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AN ITERATIVE PENALTY METHOD FOR A PROBLEM WITH CONSTRAINTS ON THE INNER BOUNDARY†)

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Under study is some variational inequality describing equilibrium of an elastic membrane in a domain with a slit. The unpermeability boundary condition in the form of an inequality is taken on the edges of the slit. We construct a penalized equation and an iterative linear equation in integral and differential forms. We prove some results upon convergence of solutions and establish an error estimate.

Let a membrane occupy a bounded domain $\Omega_0 \subset R^2$ with smooth boundary $\partial\Omega_0$. Inside Ω_0 there is a slit Γ given by a sufficiently smooth function. A fixed vector $\nu = (\nu^1, \nu^2)$ normal to Γ determines the edges of the slit, positive Γ^+ and negative Γ^-. Denote $\Omega = \Omega_0 \backslash \Gamma$. The sought vector $u = (U, w)$ of horizontal $U = (u^1, u^2)$ and vertical w displacements of the points of the middle surface of the membrane meets the following boundary conditions: first, the rigid clamping condition on the outer boundary:

$$w = \frac{\partial w}{\partial n} = u^1 = u^2 = 0 \quad \text{on} \quad \partial\Omega_0, \tag{1}$$

with n standing for the normal to $\partial\Omega_0$; second, the unpermeability condition on the edges of the slit [1]:

$$[U]\nu \geq \left|\left[\frac{\partial w}{\partial \nu}\right]\right|, \tag{2}$$

with $[s]$ denoting the jump of the function across Γ, i.e. $[s] = s|_{\Gamma^+} - s|_{\Gamma^-}$. Construct the linear functions

$$\varphi(u) = [U]\nu + \left[\frac{\partial w}{\partial \nu}\right], \quad \psi(u) = [U]\nu - \left[\frac{\partial w}{\partial \nu}\right].$$

Then condition (2) amounts to the following: $\varphi(u) \geq 0$ and $\psi(u) \geq 0$. Define the main Hilbert space

$$X = \{u \in \left(H^1(\Omega)\right)^2 \times H^2(\Omega),\ u \ \text{ satisfies } (1)\}$$

and the closed convex set

$$K = \{u \in X,\ \varphi(u) \geq 0,\ \psi(u) \geq 0\}.$$

Denote by $X^\star$ the dual space of X. Introduce the following bilinear forms which are well known in elasticity:

$$a(u,v) = a_1(U,V) + a_2(w,\omega), \quad v = (V,\omega), V = (v^1, v^2),$$

$$a_1(U,V) = \int\limits_\Omega \left(u_x^1 v_x^1 + u_y^2 v_y^2 + \kappa(u_x^1 v_y^2 + u_y^2 v_x^1) + \frac{1-\kappa}{2}(u_y^1 + u_x^2)(v_y^1 + v_x^2)\right) d\Omega,$$

$$a_2(w,\omega) = \int\limits_\Omega (w_{xx}\omega_{xx} + w_{yy}\omega_{yy} + \kappa(w_{xx}\omega_{yy} + w_{yy}\omega_{xx}) + 2(1-\kappa) w_{xy}\omega_{xy})\, d\Omega.$$

The constant $0 < \kappa < 0.5$ is given. By the Korn first inequality, we have

$$a_1(U,U) \geq c_1 \|U\|_1^2, \quad c_1 > 0.$$

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In the same way, we can deduce that

$$a_2(w, w) \geq c_2 \|w\|_2^2, \quad c_2 > 0,$$

with $\|\cdot\|_k$ the norm in $H^k(\Omega)$. Therefore, we have the inequality

$$a(u, u) \geq M\|u\|^2, \quad M > 0, \tag{3}$$

with $\|\cdot\|$ the norm in X. By coercivity and obvious boundedness of $a(\cdot,\cdot)$, we may introduce the inner product

$$(u, v) = a(u, v) + \int\limits_{\Gamma} (\varphi(u)\varphi(v) + \psi(u)\psi(v))\, d\Gamma$$

and the equivalent norm $\|u\|^2 = (u, u)$ in X. Let $f = (f^1, f^2, f^3) \in L_2(\Omega)$ be some given functions of the outer loading. The equilibrium problem for a membrane with a slit is formulated as the following variational inequality [1, 2]:

$$u \in K, \quad a(u, v - u) \geq \langle f, v - u\rangle \quad \forall v \in K. \tag{4}$$

The angular brackets $\langle\cdot,\cdot\rangle$ denote the integration over Ω. In virtue of linearity, boundedness, and coercivity of the form $a(\cdot,\cdot)$, there exists a unique solution to (4) (see [3]).

Formally integrating by parts $a(u, v)$, $u, v \in X$, we obtain

$$a(u, v) = \langle Au, v\rangle - \int\limits_{\Gamma} \left[\sigma(U)\nu V\nu + \sigma(U)\tau V\tau + K(w)\frac{\partial\omega}{\partial\nu} - T(w)\omega\right] d\Gamma,$$

where

$$Au = \left(-\triangle u^1 + \frac{1+\kappa}{2}(u^1_{yy} - u^2_{xy}), -\triangle u^2 + \frac{1+\kappa}{2}(u^2_{xx} - u^1_{xy}), \triangle^2 w\right),$$

$$\sigma(U) = \left((u^1_x + \kappa u^2_y)\nu^1 + \frac{1-\kappa}{2}(u^1_y + u^2_x)\nu^2, (u^2_y + \kappa u^1_x)\nu^2 + \frac{1-\kappa}{2}(u^1_y + u^2_x)\nu^1\right),$$

$$K(w) = \kappa \triangle w + (1-\kappa)\frac{\partial^2 w}{\partial \nu^2}, \quad T(w) = \frac{\partial(\Delta w + (1-\kappa)\frac{\partial^2 w}{\partial \tau^2})}{\partial\nu}.$$

Here $\tau = (-\nu^2, \nu^1)$ is the tangent vector to Γ. Let a solution u possess sufficient smoothness. Rewrite (4) as

$$\langle Au - f, v - u\rangle - \int\limits_{\Gamma}\Bigg([\sigma(U)\nu(V-U)\nu] + \left[K(w)\frac{\partial(\omega - w)}{\partial\nu}\right]$$
$$+[\sigma(U)\tau(V-U)\tau] - [T(w)(\omega - w)]\Bigg)\, d\Gamma \geq 0 \quad \forall v \in K.$$

Further, we vary the test functions v and use the relation

$$\sigma(U)\nu[V-U]\nu + K(w)\left[\frac{\partial(\omega - w)}{\partial\nu}\right]$$
$$= \frac{1}{2}(\sigma(U)\nu + K(w))\varphi(v-u) + \frac{1}{2}(\sigma(U)\nu - K(w))\psi(v-u).$$

Then a solution of variational inequality (4) may be interpreted as a solution $u \in X$ of the following boundary value problem:

$$Au = f \quad \text{in } \Omega,$$
$$[\sigma(U)] = 0, \quad [K(w)] = 0, \quad [T(w)] = 0,$$
$$\sigma(U)\tau = 0, \quad T(w) = 0 \quad \text{on } \Gamma,$$
$$[U]\nu \geq \left|\left[\frac{\partial w}{\partial \nu}\right]\right|, \quad -\sigma(U)\nu \geq |K(w)|,$$
$$(\sigma(U)\nu + K(w))\varphi(u) = 0, \quad (\sigma(U)\nu - K(w))\psi(u) = 0.$$

We turn to considering a penalized problem. To this end, define the penalty operator $\beta : X \to X^\star$ by the rule

$$\langle \beta(u), v\rangle = -\int\limits_\Gamma (\varphi^-(u)\varphi(v) + \psi^-(u)\psi(v))\, d\Gamma.$$

Here the angular brackets denote the duality between X and $X^\star$, and a superscript minus signifies the negative part of a function, i.e. $s = s^+ - s^-$, $s^+, s^- \geq 0$. It is seen that β is a monotone operator. Denote by $u^\varepsilon \in X$ a solution of the equation

$$a(u^\varepsilon, v) + \varepsilon^{-1}\langle \beta(u^\varepsilon), v\rangle = \langle f, v\rangle \quad \forall v \in X \tag{5}$$

depending on a small parameter $\varepsilon > 0$. Given a sufficiently smooth solution u^ε, penalized equation (5) is equivalent to the following boundary value problem:

$$Au^\varepsilon = f \quad \text{in } \Omega,$$
$$[\sigma(U^\varepsilon)] = 0, \quad [K(w^\varepsilon)] = 0, \quad [T(w^\varepsilon)] = 0,$$
$$\sigma(U^\varepsilon)\tau = 0, \quad T(w^\varepsilon) = 0 \quad \text{on } \Gamma,$$
$$\sigma(U^\varepsilon)\nu = -\varepsilon^{-1}(\varphi^-(u^\varepsilon) + \psi^-(u^\varepsilon)), \quad K(w^\varepsilon) = -\varepsilon^{-1}(\varphi^-(u^\varepsilon) - \psi^-(u^\varepsilon)).$$

Fix ε. To linearize the left-hand side of (5), given an arbitrary $u^{\varepsilon,0} \in X$, $n = 0, 1, 2, \ldots$, construct the following iterative procedure [4]:

$$a(u^{\varepsilon,n+1}, v) + \varepsilon^{-1}(u^{\varepsilon,n+1}, v) = \langle f, v\rangle + \varepsilon^{-1}(u^{\varepsilon,n}, v) - \varepsilon^{-1}\langle \beta(u^{\varepsilon,n}), v\rangle. \tag{6}$$

By the above-indicated properties of the operator of equation (6), there is a unique solution $u^{\varepsilon,n+1} \in X$ to (6). Then, for $u^{\varepsilon,n+1}$ sufficiently smooth, the corresponding boundary value problem takes the form

$$A(u^{\varepsilon,n+1} + \varepsilon^{-1}(u^{\varepsilon,n+1} - u^{\varepsilon,n})) = f \quad \text{in } \Omega,$$
$$[\sigma(U^{\varepsilon,n+1} + \varepsilon^{-1}(U^{\varepsilon,n+1} - U^{\varepsilon,n}))] = 0, \quad [K(w^{\varepsilon,n+1} + \varepsilon^{-1}(w^{\varepsilon,n+1} - w^{\varepsilon,n}))] = 0,$$
$$[T(w^{\varepsilon,n+1} + \varepsilon^{-1}(w^{\varepsilon,n+1} - w^{\varepsilon,n}))] = 0, \quad \sigma(U^{\varepsilon,n+1} + \varepsilon^{-1}(U^{\varepsilon,n+1} - U^{\varepsilon,n}))\tau = 0,$$
$$T(w^{\varepsilon,n+1} + \varepsilon^{-1}(w^{\varepsilon,n+1} - w^{\varepsilon,n})) = 0 \quad \text{on } \Gamma,$$

$$(\sigma(U^{\varepsilon,n+1} + \varepsilon^{-1}(U^{\varepsilon,n+1} - U^{\varepsilon,n})) - 2\varepsilon^{-1}[U^{\varepsilon,n+1} - U^{\varepsilon,n}])\nu$$
$$= -\varepsilon^{-1}(\varphi^-(u^{\varepsilon,n}) + \psi^-(u^{\varepsilon,n})),$$

$$K(w^{\varepsilon,n+1} + \varepsilon^{-1}(w^{\varepsilon,n+1} - w^{\varepsilon,n})) - 2\varepsilon^{-1}\left[\frac{\partial(w^{\varepsilon,n+1} - w^{\varepsilon,n})}{\partial \nu}\right]$$
$$= -\varepsilon^{-1}(\varphi^-(u^{\varepsilon,n}) - \psi^-(u^{\varepsilon,n})).$$

Theorem. *The sequence $u^{\varepsilon,n+1} \to u^\varepsilon$ converges strongly in X as $n \to \infty$. Moreover,*

$$\|u^{\varepsilon,n+1} - u^\varepsilon\|^2 \leq (1 + 2M\varepsilon)^{-(n+1)}\|u^{\varepsilon,0} - u^\varepsilon\|^2 \tag{7}$$

and $u^\varepsilon \to u$ strongly in X as $\varepsilon \to 0$.

PROOF. Subtract (5) from (6) and $\varepsilon^{-1}(u^\varepsilon, v)$ from the two sides to obtain

$$a(u^{\varepsilon,n+1} - u^\varepsilon, v) + \varepsilon^{-1}(u^{\varepsilon,n+1} - u^\varepsilon, v) = \varepsilon^{-1}a(u^{\varepsilon,n} - u^\varepsilon, v)$$
$$+\varepsilon^{-1}\int_\Gamma((\varphi(u^{\varepsilon,n}) - \varphi(u^\varepsilon) + \varphi^-(u^{\varepsilon,n}) - \varphi^-(u^\varepsilon))\varphi(v)$$
$$+(\psi(u^{\varepsilon,n}) - \psi(u^\varepsilon) + \psi^-(u^{\varepsilon,n}) - \psi^-(u^\varepsilon))\psi(v))\,d\Gamma. \tag{8}$$

Since $s_1 - s_2 + s_1^- - s_2^- = s_1^+ - s_2^+ \leq |s_1 - s_2|$; using the Hölder inequality, we can estimate the right-hand part of (8) from above by

$$(2\varepsilon)^{-1}\Big(a(u^{\varepsilon,n} - u^\varepsilon, u^{\varepsilon,n} - u^\varepsilon) + a(v,v)$$
$$+\int_\Gamma(\varphi^2(u^{\varepsilon,n} - u^\varepsilon) + \varphi^2(v) + \psi^2(u^{\varepsilon,n} - u^\varepsilon) + \psi^2(v))d\Gamma\Big)$$
$$= (2\varepsilon)^{-1}(\|u^{\varepsilon,n} - u^\varepsilon\|^2 + \|v\|^2).$$

Consider this relation at the element $v = u^{\varepsilon,n+1} - u^\varepsilon$. By (3), from (8) we have the inequality

$$(M + \varepsilon^{-1})\|u^{\varepsilon,n+1} - u^\varepsilon\|^2 \leq (2\varepsilon)^{-1}(\|u^{\varepsilon,n} - u^\varepsilon\|^2 + \|u^{\varepsilon,n+1} - u^\varepsilon\|^2).$$

Thus,

$$\|u^{\varepsilon,n+1} - u^\varepsilon\|^2 \leq (1 + 2M\varepsilon)^{-1}\|u^{\varepsilon,n} - u^\varepsilon\|^2.$$

Repeating the estimate for $n, n-1, \dots, 0$, we obtain (7) and the first convergence.

In a standard way (see [3]), the properties of the operators $a(\cdot,\cdot)$ and $\beta(\cdot)$ imply that

$$u^\varepsilon \to u \quad \text{weakly in } X \text{ as } \varepsilon \to 0. \tag{9}$$

Subtract $a(u,v)$ from (8) and consider the so-obtained equation at the element $v = u^\varepsilon - u$. We have

$$a(u^\varepsilon - u, u^\varepsilon - u) - \varepsilon^{-1}\int_\Gamma(\varphi^-(u^\varepsilon)\varphi(u^\varepsilon - u) + \psi^-(u^\varepsilon)\psi(u^\varepsilon - u))d\Gamma$$
$$= \langle f, u^\varepsilon - u\rangle - a(u, u^\varepsilon - u).$$

We estimate the left-hand side from below:

$$M\|u^\varepsilon - u\|^2 + \varepsilon^{-1}\int_\Gamma((\varphi^-(u^\varepsilon))^2 + (\psi^-(u^\varepsilon))^2)d\Gamma \leq \langle f, u^\varepsilon - u\rangle - a(u, u^\varepsilon - u).$$

Therefore, (9) implies the second result on strong convergence. The theorem is proven.

Unlike (5), the constructed iterative equation (6) is linear, which allows us to apply the standard numerical methods to it. Analogous approaches to the contact problems of elastic and plastic membranes with obstacle were treated in [5, 6].

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