b determines the inner product $(u, v)_X = b(u, v)$ and the norm $\|v\|_X^2 = b(v, v)$ on X. Suppose that $f \in L_2(\Omega)$ and $g \in L_2(\Gamma)$ are given functions such that $g \geq 0$ a.e. on Γ. Here f describes the load in Ω and g is the friction function on the boundary Γ. Define the functional

$$J(v) = \frac{1}{2}\|v\|_X^2 + \int_\Gamma g|[v]|\, d\Gamma - \int_\Omega fv\, d\Omega.$$

Observe that J is not differentiable in view of nonsmoothness of the boundary summand describing friction. We can state the problem of finding the function $u(x)$ of vertical displacements of the points x of the plate Ω with the cut Γ under the action of the force f and in the presence of the friction force g on the boundary Γ as follows [6–8]:

$$J(u) = \inf_{v \in X} J(v) \tag{1}$$

or, equivalently, as the variational inequality

$$(u, v-u)_X + \int_\Gamma g(|[v]| - |[u]|)\, d\Gamma \geq \int_\Omega f(v-u)\, d\Omega \quad \text{for all } v \in X. \tag{2}$$

We see that the functional J is coercive (by positivity of g), convex, and weakly lower semicontinuous; therefore, the problem (1) (or (2)) has a solution $u \in X$ [12]. Uniqueness of a solution follows easily from (2). Indeed, if u_1 and u_2 are two solutions; i.e.,

$$(u_1, v_1-u_1)_X + \int_\Gamma g(|[v_1]| - |[u_1]|)\, d\Gamma \geq \int_\Omega f(v_1-u_1)\, d\Omega \quad \text{for all } v_1 \in X,$$

$$(u_2, v_2-u_2)_X + \int_\Gamma g(|[v_2]| - |[u_2]|)\, d\Gamma \geq \int_\Omega f(v_2-u_2)\, d\Omega \quad \text{for all } v_2 \in X,$$

then, putting $v_1 = u_2$ and $v_2 = u_1$ and summing the results, we obtain $\|u_1 - u_2\|_X^2 \leq 0$.

2. Boundary conditions. Define the following space [13, p. 85]:

$$H_{00}^{\mu+1/2}(\Gamma) = \{v(x) \in H_0^{\mu+1/2}(\Gamma),\ \rho(x)^{-1/2} d^\mu v(x) \in L_2(\Gamma)\},$$

where $\mu \geq 0$ is an integer. Furnish this space with the norm

$$\|v\|^2_{H_{00}^{\mu+1/2}(\Gamma)} = \|v\|^2_{H^{\mu+1/2}(\Gamma)} + \|\rho^{-1/2} d^\mu v\|^2_{L_2(\Gamma)}.$$

Here d^μ denotes the derivative of order μ and the function $\rho(x)$ is infinitely differentiable in $\bar{\Gamma}$, is positive on Γ, vanishes on $\partial\Gamma$, and

$$\lim_{x \to x_0} \frac{\rho(x)}{\rho(x, \partial\Gamma)} = \rho \neq 0 \quad \text{for all } x_0 \in \partial\Gamma,$$

where $\rho(x, \partial\Gamma)$ is the distance from x to the boundary $\partial\Gamma$ of Γ.

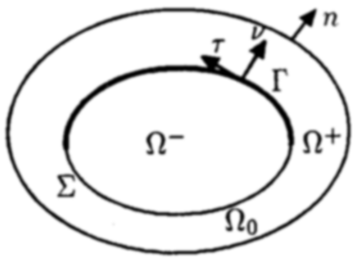

Fig. 1

Let Σ be an arbitrary closed smooth curve inside Ω_0 including Γ (Fig. 1).

Lemma 1. *A function v belongs to $H_{00}^{\mu+1/2}(\Gamma)$ if and only if the extension $\tilde{v}$ of v by zero to Σ belongs to $H^{\mu+1/2}(\Sigma)$.*

PROOF. Take the variable s to be the normalized arclength along Γ. Then Γ goes into the interval $(0,1)$ and Σ, into the real axis $\mathbb{R}$. Therefore, to prove the lemma, it suffices to consider the functions $v(s) \in H_{00}^{\mu+1/2}(0,1)$ and

$$\tilde{v}(s) = \begin{cases} v(s) & \text{for } s \in (0,1), \\ 0 & \text{for } s \in (-\infty, 0] \cup [1, \infty). \end{cases}$$

First, suppose that $\mu = 0$. Define the norms

$$\|v\|^2_{H^{1/2}(0,1)} = \int_0^1 \int_0^1 |s-t|^{-2} |v(s) - v(t)|^2 \, ds dt + \|v\|^2_{L_2(0,1)},$$

$$\|\tilde{v}\|^2_{H^{1/2}(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} |s-t|^{-2} |\tilde{v}(s) - \tilde{v}(t)|^2 \, ds dt + \|\tilde{v}\|^2_{L_2(\mathbb{R})}.$$

Since $\tilde{v}$ vanishes on $(-\infty, 0] \cup [1, \infty)$, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |s-t|^{-2} |\tilde{v}(s) - \tilde{v}(t)|^2 \, ds dt = \int_0^1 \int_0^1 |s-t|^{-2} |v(s) - v(t)|^2 \, ds dt$$
$$+ 2 \int_0^1 \int_{-\infty}^0 |s-t|^{-2} v^2(t) \, ds dt + 2 \int_0^1 \int_1^{\infty} |s-t|^{-2} v^2(t) \, ds dt.$$

We can calculate the last two integrals:

$$\int_0^1 v^2(t) \left\{ \int_{-\infty}^0 (t-s)^{-2} \, ds + \int_1^{\infty} (s-t)^{-2} \, ds \right\} dt = \int_0^1 v^2(t) \left(\frac{1}{t} + \frac{1}{1-t} \right) dt.$$

We have thus obtained the estimate

$$\|\tilde{v}\|^2_{H^{1/2}(\mathbb{R})} = \|v\|^2_{H^{1/2}(0,1)} + \int_0^1 \rho^{-1}(t) v^2(t) \, dt, \tag{3}$$

where the function $\rho(t) = t(1-t)/2$ satisfies all necessary requirements in the definition of the spaces "with two zeros." Since $H^{1/2}(0,1) = H_0^{1/2}(0,1)$ [13, p. 71], the estimate (3) proves the assertion of the lemma for $\mu = 0$.

Now, assume $\mu > 0$. Take $v \in H_{00}^{\mu+1/2}(0,1)$. It follows from the definition of the space that $d^\mu v \in H^{1/2}(0,1)$ and $\rho^{-1/2} d^\mu v \in L_2(0,1)$. Then, by analogy with (3), we derive the estimate

$$\|d^\mu \tilde{v}\|^2_{H^{1/2}(\mathbb{R})} = \|d^\mu v\|^2_{H^{1/2}(0,1)} + \|\rho^{-1/2} d^\mu v\|^2_{L_2(0,1)}. \tag{4}$$

This means that

$$d^\mu \tilde{v} \in H^{1/2}(\mathbb{R}). \tag{5}$$

Since $H_0^{\mu+1/2}(0,1)$ is embedded in $C([0,1])$, v is a continuous function on $[0,1]$ and $v(0) = v(1) = 0$. Hence, v extends continuously by zero to $\mathbb{R}$; i.e.,

$$\tilde{v} \in H^{1/2}(\mathbb{R}). \tag{6}$$

The conditions (5) and (6) are equivalent to the fact that $\tilde{v} \in H^{\mu+1/2}(\mathbb{R})$.

Conversely, suppose that $\tilde{v} \in H^{\mu+1/2}(\mathbb{R})$. Show that $v \in H_{00}^{\mu+1/2}(0,1)$. From (4) we obtain $\rho^{-1/2}(t)d^\mu v(t) \in L_2(0,1)$ and $v \in H^{\mu+1/2}(0,1)$. It remains to verify that

$$d^m v(0) = d^m v(1) = 0 \quad \text{for all } m\colon 0 \le m \le \mu - 1. \tag{7}$$

Indeed, we have $H^{\mu+1/2}(\mathbb{R}) \subset C^{\mu-1}(\mathbb{R})$ by embedding theorems. Consequently, all derivatives $d^m\tilde{v}$, $0 \le m \le \mu - 1$, are continuous functions on $\mathbb{R}$ and, since they vanish on $\mathbb{R} \setminus (0,1)$ by construction, (7) is valid. Hence, $v \in H_0^{\mu+1/2}(0,1)$, which completes the proof of the lemma.

Suppose that $v \in X$ is an arbitrary function. Then the traces of v on Σ in Ω^+ and Ω^- are well defined:

$$v^\pm \in H^{3/2}(\Sigma), \quad \partial v^\pm/\partial\nu \in H^{1/2}(\Sigma).$$

These functions have zero jumps on the part $\Sigma \setminus \Gamma$ of the boundary. Then, by Lemma 1,

$$[v] \in H_{00}^{3/2}(\Gamma), \quad [\partial v/\partial\nu] \in H_{00}^{1/2}(\Gamma) \quad \text{for all } v \in X.$$

In particular, for a solution $u \in X$ to (2) we have $[u] \in H_{00}^{3/2}(\Gamma)$ and $[\partial u/\partial\nu] \in H_{00}^{1/2}(\Gamma)$. We now define the bending moment tensor and the crosscutting force tensor on Γ:

$$T(v) = D\frac{\partial}{\partial\nu}\left(\Delta v + (1-\kappa)\frac{\partial^2 v}{\partial\tau^2}\right), \quad M(v) = D\left(\kappa\,\Delta\, v + (1-\kappa)\frac{\partial^2 v}{\partial\nu^2}\right),$$

where $\tau = (-\nu_2, \nu_1)$ is the vector tangent to Γ.

Now, we take $v = u \pm \xi$, $\xi \in C_0^\infty(\Omega)$, as test functions and insert them in (2). From the Green's formula we infer that the equilibrium equation

$$D\,\Delta^2\,u = f \tag{8}$$

is satisfied almost everywhere in Ω. Thus, the relation $\Delta^2 u \in L_2(\Omega)$ holds for a solution $u \in X$ to (2). Divide Ω_0 into $\Omega^+ \cup \Omega^- \cup \Sigma$ as above. Then in $\Omega^\pm$ we have $u \in H^2(\Omega^\pm)$ and $\Delta^2 u \in L_2(\Omega^\pm)$. Consequently [13, Chapter 2], the continuous linear functionals $T^\pm(u) \in H^{-3/2}(\Sigma)$ and $M^\pm(u) \in H^{-1/2}(\Sigma)$ are well defined on Σ and the following equality holds:

$$\begin{aligned}(u,v)_X = D\int_\Omega \Delta^2 u\, v\, d\Omega + (\langle T^+(u), v^+\rangle_{3/2,\Sigma} - \langle T^-(u), v^-\rangle_{3/2,\Sigma})\\ -(\langle M^+(u), \partial v^+/\partial\nu\rangle_{1/2,\Sigma} - \langle M^-(u), \partial v^-/\partial\nu\rangle_{1/2,\Sigma}) \quad \text{for all } v \in X.\end{aligned} \tag{9}$$

Here $\langle\cdot,\cdot\rangle_{\mu+1/2,\Sigma}$, $\mu = 0,1$, denotes the duality between $H^{\mu+1/2}(\Sigma)$ and $H^{-(\mu+1/2)}(\Sigma)$. Inserting (8) and (9) in (2), we obtain

$$\begin{aligned}(\langle T^+(u), v^+ - u^+\rangle_{3/2,\Sigma} - \langle T^-(u), v^- - u^-\rangle_{3/2,\Sigma}) + \int_\Gamma g(|[v]| - |[u]|)\, d\Gamma\\ -\left(\left\langle M^+(u), \frac{\partial(v^+ - u^+)}{\partial\nu}\right\rangle_{1/2,\Sigma} - \left\langle M^-(u), \frac{\partial(v^- - u^-)}{\partial\nu}\right\rangle_{1/2,\Sigma}\right) \ge 0.\end{aligned} \tag{10}$$

Take $v = u \pm \xi$, $\xi \in C_0^\infty(\Omega_0)$. Then $[\xi] = [\partial\xi/\partial\nu] = 0$ and (10) yields the equality

$$\langle T^+(u) - T^-(u), \xi\rangle_{3/2,\Sigma} - \langle M^+(u) - M^-(u), \partial\xi/\partial\nu\rangle_{1/2,\Sigma} = 0 \quad \text{for all } \xi \in C_0^\infty(\Omega_0).$$

In view of the arbitrariness of the values ξ and $\partial\xi/\partial\nu$ on Γ, from the last relation we conclude that

$$T^+(u) = T^-(u), \quad M^+(u) = M^-(u).$$

Henceforth we denote the values $T^\pm(u)$ and $M^\pm(u)$ by $T(u) \in H^{-3/2}(\Sigma)$ and $M(u) \in H^{-1/2}(\Sigma)$. By the above lemma, we can uniquely determine the elements $t(u) \in \left(H_{00}^{3/2}(\Gamma)\right)^*$ and $m(u) \in \left(H_{00}^{1/2}(\Gamma)\right)^*$ in the conventional conjugate spaces (i.e., the spaces of continuous linear functionals) by the rules

$$\langle t(u), v\rangle_\Gamma = \langle T(u), \tilde{v}\rangle_{3/2,\Sigma} \quad \text{for all } v \in H_{00}^{3/2}(\Gamma),$$
$$\langle m(u), w\rangle_\Gamma = \langle M(u), \tilde{w}\rangle_{1/2,\Sigma} \quad \text{for all } w \in H_{00}^{1/2}(\Gamma),$$

where $\tilde{v}$ and $\tilde{w}$ stand for the extensions of v and w by zero to Σ. Here and in the sequel, we briefly denote by $\langle \cdot, \cdot\rangle_\Gamma$ the standard duality between the conjugate spaces on Γ. Observe that $t(u) = T(u)$ and $m(u) = M(u)$ if these functions are sufficiently smooth.

According to the above notations, we can rewrite (10) as

$$\langle t(u), [v-u]\rangle_\Gamma - \left\langle m(u), \left[\frac{\partial(v-u)}{\partial\nu}\right]\right\rangle_\Gamma + \int_\Gamma g(|[v]| - |[u]|)\, d\Gamma \geq 0 \quad \text{for all } v \in X. \tag{11}$$

Construct a function $\xi \in X$ as follows: Suppose that $B(x_0)$ is a sufficiently small disk with center $x_0 \in \Gamma$ such that Γ divides the disk into two domains $B^+(x_0)$ and $B^-(x_0)$. Assume that $\xi \equiv 0$ in $\Omega_0 \setminus B(x_0)$, $\xi \equiv 0$ in $B^-(x_0)$, $\xi \in C^\infty(B^+(x_0))$, and ξ possesses the following properties on the boundary of $B^+(x_0)$: the function ξ vanishes together with all its derivatives near the outer boundary $\partial B(x_0)$ and $\xi = 0$ on the inner boundary $\Gamma \cap B^+(x_0)$. Denote the value of the normal derivative of ξ on the boundary $\Gamma \cap B^+(x_0)$ by $\phi \in C_0^\infty(\Gamma \cap B^+(x_0))$. By construction, $[\xi] = 0$ and $[\partial\xi/\partial\nu] = \tilde{\phi}$ belongs to $C_0^\infty(\Gamma)$, where

$$\tilde{\phi} = \begin{cases} \phi & \text{on } \Gamma \cap B^+(x_0), \\ 0 & \text{otherwise.} \end{cases}$$

Inserting $v = u \pm \xi$ in (11), we obtain $\mp\langle m(u), \tilde{\phi}\rangle_\Gamma \geq 0$. Hence, in view of the arbitrariness of ϕ and $B(x_0)$, we conclude that

$$m(u) = 0 \quad \text{on } \Gamma \tag{12}$$

as an element of the space $\left(H_{00}^{1/2}(\Gamma)\right)^*$. Thus, it follows from (11) and (12) that

$$\langle t(u), [v-u]\rangle_\Gamma + \int_\Gamma g(|[v]| - |[u]|)\, d\Gamma \geq 0 \quad \text{for all } v \in X.$$

Here we replace v with $\pm\lambda v$, where $\lambda \geq 0$ is a constant. Then

$$\lambda\Big(\pm\langle t(u), [v]\rangle_\Gamma + \int_\Gamma g|[v]|\, d\Gamma\Big) - \Big(\langle t(u), [u]\rangle_\Gamma + \int_\Gamma g|[u]|\, d\Gamma\Big) \geq 0 \quad \text{for all } v \in X,\ \lambda \geq 0.$$

Hence, in view of the arbitrariness of λ and v, we conclude that the second summand must vanish:

$$\langle t(u), [u]\rangle_\Gamma + \int_\Gamma g|[u]|\, d\Gamma = 0, \tag{13}$$

and the first is positive; i.e.,

$$|\langle t(u), [v] \rangle_\Gamma| \le \int\limits_\Gamma g|[v]|\, d\Gamma. \tag{14}$$

Thus, the boundary value problem (1) (or (2)) on equilibrium of a plate with friction on a cut can be equivalently stated as follows:

$$\begin{gathered} D\,\Delta^2 u = f \text{ in } \Omega, \\ m(u) = 0, \quad |t(u)| \le g, \quad t(u)\,[u] + g|[u]| = 0 \text{ on } \Gamma. \end{gathered} \tag{15}$$

The exact meaning of the conditions on the boundary is given in (12)–(14).

REMARK 1. We can obviously rewrite the last two relations on Γ in the form

$$|t(u)| < g \Rightarrow [u] = 0, \quad t(u) = g \Rightarrow [u] \le 0, \quad t(u) = -g \Rightarrow [u] \ge 0.$$

REMARK 2. If $g \equiv 0$ a.e. on Γ then from (15) we obtain the natural boundary conditions $m(u) = t(u) = 0$.

3. The associate penalized problem. Define the function $B \in H^2(\mathbb{R})$ by the rule

$$B(t) = \begin{cases} t & \text{for } t > 1, \\ -t & \text{for } t < -1, \\ (1+t^2)/2 & \text{for } |t| \le 1. \end{cases}$$

Its derivative $B' \in H^1(\mathbb{R})$ has the form

$$B'(t) = \begin{cases} 1 & \text{for } t > 1, \\ -1 & \text{for } t < -1, \\ t & \text{for } |t| \le 1. \end{cases}$$

Observe that these functions are Lipschitz continuous in view of the obvious inequalities

$$|B(t) - B(s)| \le |t-s|, \quad |B'(t) - B'(s)| \le |t-s| \quad \text{for all } t, s \in \mathbb{R}.$$

We use the following result [14, p. 50]: suppose that $\theta(t)$ is a Lipschitz continuous function whose derivative has finitely many discontinuity points; if $\phi \in H^1(\Gamma)$ then $\theta(\phi) \in H^1(\Gamma)$. Thus, for an arbitrary $\phi \in H^1(\Gamma)$ we have $B(\phi), B'(\phi) \in H^1(\Gamma)$.

Define the following penalized functional J_ε on X:

$$J_\varepsilon(w) = \frac{1}{2}\|w\|_X^2 + \varepsilon \int\limits_\Gamma gB\left(\frac{[w]}{\varepsilon}\right) d\Gamma - \int\limits_\Omega fw\, d\Omega \quad \text{for all } w \in X,$$

where $\varepsilon > 0$ is a small parameter. Since by Lemma 1 $[w] \in H_{00}^{3/2}(\Gamma) \subset H^1(\Gamma)$ for all $w \in X$, we have $B([w]/\varepsilon) \in H^1(\Gamma)$. Recalling that $g \in L_2(\Gamma)$, we conclude that the boundary integral in J_ε makes sense. The functional J_ε is coercive (for $g \ge 0$ and $B(t) \ge 0$), convex, and weakly lower semicontinuous; therefore, there is a solution $u_\varepsilon \in X$ to the minimization problem

$$J_\varepsilon(u_\varepsilon) = \inf_{w \in X} J_\varepsilon(w) \tag{16}$$

which is equivalent to the variational inequality

$$(u_\varepsilon, w - u_\varepsilon)_X + \varepsilon \int\limits_\Gamma g\left(B\left(\frac{[w]}{\varepsilon}\right) - B\left(\frac{[u_\varepsilon]}{\varepsilon}\right)\right) d\Gamma \ge \int\limits_\Omega f(w - u_\varepsilon)\, d\Omega \tag{17}$$

for $w \in X$. Proceeding routinely by way of contradiction, we can prove that a solution to (17) is unique.

Theorem 1. *We have $u_\varepsilon \to u$ strongly in X as $\varepsilon \to 0$, where u_ε is a solution to the penalized problem* (17) *and u is a solution to* (2). *Moreover, the following estimate is valid:*

$$\|u_\varepsilon - u\|_X \le C\sqrt{\varepsilon}$$

with $C^2 = 1/2\,\|g\|_{L_1(\Gamma)}$.

PROOF. We can easily obtain the estimate $0 \le \varepsilon B(t/\varepsilon) - |t| \le \varepsilon/2$ for all $t \in \mathbb{R}$. It remains valid for an arbitrary continuous function ϕ:

$$0 \le \varepsilon B(\phi(x)/\varepsilon) - |\phi(x)| \le \varepsilon/2 \quad \text{for all } \ x. \tag{18}$$

Take $v = u_\varepsilon$ in (2), put $w = u$ in (17), and sum these two inequalities to obtain

$$(u_\varepsilon - u, u_\varepsilon - u)_X \le \int\limits_\Gamma g\left\{\left(\varepsilon B\left(\frac{[u]}{\varepsilon}\right) - |[u]|\right) - \left(\varepsilon B\left(\frac{[u_\varepsilon]}{\varepsilon}\right) - |[u_\varepsilon]|\right)\right\} d\Gamma.$$

Since $[u], [u_\varepsilon] \in H_{00}^{3/2}(\Gamma) \subset C(\bar{\Gamma})$, we can use (18) and positivity of g to derive the estimate

$$\|u_\varepsilon - u\|_X^2 \le \int\limits_\Gamma g(\varepsilon/2 - 0)\, d\Gamma = \varepsilon/2\|g\|_{L_1(\Gamma)}.$$

The theorem is proven.

Now, from (16) we obtain the penalized boundary value problem. First, observe that J_ε is a differentiable functional. Indeed, for arbitrary $\phi, h \in C(\bar{\Gamma})$ there is a limit

$$\lim_{\lambda\to 0} \lambda^{-1}(B(\phi(x) + \lambda h(x)) - B(\phi(x))) = B'(\phi(x))h(x).$$

Then the functional $I(\phi) = \int_\Gamma gB(\phi)\, d\Gamma$ has the derivative at $\phi \in C(\bar{\Gamma})$: $I'_\phi(h) = \int_\Gamma gB'(\phi)h\, d\Gamma$ for all $h \in C(\bar{\Gamma})$. Since $[v], [w] \in H_{00}^{3/2}(\Gamma) \subset C(\bar{\Gamma})$ for arbitrary $v, w \in X$, we can calculate the derivative of J_ε at v:

$$(J_\varepsilon)'_v(w) = (v, w)_X + \int\limits_\Gamma gB'\left(\frac{[v]}{\varepsilon}\right)[w]\, d\Gamma - \int\limits_\Omega fw\, d\Omega \quad \text{for all } \ w \in X.$$

The integral over Γ is well defined due to the smoothness of the functions g and $[w]$ on Γ and the fact that $B'([v]/\varepsilon) \in H^1(\Gamma)$. Thus, (16) and (17) are equivalent to the Euler equation $(J_\varepsilon)'_{u_\varepsilon}(w) = 0$ which has the form

$$(u_\varepsilon, w)_X + \int\limits_\Gamma gB'\left(\frac{[u_\varepsilon]}{\varepsilon}\right)[w]\, d\Gamma = \int\limits_\Omega fw\, d\Omega \quad \text{for all } \ w \in X. \tag{19}$$

As in the preceding section, the functionals $m(u_\varepsilon) \in \left(H_{00}^{1/2}(\Gamma)\right)^*$ and $t(u_\varepsilon) \in \left(H_{00}^{3/2}(\Gamma)\right)^*$ are well defined for a solution u_ε to (19) and the Green's formula is valid:

$$(u_\varepsilon, w)_X = D\int\limits_\Omega \Delta^2 u_\varepsilon w\, d\Omega + \langle t(u_\varepsilon), [w]\rangle_\Gamma - \left\langle m(u_\varepsilon), \left[\frac{\partial w}{\partial \nu}\right]\right\rangle_\Gamma \quad \text{for all } \ w \in X.$$

Hence, the penalized problem (19) is representable as

$$\begin{gathered} D\,\Delta^2 u_\varepsilon = f \text{ in } \Omega, \\ m(u_\varepsilon) = 0, \quad t(u_\varepsilon) + gB'([u_\varepsilon]/\varepsilon) = 0 \text{ on } \Gamma. \end{gathered} \tag{20}$$

Here the last condition on Γ is satisfied in the following sense:

$$\langle t(u_\varepsilon),[w]\rangle_\Gamma+\int_\Gamma gB'([u_\varepsilon]/\varepsilon)[w]\,d\Gamma=0\quad\text{for all }\ w\in X.$$

4. The iterative penalty method. Fix ε. To find a solution to the penalized nonlinear equation (19), we construct the following linear iterative process for arbitrary $u_\varepsilon^0\in X$ and $n\ge 0$:

$$\begin{aligned}(u_\varepsilon^{n+1},w)_X&+\int_\Gamma g\left(\frac{[u_\varepsilon^{n+1}-u_\varepsilon^n]}{\varepsilon}\right)[w]\,d\Gamma\\&=\int_\Omega fw\,d\Omega-\int_\Gamma gB'\left(\frac{[u_\varepsilon^n]}{\varepsilon}\right)[w]\,d\Gamma\quad\text{for all }\ w\in X.\end{aligned}\tag{21}$$

It is obvious that a solution $u_\varepsilon^{n+1}\in X$ exists and is unique for each n. Using the Green's formula of Section 3, we similarly obtain the following boundary value problem for (21):

$$\begin{gathered}D\,\Delta^2\,u_\varepsilon^{n+1}=f\text{ in }\Omega,\\ m(u_\varepsilon^{n+1})=0,\quad t(u_\varepsilon^{n+1})+g[u_\varepsilon^{n+1}]/\varepsilon=g([u_\varepsilon^n]/\varepsilon-B'([u_\varepsilon^n]/\varepsilon))\text{ on }\Gamma,\end{gathered}$$

where $m(u_\varepsilon^{n+1})\in(H_{00}^{1/2}(\Gamma))^*$ and $t(u_\varepsilon^{n+1})\in(H_{00}^{3/2}(\Gamma))^*$.

Theorem 2. *The strong convergence $u_\varepsilon^{n+1}\to u_\varepsilon$ in X holds as $n\to\infty$.*

PROOF. Subtract (19) from (21) and put $w=u_\varepsilon^{n+1}-u_\varepsilon$. Then

$$\begin{aligned}\|u_\varepsilon^{n+1}-u_\varepsilon\|_X^2&+\int_\Gamma\frac{g[u_\varepsilon^{n+1}-u_\varepsilon]^2}{\varepsilon}\,d\Gamma\\&=\int_\Gamma g\left(\frac{[u_\varepsilon^n]}{\varepsilon}-\frac{[u_\varepsilon]}{\varepsilon}-B'\left(\frac{[u_\varepsilon^n]}{\varepsilon}\right)+B'\left(\frac{[u_\varepsilon]}{\varepsilon}\right)\right)[u_\varepsilon^{n+1}-u_\varepsilon]\,d\Gamma.\end{aligned}$$

We can obtain the inequality $|t-s-B'(t)+B'(s)|\le|t-s|$ for all $t,s\in\mathbb{R}$. Therefore, by Hölder's inequality, from the above estimate we infer that

$$2\|u_\varepsilon^{n+1}-u_\varepsilon\|_X^2+\frac{1}{\varepsilon}\int_\Gamma g[u_\varepsilon^{n+1}-u_\varepsilon]^2\,d\Gamma\le\frac{1}{\varepsilon}\int_\Gamma g[u_\varepsilon^n-u_\varepsilon]^2\,d\Gamma.\tag{22}$$

Divide Ω_0 into $\Omega^+\cup\Omega^-\cup\Sigma$ as above. Then for each of the domains $\Omega^\pm$ there exist positive constants $c^\pm$ such that

$$c^\pm\int_\Sigma|v^\pm|^2\,d\Sigma\le\|v\|_{H^2(\Omega^\pm)}^2\quad\text{for all }\ v\in X.$$

Since $v^+=v^-$ on $\Sigma\setminus\Gamma$ and g is nonnegative. we have

$$c\int_\Gamma g[v]^2\,d\Gamma\le\|v\|_X^2\quad\text{for all }\ v\in X,$$

where c is some positive constant. Then it follows from (22) that

$$\int_\Gamma g\left[u_\varepsilon^{n+1} - u_\varepsilon\right]^2 d\Gamma \le (1+2c\varepsilon)^{-1} \int_\Gamma g\left[u_\varepsilon^{n} - u_\varepsilon\right]^2 d\Gamma \le (1+2c\varepsilon)^{-(n+1)} \int_\Gamma g\left[u_\varepsilon^{0} - u_\varepsilon\right]^2 d\Gamma.$$

The expression on the right-hand side of the above inequality vanishes as $n \to \infty$ due to the estimate $(1+2c\varepsilon)^{-1} < 1$; therefore,

$$\int_\Gamma g\left[u_\varepsilon^{n+1} - u_\varepsilon\right]^2 d\Gamma \to 0 \text{ as } n \to \infty.$$

But then $\left\|u_\varepsilon^{n+1} - u_\varepsilon\right\|^2 \to 0$ as $n \to \infty$ by (22). The theorem is proven.

REMARK 3. Observe that the estimate (22) degenerates as $\varepsilon \to 0$.

5. The saddle point problem. By the construction of B, we have the inequality $|B'([u_\varepsilon(x)]/\varepsilon)| \le 1$, $x \in \Gamma$, which is valid for each fixed ε. Then the net $\{B'([u_\varepsilon]/\varepsilon)\} \in H^1(\Gamma)$ is bounded in $L_2(\Gamma)$. Consequently, there is a subnet such that

$$B'([u_\varepsilon]/\varepsilon) \to p \text{ weakly in } L_2(\Gamma) \text{ as } \varepsilon \to 0 \tag{23}$$

and p belongs to $L_2(\Gamma)$ and remains in a bounded set; i.e., $|p| \le 1$ a.e. on Γ. Passing to the limit in (19) and using Theorem 1 and (23), we infer that a solution u to (2) satisfies the equation

$$(u,w)_X + \int_\Gamma gp[w]\, d\Gamma = \int_\Omega fw\, d\Omega \quad \text{for all } w \in X. \tag{24}$$

Here we take $w = u$. Then

$$(u,u)_X + \int_\Gamma gp[u]\, d\Gamma = \int_\Omega fu\, d\Omega.$$

On the other hand, inserting $v = 0$ and $v = 2u$ in (2), we obtain

$$(u,u)_X + \int_\Gamma g|[u]|\, d\Gamma = \int_\Omega fu\, d\Omega.$$

The last two equations imply that

$$\int_\Gamma g(|[u]| - p[u])\, d\Gamma = 0. \tag{25}$$

Since $|p(x)| \le 1$ for a.e. $x \in \Gamma$, we have

$$p(x)[u(x)] \le |p(x)[u(x)]| \le |p(x)|\,|[u(x)]| \le |[u(x)]|;$$

i.e., $|[u]| - p[u] \ge 0$ a.e. on Γ. Consequently, from (25) we conclude that

$$|[u]| - p[u] = 0 \quad \text{a.e. on } \Gamma.$$

Thus, there is a function $p \in L_2(\Gamma)$ such that $|p| \le 1$ and $p[u] = |[u]|$; moreover, the solution u to (2) satisfies the equation (24) which can be rewritten as the following boundary value problem (whose exact meaning is discussed in the preceding sections):

$$D\,\Delta^2 u = f \text{ in } \Omega,$$

$$m(u) = 0, \quad t(u) + gp[u] = 0 \text{ on } \Gamma.$$

Define the closed convex set

$$K = \{q \in L_2(\Gamma), \quad |q| \leq 1 \text{ a.e. on } \Gamma\}$$

and the Lagrangian

$$\mathcal{L}(v,q) = \frac{1}{2}\|v\|_X^2 + \int_\Gamma gq[v]\, d\Gamma - \int_\Omega fv\, d\Omega, \quad v \in X,\ q \in K.$$

Theorem 3. *The pair (u,p), where u is a solution to (2) and p is defined by (23), is a unique saddle point of the Lagrangian $\mathcal{L}(\cdot,\cdot)$ on the set $X \times K$.*

PROOF. We have shown that $p[u] = |[u]|$; therefore, $\mathcal{L}(u,p) = J(u)$. Furthermore, (24) is the Euler equation $\mathcal{L}'_u(w,p) = 0$, $w \in X$, for the problem $\mathcal{L}(v,p) \xrightarrow{v}$ extr. Since $\sup_{v\in X} \mathcal{L}(v,p) = \infty$, we have

$$\mathcal{L}(u,p) = \inf_{v\in X} \mathcal{L}(v,p).$$

On the other hand, for an arbitrary $q \in K$ we have $|q(x)| \leq 1$ for a.e. $x \in \Gamma$; therefore, $\sup_{q\in K}(q(x)[u(x)]) = |[u(x)]| = p(x)[u(x)]$ for a.e. $x \in \Gamma$; i.e.,

$$\mathcal{L}(u,p) = \sup_{q\in K} \mathcal{L}(u,q)$$

by positivity of g. We have thus obtained the chain of the inequalities

$$\mathcal{L}(u,q) \leq \mathcal{L}(u,p) = J(u) \leq \mathcal{L}(v,p) \quad \text{for all } v \in X, q \in K,$$

which implies that (u,p) is a saddle point of the Lagrangian [12]. Uniqueness of u was demonstrated earlier in Section 1. Show uniqueness of p. Suppose that there exist $p_1, p_2 \in K$ such that $p_1[u] = |[u]| = p_2[u]$ a.e. on Γ. Then (24) is valid for each p_i, $i = 1,2$:

$$(u,w)_X + \int_\Gamma gp_1[w]\, d\Gamma = \int_\Omega fw\, d\Omega \quad \text{for all } w \in X,$$

$$(u,w)_X + \int_\Gamma gp_2[w]\, d\Gamma = \int_\Omega fw\, d\Omega \quad \text{for all } w \in X.$$

Subtracting one equation from the other, we obtain

$$\int_\Gamma g(p_1 - p_2)[w]\, d\Gamma = 0 \quad \text{for all } w \in X.$$

By positivity of g and arbitrariness of w, the above equality means that $p_1 - p_2 = 0$ a.e. on Γ. The theorem is proven.

REMARK 4. We see that the so-constructed function $B'(\cdot)$ is the projection from $L_2(\Gamma)$ onto the set K: $B'(\phi) = P_K(\phi)$ for all $\phi \in L_2(\Gamma)$; i.e.,

$$\|\phi - B'(\phi)\|_{L_2(\Gamma)} \leq \|\phi - \psi\|_{L_2(\Gamma)} \quad \text{for all } \psi \in K.$$

6. Analytical solution in the one-dimensional case. Suppose that $\Omega_0 = (0,1)$, Γ represents some point y, $0 < y < 1$, and $\Omega = (0, y-0) \cup (y+0, 1)$. This is the model of a rod with a cross-section. The above-introduced bilinear form b takes the shape

$$b(u,v) \equiv (u,v)_X = D \int_\Omega u_{xx} v_{xx}\, dx, \quad u,v \in X,$$

for the space

$$X = \{v \in H^2(\Omega),\ v(x) = v_x(x) = 0 \text{ for } x = 0,1\}.$$

Suppose that the external load function $f \in L_2(\Omega)$ and the friction $g \geq 0$ are given. The problem of finding the function $u(x) \in X$ of vertical displacements of the points x of the rod Ω rigidly clamped at the endpoints and having a cut at y under the action of the force f and the friction g on the cut is stated as the variational inequality

$$D \int_\Omega u_{xx}(v_{xx} - u_{xx})\, dx + g(|[v]| - |[u]|) \geq \int_\Omega f(v-u)\, dx \quad \text{for all} \quad v \in X. \tag{26}$$

Here the jump of a function is understood in the sense $[v] = v(y+0) - v(y-0)$. Take $v = u \pm \xi$, $\xi \in C_0^\infty(\Omega)$, and insert it in (26). We infer that the equation

$$Du_{xxxx} = f \tag{27}$$

is valid a.e. in Ω. Since $f \in L_2(\Omega)$, (27) implies that the solution u to (26) belongs to $X \cap H^4(\Omega) \subset C^3(\bar{\Omega})$. Therefore, the quantities $T^\pm(u) = Du_{xxx}(y \pm 0)$ and $M^\pm(u) = Du_{xx}(y \pm 0)$ are well defined. We have shown earlier in Section 2 that $[T(u)] = [M(u)] = 0$; consequently, their limit values at y from the left and right coincide and equal

$$t(u) = Du_{xxx}(y), \quad m(u) = Du_{xx}(y).$$

Then the Green's formula

$$\int_\Omega u_{xx} v_{xx}\, dx = \int_\Omega u_{xxxx} v\, dx - u_{xx}(y)[v_x] + u_{xxx}(y)[v], \quad v \in X,$$

is valid by the smoothness of a solution. Using this circumstance, we can rewrite the boundary value problem (15) in the one-dimensional case as

$$Du_{xxxx} = f \text{ in } \Omega,$$

$$u_{xx}(y) = 0, \quad D|u_{xxx}(y)| \leq g, \quad Du_{xxx}(y)[u] + g|[u]| = 0.$$

For finding a solution to (26) we propose the duality method. Construct the Lagrangian

$$\mathcal{L}(v,q) = \frac{D}{2} \int_\Omega v_{xx}^2\, dx + gq[v] - \int_\Omega fv\, dx$$

on the set $v \in X$, $q \in K = \{t \in R, |t| \leq 1\}$. According to Theorem 3, the problem (26) is equivalent to finding a unique saddle point $(u,p) \in X \times K$ such that

$$\mathcal{L}(u,p) = \inf_{v \in X} \sup_{q \in K} \mathcal{L}(v,q) = \sup_{q \in K} \inf_{v \in X} \mathcal{L}(v,q); \tag{28}$$

moreover, as was shown, $p[u] = |[u]|$. Consider the second (dual) problem (28). We consider the number $q \in K$ as a parameter and find a solution $u^q \in X$ to the problem

$$\mathcal{L}(u^q, q) = \inf_{v \in X} \mathcal{L}(v, q)$$

which is equivalent to the following equation:

$$D \int_\Omega u^q_{xx} v_{xx}\, dx + gq[v] = \int_\Omega fv\, dx \quad \text{for all } v \in X. \tag{29}$$

We seek a solution to (29) in the form of a sum $u^q = w + w^q$, where

$$D \int_\Omega w_{xx} v_{xx}\, dx = \int_\Omega fv\, dx \quad \text{for all } v \in X, \tag{30}$$

$$D \int_\Omega w^q_{xx} v_{xx}\, dx + gq[v] = 0 \quad \text{for all } v \in X. \tag{31}$$

A formal application of the Green's formula for smooth solutions implies that the equation (30) in X is equivalent to the boundary value problem

$$Dw_{xxxx} = f \text{ in } \Omega, \quad w_{xx}(y) = w_{xxx}(y) = 0.$$

It has a unique solution $w \in X \cap H^4(\Omega)$, since it actually splits into two simple problems

$$\begin{aligned} Dw_{xxxx} &= f \quad \text{for a.e. } (0, y), \quad w_{xx}(y) = w_{xxx}(y) = 0, \quad w(0) = w_x(0) = 0, \\ Dw_{xxxx} &= f \quad \text{for a.e. } (y, 1), \quad w_{xx}(y) = w_{xxx}(y) = 0, \quad w(1) = w_x(1) = 0. \end{aligned}$$

Now, consider the equation (31). On assuming that (31) has a smooth solution in X, from the Green's formula we infer that

$$w^q_{xxxx} = 0 \text{ in } \Omega, \quad w^q_{xx}(y) = 0, \quad Dw^q_{xxx}(y) + gq = 0.$$

It is easy to construct a solution $w^q \in X \cap H^4(\Omega)$. To this end, observe that w^q is representable as $w^q(x) = -gqD^{-1}\alpha(x)$ with a function $\alpha \in X$ satisfying the relations

$$\begin{gathered} \alpha_{xxxx} = 0 \text{ in } \Omega, \quad \alpha_{xx}(y) = 0, \quad \alpha_{xxx}(y) = 1, \\ \alpha(0) = \alpha_x(0) = \alpha(1) = \alpha_x(1) = 0. \end{gathered}$$

Hence, we find that

$$\alpha(x) = \frac{1}{6} \begin{cases} x^3 - 3yx^2 & \text{for } x \in (0, y - 0), \\ (x-1)^3 - 3(y-1)(x-1)^2 & \text{for } x \in (y + 0, 1). \end{cases} \tag{32}$$

It follows from the construction that $\alpha \in X \cap C^\infty(\Omega)$ and $[\alpha] = (3y^2 - 3y + 1)/3 > 0$. Thus, the function $u^q(x) = w(x) - gqD^{-1}\alpha(x)$ is a unique solution to (29).

To find a solution $u = u^p$ to (28) (and thereby to (26)), we have to choose the parameter $q = p$ in (29) so as to fulfill the necessary relations

$$p[u^p] = |[u^p]|, \quad |p| \le 1, \tag{33}$$

according to the arguments of Section 5. By the above constructions, (33) takes the form

$$p([w] - gpD^{-1}[\alpha]) = |[w] - gpD^{-1}[\alpha]|, \quad |p| \le 1.$$

These conditions mean that $p = 1$ if $D[w](g[\alpha])^{-1} > 1$, $p = -1$ if $D[w](g[\alpha])^{-1} < -1$, and $p = D[w](g[\alpha])^{-1}$ if $|D[w](g[\alpha])^{-1}| \le 1$ or, shortly, $p = B'(D[w](g[\alpha])^{-1})$. We have thus proven the following theorem:

Theorem 4. *A solution $u \in X \cap H^4(\Omega)$ to (26) is representable as*

$$u(x) = w(x) - \frac{g}{D} B' \left(\frac{3D[w]}{g(3y^2 - 3y + 1)} \right) \alpha(x),$$

where w is a solution to the linear equation (30) and the function α is given explicitly by (32).

From the smoothness of α and w we derive the following corollary:

Corollary 1. *If $f \in H^m(\Omega)$ then $u \in X \cap H^{m+4}(\Omega)$, $m \geq 0$.*

Corollary 2. *The solution u is fused at the point y, i.e., $u \in X \cap H^4(\Omega_0)$, if*

$$|[w]| \leq \frac{g(3y^2 - 3y + 1)}{3D} \quad \text{and} \quad [w_x] + \frac{3(1-2y)}{2(3y^2 - 3y + 1)}[w] = 0.$$

Indeed, from Theorem 4 we obtain

$$[u] = \begin{cases} -g[\alpha]/D + [w], & [w] > g[\alpha]/D, \\ g[\alpha]/D + [w], & [w] < -g[\alpha]/D, \\ 0, & |[w]| \leq g[\alpha]/D, \end{cases}$$

$$[u_x] = [w_x] + \begin{cases} -g[\alpha_x]/D, & [w] > g[\alpha]/D, \\ g[\alpha_x]/D, & [w] < -g[\alpha]/D, \\ -[w][\alpha_x]/[\alpha], & |[w]| \leq g[\alpha]/D, \end{cases}$$

whereas $[u_{xx}] = [u_{xxx}] = 0$ is always valid.

EXAMPLE. Suppose that $f(x) \equiv kD$, where k is a constant. A solution to the boundary value problem for the equation (30) has the form

$$w(x) = \frac{k}{24} \begin{cases} x^4 - 4yx^3 + 6y^2x^2 & \text{for } x \in (0, y-0), \\ (x-1)^4 - 4(y-1)(x-1)^3 + 6(y-1)^2(x-1)^2 & \text{for } x \in (y+0, 1); \end{cases}$$

and for its jump we have $[w] = k((y-1)^4 - y^4)/8$. Then we can calculate the number

$$A \equiv \frac{g}{D} B' \left(\frac{D[w]}{g[\alpha]} \right) = \frac{g}{D} B' \left(\frac{3kD((y-1)^4 - y^4)}{8g(3y^2 - 3y + 1)} \right)$$

and find the function u solving the problem (26):

$$u(x) = \frac{1}{24} \begin{cases} kx^4 - 4(ky + A)x^3 + 6(ky + 2A)yx^2, \\ k(x-1)^4 - 4(k(y-1) + A)(x-1)^3 + 6(k(y-1) + 2A)(y-1)(x-1)^2. \end{cases}$$

References

1. S. A. Nazarov and B. A. Plamenevskiĭ, Elliptic Problems in Domains with Piecewise Smooth Boundary [in Russian], Nauka, Moscow (1991).
2. V. A. Kondrat′ev and O. A. Oleĭnik, "Boundary value problems for partial differential equations in nonsmooth domains," Uspekhi Mat. Nauk, **38**, No. 2, 3–76 (1983).
3. V. G. Maz′ya, N. F. Morozov, and B. A. Plamenevskiĭ, "Nonlinear bending of a plate with a crack," in: Differential and Integral Equations. Boundary Value Problems [in Russian], Tbilis. Univ., Tbilisi, 1979, pp. 145–163.
4. R. Duduchava and W. Wendland, "The Wiener–Hopf method for systems of pseudodifferential equations with an application to crack problems," Integral Equations Operator Theory, **23**, No. 1, 294–335 (1995).

5. A. M. Khludnev, "A contact problem for a shallow shell with a crack," Prikl. Mat. Mekh., **59**, No. 2, 318–326 (1995).
6. G. Devaut and J.-L. Lions, Inequalities in Mechanics and Physics [Russian translation], Nauka, Moscow (1980).
7. R. Glowinski, J.-L. Lions, and R. Tremolieres, Numerical Analysis of Variational Inequalities [Russian translation], Mir, Moscow (1979).
8. I. Hlavaček, J. Haslinger, J. Nečas, and J. Lovišek, Solution of Variational Inequalities in Mechanics [Russian translation], Mir, Moscow (1986).
9. V. A. Kovtunenko, "An iterative penalty method for a problem with constraints on the inner boundary," Sibirsk. Mat. Zh., **37**, No. 3, 585–591 (1996).
10. V. Kovtunenko, "Iterative approximations of penalty operators," Numer. Funct. Anal. Optim., **18**, No. 3–4, 383–387 (1997).
11. V. Kovtunenko, "Analytical solution of a variational inequality for a cutted bar," Control Cybernet., **25**, No. 4, 801–808 (1996).
12. I. Ekeland and R. Temam, Convex Analysis and Variational Problems [Russian translation], Mir, Moscow (1979).
13. J.-L. Lions and E. Magenes, Inhomogeneous Boundary Value Problems and Their Applications [Russian translation], Mir, Moscow (1971).
14. D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications [Russian translation], Mir, Moscow (1983).