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AN ITERATIVE PENALTY METHOD FOR VARIATIONAL INEQUALITIES WITH STRONGLY MONOTONE OPERATORS

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We consider penalized operator equations approximating variational inequalities. For equations with strongly monotone operators we construct an iterative method, prove convergence of solutions, and obtain error estimates.

Introduce the following notations: V, a reflexive Banach space; V^*, its dual; $\langle\cdot,\cdot\rangle$, the duality between V and V^*; $\|\cdot\|$ and $\|\cdot\|_*$, the respective norms in V and V^*. Let K be a closed convex subset in V, and let an operator A act from V into V^*; and let $f \in V^*$ be given. We study the variational inequality

$$u \in K, \quad \langle Au, v-u\rangle \geq \langle f, v-u\rangle \quad \forall v \in K. \tag{1}$$

Construct a penalized equation which approximates (1) as indicated in [1]. To this end, impose the following assumption: there is a Hilbert space W such that $V \subseteq W \subseteq V^*$, with $(\cdot,\cdot)$ the scalar product in W and $\|v\|_W^2 = (v,v)$ the norm of v in W, and there is a constant $c > 0$ such that

$$\|\cdot\|_W \leq c\|\cdot\|. \tag{2}$$

Suppose that K is a closed convex set in W as well. Define P to be the projection of W onto K; i.e., $Pa \in K$ is the projection of $a \in W$ if

$$\|a - Pa\|_W \leq \|a - s\|_W \quad \forall s \in K. \tag{3}$$

Lemma. *The following inequality holds for all $a, b \in W$:*

$$\|Pa - Pb\|_W \leq \|a-b\|_W.$$

PROOF. In view of convexity of K, in (3) we can take

$$s = (1-\lambda)Pa + \lambda d \quad \forall d \in K,\ 0 \leq \lambda \leq 1.$$

Then

$$\|a - Pa\|_W \leq \|a - Pa + \lambda(Pa - d)\|_W \quad \forall d \in K. \tag{4}$$

We have the following obvious inequality for arbitrary $u, v \in W$:

$$(u, v-u) \leq 0.5(\|v\|_W + \|u\|_W)(\|v\|_W - \|u\|_W). \tag{5}$$

Take $u = a - Pa + \lambda(Pa - d)$ and $v = a - Pa$ and substitute them into (5). From (4) we obtain $\lambda(a - Pa + \lambda(Pa-d), d - Pa) \leq 0$. Canceling the factor λ in the inequality and tending λ to zero afterwards, we obtain $(a - Pa, d - Pa) \leq 0 \quad \forall d \in K$. In a similar way, for $b \in W$ we obtain $(b - Pb, e - Pb) \leq 0 \quad \forall e \in K$. Take $d = Pb$ and $e = Pa$ and sum the last two inequalities:

$$(a - b - Pa + Pb, Pb - Pa) \leq 0. \tag{6}$$

Consider the norm $\|Pb - Pa\|_W^2$. By (6) and Hölder's inequality, we have

$$\|Pb - Pa\|_W^2 = (a - b - Pa + Pb, Pb - Pa) + (b - a, Pb - Pa) \leq \|Pb - Pa\|_W \|b - a\|_W.$$

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The lemma is proved.

Introduce the penalty operator $\beta : W \to W$ by the rule $\beta(v) = v - Pv$ for $v \in W$. The operator β is monotone by [1, p. 384] and continuous by the lemma. Given a small parameter $\varepsilon > 0$, we write down the penalized equation

$$Au + \varepsilon^{-1}\beta(u) = f \tag{7}$$

which is to be understood as follows:

$$\langle Au, v\rangle + \varepsilon^{-1}(\beta(u), v) = \langle f, v\rangle \quad \forall v \in V.$$

Here and in the sequel, we do not indicate the dependence of functions on ε for simplicity. Fix ε and construct the sequence of equations

$$Au^{n+1} + \varepsilon^{-1}u^{n+1} = f + \varepsilon^{-1}Pu^n, \tag{8}$$

where $n = 0, 1, 2, \dots$ and $u^0 \in V$ is arbitrary. Suppose that the operator A is

(1) radially continuous, i.e., the function $s \to \langle A(u + sv), v\rangle$ is continuous on $[0,1]$ for all fixed $u, v \in V$;

(2) strongly monotone, i.e., there is a constant $M > 0$ such that

$$\langle Au - Av, u - v\rangle \geq M\|u - v\|^2 \quad \forall u, v \in V.$$

Property (2) obviously implies coercivity and strong monotonicity of A. The right-hand side of (8) belongs to V^*, since $W \subset V^*$. Then, by Browder's theorem [2, p. 95, Theorems 2.1 and 2.2], there exists a unique solution $u^{n+1} \in V$, $n = 0, 1, 2, \dots$, to problem (8).

Theorem 1. *Under the above assumptions,*

$$u^{n+1} \to u \text{ strongly in } V \text{ as } n \to \infty,$$

where u is a solution to problem (7); *moreover, the following estimate holds:*

$$\|u - u^{n+1}\| \leq c(\varepsilon M^2)^{-1}\rho^{0.5n}\|f - Au^0 - \varepsilon^{-1}\beta(u^0)\|_*,$$

with $\rho = c^2(c^2 + 2\varepsilon M)^{-1} < 1$.

PROOF. Take (8) at the preceding step in n:

$$Au^n + \varepsilon^{-1}u^n = f + \varepsilon^{-1}Pu^{n-1},$$

subtract this equality from (8), and consider the so-obtained difference at the element $u^{n+1} - u^n \in V$:

$$\langle Au^{n+1} - Au^n, u^{n+1} - u^n\rangle + \varepsilon^{-1}\|u^{n+1} - u^n\|_W^2 = \varepsilon^{-1}(Pu^n - Pu^{n-1}, u^{n+1} - u^n).$$

By making use of strong monotonicity of A, Hölder's inequality, and the lemma, we obtain the estimate

$$2\|u^{n+1} - u^n\|^2 + (\varepsilon M)^{-1}\|u^{n+1} - u^n\|_W^2 \leq (\varepsilon M)^{-1}\|u^n - u^{n-1}\|_W^2. \tag{9}$$

Introduce the equivalent norm in V as follows: $[\,\cdot\,]^2 = \|\cdot\|^2 + (\varepsilon M)^{-1}\|\cdot\|_W^2$. By (2), from (9) we obtain

$$[u^{n+1} - u^n]^2 \leq \rho[u^n - u^{n-1}]^2 \leq \dots \leq \rho^n[u^1 - u^0]^2. \tag{10}$$

Thus,

$$u^{n+1} - u^n \to 0 \text{ strongly in } V \text{ as } n \to \infty, \tag{11}$$

It follows that there exists an element $u \in V$ such that

$$u^{n+1} \to u \text{ strongly in } V \text{ as } n \to \infty. \tag{12}$$

Properties (1) and (2) of the operator A imply that A acts continuously from V with strong topology into V^* with weak topology [2, p. 84]:

$$Au^{n+1} \to Au \text{ weakly in } V \text{ as } n \to \infty. \tag{13}$$

From the continuity of the penalty operator it follows that

$$\beta(u^n) \to \beta(u) \text{ strongly in } W \text{ as } n \to \infty. \tag{14}$$

Representing (8) as

$$Au^{n+1} + \varepsilon^{-1}\beta(u^n) + \varepsilon^{-1}(u^{n+1} - u^n) = f,$$

passing to the limit as $n \to \infty$, and using (11)–(14), we obtain (7). Now we estimate the error. Consider (8) for $n = 0$:

$$Au^1 - Au^0 + \varepsilon^{-1}(u^1 - u^0) = f - Au^0 - \varepsilon^{-1}\beta(u^0)$$

at the element $u^1 - u^0$; Hölder's inequality implies the estimate

$$[u^1 - u^0] \leq M^{-1}\|f - Au^0 - \varepsilon^{-1}\beta(u^0)\|_*. \tag{15}$$

Now represent (8) as

$$Au^{n+1} + \varepsilon^{-1}\beta(u^{n+1}) = f + \varepsilon^{-1}(Pu^n - Pu^{n+1}),$$

subtract it from (7), and consider the difference at the element $u - u^{n+1}$:

$$\langle Au - Au^{n+1}, u - u^{n+1}\rangle + \varepsilon^{-1}(\beta(u) - \beta(u^{n+1}), u - u^{n+1}) = \varepsilon^{-1}(Pu^{n+1} - Pu^n, u - u^{n+1}).$$

Applying Hölder's inequality and making use of strong monotonicity of A, the monotonicity of β, the lemma, and the estimates for the norms, we obtain

$$\|u - u^{n+1}\| \leq c(\varepsilon M)^{-1}[u^{n+1} - u^n]. \tag{16}$$

Combining (10), (15), and (16), we arrive at the required estimate. The theorem is proved.

REMARK 1. In the general case the constant M depends on ε as well. If M is independent of ε, then the stated requirements are sufficient for solutions to equations (7) (depending on ε) converge to a solution of variational inequality (1) as $\varepsilon \to 0$ [1, p. 385, Theorem 5.2]. So solutions to equations (8) too converge to a solution of inequality (1) as $n \to \infty$ and $\varepsilon \to 0$.

Let $B : V \to V^*$ be a Lipschitz-continuous operator, i.e., there is a constant $\sigma > 0$ such that

$$\|Bu - Bv\|_* \leq \sigma\|u - v\| \quad \forall u, v \in V.$$

Consider the operator $A + B$ instead of A. Then (7) takes the form

$$Au + Bu + \varepsilon^{-1}\beta(u) = f. \tag{17}$$

Consider the iterative sequence of equations

$$Au^{n+1} + \varepsilon^{-1}u^{n+1} = f - Bu^n + \varepsilon^{-1}Pu^n, \tag{18}$$

where $n = 0, 1, 2, \ldots$ and $u^0 \in V$ is arbitrary. By analogy with (8), there exists a unique solution $u^{n+1} \in V$ to problem (18).

Theorem 2. *If $M - \sigma = \varkappa > 0$, then*

$$u^{n+1} \to u \text{ strongly in } V \text{ as } n \to \infty;$$

u is a solution to problem (17); *and the following estimate is valid:*

$$\|u - u^{n+1}\| \le kR^{0.5n}\|f - Au^0 - Bu^0 - \varepsilon^{-1}\beta(u^0)\|_*,$$

where $R = (c^2 + \varepsilon\sigma)(c^2 + \varepsilon\sigma + 2\varepsilon\varkappa)^{-1} < 1$ and $k = \varkappa^{-1}\{1 - \varepsilon\varkappa(c^2 + \varepsilon M)^{-1}\}\{1 + c^2[\varepsilon\sigma(c^2 + \varepsilon\sigma)]^{-0.5}\}$.

PROOF. From (18) we obtain

$$Au^{n+1} - Au^n + \varepsilon^{-1}(u^{n+1} - u^n) = Bu^{n-1} - Bu^n + \varepsilon^{-1}(Pu^n - Pu^{n-1}).$$

Consider this equation at the element $u^{n+1} - u^n \in V$. Employ Hölder's inequality together with strong monotonicity of A, Lipschitz-continuity of B, and the lemma. As a result, obtain the estimate

$$M\|u^{n+1} - u^n\|^2 + \varepsilon^{-1}\|u^{n+1} - u^n\|_W^2$$
$$\le \sigma\|u^{n+1} - u^n\|\|u^n - u^{n-1}\| + \varepsilon^{-1}\|u^{n+1} - u^n\|_W\|u^n - u^{n-1}\|_W,$$

or, after estimating the right-hand side by means of the squares of the norms, the estimate

$$(2\varkappa + \sigma)\|u^{n+1} - u^n\|^2 + \varepsilon^{-1}\|u^{n+1} - u^n\|_W^2 \le \sigma\|u^n - u^{n-1}\|^2 + \varepsilon^{-1}\|u^n - u^{n-1}\|_W^2.$$

Introduce the equivalent norm in V: $[\cdot]^2 = \sigma\|\cdot\|^2 + \varepsilon^{-1}\|\cdot\|_W^2$. Then the following estimates hold: $\sigma^{0.5}\|\cdot\| \le [\cdot] \le (\sigma + c^2\varepsilon^{-1})^{0.5}\|\cdot\|$. Arguing as in the proof of Theorem 1, we obtain the relations $[u^{n+1} - u^n]^2 \le R[u^n - u^{n-1}]^2 \le \dots \le R^n[u^1 - u^0]^2$. Since $M - \sigma = \varkappa > 0$; therefore,

$$[u^1 - u^0] \le k_1\|f - Au^0 - Bu^0 - \varepsilon^{-1}\beta(u^0)\|_*,$$

where $k_1 = \sigma^{-0.5}\{1 - \varepsilon\varkappa(c^2 + \varepsilon M)^{-1}\}$. Using the estimate $\|\cdot\|_W \le \{\varepsilon c^2(c^2 + \varepsilon\sigma)^{-1}\}^{0.5}[\cdot]$ for the norm, we obtain $\varkappa\|u - u^{n+1}\| \le k_2[u^{n+1} - u^n]$, where $k_2 = \sigma^{0.5}\{1 + c^2[\varepsilon\sigma(c^2 + \varepsilon\sigma)]^{-0.5}\}$. Whence the assertion of the theorem follows.

REMARK 2. The constant σ can depend on ε, too. Since

$$\langle (A + B)u - (A + B)v, u - v\rangle$$
$$\ge \langle Au - Av, u - v\rangle - \|Bu - Bv\|_*\|u - v\| \ge \varkappa\|u - v\|^2 \quad \forall u, v \in V$$

under the made assumptions, the operator $A + B$ too is strongly monotone and radially continuous. Therefore, if $\varkappa$ is independent of ε, then solutions to equation (17) converge as $\varepsilon \to 0$ to a solution of variational inequality (1) with the corresponding operator $A + B$.

REMARK 3. If the operator A is linear, then equations (8) and (18) are linear too.

REMARK 4. The estimates obtained in Theorems 1 and 2 immediately yield the superlinear rate of convergence for solutions to the iterative problems.

The approximate method developed is constructive in the following sense: if A is a linear operator, then equations (8) and (18) can be solved by standard numerical methods. The necessity of studying penalized equations (7) and (17) appears in consideration of a number of boundary value problems with constraints, for example, contact [3] and plasticity [4] problems of mechanics.

References

1. J.-L. Lions, Some Methods for Solving Nonlinear Boundary Value Problems [Russian translation], Mir, Moscow (1972).
2. H. Gajewski, K. Gröger, and K. Zacharias, Nonlinear Operator Equations and Operator Differential Equations [Russian translation], Mir, Moscow (1978).
3. A. M. Khludnev, "Optimal control of a plate over an obstacle," Sibirsk. Mat. Zh., **31**, No. 1, 172–179 (1990).
4. A. M. Khludnev, "Variational inequalities in contact plastic problems," Differentsial'nye Uravneniya, No. 9, 1622–1628 (1988).

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