

SHAPE DIFFERENTIABILITY OF LAGRANGIANS AND APPLICATION TO STOKES PROBLEM*

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Abstract. A class of convex constrained minimization problems over polyhedral cones for geometry-dependent quadratic objective functions is considered in a functional analysis framework. Shape differentiability of the primal minimization problem needs a bijective property for mapping of the primal cone. This restrictive assumption is relaxed to bijection of the dual cone within the Lagrangian formulation as a primal-dual minimax problem. In this paper, we give results on primal-dual shape sensitivity analysis that extends the class of shape-differentiable problems supported by an explicit formula of the shape derivative. We apply the results to the Stokes problem under mixed Dirichlet–Neumann boundary conditions subject to the divergence-free constraint.

Key words. shape derivative, velocity method, constrained minimization, primal and dual cone, Lagrangian, primal-dual minimax problem, Stokes problem

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1. Introduction. We aim at shape differentiability for a class of convex constrained minimization problems over polyhedral cones, where the objective functions are assumed quadratic and depend on a geometry.

Typical examples are contact problems in solid mechanics (see [27, 28]) and other elliptic partial differential equations in variable domains with equality- and inequality-type constraints (see [16, 31, 39, 42]). Our special interest concerns nonlinear crack problems in fracture mechanics due to nonpenetration between crack faces, which are developed in [21, 22, 23] and other works by the authors. By this, shape variations may imply regular perturbations along a predefined crack path (see [2, 20, 26]), as well as singular perturbations due to kink of the crack (see [24, 25]). A recent result of [33] concerns shape-topological control by posing a small defect in the cracked domain.

From the point of view of shape and topology optimization, a shape sensitivity analysis of the problem is performed with the help of the velocity method. Introducing a proper kinematic velocity (see, e.g., [32]), a general perturbation of quadratic constrained minimization problems over convex cones in Hilbert spaces is established in [18]. An explicit formula of the shape derivative is provided by bijective properties of the velocity-based diffeomorphic flow of a geometry. However, this result restricts the primal cone to be a bijection within the flow. The bijection fails for constraints involving normal vectors on curves (e.g., Signorini conditions) or having an integral, gradient, divergence operator, etc. This is rather restrictive, not even a complete list.

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In the case of Signorini-type constraints imposed on curvilinear manifolds implying cracks, the shape differentiability result is improved in [36, 45, 46] relying on a Γ -convergence of the primal cones. For this specific problem, in [29, 30] the assumption of bijection is relaxed further to the dual cone within a Lagrangian formalism. See another specific example of shape sensitivity of a Lagrangian associated with an inhomogeneous Dirichlet problem in [11], and the general Lagrangian method together with related primal-dual minimax problems in [19].

For another example of such a non-bijective primal cone, in the present work we consider a Stokes problem under mixed Dirichlet–Neumann boundary conditions subject to the divergence-free constraint. We refer the reader to [8, 14, 35] for the Stokes problem, and to [7, 17] for its shape sensitivity. It is worth stressing that the divergence-free constraint is not preserved by transport. The treatment of the incompressibility within the dynamical shape control of Navier–Stokes equations is discussed in [40, section 5]. It employs special transforms (Piola transformation, transverse map) and a hold-all domain assumption, but has a lack of rigorous mathematical justification [40, p. 142].

In section 2 we develop our concept of the shape differentiability of Lagrangians in a functional analysis framework. Based on the Lagrangian setting which implies a primal-dual minimax problem, we relax the bijection assumption from the primal cone K (in the space of primal variable) to the dual cone K^* (in the space of dual variable) (see (2.20c)). This relaxation allows us to lead the primal-dual shape sensitivity analysis and to obtain the shape derivative explicitly. The improvement of the previous shape sensitivity results is attained with respect to non-bijective primal cones, thus extending the class of shape-differentiable problems.

It is important to put our investigation in the classic context of optimal value functions adopted in optimization. The directional differentiability of optimal value Lagrangians in abstract formulation was established in [9] (see also [4, Chapter 4.3.2]), and extended to the shape optimization framework in [10]. For a concept of directional differentiability of metric projections onto polyhedral sets corresponding to shape derivatives we refer the reader to [37] and references therein.

The abstract optimal value Lagrangian function used for shape optimization in a time-dependent domain Ω_t with parameter t can be defined by a general map of the form

$$(OVF) \quad \mathbb{R} \mapsto \mathbb{R}, \quad t \mapsto \mathcal{L}(u_t, \lambda_t; \Omega_t),$$

where a saddle point $(u_t, \lambda_t) \in V(\Omega_t) \times K^*(\Omega_t)$ satisfies

$$(SP) \quad \mathcal{L}(u_t, p; \Omega_t) \leq \mathcal{L}(u_t, \lambda_t; \Omega_t) \leq \mathcal{L}(w, \lambda_t; \Omega_t) \quad \forall (w, p) \in V(\Omega_t) \times K^*(\Omega_t)$$

for a Lagrangian

$$(L) \quad (u, \lambda) \mapsto \mathcal{L}(u, \lambda; \Omega_t) : V(\Omega_t) \times K^*(\Omega_t) \mapsto \mathbb{R},$$

defined over topological vector spaces $V(\Omega_t)$ and $K^*(\Omega_t)$ (the upper star to be explained later on). The aim is to find the *directional derivative*:

$$(DD) \quad \partial_t \mathcal{L}(u_t, \lambda_t; \Omega_t) := \lim_{s \rightarrow 0} \frac{\mathcal{L}(u_{t+s}, \lambda_{t+s}; \Omega_{t+s}) - \mathcal{L}(u_t, \lambda_t; \Omega_t)}{s}.$$

Since the perturbed optimal value function $\mathcal{L}(u_{t+s}, \lambda_{t+s}; \Omega_{t+s})$ in (DD) is given by the perturbed Lagrangian

$$(PL) \quad (v, \mu) \mapsto \mathcal{L}(v, \mu; \Omega_{t+s}) : V(\Omega_{t+s}) \times K^*(\Omega_{t+s}) \mapsto \mathbb{R},$$

which is defined over s -dependent spaces $V(\Omega_{t+s}) \times K^*(\Omega_{t+s})$, then the usual trick in shape optimization is to use a coordinate transformation

$$(CT) \quad \phi_s : \Omega_t \mapsto \Omega_{t+s}, \quad \phi_s^{-1} : \Omega_{t+s} \mapsto \Omega_t$$

that maps (PL) to a transformed perturbed Lagrangian

$$(TPL) \quad (s, u, \lambda) \mapsto \mathcal{L}_s(u, \lambda; \Omega_t) : \mathbb{R} \times V(\Omega_t) \times K^*(\Omega_t) \mapsto \mathbb{R}$$

over fixed spaces $V(\Omega_t) \times K^*(\Omega_t)$ such that $\mathcal{L}_0 = \mathcal{L}$ and

$$(BL) \quad \mathcal{L}_s(v \circ \phi_s, \mu \circ \phi_s; \Omega_t) = \mathcal{L}(v, \mu; \Omega_{t+s})$$

for all $(v, \mu) \in V(\Omega_{t+s}) \times K^*(\Omega_{t+s})$. This needs the fulfillment of the bijective property between the function spaces

$$(BS) \quad [v \mapsto v \circ \phi_s] : V(\Omega_{t+s}) \mapsto V(\Omega_t), \quad [\mu \mapsto \mu \circ \phi_s] : K^*(\Omega_{t+s}) \mapsto K^*(\Omega_t)$$

and allows us to rewrite (DD) in the equivalent form

$$(DD') \quad \partial_s \mathcal{L}_s(0, u_t, \lambda_t; \Omega_t) = \lim_{s \rightarrow 0} \frac{\mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t) - \mathcal{L}(u_t, \lambda_t; \Omega_t)}{s}.$$

The bijection (BS) is central in this work.

In the constrained optimization context, K^* is associated to a dual cone compared with its primal counterpart K . For the divergence-free constraint, in section 3 we give an example of the space $K^*(\Omega_{t+s})$ where the bijection of dual cones (see (2.20c)) fails. Namely, considering the Stokes problem under the no-slip Dirichlet condition, the integral identity $\int_{\Omega_{t+s}} v(y) dy = 0$ characterizing the space $L_0^2(\Omega_{t+s})$ (see (3.8)) is not preserved by the transport $y = \phi_s(x)$ in general; thus, the equivalence between (DD) and (DD') is not true. A possible remedy is to use special area-preserving maps. In the current paper, we suggest to consider the Stokes problem under mixed Dirichlet–Neumann boundary conditions such that the bijective property (BS) holds true.

2. Shape derivative of Lagrangians for polyhedral cones. We start the investigation with a family of *time-dependent geometric sets* $t \mapsto \Omega_t \subset \mathbb{R}^d$, $d \in \mathbb{N}$.

For every fixed time $t \in \mathbb{R}$, we consider two *geometry-dependent Hilbert spaces* $V(\Omega_t)$ and $H(\Omega_t)$ with the *dual spaces* $V^*(\Omega_t)$ and $H^*(\Omega_t)$. Let a linear operator $A : V(\Omega_t) \mapsto V^*(\Omega_t)$ be *strongly monotone* such that

$$(2.1) \quad \langle Au, u \rangle_{\Omega_t} \geq \underline{c}_A \|u\|_{V(\Omega_t)}^2, \quad \underline{c}_A > 0, \quad u \in V(\Omega_t),$$

with the duality pairing $\langle \cdot, \cdot \rangle_{\Omega_t}$ between $V^*(\Omega_t)$ and $V(\Omega_t)$, and *continuous* such that

$$(2.2) \quad \|Au\|_{V^*(\Omega_t)} \leq \bar{c}_A \|u\|_{V(\Omega_t)}, \quad \bar{c}_A \geq \underline{c}_A > 0, \quad u \in V(\Omega_t),$$

uniformly in a time interval $t \in (t_0, t_1)$ with fixed $t_0 < t_1$. Let a linear operator $B : V(\Omega_t) \mapsto H(\Omega_t)$ be *surjective* (i.e., for every $\zeta \in H(\Omega_t)$ there is at least one $u \in V(\Omega_t)$ such that $Bu = \zeta$) and *continuous* with the estimate

$$(2.3) \quad \|Bu\|_{H(\Omega_t)} \leq \bar{c}_B \|u\|_{V(\Omega_t)}, \quad \bar{c}_B > 0, \quad u \in V(\Omega_t),$$

that holds uniformly for all $t \in (t_0, t_1)$.

Using the order relation for measured functions in $H(\Omega_t)$, we define the *primal cone* as a polyhedral cone as follows:

$$(2.4) \quad K(\Omega_t) := \{u \in V(\Omega_t) \mid Bu \geq 0\}$$

which is convex and closed. For a stationary right-hand side f such that $f \in \bigcap_{t \in (t_0, t_1)} V^*(\Omega_t)$, let the *geometry-dependent objective function* $\mathcal{E} : V(\Omega_t) \mapsto \mathbb{R}$ be given by

$$(2.5) \quad \mathcal{E}(u; \Omega_t) := \langle \frac{1}{2}Au - f, u \rangle_{\Omega_t}$$

that is quadratic, bounded due to (2.2), and coercive due to (2.1).

We consider the *primal constrained minimization problem*: Find $u_t \in K(\Omega_t)$ such that

$$(2.6) \quad \mathcal{E}(u_t; \Omega_t) = \min_{w \in K(\Omega_t)} \mathcal{E}(w; \Omega_t).$$

The unique solution to (2.6) exists and satisfies the first order *optimality condition* in the form of a variational inequality due to (2.5) and (2.6):

$$(2.7) \quad \langle Au_t - f, w - u_t \rangle_{\Omega_t} \geq 0 \quad \forall w \in K(\Omega_t),$$

which is a necessary and sufficient condition for (2.6). For a general theory of pseudo-monotone variational inequalities see [15].

Now we define the *dual cone* (in the space of dual variable) as follows:

$$(2.8) \quad K^*(\Omega_t) := \{\lambda \in H^*(\Omega_t) \mid (\lambda, Bu)_{\Omega_t} \geq 0 \quad \forall u \in K(\Omega_t)\},$$

where $(\cdot, \cdot)_{\Omega_t}$ stands for the duality pairing between $H^*(\Omega_t)$ and $H(\Omega_t)$. It is important to note that, due to surjection of B , the dual cone in (2.8) can be restated equivalently in the form

$$(2.8') \quad K^*(\Omega_t) = \{\lambda \in H^*(\Omega_t) \mid (\lambda, \zeta)_{\Omega_t} \geq 0 \quad \forall \zeta \in H(\Omega_t), \zeta \geq 0\}.$$

The corresponding *primal-dual minimax problem* reads as follows: Find the pair $(u_t, \lambda_t) \in V(\Omega_t) \times K^*(\Omega_t)$ such that

$$(2.9) \quad \mathcal{L}(u_t, \lambda_t; \Omega_t) = \min_{w \in V(\Omega_t)} \max_{p \in K^*(\Omega_t)} \mathcal{L}(w, p; \Omega_t)$$

with the *Lagrangian function* $\mathcal{L} : V(\Omega_t) \times H^*(\Omega_t) \mapsto \mathbb{R}$ given by

$$(2.10) \quad \mathcal{L}(u, \lambda; \Omega_t) := \mathcal{E}(u; \Omega_t) - (\lambda, Bu)_{\Omega_t}.$$

Well-posedness and optimality properties of (2.9) are gathered in the following theorem.

THEOREM 2.1. (i) *There exists a solution of the minimax problem (2.9) which implies that $(u_t, \lambda_t) \in V(\Omega_t) \times K^*(\Omega_t)$ is a saddle point:*

$$(2.9') \quad \mathcal{L}(u_t, p; \Omega_t) \leq \mathcal{L}(u_t, \lambda_t; \Omega_t) \leq \mathcal{L}(w, \lambda_t; \Omega_t) \quad \forall (w, p) \in V(\Omega_t) \times K^*(\Omega_t).$$

Further, this solution satisfies the primal-dual optimality conditions

$$(2.11a) \quad \langle Au_t - f, w \rangle_{\Omega_t} - (\lambda_t, Bw)_{\Omega_t} = 0 \quad \forall w \in V(\Omega_t),$$

$$(2.11b) \quad (p - \lambda_t, Bu_t)_{\Omega_t} \geq 0 \quad \forall p \in K^*(\Omega_t).$$

The primal component $u_t \in K(\Omega_t)$ is the unique solution of the primal problem (2.6). If the Ladyzhenskaya–Babuška–Brezzi (LBB) condition holds for $\lambda \in H^*(\Omega_t)$, i.e.,

$$(2.12) \quad \sup_{u \in V(\Omega_t) \setminus \{0\}} \frac{(\lambda, Bu)_{\Omega_t}}{\|u\|_{V(\Omega_t)}} \geq \underline{c}_B \|\lambda\|_{H^*(\Omega_t)}, \quad 0 < \underline{c}_B \leq \bar{c}_B,$$

then the dual component λ_t is unique.

(ii) The optimal value objective function $t \mapsto \mathcal{E}(u_t; \Omega_t)$ defined by (2.6) and the optimal value Lagrangian function $t \mapsto \mathcal{L}(u_t, \lambda_t; \Omega_t)$ given in (2.9) are equal:

$$(2.13) \quad \min_{w \in V(\Omega_t)} \mathcal{E}(w; \Omega_t) = \min_{w \in V(\Omega_t)} \max_{p \in K^*(\Omega_t)} \mathcal{L}(w, p; \Omega_t).$$

Proof. Indeed, based on (2.1)–(2.10), existence of a solution to the minimax problem follows from, e.g., [28, Theorem 3.11]. The inclusion $u_t \in K(\Omega_t)$ is a consequence of the bipolar theorem (see, e.g., [44, Theorem 14.1]) due to surjection of B . The optimality conditions (2.11) and the uniqueness assertion under the LBB condition (2.12) are stated, e.g., in [28, Theorem 3.14]. The cone $K^*(\Omega_t)$ is convex and $V(\Omega_t)$ is linear, the Lagrangian \mathcal{L} is convex-concave and Gâteaux differentiable, so that (2.11) is equivalent to (see [13, Proposition 1.5])

$$\begin{aligned} \langle \partial_u \mathcal{L}(u_t, \lambda_t; \Omega_t), w \rangle_{\Omega_t} &= 0 \quad \forall w \in V(\Omega_t), \\ \langle \partial_\lambda \mathcal{L}(u_t, \lambda_t; \Omega_t), p - \lambda_t \rangle_{\Omega_t} &\geq 0 \quad \forall p \in K^*(\Omega_t), \end{aligned}$$

and the pair $(u_t, \lambda_t) \in V(\Omega_t) \times K^*(\Omega_t)$ also satisfies (2.9'), implying the saddle point (see [13, Definition 1.1]).

To prove the assertion (ii), we test (2.11b) with $p = 0$ and $p = 2\lambda_t$ yielding $(\lambda_t, Bu_t)_{\Omega_t} = 0$; hence $\mathcal{E}(u_t; \Omega_t) = \mathcal{L}(u_t, \lambda_t; \Omega_t)$, in turn implying (2.13). \square

In the following we lead a shape sensitivity analysis of the problem.

2.1. Primal-dual shape sensitivity analysis. For fixed $t \in (t_0, t_1)$ and a small perturbation parameter $s \in (t_0 - t, t_1 - t)$, let given vector-functions

$$(2.14a) \quad [s \mapsto \phi_s], [s \mapsto \phi_s^{-1}] \in C^1([t_0 - t, t_1 - t]; W_{\text{loc}}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d))$$

associate the coordinate transformation $y = \phi_s(x)$ and the inverse mapping $x = \phi_s^{-1}(y)$ such that its composition satisfies

$$(2.14b) \quad (\phi_s^{-1} \circ \phi_s)(x) = x, \quad (\phi_s \circ \phi_s^{-1})(y) = y.$$

Then the shape perturbation

$$(2.15) \quad \Omega_{t+s} := \{y \in \mathbb{R}^d \mid y = \phi_s(x), x \in \Omega_t\}$$

builds the diffeomorphism

$$(2.16) \quad \phi_s : \Omega_t \mapsto \Omega_{t+s}, x \mapsto y; \quad \phi_s^{-1} : \Omega_{t+s} \mapsto \Omega_t, y \mapsto x.$$

We reset the perturbed primal constrained minimization problem: Find $u_{t+s} \in K(\Omega_{t+s})$ such that

$$(2.17) \quad \mathcal{E}(u_{t+s}; \Omega_{t+s}) = \min_{v \in K(\Omega_{t+s})} \mathcal{E}(v; \Omega_{t+s}).$$

Then we reset the corresponding *perturbed primal-dual minimax problem*: Find the pair $(u_{t+s}, \lambda_{t+s}) \in V(\Omega_{t+s}) \times K^*(\Omega_{t+s})$ such that

$$(2.18) \quad \mathcal{L}(u_{t+s}, \lambda_{t+s}; \Omega_{t+s}) = \min_{v \in V(\Omega_{t+s})} \max_{\mu \in K^*(\Omega_{t+s})} \mathcal{L}(v, \mu; \Omega_{t+s})$$

with the *perturbed Lagrangian and objective functions*, respectively,

$$(2.19a) \quad \mathcal{L}(v, \mu; \Omega_{t+s}) = \mathcal{E}(v; \Omega_{t+s}) - (\mu, Bv)_{\Omega_{t+s}},$$

$$(2.19b) \quad \mathcal{E}(v; \Omega_{t+s}) = \langle \frac{1}{2}Av - f, v \rangle_{\Omega_{t+s}}.$$

These functions are defined for $v \in V(\Omega_{t+s})$ and $\mu \in H^*(\Omega_{t+s})$ with the duality pairings $\langle \cdot, \cdot \rangle_{\Omega_{t+s}}$ between $V^*(\Omega_{t+s})$ and $V(\Omega_{t+s})$, and $(\cdot, \cdot)_{\Omega_{t+s}}$ between $H^*(\Omega_{t+s})$ and $H(\Omega_{t+s})$.

Within the kinematic flow (2.14)–(2.16), we employ the following assumptions: The map $[v \mapsto v \circ \phi_s]$ is bijective in the function spaces

$$(2.20a) \quad V(\Omega_{t+s}) \mapsto V(\Omega_t), \quad V^*(\Omega_{t+s}) \mapsto V^*(\Omega_t),$$

$$(2.20b) \quad H(\Omega_{t+s}) \mapsto H(\Omega_t), \quad H^*(\Omega_{t+s}) \mapsto H^*(\Omega_t),$$

and $[\mu \mapsto \mu \circ \phi_s]$ is bijective in the dual cones

$$(2.20c) \quad K^*(\Omega_{t+s}) \mapsto K^*(\Omega_t).$$

As $s \rightarrow 0$, let the asymptotic representations hold for the operator A :

$$(2.20d) \quad \langle Av, \chi \rangle_{\Omega_{t+s}} = \langle [A + sA^1 + A_s^2](v \circ \phi_s), \chi \circ \phi_s \rangle_{\Omega_t}$$

with linear bounded operators $A^1, A_s^2 : V(\Omega_t) \mapsto V^*(\Omega_t)$ and the residual A_s^2 such that

$$(2.20e) \quad \|A_s^2 u\|_{V^*(\Omega_t)} \leq c_{RA}(s) \|u\|_{V(\Omega_t)}, \quad 0 \leq c_{RA}(s) = o(s);$$

for the operator B :

$$(2.20f) \quad (\mu, Bv)_{\Omega_{t+s}} = (\mu \circ \phi_s, [B + sB^1 + B_s^2](v \circ \phi_s))_{\Omega_t}$$

with linear bounded operators $B^1, B_s^2 : V(\Omega_t) \mapsto H(\Omega_t)$ such that $B + sB^1 + B_s^2$ is surjective and the residual B_s^2 satisfies

$$(2.20g) \quad \|B_s^2 u\|_{H(\Omega_t)} \leq c_{RB}(s) \|u\|_{V(\Omega_t)}, \quad 0 \leq c_{RB}(s) = o(s);$$

and for the right-hand side f :

$$(2.20h) \quad \langle f, v \rangle_{\Omega_{t+s}} = \langle f + sf^1 + f_s^2, v \circ \phi_s \rangle_{\Omega_t}$$

with $f^1, f_s^2 \in V^*(\Omega_t)$ and the residual f_s^2 such that

$$(2.20i) \quad \|f_s^2\|_{V^*(\Omega_t)} \leq c_{Rf}(s), \quad 0 \leq c_{Rf}(s) = o(s)$$

for test functions $v, \chi \in V(\Omega_{t+s})$, $\mu \in H^*(\Omega_{t+s})$, $u \in V(\Omega_t)$, uniformly for all $s \in (t_0 - t, t_1 - t)$ and $t \in (t_0, t_1)$.

THEOREM 2.2. Under the assumptions (2.20), the optimal value function $\mathbb{R} \mapsto \mathbb{R}, t \mapsto \mathcal{E}(u_t; \Omega_t)$ of the objective \mathcal{E} given in (2.5) and (2.6) is shape-differentiable such that

$$(2.21) \quad \frac{d}{dt} \mathcal{E}(u_t; \Omega_t) := \lim_{s \rightarrow 0} \frac{\mathcal{E}(u_{t+s}; \Omega_{t+s}) - \mathcal{E}(u_t; \Omega_t)}{s} = \mathcal{L}^1(u_t, \lambda_t; \Omega_t)$$

with the shape derivative $\mathcal{L}^1(u_t, \lambda_t; \Omega_t)$ determined by

$$(2.22a) \quad \mathcal{L}^1(u, \lambda; \Omega_t) := \mathcal{E}^1(u; \Omega_t) - (\lambda, B^1 u)_{\Omega_t},$$

$$(2.22b) \quad \mathcal{E}^1(u; \Omega_t) := \langle \frac{1}{2} A^1 u - f^1, u \rangle_{\Omega_t}.$$

Proof. We apply to (2.18) the asymptotic formula (2.20d), (2.20f), (2.20h) and use the assumptions (2.20a)–(2.20c) to get the transformed solution pair $(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s) \in V(\Omega_t) \times K^*(\Omega_t)$ which solves the minimax problem

$$(2.23) \quad \mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t) = \min_{w \in V(\Omega_t)} \max_{p \in K^*(\Omega_t)} \mathcal{L}_s(w, p; \Omega_t),$$

implying a saddle point (see (2.9')):

$$(2.23') \quad \begin{aligned} \mathcal{L}_s(u_{t+s} \circ \phi_s, p; \Omega_t) &\leq \mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t) \\ &\leq \mathcal{L}_s(w, \lambda_{t+s} \circ \phi_s; \Omega_t) \quad \forall (w, p) \in V(\Omega_t) \times K^*(\Omega_t). \end{aligned}$$

The transformed Lagrangian $\mathcal{L}_s : V(\Omega_t) \times H(\Omega_t) \mapsto \mathbb{R}$ is defined via

$$(2.24a) \quad \mathcal{L}_s(v \circ \phi_s, \mu \circ \phi_s; \Omega_t) = \mathcal{L}(v, \mu; \Omega_{t+s}) \quad (\text{with } \mathcal{L}_0 = \mathcal{L})$$

for all $(v, \mu) \in V(\Omega_{t+s}) \times K^*(\Omega_{t+s})$ and yields the expansion

$$(2.24b) \quad \mathcal{L}_s(u, \lambda; \Omega_t) := \mathcal{L}(u, \lambda; \Omega_t) + s \mathcal{L}^1(u, \lambda; \Omega_t) + \mathcal{L}_s^2(u, \lambda; \Omega_t),$$

where the first asymptotic term $\mathcal{L}^1(u, \lambda; \Omega_t)$ is given in (2.22a), and the residual

$$(2.24c) \quad \mathcal{L}_s^2(u, \lambda; \Omega_t) := \langle \frac{1}{2} A_s^2 u - f_s^2, u \rangle_{\Omega_t} - (\lambda, B_s^2 u)_{\Omega_t}.$$

Based on Theorem 2.1, optimality conditions for (2.23) are

$$(2.25a) \quad \begin{aligned} \langle [A + sA^1 + A_s^2](u_{t+s} \circ \phi_s) - (f + sf^1 + f_s^2), w \rangle_{\Omega_t} \\ - (\lambda_{t+s} \circ \phi_s, [B + sB^1 + B_s^2]w)_{\Omega_t} = 0 \quad \forall w \in V(\Omega_t), \end{aligned}$$

$$(2.25b) \quad (p - \lambda_{t+s} \circ \phi_s, [B + sB^1 + B_s^2](u_{t+s} \circ \phi_s))_{\Omega_t} \geq 0 \quad \forall p \in K^*(\Omega_t).$$

Taking the test function $w = u_{t+s} \circ \phi_s$ in (2.25a), using the complementarity

$$(2.26) \quad (\lambda_{t+s} \circ \phi_s, [B + sB^1 + B_s^2]u_{t+s} \circ \phi_s)_{\Omega_t} = 0$$

which follows from (2.25b), the strong monotony (2.1) of A , and the residual estimates (2.20e), (2.20g), (2.20i), for $|s| \in (0, s_0)$ with sufficiently small $s_0 > 0$ and $t \in (t_0, t_1)$ we get the uniform estimate

$$(2.27a) \quad \|u_{t+s} \circ \phi_s\|_{V(\Omega_t)} \leq \text{const.}$$

Similarly, from (2.25a) we derive the uniform estimate in the dual space:

$$(2.27b) \quad \|\lambda_{t+s} \circ \phi_s\|_{H^*(\Omega_t)} \leq \text{const}$$

for $|s| \in (0, s_1)$ with sufficiently small $0 < s_1 \leq s_0$ and $t \in (t_0, t_1)$.

From (2.27) follows the existence of $(\bar{u}, \bar{\lambda}) \in V(\Omega_t) \times H^*(\Omega_t)$ and a subsequence denoted by s_k such that as $s_k \rightarrow 0$,

$$(2.28a) \quad u_{t+s_k} \circ \phi_{s_k} \rightharpoonup \bar{u} \quad \text{weakly in } V(\Omega_t),$$

$$(2.28b) \quad \lambda_{t+s_k} \circ \phi_{s_k} \rightharpoonup \bar{\lambda} \quad \star\text{-weakly in } H^*(\Omega_t).$$

Every linear and continuous operator B is weak-to-weak continuous (see [5, Theorem 3.10]); therefore

$$(2.28c) \quad B(u_{t+s_k} \circ \phi_{s_k}) \rightharpoonup B\bar{u} \quad \text{weakly in } H(\Omega_t).$$

In accordance with (2.20c) the inclusion $\lambda_{t+s} \circ \phi_s \in K^*(\Omega_t)$ holds, the convex closed set $K^*(\Omega_t)$ is \star -weakly closed, and hence $\bar{\lambda} \in K^*(\Omega_t)$. Since a quadratic form is weakly lower semicontinuous, we pass to the limit in (2.23') using the weak convergences in (2.28) and get

$$\begin{aligned} \mathcal{L}(\bar{u}, p; \Omega_t) &\leq \liminf_{s_k \rightarrow 0} \mathcal{L}_{s_k}(u_{t+s_k} \circ \phi_{s_k}, p; \Omega_t) \leq \mathcal{L}(\bar{u}, \bar{\lambda}; \Omega_t) \\ &\leq \limsup_{s_k \rightarrow 0} \mathcal{L}_{s_k}(w, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) \leq \mathcal{L}(w, \bar{\lambda}; \Omega_t) \end{aligned}$$

for arbitrary $(w, p) \in V(\Omega_t) \times K^*(\Omega_t)$. Therefore, $(\bar{u}, \bar{\lambda}) = (u_t, \lambda_t)$ is a saddle point satisfying (2.9') and thus solves (2.9).

In order to estimate the solution difference in the norm, we start with the inequality (2.1) and rearrange the terms such that

$$\begin{aligned} \frac{\epsilon_A}{2} \|u_{t+s} \circ \phi_s - u_t\|_{V(\Omega_t)}^2 &\leq \frac{1}{2} \langle A(u_{t+s} \circ \phi_s - u_t), u_{t+s} \circ \phi_s - u_t \rangle_{\Omega_t} \\ &= -\langle A(u_{t+s} \circ \phi_s - u_t), u_t \rangle_{\Omega_t} - \frac{1}{2} \langle Au_t, u_t \rangle_{\Omega_t} + \frac{1}{2} \langle A(u_{t+s} \circ \phi_s), u_{t+s} \circ \phi_s \rangle_{\Omega_t} \\ &= -\langle A(u_{t+s} \circ \phi_s - u_t), u_t \rangle_{\Omega_t} + \langle f, u_{t+s} \circ \phi_s - u_t \rangle_{\Omega_t} \\ &+ \mathcal{L}(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t) - \mathcal{L}(u_t, \lambda_t; \Omega_t) + (\lambda_{t+s} \circ \phi_s, [sB^1 + B_s^2]u_{t+s} \circ \phi_s)_{\Omega_t} \end{aligned}$$

due to the orthogonality relations $(\lambda_t, Bu_t)_{\Omega_t} = 0$ and (2.26). Further, using

$$\begin{aligned} &\limsup_{s_k \rightarrow 0} \{ \mathcal{L}(u_{t+s_k} \circ \phi_{s_k}, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) - \mathcal{L}(u_t, \lambda_t; \Omega_t) \} \\ &= \limsup_{s_k \rightarrow 0} \{ \mathcal{L}_{s_k}(u_{t+s_k} \circ \phi_{s_k}, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) - \mathcal{L}_{s_k}(u_t, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) \} \leq 0 \end{aligned}$$

because of (2.23') with $w = u_t$ and (2.28), we conclude that

$$(2.29a) \quad \frac{\epsilon_A}{2} \limsup_{s_k \rightarrow 0} \|u_{t+s_k} \circ \phi_{s_k} - u_t\|_{V(\Omega_t)}^2 \leq 0.$$

Therefore, from (2.3) it follows that as $s_k \rightarrow 0$,

$$(2.29b) \quad \|B(u_{t+s_k} \circ \phi_{s_k} - u_t)\|_{H(\Omega_t)} \rightarrow 0.$$

From (2.11a) and (2.25a) we arrive at

$$(\lambda_{t+s} \circ \phi_s - \lambda_t, Bw)_{\Omega_t} = \langle A(u_{t+s} \circ \phi_s - u_t), w \rangle_{\Omega_t} + O(s)$$

for all $w \in V(\Omega_t)$; henceforth the surjection of B provides that

$$(2.29c) \quad \|\lambda_{t+s_k} \circ \phi_{s_k} - \lambda_t\|_{H^*(\Omega_t)} \rightarrow 0.$$

The relations (2.29) imply the strong convergences in (2.28).

Based on the asymptotic formula (2.24) we find the lower bound

$$(2.30a) \quad \begin{aligned} \mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t) - \mathcal{L}(u_t, \lambda_t; \Omega_t) \\ \geq \mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_t; \Omega_t) - \mathcal{L}(u_{t+s} \circ \phi_s, \lambda_t; \Omega_t) \\ = s\mathcal{L}^1(u_{t+s} \circ \phi_s, \lambda_t; \Omega_t) + \mathcal{L}_s^2(u_{t+s} \circ \phi_s, \lambda_t; \Omega_t) \end{aligned}$$

using the maximum in (2.23') with the test function $p = \lambda_t$, and the minimum in (2.9') with the test function $w = u_{t+s} \circ \phi_s$. Similarly, we calculate the upper bound

$$(2.30b) \quad \begin{aligned} \mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t) - \mathcal{L}(u_t, \lambda_t; \Omega_t) \\ \leq \mathcal{L}_s(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t) - \mathcal{L}(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t) \\ = s\mathcal{L}^1(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t) + \mathcal{L}_s^2(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t) \end{aligned}$$

utilizing the minimum in (2.23') with the test function $w = u_t$, and the maximum in (2.9') with the test function $p = \lambda_{t+s} \circ \phi_s$. The strong convergences (2.29) provide the asymptotic order of the residuals

$$\mathcal{L}_{s_k}^2(u_t, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) = o(s_k), \quad \mathcal{L}_{s_k}^2(u_{t+s_k} \circ \phi_{s_k}, \lambda_t; \Omega_t) = o(s_k);$$

hence from (2.30) divided by s follows existence of the limit

$$(2.31) \quad \lim_{s_k \rightarrow 0} \frac{\mathcal{L}(u_{t+s_k}, \lambda_{t+s_k}; \Omega_{t+s_k}) - \mathcal{L}(u_t, \lambda_t; \Omega_t)}{s_k} = \mathcal{L}^1(u_t, \lambda_t; \Omega_t)$$

because of the identity $\mathcal{L}(u_{t+s}, \lambda_{t+s}; \Omega_{t+s}) = \mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t)$ due to (2.24a). The optimal value Lagrangian and objective functions are equal (see (2.13) and the similar identity $\mathcal{L}(u_{t+s}, \lambda_{t+s}; \Omega_{t+s}) = \mathcal{E}(u_{t+s}; \Omega_{t+s})$); then (2.31) coincides with formula (2.21) of the shape derivative and completes the proof. \square

Remark 2.1. Theorem 2.2 presents a direct proof of the shape differentiability. Since the bijection (2.20a)–(2.20c) holds, then the Correa–Seeger theorem on directional differentiability can be applied by checking hypotheses (H1)–(H4) in [12, Chapter 10, Theorem 5.1].

To formulate the hypotheses, let us define the optimal values

$$l_t := \sup_{p \in K^*(\Omega_t)} \inf_{w \in V(\Omega_t)} \mathcal{L}(w, p; \Omega_t) \leq \inf_{w \in V(\Omega_t)} \sup_{p \in K^*(\Omega_t)} \mathcal{L}(w, p; \Omega_t) =: l^t$$

and the solution sets

$$\begin{aligned} V_t &= \left\{ u \in V(\Omega_t) \mid \sup_{p \in K^*(\Omega_t)} \mathcal{L}(u, p; \Omega_t) = l^t \right\}, \\ K_t^* &= \left\{ \lambda \in K^*(\Omega_t) \mid \inf_{w \in V(\Omega_t)} \mathcal{L}(w, \lambda; \Omega_t) = l_t \right\} \quad \text{for } t \in (t_0, t_1). \end{aligned}$$

(H1) The solution sets are nonempty due to Theorem 2.1. Moreover, $l_t = l^t$ and $V_t = \{u_t\}$, $K_t^* = \{\lambda_t\}$ are singletons.

(H2) For $t \in (t_0, t_1)$ there exists the partial derivative

$$(2.32a) \quad \lim_{s \rightarrow 0} \frac{\mathcal{L}_s(u, \lambda; \Omega_t) - \mathcal{L}(u, \lambda; \Omega_t)}{s} = \mathcal{L}^1(u, \lambda; \Omega_t) \\ \forall (u, \lambda) \in \left(\bigcup_{\tau \in (t_0, t_1)} V_\tau \times K_\tau^* \right) \cup \left(V_t \times \bigcup_{\tau \in (t_0, t_1)} K_\tau^* \right)$$

within the asymptotic expansion (2.24b), which is uniform with respect to (u, λ) . This hypothesis holds due to assumptions (2.20d)–(2.20i).

(H3) There exist an accumulation point $\bar{u} \in V_t$ and a subsequence $u_{t+s_k} \circ \phi_{s_k} \in V_t$ denoted by s_k such that

$$(2.32b) \quad \|u_{t+s_k} \circ \phi_{s_k} - \bar{u}\|_{V(\Omega_t)} \rightarrow 0 \quad \text{as } s_k \rightarrow 0,$$

which is proved in (2.28a) with $\bar{u} = u_t$, and

$$(2.32c) \quad \liminf_{s_k \rightarrow 0} \mathcal{L}^1(u_{t+s_k} \circ \phi_{s_k}, p; \Omega_t) \geq \mathcal{L}^1(\bar{u}, p; \Omega_t) \quad \forall p \in K_t^*,$$

which holds due to continuity in the strong topology of the bilinear mapping $w \mapsto \mathcal{L}^1(w, p; \Omega_t)$.

(H4) There exist an accumulation point $\bar{\lambda} \in K_t^*$ and a subsequence $\lambda_{t+s_k} \circ \phi_{s_k} \in K_t^*$ denoted by s_k such that

$$(2.32d) \quad \|\lambda_{t+s_k} \circ \phi_{s_k} - \bar{\lambda}\|_{H^*(\Omega_t)} \rightarrow 0 \quad \text{as } s_k \rightarrow 0,$$

with $\bar{\lambda} = \lambda_t$ according to (2.28c), and

$$(2.32e) \quad \limsup_{s_k \rightarrow 0} \mathcal{L}^1(w, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) \leq \mathcal{L}^1(w, \bar{\lambda}; \Omega_t) \quad \forall w \in V_t,$$

provided by the weak continuity of the linear mapping $p \mapsto \mathcal{L}^1(w, p; \Omega_t)$.

Indeed, testing (2.23') with $(w, p) = (u_t, \lambda_t)$ and (2.9') with $(w, p) = (u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s)$ gives

$$\frac{\mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_t; \Omega_t) - \mathcal{L}(u_{t+s} \circ \phi_s, \lambda_t; \Omega_t)}{s} \leq \frac{\mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t) - \mathcal{L}(u_t, \lambda_t; \Omega_t)}{s} =: \Delta(s) \\ \leq \frac{\mathcal{L}_s(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t) - \mathcal{L}(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t)}{s}.$$

Since we show that the expansion (2.24b) holds, we get

$$\mathcal{L}^1(u_{t+s} \circ \phi_s, \lambda_t; \Omega_t) + \frac{1}{s} \mathcal{L}_s^2(u_{t+s} \circ \phi_s, \lambda_t; \Omega_t) \leq \Delta(s) \\ \leq \mathcal{L}^1(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t) + \frac{1}{s} \mathcal{L}_s^2(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t)$$

and use (2.32c) and (2.32e) to pass it to the limit as $s_k \rightarrow 0$, which is essentially the idea of the theorem of Correa and Seeger.

Remark 2.2. The assumptions (2.20d)–(2.20i) on the asymptotic expansion can be relaxed in Theorem 2.2 to the abstract conditions (2.32).

We note the important special cases in two corollaries. The first corollary relates the assumption (2.20c) of the dual cones to the primal cones; see [18, Theorem 3.4].

COROLLARY 2.1. *If the primal cone (2.4) is such that $K^*(\Omega_t) = K(\Omega_t)$, then the assumption (2.20c) is equivalent to bijection of the primal cones*

$$(2.33) \quad K(\Omega_t) \mapsto K(\Omega_{t+s})$$

and the formula of the shape derivative (2.21) implies the equality

$$\frac{d}{dt}\mathcal{E}(u_t; \Omega_t) = \mathcal{E}^1(u_t; \Omega_t), \quad (\lambda_t, B^1 u_t)_{\Omega_t} = 0$$

under the assumptions (2.20) used in Theorem 2.2.

The second corollary extends the result to equality constraints.

COROLLARY 2.2. *The inequality constraint in (2.4) can be replaced with the equality constraint resulting in the following primal and dual cones:*

$$(2.34) \quad K(\Omega_t) = \{u \in V(\Omega_t) \mid Bu = 0\}, \quad K^*(\Omega_t) = H^*(\Omega_t).$$

Then the assumption (2.20c) is satisfied within (2.20b) in Theorem 2.2.

In the next section we show in an application of Corollary 2.2 to the Stokes problem with the divergence-free equality constraint that mapping is not a bijection again.

3. Example of shape derivative: Stokes problem. Let Ω_t be a domain with Lipschitz continuous boundary, denote by $n_t = ((n_t)_1, \dots, (n_t)_d)$ the outward unit normal vector, and let the boundary $\partial\Omega_t$ consist of two disjoint sets Γ_t^D and Γ_t^N . For a given stationary external force $f = (f_1, \dots, f_d) \in H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^d)$, we consider the *Stokes problem*: Find a vector-valued field of flow velocity $u_t = ((u_t)_1, \dots, (u_t)_d)$ and a scalar-valued λ_t implying the pressure such that

$$(3.1a) \quad -\Delta u_t + \nabla \lambda_t = f \quad \text{in } \Omega_t,$$

$$(3.1b) \quad \operatorname{div} u_t = 0 \quad \text{in } \Omega_t,$$

$$(3.1c) \quad u_t = 0 \quad \text{on } \Gamma_t^D,$$

$$(3.1d) \quad \frac{\partial}{\partial n_t} u_t - \lambda_t n_t = 0 \quad \text{on } \Gamma_t^N.$$

The mixed boundary conditions imply no-slip (3.1c) and a Neumann-type condition (3.1d). For mixed boundary conditions appropriate for the Stokes equation see [6], [34, Chapter 6].

Corresponding to (3.1), the primal minimization problem reads as follows: Find $u_t \in V(\Omega_t)$ such that $\operatorname{div} u_t = 0$ and

$$(3.2) \quad \mathcal{E}(u_t; \Omega_t) = \min_{w \in K(\Omega_t)} \mathcal{E}(w; \Omega_t),$$

minimizing the objective function of the energy

$$(3.3) \quad \mathcal{E}(w; \Omega_t) = \int_{\Omega_t} \sum_{i=1}^d \left(\frac{1}{2} |\nabla w_i|^2 - f_i w_i \right) dx$$

over the primal cone determined by the divergence-free constraint

$$(3.4) \quad K(\Omega_t) = \{w \in V(\Omega_t) \mid \operatorname{div} w = 0 \text{ a.e. } \Omega_t\}$$

in the function space

$$(3.5) \quad V(\Omega_t) = \{w = (w_1, \dots, w_d) \in H^1(\Omega_t; \mathbb{R}^d) \mid w = 0 \text{ a.e. } \Gamma_t^D\}.$$

The operators $A = -\Delta$ and $B = \text{div}$ constitute the respective duality pairings

$$(3.6a) \quad \langle Au, w \rangle_{\Omega_t} = \int_{\Omega_t} \sum_{i=1}^d (\nabla u_i)^\top \nabla w_i \, dx, \quad u, w \in V(\Omega_t),$$

$$(3.6b) \quad (\lambda, Bu)_{\Omega_t} = \int_{\Omega_t} \lambda \text{div} u \, dx, \quad \lambda \in H(\Omega_t),$$

and the dual cone

$$(3.7) \quad K^*(\Omega_t) = \{\lambda \in H^*(\Omega_t) \mid (\lambda, Bu)_{\Omega_t} = 0 \quad \forall u \in K(\Omega_t)\},$$

where $H(\Omega_t) = H^*(\Omega_t) = L^2(\Omega_t; \mathbb{R})$.

If the surface measure $\text{meas}(\Gamma_t^N) > 0$, then the LBB condition (2.12) holds [28, Theorem 7.2], which means that $B : V(\Omega_t) \mapsto H(\Omega_t)$ is surjective and $K^*(\Omega_t) = H^*(\Omega_t)$. So we can apply Corollary 2.2.

If $\text{meas}(\Gamma_t^N) = 0$, then $B : H_0^1(\Omega_t; \mathbb{R}^d) \mapsto L_0^2(\Omega_t; \mathbb{R})$, where

$$(3.8) \quad L_0^2(\Omega_t; \mathbb{R}) = \{\lambda \in H(\Omega_t) \mid (\lambda, 1)_{\Omega_t} = 0\}$$

and its dual space excludes constants. In this case we cannot apply Corollary 2.2. In fact, the bijection in (2.20c) between $L_0^2(\Omega_t; \mathbb{R})$ and

$$L_0^2(\Omega_{t+s}; \mathbb{R}) = \{\mu \in H(\Omega_{t+s}) \mid (\mu, 1)_{\Omega_{t+s}} = 0\}$$

fails because $(\mu, 1)_{\Omega_{t+s}} \neq (\mu \circ \phi_s, 1)_{\Omega_t}$ according to the transformation formula (2.20f).

The primal-dual formulation of (3.2) consists in finding the pair $(u_t, \lambda_t) \in V(\Omega_t) \times L^2(\Omega_t; \mathbb{R})$ which is a saddle point,

$$(3.9) \quad \mathcal{L}(u_t, \lambda_t; \Omega_t) = \min_{w \in V(\Omega_t)} \max_{p \in L^2(\Omega_t; \mathbb{R})} \mathcal{L}(w, p; \Omega_t),$$

of the Lagrangian

$$(3.10) \quad \mathcal{L}(w, p; \Omega_t) = \mathcal{E}(w; \Omega_t) - \int_{\Omega_t} p \text{div} w \, dx,$$

where the dual cone $K^*(\Omega_t) = L^2(\Omega_t; \mathbb{R})$ according to (3.7). The optimality conditions (2.11) for problems (3.3) and (3.10) have the form

$$(3.11a) \quad \int_{\Omega_t} \left(\sum_{i=1}^d (\nabla(u_t)_i)^\top \nabla w_i - f_i w_i - \lambda_t \text{div} w \right) dx = 0 \quad \forall w \in V(\Omega_t),$$

$$(3.11b) \quad \int_{\Omega_t} p \text{div} u_t \, dx = 0 \quad \forall p \in L^2(\Omega_t; \mathbb{R}).$$

The solution pair is unique since the LBB condition holds in this case.

For a stationary *kinematic velocity* $\Lambda = (\Lambda_1, \dots, \Lambda_d) \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$, the unique solutions $[s \mapsto \phi_s], [s \mapsto \phi_s^{-1}] \in C^1([-T, T]; W_{\text{loc}}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d))$ of the *autonomous ODE systems* with some $T > 0$,

$$\begin{cases} \frac{d}{ds}\phi_s = \Lambda(\phi_s) & \text{for } s \neq 0, \\ \phi_s = x & \text{for } s = 0, \end{cases} \quad \begin{cases} \frac{d}{ds}\phi_s^{-1} = -\Lambda(\phi_s^{-1}) & \text{for } s \neq 0, \\ \phi_s^{-1} = y & \text{for } s = 0, \end{cases}$$

satisfy (2.14) and build the diffeomorphism (2.16); see [18, Lemma 2.2]. In the non-stationary case, the velocity $\Lambda \in C([-T, T]; W_{\text{loc}}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d))$ is defined by $\Lambda(t+s, y) = \frac{d}{ds}\phi_s(\phi_s^{-1}(y))$; see [47, section 2.9]. By this, the *transformation matrix* $\nabla_y \phi_s^{-1} := \{(\phi_s^{-1})_{i,j}\}_{i,j=1}^d$, where $(\phi_s^{-1})_{i,j} = \frac{\partial(\phi_s^{-1})_i}{\partial y_j}$, and the Jacobian determinant $\det(\nabla \phi_s)$ of the matrix $\nabla \phi_s := \{(\phi_s)_{i,j}\}_{i,j=1}^d$, where $(\phi_s)_{i,j} = \frac{\partial(\phi_s)_i}{\partial x_j}$, admit the following asymptotic expansion as $s \rightarrow 0$:

$$(3.12) \quad \nabla_y \phi_s^{-1}(\phi_s) = I - s \nabla \Lambda + r_s^1, \quad \det(\nabla \phi_s) = 1 + s \operatorname{div} \Lambda + r_s^2,$$

with the uniform estimate of the residual $\|r_s^1\|_{C([-T, T]; L_{\text{loc}}^\infty(\mathbb{R}^{d \times d}))} = o(s)$ and the residual $\|r_s^2\|_{C([-T, T]; L_{\text{loc}}^\infty(\mathbb{R}))} = o(s)$, where $\nabla \Lambda = \{\frac{\partial \Lambda_i}{\partial x_j}\}_{i,j=1}^d$ and I stands for the $d \times d$ identity matrix.

We apply the coordinate transformation $y = \phi_s(x)$ to the duality pairings in (3.6) rewritten over the perturbed domain Ω_{t+s} according to (2.15). As the result, using the chain rule $\nabla_y = (\nabla_y \phi_s^{-1}(\phi_s))^\top \nabla_x$ and (3.12), we derive the asymptotic expansions (3.13a)–(3.13f) corresponding to the assumptions (2.20d)–(2.20i). Indeed, the operator A is expanded as follows for $v, \chi \in H^1(\Omega_{t+s}; \mathbb{R}^d)$:

$$\begin{aligned} (3.13a) \quad \langle Av, \chi \rangle_{\Omega_{t+s}} &= \int_{\Omega_{t+s}} \sum_{i=1}^d (\nabla_y v_i)^\top \nabla_y \chi_i \, dy \\ &= \int_{\Omega_t} \sum_{i=1}^d (\nabla(v_i \circ \phi_s))^\top \nabla_y \phi_s^{-1}(\phi_s) (\nabla_y \phi_s^{-1}(\phi_s))^\top \nabla(\chi_i \circ \phi_s) \det(\nabla \phi_s) \, dx \\ &= \int_{\Omega_t} \sum_{i=1}^d (\nabla(v_i \circ \phi_s))^\top (I + s\{(\operatorname{div} \Lambda)I - \nabla \Lambda - (\nabla \Lambda)^\top\}) \nabla(\chi_i \circ \phi_s) \, dx + o(s) \end{aligned}$$

implying (2.20d) and (2.20e) with the first asymptotic term

$$(3.13b) \quad \langle A^1 u, w \rangle_{\Omega_t} = \int_{\Omega_t} \sum_{i=1}^d (\nabla u_i)^\top \{(\operatorname{div} \Lambda)I - \nabla \Lambda - (\nabla \Lambda)^\top\} \nabla w_i \, dx.$$

Accordingly, for $\mu \in L^2(\Omega_{t+s}; \mathbb{R})$ the operator B is expanded as

$$\begin{aligned} (3.13c) \quad (\mu, Bv)_{\Omega_{t+s}} &= \int_{\Omega_{t+s}} \mu \operatorname{div}_y v \, dy \\ &= \int_{\Omega_t} (\mu \circ \phi_s) \sum_{i,j=1}^d (\phi_s^{-1})_{j,i} (v \circ \phi_s)_{i,j} \det(\nabla \phi_s) \, dx, \end{aligned}$$

which implies (2.20f) and (2.20g) with

$$(3.13d) \quad (\lambda, B^1 u)_{\Omega_t} = \int_{\Omega_t} \lambda \left\{ (\operatorname{div} \Lambda)(\operatorname{div} u) - \sum_{i,j=1}^d \Lambda_{j,i} u_{i,j} \right\} dx$$

for $u, w \in H^1(\Omega_t; \mathbb{R}^d)$ and $\lambda \in L^2(\Omega_t; \mathbb{R})$. Then the transformation

$$(3.13e) \quad \langle f, v \rangle_{\Omega_{t+s}} = \int_{\Omega_{t+s}} \sum_{i=1}^d f_i v_i \, dy = \int_{\Omega_t} \sum_{i=1}^d (f_i \circ \phi_s)(v_i \circ \phi_s) \det(\nabla \phi_s) \, dx$$

due to (3.12) and $f_i \circ \phi_s = f_i + s\Lambda^\top \nabla f_i + o(s)$ leads to (2.20h) and (2.20i) with the first asymptotic term

$$(3.13f) \quad \langle f^1, u \rangle_{\Omega_t} = \int_{\Omega_t} \sum_{i=1}^d ((\operatorname{div} \Lambda) f_i + \Lambda^\top \nabla f_i) u_i \, dx.$$

The decompositions (3.13) agree with the assumptions (2.20a) and (2.20b).

The assumption of bijection (2.33) is not true for the primal cone (3.4) because of the transformation of the divergence (see formula (3.13c)). Nevertheless, the bijection of the dual cone allows us to apply Theorem 2.2 in the form of Corollary 2.2. The shape differentiability of the Stokes problem based on (3.13) and using $\operatorname{div} u_t = 0$ is established in the next theorem.

THEOREM 3.1. *The Stokes problem given in (3.2)–(3.4) has the shape derivative $\frac{d}{dt} \mathcal{E}(u_t; \Omega_t) = \mathcal{L}^1(u_t, \lambda_t; \Omega_t)$, which is defined in (2.21) and calculated according to formula (2.22) as follows:*

$$(3.14a) \quad \mathcal{L}^1(u_t, \lambda_t; \Omega_t) = \mathcal{E}^1(u_t; \Omega_t) + \int_{\Omega_t} \lambda_t \sum_{i,j=1}^d \Lambda_{j,i}(u_t)_{i,j} \, dx,$$

$$(3.14b) \quad \mathcal{E}^1(u_t; \Omega_t) = \int_{\Omega_t} \sum_{i=1}^d \left(\frac{1}{2} (\operatorname{div} \Lambda) |\nabla(u_t)_i|^2 - \sum_{k,j=1}^d (u_t)_{i,k} \Lambda_{k,j}(u_t)_{i,j} - ((\operatorname{div} \Lambda) f_i + \Lambda^\top \nabla f_i)(u_t)_i \right) dx.$$

We note the singularity at the intersection $\overline{\Gamma_t^D} \cap \overline{\Gamma_t^N}$ (see, e.g., [3]) such that (u_t, λ_t) is generally not in $H^2(\Omega_t; \mathbb{R}^d) \times H^1(\Omega_t; \mathbb{R})$ as shown in [41, Theorem 1.3.2]. Let the singular points be contained locally in a domain $\overline{\omega_t} \subset \overline{\Omega_t}$ such that $(u_t, \lambda_t) \in H^2(\Omega_t \setminus \omega_t; \mathbb{R}^d) \times H^1(\Omega_t \setminus \omega_t; \mathbb{R})$, and $f, \Lambda \equiv \text{const}$ in ω_t . In this case, using integration of (3.14) by parts, we get the following expression over the boundary of $\Omega_t \setminus \omega_t$:

$$\mathcal{L}^1(u_t, \lambda_t; \Omega_t) = \int_{\partial(\Omega_t \setminus \omega_t)} \sum_{i=1}^d \left((\Lambda^\top n_t) \left(\frac{1}{2} |\nabla(u_t)_i|^2 - f_i(u_t)_i \right) - (\Lambda^\top \nabla(u_t)_i) \left(\frac{\partial}{\partial n_t} (u_t)_i - \lambda_t(n_t)_i \right) \right) dS_x,$$

which implies the generalized J-integral (see [2, 41]).

In the case of $\Gamma_t^N = \emptyset$, to preserve the integral (see (3.8)), we need special area-preserving maps that form special linear group $\text{SL}(d)$ as stated in the following corollary.

COROLLARY 3.1. *Let problem (3.1) be stated under solely no-slip Dirichlet condition $u_t = 0$ on $\partial\Omega_t = \Gamma_t^D$. If the transformation $y = \phi_s(x)$ is characterized by the Jacobian determinant $\det(\nabla \phi_s) = 1$, then formula (3.14) in Theorem 3.1 still holds true with $\operatorname{div} \Lambda = 0$.*

Examples of such area-preserving bijection are translation and rotation of bodies obeying circular or cylindrical symmetry that maps the body into itself.

4. Conclusion. The result of the shape sensitivity analysis is useful in structure optimization; see, e.g., [1]. In particular, a positive/negative sign of the shape derivative forces, respectively, either increase or decay of the objective function \mathcal{E} of the energy.

For further development in the shape differentiability of Lagrangians, we may suggest combining Theorem 2.2 with Corollary 2.2 in order to account simultaneously for both equality- and inequality-type constraints within polyhedral cones. The example is the Stokes problem under the threshold slip boundary condition; see [38, 43].

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