AN ITERATIVE PENALTY METHOD FOR VARIATIONAL INEQUALITIES WITH STRONGLY MONOTONE OPERATORS

V. A. Kovtunenko

We consider penalized operator equations approximating variational inequalities. For equations with strongly monotone operators we construct an iterative method, prove convergence of solutions, and obtain error estimates.

Introduce the following notations: $V$, a reflexive Banach space; $V^*$, its dual; $\langle \cdot, \cdot \rangle$, the duality between $V$ and $V^*$; $\| \cdot \|$ and $\| \cdot \|_*$, the respective norms in $V$ and $V^*$. Let $K$ be a closed convex subset in $V$, and let an operator $A$ act from $V$ into $V^*$; and let $f \in V^*$ be given. We study the variational inequality

$$\forall v \in K, \quad \langle Au, v - u \rangle \geq \langle f, v - u \rangle.$$  

Construct a penalized equation which approximates (1) as indicated in [1]. To this end, impose the following assumption: there is a Hilbert space $W$ such that $V \subseteq W \subseteq V^*$, with $\langle \cdot, \cdot \rangle$ the scalar product in $W$ and $\|v\|_W^2 = (v, v)$ the norm of $v$ in $W$, and there is a constant $c > 0$ such that

$$\| \cdot \|_W \leq c \| \cdot \|.$$  

Suppose that $K$ is a closed convex set in $W$ as well. Define $P$ to be the projection of $W$ onto $K$; i.e., $Pa \in K$ is the projection of $a \in W$ if

$$\|a - Pa\|_W \leq \|a - s\|_W \quad \forall s \in K.$$  

Lemma. The following inequality holds for all $a, b \in W$:

$$\|Pa - Pb\|_W \leq \|a - b\|_W.$$  

Proof. In view of convexity of $K$, in (3) we can take

$$s = (1 - \lambda)Pa + \lambda d \quad \forall d \in K, \ 0 \leq \lambda \leq 1.$$  

Then

$$\|a - Pa\|_W \leq \|a - Pa + \lambda(Pa - d)\|_W \quad \forall d \in K.$$  

We have the following obvious inequality for arbitrary $u, v \in W$:

$$\langle u, v - u \rangle \leq 0.5(||v||_W + \|u\|_W)(||v||_W - ||u||_W).$$  

Take $u = a - Pa + \lambda(Pa - d)$ and $v = a - Pa$ and substitute them into (5). From (4) we obtain

$$\lambda(a - Pa + \lambda(Pa - d), d - Pa) \leq 0.$$  

Canceling the factor $\lambda$ in the inequality and tending $\lambda$ to zero afterwards, we obtain $\langle a - Pa, d - Pa \rangle \leq 0 \quad \forall d \in K$. In a similar way, for $b \in W$ we obtain $\langle b - Pb, e - Pb \rangle \leq 0 \quad \forall e \in K$. Take $d = Pb$ and $e = Pa$ and sum the last two inequalities:

$$(a - b - Pa + Pb, Pb - Pa) \leq 0.$$  

Consider the norm $\|Pb - Pa\|_W^2$. By (6) and Hölder's inequality, we have

$$\|Pb - Pa\|_W^2 = (a - b - Pa + Pb, Pb - Pa) + (b - a, Pb - Pa) \leq \|Pb - Pa\|_W \|b - a\|_W.$$
The lemma is proved.

Introduce the penalty operator $\beta : W \to W$ by the rule $\beta(v) = v - Pv$ for $v \in W$. The operator $\beta$ is monotone by [1, p. 384] and continuous by the lemma. Given a small parameter $\varepsilon > 0$, we write down the penalized equation

$$ Au + \varepsilon^{-1} \beta(u) = f $$

which is to be understood as follows:

$$ (Au, v) + \varepsilon^{-1}(\beta(u), v) = (f, v) \quad \forall v \in V. $$

Here and in the sequel, we do not indicate the dependence of functions on $\varepsilon$ for simplicity. Fix $\varepsilon$ and construct the sequence of equations

$$ Au^{n+1} + \varepsilon^{-1}u^{n+1} = f + \varepsilon^{-1}Pu^n, $$

where $n = 0, 1, 2, \ldots$ and $u^0 \in V$ is arbitrary. Suppose that the operator $A$ is

1. radially continuous, i.e., the function $s \mapsto \langle A(u + sv), v \rangle$ is continuous on $[0, 1]$ for all fixed $u, v \in V$;
2. strongly monotone, i.e., there is a constant $M > 0$ such that

$$ (Au - Av, u - v) \geq M\|u - v\|^2 \quad \forall u, v \in V. $$

Property (2) obviously implies coercivity and strong monotonicity of $A$. The right-hand side of (8) belongs to $V^*$, since $W \subset V^*$. Then, by Browder's theorem [2, p. 95, Theorems 2.1 and 2.2], there exists a unique solution $u^{n+1} \in V, n = 0, 1, 2, \ldots$, to problem (8).

**Theorem 1.** Under the above assumptions,

$$ u^{n+1} \to u \text{ strongly in } V \text{ as } n \to \infty, $$

where $u$ is a solution to problem (7); moreover, the following estimate holds:

$$ \|u - u^{n+1}\| \leq c(\varepsilon M^2)^{-1}\rho^{0.5n}\|f - Au^0 - \varepsilon^{-1}\beta(u^0)\|_{*}, $$

with $\rho = c^2(\varepsilon^2 + 2\varepsilon M)^{-1} < 1$.

**Proof.** Take (8) at the preceding step in $n$:

$$ Au^n + \varepsilon^{-1}u^n = f + \varepsilon^{-1}Pu^n, $$

subtract this equality from (8), and consider the so-obtained difference at the element $u^{n+1} - u^n \in V$:

$$ \langle Au^{n+1} - Au^n, u^{n+1} - u^n \rangle + \varepsilon^{-1}\|u^{n+1} - u^n\|^2_W = \varepsilon^{-1}(Pu^n - Pu^{n-1}, u^{n+1} - u^n). $$

By making use of strong monotonicity of $A$, Hölder's inequality, and the lemma, we obtain the estimate

$$ 2\|u^{n+1} - u^n\|^2 + (\varepsilon M)^{-1}\|u^{n+1} - u^n\|^2_W \leq (\varepsilon M)^{-1}\|u^n - u^{n-1}\|^2_W. $$

Introduce the equivalent norm in $V$ as follows: $[\cdot]^2 = \|\cdot\|^2 + (\varepsilon M)^{-1}\|\cdot\|_W^2$. By (2), from (9) we obtain

$$ [u^{n+1} - u^n]^2 \leq \rho[u^n - u^{n-1}]^2 \leq \cdots \leq \rho^n[u^1 - u^0]^2. $$

Thus,

$$ u^{n+1} - u^n \to 0 \text{ strongly in } V \text{ as } n \to \infty, $$

It follows that there exists an element $u \in V$ such that

$$ u^{n+1} \to u \text{ strongly in } V \text{ as } n \to \infty. $$
Properties (1) and (2) of the operator $A$ imply that $A$ acts continuously from $V$ with strong topology into $V^*$ with weak topology [2, p. 84]:

\[ Au^{n+1} \rightharpoonup Au \text{ weakly in } V \text{ as } n \to \infty. \quad (13) \]

From the continuity of the penalty operator it follows that

\[ \beta(u^n) \rightharpoonup \beta(u) \text{ strongly in } W \text{ as } n \to \infty. \quad (14) \]

Representing (8) as

\[ Au^{n+1} + \varepsilon^{-1}\beta(u^n) = f - \varepsilon^{-1}(u^{n+1} - u^n), \]

passing to the limit as $n \to \infty$, and using (11)-(14), we obtain (7). Now we estimate the error. Consider (8) for $n = 0$:

\[ Au^1 - Au^0 + \varepsilon^{-1}(u^1 - u^0) = f - Au^0 - \varepsilon^{-1}\beta(u^0) \]

at the element $u^1 - u^0$; Hölder’s inequality implies the estimate

\[ |u^1 - u^0| \leq M^{-1}\|f - Au^0 - \varepsilon^{-1}\beta(u^0)\|_* . \quad (15) \]

Now represent (8) as

\[ Au^{n+1} + \varepsilon^{-1}\beta(u^{n+1}) = f + \varepsilon^{-1}(Pu^n - Pu^{n+1}), \]

subtract it from (7), and consider the difference at the element $u - u^{n+1}$:

\[ (Au - Au^{n+1}, u - u^{n+1}) + \varepsilon^{-1}(\beta(u) - \beta(u^{n+1}), u - u^{n+1}) = \varepsilon^{-1}(Pu^{n+1} - Pu^n, u - u^{n+1}). \]

Applying Hölder’s inequality and making use of strong monotonicity of $A$, the monotonicity of $\beta$, the lemma, and the estimates for the norms, we obtain

\[ \|u - u^{n+1}\| \leq c(\varepsilon M)^{-1}\|u^{n+1} - u^n\|. \quad (16) \]

Combining (10), (15), and (16), we arrive at the required estimate. The theorem is proved.

**Remark 1.** In the general case the constant $M$ depends on $\varepsilon$ as well. If $M$ is independent of $\varepsilon$, then the stated requirements are sufficient for solutions to equations (7) (depending on $\varepsilon$) converge to a solution of variational inequality (1) as $\varepsilon \to 0$ [1, p. 385, Theorem 5.2]. So solutions to equations (8) too converge to a solution of inequality (1) as $n \to \infty$ and $\varepsilon \to 0$.

Let $B : V \to V^*$ be a Lipschitz-continuous operator, i.e., there is a constant $\sigma > 0$ such that

\[ \|Bu - Bv\|_* \leq \sigma\|u - v\| \quad \forall u, v \in V. \]

Consider the operator $A + B$ instead of $A$. Then (7) takes the form

\[ Au + Bu + \varepsilon^{-1}\beta(u) = f. \quad (17) \]

Consider the iterative sequence of equations

\[ Au^{n+1} + \varepsilon^{-1}u^{n+1} = f - Bu^n + \varepsilon^{-1}Pu^n, \quad (18) \]

where $n = 0, 1, 2, \ldots$ and $u^0 \in V$ is arbitrary. By analogy with (8), there exists a unique solution $u^{n+1} \in V$ to problem (18).
Theorem 2. If \( M - \sigma = \kappa > 0 \), then
\[
u^{n+1} \to u \text{ strongly in } V \text{ as } n \to \infty;
\]
u is a solution to problem (17); and the following estimate is valid:
\[
\|u - u^{n+1}\| \leq kR^{0.5n}\|f - Au^0 - Bu^0 - \varepsilon^{-1}\beta(u^0)\|_*,
\]
where \( R = (c^2 + \varepsilon\sigma)(c^2 + \varepsilon\sigma + 2\varepsilon\kappa)^{-1} < 1 \) and \( k = \kappa^{-1}\{1 - \varepsilon\kappa(c^2 + \varepsilon M)^{-1}\}\{1 + c^2[\varepsilon\sigma(c^2 + \varepsilon\sigma)]^{-0.5}\} \).

**Proof.** From (18) we obtain
\[
Au^{n+1} - Au^n + \varepsilon^{-1}(u^{n+1} - u^n) = Bu^{n-1} - Bu^n + \varepsilon^{-1}(Pu^n - Pu^{n-1}).
\]
Consider this equation at the element \( u^{n+1} - u^n \in V \). Employ Hölder’s inequality together with strong monotonicity of \( A \), Lipschitz-continuity of \( B \), and the lemma. As a result, obtain the estimate
\[
M\|u^{n+1} - u^n\|^2 + \varepsilon^{-1}\|u^{n+1} - u^n\|^2 \leq \sigma\|u^n - u^{n-1}\|^2 + \varepsilon^{-1}\|u^n - u^{n-1}\|_W^2,
\]
or, after estimating the right-hand side by means of the squares of the norms, the estimate
\[
(2\kappa + \sigma)\|u^{n+1} - u^n\|^2 + \varepsilon^{-1}\|u^{n+1} - u^n\|^2 \leq \sigma\|u^n - u^{n-1}\|^2 + \varepsilon^{-1}\|u^n - u^{n-1}\|_W^2.
\]
Introduce the equivalent norm in \( V \):
\[
|.|^2 = \sigma|.|^2 + \varepsilon^{-1}|.|^2. \]
Then the following estimates hold:
\[
\sigma^{0.5}|.| \leq [(\sigma + c^2\varepsilon^{-1})^{0.5}] \cdot |.|. \]
Arguing as in the proof of Theorem 1, we obtain the relations
\[
[u^{n+1} - u^n]^2 \leq R[u^n - u^{n-1}]^2 \leq \cdots \leq R^n[u^1 - u^0]^2. \]
Since \( M - \sigma = \kappa > 0 \); therefore,
\[
|u^1 - u^0|^2 \leq k_1|f - Au^0 - Bu^0 - \varepsilon^{-1}\beta(u^0)|_*,
\]
where \( k_1 = \sigma^{-0.5}\{1 - \varepsilon\kappa(c^2 + \varepsilon M)^{-1}\}\). Using the estimate \( |.|_W \leq \{\varepsilon^2(c^2 + \varepsilon\sigma)^{-1}\}^{0.5}|.| \) for the norm, we obtain
\[
\varepsilon\|u - u^{n+1}\| \leq k_2[u^{n+1} - u^n], \]
where \( k_2 = \sigma^{0.5}\{1 + c^2[\varepsilon\sigma(c^2 + \varepsilon\sigma)]^{-0.5}\} \). Whence the assertion of the theorem follows.

**Remark 2.** The constant \( \sigma \) can depend on \( \varepsilon \), too. Since
\[
\langle (A + B)u - (A + B)v, u - v \rangle \geq \langle Au - Av, u - v \rangle - \|Bu - Bv\|_*\|u - v\| \geq \varepsilon\|u - v\|^2 \quad \forall u, v \in V
\]
under the made assumptions, the operator \( A + B \) too is strongly monotone and radially continuous. Therefore, if \( \kappa \) is independent of \( \varepsilon \), then solutions to equation (17) converge as \( \varepsilon \to 0 \) to a solution of variational inequality (1) with the corresponding operator \( A + B \).

**Remark 3.** If the operator \( A \) is linear, then equations (8) and (18) are linear too.

**Remark 4.** The estimates obtained in Theorems 1 and 2 immediately yield the superlinear rate of convergence for solutions to the iterative problems.

The approximate method developed is constructive in the following sense: if \( A \) is a linear operator, then equations (8) and (18) can be solved by standard numerical methods. The necessity of studying penalized equations (7) and (17) appears in consideration of a number of boundary value problems with constraints, for example, contact [3] and plasticity [4] problems of mechanics.

**References**


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