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# A SHAPE-TOPOLOGICAL CONTROL PROBLEM FOR NONLINEAR CRACK-DEFECT INTERACTION: THE ANTIPLANE VARIATIONAL MODEL*

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**Abstract.** We consider the shape-topological control of a singularly perturbed variational inequality. The geometry-dependent state problem that we address in this paper concerns a heterogeneous medium with a micro-object (defect) and a macro-object (crack) modeled in two dimensions. The corresponding nonlinear optimization problem subject to inequality constraints at the crack is considered within a general variational framework. For the reason of asymptotic analysis, singular perturbation theory is applied, resulting in the topological sensitivity of an objective function representing the release rate of the strain energy. In the vicinity of the nonlinear crack, the antiplane strain energy release rate is expressed by means of the mode-III stress intensity factor that is examined with respect to small defects such as microcracks, holes, and inclusions of varying stiffness. The result of shape-topological control is useful either for arrests or rise of crack growth.

**Key words.** shape-topological control, topological derivative, singular perturbation, variational inequality, crack-defect interaction, nonlinear crack with nonpenetration, antiplane stress intensity factor, strain energy release rate, dipole tensor

**AMS subject classifications.** 35B25, 49J40, 49Q10, 74G70

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## 1. Introduction.

This paper focuses on shape-topological control of geometry-dependent variational inequalities, which are motivated by application to nonlinear cracking phenomena.

From a physical point of view, both cracks and defects appear in heterogeneous media and composites in the context of fracture. We refer the reader to [32] for a phenomenological approach to fracture with and without defects. Particular cases for the linear model of a stress-free crack interacting with inhomogeneities and microdefects were considered in [12, 31, 33]. In the present paper we investigate the sensitivity of a nonlinear crack with respect to a small object (called defect) of arbitrary physical and geometric nature.

While the classic model of a crack is assumed linear, the physical consistency needs nonlinear modeling. Nonlinear crack models subject to nonpenetration (contact) conditions have been developed in [9, 16, 21, 22, 23, 25] and other works. Recently, nonlinear cracks were bridged with thin inclusions under nonideal contact; see [15, 19, 20]. In the present paper we confine ourselves to the antiplane model simplification; for this case, inequality-type constraints at the plane crack are examined in [17, 18]. The linear crack also is included in this model as a particular case.

From a mathematical point of view, a topology perturbation problem is considered

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by varying defects posed in a cracked domain. For shape and topology optimization of cracks, we refer the reader to [3, 5, 10] and to [35] for shape perturbations in a general context. As the size of the defect tends to zero, we have to employ singular perturbation theory. The respective asymptotic methods were developed in [1, 14, 30], mostly for linear partial differential equations (PDEs) stated in singularly perturbed domains. Nevertheless, nonlinear boundary conditions are admissible to impose at those boundaries, which are separated from the varying object, as described in [6, 11].

From the point of view of shape and topology optimization, we investigate a novel setting of interaction problems between dilute geometric objects. In a broad scope, we consider a new class of geometry-dependent objective functions $J$ which are perturbed by at least two interacting objects $\Gamma$ and $\omega$ such that

$$J : \{\Gamma\} \times \{\omega\} \mapsto \mathbb{R}, \quad J = J(\Gamma, \omega).$$

In particular, we examine how a perturbation of one geometric object, say $\omega$, will affect a topology sensitivity, which here is the derivative of $J$ with respect to another geometric object $\Gamma$. In our particular setting of the interaction problem, the symbol $\Gamma$ refers to a crack and $\omega$ refers to an inhomogeneity (defect) in a heterogeneous medium.

The principal difficulty is that $\Gamma$ and $\omega$ enter the objective $J$ in a fully implicit way through a solution of a state (PDE) geometry-dependent problem. Therefore, to get an explicit formula, we rely on asymptotic modeling which concerns the smallness of $\omega$. Moreover, we generalize the state problem by allowing it to be a variational inequality. In fact, the variational approach to the perturbation problem allows us to incorporate nonlinear boundary conditions stated at the crack $\Gamma$.

The outline of the paper is as follows.

To gain insight into the mathematical problem, in section 2 we start with a general concept of shape-topological control for singular perturbations of abstract variational inequalities. In sections 3 and 4 this concept is specified for the nonlinear dipole problem of crack-defect interaction in two dimensions (2d).

For the antiplane model introduced in section 3, and further in section 4, we provide the topological sensitivity of an objective function expressing the strain energy release rate $J_{\mathrm{SERR}}$ by means of the mode-III stress intensity factor $J_{\mathrm{SIF}}$, which is of primary importance in engineering. The first order asymptotic term determines the so-called topological derivative of the objective function with respect to diminishing defects such as holes and inclusions of varying stiffness. We prove its semi-analytic expression by using a dipole representation of the crack tip—the defect center with the help of a Green-type (weight) function. The respective dipole matrix is inherently related to polarization and virtual mass matrices; see [34].

Within an equivalent ellipse concept (see, for example, [8, 33]), we further derive explicit formulae of the dipole matrix for the particular cases of the ellipse-shaped defects. Holes and rigid inclusions are accounted for here as the two limit cases of the stiffness parameters $\delta \searrow +0$ and $\delta \nearrow \infty$, respectively (see Appendix A).

The asymptotic result of shape-topological control is useful to force either shielding or amplification of an incipient crack by posing trial inhomogeneities (defects) in the test medium.

**2. Shape-topological control.** In the abstract context of shape-topological differentiability (see, e.g., [28, 29]), our construction can be outlined as follows.

We deal with variational inequalities of the following type: Find $u^0 \in K$ such that

$$\langle Au^0 - G, v - u^0 \rangle \geq 0 \quad \text{for all } v \in K, \tag{1}$$

with a linear strongly monotone operator $A : H \mapsto H^\star$, fixed $G \in H^\star$, and a polyhedric cone $K \subset H$, which are defined in a Hilbert space $H$ and its dual space $H^\star$. The solution of variational inequality (1) implies a metric projection $P_K : H^\star \mapsto K$, $G \mapsto u^0$. Its differentiability properties are useful in control theory; see [28, 29].

For control in the "right-hand side" (the inhomogeneity) of (1), one employs *regular perturbations* of $G$ with a small parameter $\varepsilon > 0$ in the direction of $h \in H^\star$ as follows: Find $u^\varepsilon \in K$ such that

$$\langle Au^\varepsilon - (G + \varepsilon h), v - u^\varepsilon \rangle \geq 0 \quad \text{for all } v \in K. \tag{2}$$

Then the directional differentiability of $P_K(G + \varepsilon h)$ from the right as $\varepsilon = +0$ implies the linear asymptotic expansion

$$u^\varepsilon = u^0 + \varepsilon q + \mathrm{o}(\varepsilon) \quad \text{in } H \text{ as } \varepsilon \searrow +0, \tag{3}$$

with $q \in S(u^0)$ uniquely determined on a proper convex cone $S(u^0)$, $K \subset S(u^0) \subset H$, and depending on $u^0$ and $h$; see [28, 29] for details.

In contrast, our underlying problem implies *singular perturbations* and the control of the operator $A$ of (1); namely, find $u^\varepsilon \in K$ such that

$$\langle A_\varepsilon u^\varepsilon - G, v - u^\varepsilon \rangle \geq 0 \quad \text{for all } v \in K, \tag{4}$$

where $A_\varepsilon = A + \varepsilon F_\varepsilon$, with a bounded linear operator $F_\varepsilon : H \mapsto H^\star$ such that $A_\varepsilon$ is strongly monotone and uniformly in $\varepsilon$, and $\varepsilon \|F_\varepsilon\| = \mathrm{O}(\varepsilon)$. In this case, we arrive at the nonlinear representation in $\varepsilon \searrow +0$,

$$u^\varepsilon = u^0 + \varepsilon \tilde{q}^\varepsilon + \mathrm{O}(f(\varepsilon)) \quad \text{in } H, \qquad \varepsilon \|\tilde{q}^\varepsilon\| = \mathrm{O}(\varepsilon). \tag{5}$$

In (5) $\tilde{q}^\varepsilon$ depends on $u^0$ and $F_\varepsilon$. A typical example, $\tilde{q}^\varepsilon(x) = \tilde{q}\left(\frac{x}{\varepsilon}\right)$, implies the existence of a boundary layer, e.g., in homogenization theory. In contrast to the differential $q$ in (3), a representative $\varepsilon \tilde{q}^\varepsilon$ is not uniquely defined by $\varepsilon$ but depends also on $\mathrm{o}(f(\varepsilon))$-terms. Examples are slant derivatives. The asymptotic behavior $f(\varepsilon)$ of the residual in (5) may differ for concrete problems. Thus, in the subsequent analysis, $f(\varepsilon) = \varepsilon^2$ in 2d.

In order to find the representative $\tilde{q}^\varepsilon$ in (5), we suggest sufficient conditions (6)–(9) below.

Proposition 1. *If the following relations hold:*

$$u^0 + \epsilon \tilde{q}^\varepsilon \in K, \tag{6}$$

$$u^\varepsilon - \varepsilon \tilde{q}^\varepsilon \in K, \tag{7}$$

$$\langle A_\varepsilon \tilde{q}^\varepsilon + F_\varepsilon u^0 - R_\varepsilon, v \rangle = 0 \quad \textit{for all } v \in H, \tag{8}$$

$$\varepsilon \|R_\varepsilon\| = \mathrm{O}(f(\varepsilon)), \tag{9}$$

*then* (5) *holds for the solutions of variational inequalities* (1) *and* (4).

*Proof.* Indeed, plugging test functions $v = u^\varepsilon - \varepsilon\tilde{q}^\varepsilon \in K$ in (1) due to (7) and $v = u^0 + \varepsilon\tilde{q}^\varepsilon \in K$ in (4) due to (6), after summation

$$\langle A_\varepsilon(u^\varepsilon - u^0) + \varepsilon F_\varepsilon u^0, u^\varepsilon - u^0 - \varepsilon\tilde{q}^\varepsilon\rangle \le 0,$$

and substituting $v = u^\varepsilon - u^0 - \varepsilon\tilde{q}^\varepsilon$ in (8) multiplied by $-\varepsilon$, this yields

$$\langle A_\varepsilon(u^\varepsilon - u^0 - \varepsilon\tilde{q}^\varepsilon) + \varepsilon R_\varepsilon, u^\varepsilon - u^0 - \varepsilon\tilde{q}^\varepsilon\rangle \le 0.$$

Applying the Cauchy–Schwarz inequality here, together with (9), shows (5) and completes the proof. ∎

We consider shape-topological control by means of mathematical programs with equilibrium constraints (MPEC) as follows: Find optimal parameters $p \in P$ from a feasible set $P$ such that

$$\underset{p\in P}{\text{minimize}}\, J(u^{(\varepsilon,p)}) \quad \text{subject to } \Pi(u^{(\varepsilon,p)}) = \min_{v\in K_p} \Pi(v). \tag{10}$$

In (10) the functional $\Pi : H \mapsto \mathbb{R}$, $\Pi(v) := \langle \frac{1}{2}A_\varepsilon v - G, v\rangle$ represents the strain energy (SE) of the state problem such that variational inequality (4) implies the first order optimality condition for the minimization of $\Pi(v)$ over $K_p \subset H$. The multiparameter $p$ may include the right-hand side $G$, geometric variables, and other data of the problem. The optimal value function $J$ in (10) is motivated by underlying physics, which we will specify in examples below.

The main difficulty of the shape-topological control is that geometric parameters are involved in MPEC in a fully implicit way. In this respect, relying on asymptotic models under small variations $\varepsilon$ of geometry is helpful to linearize the optimal value function. See, e.g., the application of topological sensitivity to inverse scattering problems in [26].

In order to expand (10) in $\varepsilon \searrow +0$, the uniform asymptotic expansion (5) is useful; however, it is varied by $f(\varepsilon)$. The variability of $f(\varepsilon)$ is inherent here due to nonuniqueness of a representative $\varepsilon\tilde{q}^\varepsilon$ defined up to the $\mathrm{o}(f(\varepsilon))$-terms. As an alternative, developing a variational technique related to Green functions and truncated Fourier series, in section 4.2 we derive local asymptotic expansions in the near-field, which are uniquely determined.

Since Proposition 1 gives only sufficient conditions for (5), in the following sections we suggest a method of topology perturbation to find the correction $\tilde{q}^\varepsilon$ for the underlying variational inequality.

**3. Nonlinear problem of crack-defect interaction in 2d.** We start with the two-dimensional geometry description.

**3.1. Geometric configuration.** For $x = (x_1, x_2)^\top \in \mathbb{R}^2$ we set the semi-infinite straight crack $\Gamma_\infty = \{x \in \mathbb{R}^2 : x_1 < 0, x_2 = 0\}$, with the unit normal vector $n = (0,1)^\top$ at $\Gamma_\infty$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, with the Lipschitz boundary $\partial\Omega$ and the normal vector $n = (n_1, n_2)^\top$ at $\partial\Omega$. We assume that the origin $0 \in \Omega$ and assign it to the tip of a finite crack $\Gamma := \Gamma_\infty \cap \Omega$. An example geometric configuration is drawn in Figure 1.

Let $x^0$ be an arbitrarily fixed point in the cracked domain $\Omega \setminus \Gamma$. We associate the poles 0 and $x^0$ with two polar coordinate systems $x = \rho(\cos\theta, \sin\theta)^\top$, $\rho > 0$, $\theta \in [-\pi, \pi]$, and $x - x^0 = \rho_0(\cos\theta_0, \sin\theta_0)^\top$, $\rho_0 > 0$, $\theta_0 \in (-\pi, \pi]$. Here $x^0 = r(\cos\phi, \sin\phi)^\top$ is given by $r > 0$ and $\phi \in (-\pi, \pi)$ as depicted in Figure 1(a). We assign $x^0$ to the center of a defect $\omega_\varepsilon(x^0)$ posed in $\Omega$ as illustrated in Figure 1(b).

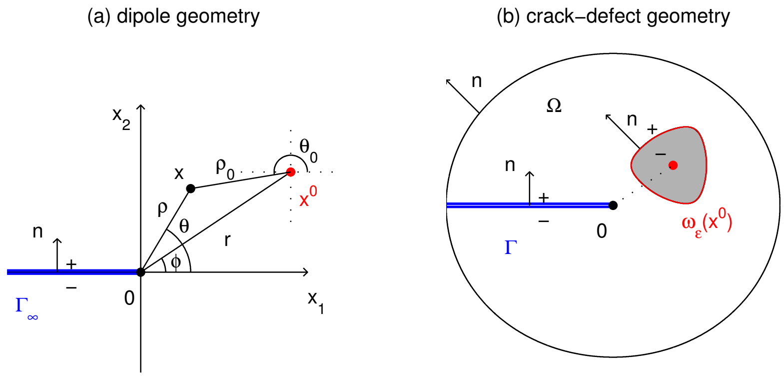


Fig. 1. *Example geometric configuration.*

More precisely, let a trial geometric object be given by the compact set $\omega_\varepsilon(x^0) = \{x \in \mathbb{R}^2 : \frac{x-x^0}{\varepsilon} \in \omega\}$ which is parametrized by an admissible triple of the shape $\omega \in \Theta$, center $x^0 \in \Omega \setminus \Gamma$, and size $\varepsilon > 0$. Let $B_\rho(x^0)$ denote the disk around $x^0$ of radius $\rho$. For admissible shapes $\Theta$ we choose a domain $\omega$ such that $0 \in \omega \subseteq B_1(0)$ and $\rho = 1$ is the minimal radius among all bounding discs $B_\rho(0) \supset \omega$. Thus, the shapes are invariant to translations and isotropic scaling, so that we express them with the equivalent notation $\omega = \omega_1(0)$. Admissible geometric parameters $(\omega, \varepsilon, x^0) \in \Theta \times \mathbb{R}_+ \times (\Omega \setminus \Gamma)$ should satisfy the consistency condition $\omega_\varepsilon(x^0) \subset B_\varepsilon(x^0) \subset \Omega \setminus \Gamma$.

We note that the motivation of inclusion $\omega \subseteq B_1(0)$ (but not $\omega \supseteq B_1(0)$) is to separate the far-field $\mathbb{R}^2 \setminus B_1(0)$ from the near-field $B_1(0) \setminus \omega$ of the object $\omega$.

In the following, we assume that the Hausdorff measure $\mathrm{meas}_2(\omega) > 0$ and the boundary $\partial\omega_\varepsilon(x^0)$ is Lipschitz continuous, and we assign $n$ to the unit normal vector at $\partial\omega_\varepsilon(x^0)$, which points outward toward $\omega_\varepsilon(x^0)$. In a particular situation, our consideration admits also the degenerate case when $\omega_\varepsilon(x^0)$ shrinks to a one-dimensional Lipschitz manifold of codimension one in $\mathbb{R}^2$, thus allowing for defects such as curvilinear inclusions. The degenerate case will appear in more detail when shrinking ellipses to line segments, as described in Appendix A.

**3.2. Variational problem.** In the reference configuration of the cracked domain $\Omega \setminus \Gamma$ with the fixed inclusion $\omega_\varepsilon(x^0)$ we state a constrained minimization problem related to a PDE, which here is a model problem with the scalar Laplace operator. Motivated by three-dimensional fracture problems with possible contact between crack faces, as described in [17], in the antiplane framework of linear elasticity we look for admissible displacements $u(x)$ in $\Omega \setminus \Gamma$ which are restricted along the crack by the inequality constraint

$$[\![u]\!] = u|_{\Gamma^+} - u|_{\Gamma^-} \geq 0 \quad \text{on } \Gamma. \tag{11}$$

The positive $\Gamma^+_\infty$ (hence, its part $\Gamma^+ = \Gamma^+_\infty \cap \Omega$) and the negative $\Gamma^-_\infty$ (hence, $\Gamma^- = \Gamma^-_\infty \cap \Omega$) crack faces are distinguished as the limit of points $(x_1, x_2)^\top$ for $x_1 < 0$ and $x_2 \to 0$, when $x_2 > 0$ and $x_2 < 0$, respectively; see Figure 1.

Now we get a variational formulation of a state problem due to the unilateral constraint (11).

Let the external boundary $\partial\Omega$ consist of two disjoint parts $\Gamma_N$ and $\Gamma_D$. We assume that the Dirichlet part has the positive measure $\mathrm{meas}_1(\Gamma_D) > 0$; otherwise, we should exclude the nontrivial kernel (the rigid displacements) for coercivity of the objective functional $\Pi$ in (13) below. The set of admissible displacements contains functions $u$ from the Sobolev space

$$H(\Omega \setminus \Gamma) = \{u \in H^1(\Omega \setminus \Gamma) :\ u = 0 \quad \text{on } \Gamma_D\}$$

such that (11) holds as follows:

$$K(\Omega \setminus \Gamma) = \{u \in H(\Omega \setminus \Gamma) :\ [\![u]\!] \geq 0 \quad \text{on } \Gamma\}.$$

This is a convex cone in $H(\Omega \setminus \Gamma)$—moreover, a polyhedric cone; see [28, 29]. We note that the jump of the traces at $\Gamma$ is well-defined in the Lions–Magenes space $[\![u]\!] \in H_{00}^{1/2}(\Gamma)$; see [16, section 1.4].

Let $\mu > 0$ be a fixed material parameter (the Lamé constant) in the homogeneous reference domain $\Omega \setminus \Gamma$. We distinguish the inhomogeneity with the help of a variable parameter $\delta > 0$ such that the characteristic function is given by

$$\chi^{\delta}_{\omega_\varepsilon(x^0)}(x) := 1 - (1-\delta)\mathbf{1}_{\omega_\varepsilon(x^0)} = \begin{cases} 1,\ x \in \Omega \setminus \omega_\varepsilon(x^0), \\ \delta,\ x \in \omega_\varepsilon(x^0). \end{cases} \tag{12}$$

In the following we use the notation $\mu\chi^{\delta}_{\omega_\varepsilon(x^0)}$, which implies, due to (12), the material parameter $\mu$ in the homogeneous domain $\Omega \setminus \omega_\varepsilon(x^0)$, and the material parameter $\mu\delta$ in $\omega_\varepsilon(x^0)$ characterizing stiffness of the inhomogeneity. The parameter $\delta$ accounts for the following three physical situations: inclusions of varying stiffness for finite $0 < \delta < \infty$, holes for $\delta \searrow +0$, and rigid inclusions for $\delta \nearrow +\infty$.

For given boundary traction $g \in L^2(\Gamma_N)$, the SE of the heterogeneous medium is described by the functional $\Pi : H(\Omega \setminus \Gamma) \mapsto \mathbb{R}$,

$$\Pi(u; \Gamma, \omega_\varepsilon(x^0)) := \frac{1}{2} \int_{\Omega\setminus\Gamma} \mu\chi^{\delta}_{\omega_\varepsilon(x^0)} |\nabla u|^2 \, dx - \int_{\Gamma_N} gu \, dS_x, \tag{13}$$

which is quadratic and strongly coercive over $H(\Omega\setminus\Gamma)$. Henceforth, the Babuška–Lax–Milgram theorem guarantees the unique solvability of the constrained minimization of $\Pi$ over $K(\Omega \setminus \Gamma)$, which implies the variational formulation of the *heterogeneous problem* as follows: Find $u^{(\omega,\varepsilon,x^0,\delta)} \in K(\Omega \setminus \Gamma)$ such that

$$\begin{aligned} &\int_{\Omega\setminus\Gamma} \mu\chi^{\delta}_{\omega_\varepsilon(x^0)} (\nabla u^{(\omega,\varepsilon,x^0,\delta)})^\top \nabla(v - u^{(\omega,\varepsilon,x^0,\delta)}) \, dx \\ &\geq \int_{\Gamma_N} g(v - u^{(\omega,\varepsilon,x^0,\delta)}) \, dS_x \quad \text{for all } v \in K(\Omega \setminus \Gamma). \end{aligned} \tag{14}$$

The variational inequality (14) describes the weak solution of the following boundary value problem:

$$-\Delta u^{(\omega,\varepsilon,x^0,\delta)} = 0 \quad \text{in } \Omega \setminus \Gamma, \tag{15a}$$

$$u^{(\omega,\varepsilon,x^0,\delta)} = 0 \quad \text{on } \Gamma_D, \qquad \mu\frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} = g \quad \text{on } \Gamma_N, \tag{15b}$$

$$\begin{aligned} &\left[\!\!\left[\frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n}\right]\!\!\right] = 0, \quad [\![u^{(\omega,\varepsilon,x^0,\delta)}]\!] \geq 0, \quad \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} \leq 0, \\ &\frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n}[\![u^{(\omega,\varepsilon,x^0,\delta)}]\!] = 0 \quad \text{on } \Gamma, \end{aligned} \tag{15c}$$

$$\text{(15d)}\qquad \left.\frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n}\right|_{\partial\omega_\varepsilon(x^0)+} - \delta \left.\frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n}\right|_{\partial\omega_\varepsilon(x^0)-} = 0, \quad [\![u^{(\omega,\varepsilon,x^0,\delta)}]\!] = 0 \quad \text{on } \partial\omega_\varepsilon(x^0).$$

In (15d) the jump across the defect boundary is defined as

$$\text{(16)}\qquad [\![u]\!] = u|_{\partial\omega_\varepsilon(x^0)+} - u|_{\partial\omega_\varepsilon(x^0)-} \quad \text{on } \partial\omega_\varepsilon(x^0),$$

where $+$ and $-$ correspond to the chosen direction of the normal $n$, which points outward toward $\omega_\varepsilon(x^0)$; see Figure 1(b).

We remark that the $L^2$-regularity of the normal derivatives at the boundaries $\Gamma_N$, $\Gamma$, and $\partial\omega_\varepsilon(x^0)$ is needed in order to have strong solutions in (15). The exact sense of the boundary conditions (15c) can be given for the traction $\frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n}$ in the dual space of $H^{1/2}_{00}(\Gamma)$, which is denoted by $H^{1/2}_{00}(\Gamma)^\star$, to (15b) for $\frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n}$ in the dual space of $H^{1/2}_{00}(\Gamma_N)$, and to (15d) for $\frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n}|_{\partial\omega_\varepsilon(x^0)\pm} \in H^{-1/2}(\partial\omega_\varepsilon(x^0))$. Moreover, the solution $u^{(\omega,\varepsilon,x^0,\delta)}$ is $H^2$-smooth away from the crack tip, boundary of defect, and possible irregular points of external boundary; for details see [16, section 2].

If $\varepsilon \searrow +0$, similarly to (14) there exists the unique solution of the *homogeneous problem* as follows: Find $u^0 \in K(\Omega\setminus\Gamma)$ such that for all $v \in K(\Omega\setminus\Gamma)$,

$$\text{(17)}\qquad \int_{\Omega\setminus\Gamma} \mu(\nabla u^0)^\top \nabla(v-u^0)\,dx \ge \int_{\Gamma_N} g(v-u^0)\,dS_x,$$

which implies the boundary value problem

$$\text{(18a)}\qquad -\Delta u^0 = 0 \quad \text{in } \Omega\setminus\Gamma,$$

$$\text{(18b)}\qquad u^0 = 0 \quad \text{on } \Gamma_D, \quad \mu\frac{\partial u^0}{\partial n} = g \quad \text{on } \Gamma_N,$$

$$\text{(18c)}\qquad \left[\!\!\left[\frac{\partial u^0}{\partial n}\right]\!\!\right] = 0, \quad [\![u^0]\!] \ge 0, \quad \frac{\partial u^0}{\partial n} \le 0, \quad \frac{\partial u^0}{\partial n}[\![u^0]\!] = 0 \quad \text{on } \Gamma,$$

$$\text{(18d)}\qquad \left[\!\!\left[\frac{\partial u^0}{\partial n}\right]\!\!\right] = 0, \quad [\![u^0]\!] = 0 \quad \text{on } \partial\omega_\varepsilon(x^0).$$

We note that (18d) is written here for comparison with (15d), and it implies that the solution $u^0$ is $C^\infty$-smooth in $B_\varepsilon(x^0) \supset \omega_\varepsilon(x^0)$ compared to $u^{(\omega,\varepsilon,x^0,\delta)}$.

Using Green's formulae separately in $(\Omega\setminus\Gamma)\setminus\omega_\varepsilon(x^0)$ and in $\omega_\varepsilon(x^0)$, the variational inequality (17) can be transformed into an equivalent variational inequality depending on the parameter $\delta$,

$$\text{(19)}\qquad \int_{\Omega\setminus\Gamma} \mu\chi^\delta_{\omega_\varepsilon(x^0)}(\nabla u^0)^\top\nabla(v-u^0)\,dx \ge \int_{\Gamma_N} g(v-u^0)\,dS_x - (1-\delta)\int_{\partial\omega_\varepsilon(x^0)} \mu\frac{\partial u^0}{\partial n}(v-u^0)\,dS_x \quad \text{for all } v \in K(\Omega\setminus\Gamma).$$

The left-hand side of (19) has the same operator as (14), and this fact will be used in section 4 for asymptotic analysis of the solution $u^{(\omega,\varepsilon,x^0,\delta)}$.

**4. Topology asymptotic analysis.** To examine the heterogeneous state (14) in comparison with the homogeneous state (17) in an explicit way, we rely on small defects; thus, passing $\varepsilon \searrow +0$ leads to the first order asymptotic analysis. First, for the solution of the state problem, we obtain a two-scale asymptotic expansion, which is related to Green's functions. For this reason we apply the singular perturbation theory and endow it with variational arguments. Second, with its help, we provide a topology sensitivity of the geometry-dependent objective functions representing the mode-III stress intensity factor (SIF) and the strain energy release rate (SERR), which are the primary physical characteristics of fracture.

**4.1. Asymptotic analysis of the solution.** We start with the inner asymptotic expansion of the solution $u^0$ of the homogeneous variational inequality (17), which is $C^\infty$-smooth in the ball $B_R(x^0)$ of the radius $R < \min\{r, \operatorname{dist}(x^0, \partial\Omega)\}$. We recall that $r$ is the distance of the defect center $x^0$ from the crack tip at the origin 0. Due to (18a), we have the following representation (see, e.g., [14, section 3]):

$$(20)\quad \begin{aligned} &u^0(x) = u^0(x^0) + \nabla u^0(x^0)^\top (x - x^0) + U_{x^0}(x) \quad \text{for } x \in B_R(x^0),\\ &\int_{-\pi}^{\pi} U_{x^0}\, d\theta_0 = \int_{-\pi}^{\pi} U_{x^0} \frac{x - x^0}{\rho_0}\, d\theta_0 = 0, \quad U_{x^0} = \mathrm{O}\big(\rho_0^2\big),\ \nabla U_{x^0} = \mathrm{O}(\rho_0). \end{aligned}$$

From (20) we infer the expansion of the traction

$$(21)\quad \frac{\partial u^0}{\partial n} = \nabla u_0(x^0)^\top n + \frac{\partial U_{x^0}}{\partial n}, \quad \frac{\partial U_{x^0}}{\partial n} = \mathrm{O}(\varepsilon) \quad \text{on } \partial\omega_\varepsilon(x^0),$$

which will be used further for expansion of the right-hand side in (19).

Moreover, to compensate the O(1)-asymptotic term $\nabla u_0(x^0)^\top n$ in (21), we will need to construct a boundary layer near $\partial\omega_\varepsilon(x^0)$. For this task, we stretch the coordinates as $y = \frac{x - x^0}{\varepsilon}$, which implies the diffeomorphic map $\omega_\varepsilon(x^0) \mapsto \omega_1(0) \subset B_1(0)$. In the following, the stretched coordinates $y = (y_1, y_2)^\top = |y|(\cos\theta_0, \sin\theta_0)^\top$ refer always to the infinite domain. In the whole $\mathbb{R}^2$ we introduce the weighted Sobolev space

$$H^1_\nu(\mathbb{R}^2) = \{v :\ \nu v, \nabla v \in L^2(\mathbb{R}^2)\}, \quad \nu(y) = \frac{1}{|y| \ln |y|} \quad \text{in } \mathbb{R}^2 \setminus B_2(0),$$

with the weight $\nu \in L^\infty(\mathbb{R}^2)$ due to the weighted Poincaré inequality in exterior domains; see [4]. In this space, excluding constant polynomials $\mathbb{P}_0$, we state the following auxiliary result, which is closely related to the generalized polarization tensors considered in [2, section 3].

LEMMA 1. *There exists the unique solution of the following variational problem: Find* $w \in \big(H^1_\nu(\mathbb{R}^2) \setminus \mathbb{P}_0\big)^2$, $w = (w_1, w_2)^\top(y)$, *such that*

$$(22)\quad \int_{\mathbb{R}^2} \chi^\delta_{\omega_1(0)} \nabla w_i^\top \nabla v\, dy = (1 - \delta) \int_{\partial\omega_1(0)} n_i v\, dS_y \quad \textit{for all } v \in H^1_\nu(\mathbb{R}^2),$$

*for* $i = 1, 2$, *which satisfies the Laplace equation in* $\mathbb{R}^2 \setminus \partial\omega_1(0)$ *and the following transmission boundary conditions across* $\partial\omega_1(0)$*:*

$$(23)\quad \frac{\partial w}{\partial n}\Big|_{\partial\omega_1(0)^+} - \delta \frac{\partial w}{\partial n}\Big|_{\partial\omega_1(0)^-} = -(1 - \delta) n, \qquad w|_{\partial\omega_1(0)^+} - w|_{\partial\omega_1(0)^-} = 0.$$

*After rescaling, the far-field representation holds as follows:*

$$
(24)\quad \begin{aligned}
&w\left(\frac{x-x^0}{\varepsilon}\right)=\frac{\varepsilon}{2\pi}A_{(\omega,\delta)}\frac{x-x^0}{\rho_0^2}+W(x)\quad \text{for } x\in\mathbb{R}^2\setminus B_\varepsilon(x^0),\\
&\int_{-\pi}^{\pi}W\,d\theta_0=\int_{-\pi}^{\pi}W\frac{x-x^0}{\rho_0}\,d\theta_0=0,\quad W=\mathrm{O}\Big(\Big(\frac{\varepsilon}{\rho_0}\Big)^2\Big),\ \nabla W=\mathrm{O}\Big(\frac{\varepsilon^2}{\rho_0^3}\Big),
\end{aligned}
$$

*where the dipole matrix $A_{(\omega,\delta)}\in\mathrm{Sym}(\mathbb{R}^{2\times 2})$ has entries ($i,j=1,2$) as follows:*

$$
(25)\qquad (A_{(\omega,\delta)})_{ij}=(1-\delta)\Big\{\delta_{ij}\,\mathrm{meas}_2(\omega_1(0))+\int_{\partial\omega_1(0)}w_i n_j\,dS_y\Big\}.
$$

*Moreover, $A_{(\omega,\delta)}\in\mathrm{Spd}(\mathbb{R}^{2\times 2})$ if $\delta\in[0,1)$ and $\mathrm{meas}_2(\omega_1(0))>0$.*

*Proof.* The existence of a solution to (22) up to a free constant follows from the results of [4]. Following [10, Lemma 3.2], below we prove the far-field pattern (25) in representation (24).

For this reason, we split $\mathbb{R}^2$ into the far-field $\mathbb{R}^2\setminus B_1(0)$ and the near-field $B_1(0)$. Since $w$ from (22) solves the Laplace equation, in the far-field it exhibits the outer asymptotic expansion

$$
(26)\quad \begin{aligned}
&w(y)=\frac{1}{2\pi}A_{(\omega,\delta)}\frac{y}{|y|^2}+W(y)\quad\text{for } y\in\mathbb{R}^2\setminus B_1(0),\\
&\int_{-\pi}^{\pi}W\,d\theta_0=\int_{-\pi}^{\pi}W\frac{y}{|y|}\,d\theta_0=0,\quad W=\mathrm{O}\Big(\Big(\frac{1}{|y|}\Big)^2\Big),\ \nabla W=\mathrm{O}\Big(\Big(\frac{1}{|y|}\Big)^3\Big),
\end{aligned}
$$

which implies (24) after rescaling $y=\frac{x-x^0}{\varepsilon}$.

In the near-field, we apply the second Green's formula for $i,j=1,2$,

$$
\begin{aligned}
0=\int_{B_1(0)}\chi^{\delta}_{\omega_1(0)}\{\Delta w_i y_j-w_i\Delta y_j\}\,dy=\int_{\partial B_1(0)}\Big\{\frac{\partial w_i}{\partial|y|}y_j-w_i\frac{\partial y_j}{\partial|y|}\Big\}\,dS_y\\
-\int_{\partial\omega_1(0)}\Big\{\Big[\frac{\partial w_i}{\partial n}|_{\partial\omega_1(0)^+}-\delta\frac{\partial w_i}{\partial n}|_{\partial\omega_1(0)^-}\Big]y_j-(1-\delta)w_i\frac{\partial y_j}{\partial n}\Big\}\,dS_y,
\end{aligned}
$$

and substitute here the transmission conditions (23) to derive that

$$
(27)\qquad -\int_{\partial B_1(0)}\Big\{\frac{\partial w_i}{\partial|y|}-w_i\Big\}\frac{y_j}{|y|}\,dS_y=(1-\delta)\int_{\partial\omega_1(0)}\{n_i y_j+w_i n_j\}\,dS_y.
$$

We apply to (27) the divergence theorem

$$
\int_{\partial\omega_1(0)}n_i y_j\,dS_y=\int_{\omega_1(0)}y_{j,i}\,dy=\delta_{ij}\,\mathrm{meas}_2(\omega_1(0))
$$

and substitute (26) to calculate the integral over $\partial B_1(0)$ as

$$
-\int_{\partial B_1(0)}\Big\{\frac{\partial w_i}{\partial|y|}-w_i\Big\}\frac{y_j}{|y|}\,dS_y=\frac{1}{\pi}\int_{-\pi}^{\pi}(A_{(\omega,\delta)})_{ik}\frac{y_k}{|y|}\frac{y_j}{|y|}\,d\theta_0=(A_{(\omega,\delta)})_{ij},
$$

which implies (25).

We now prove the symmetry and positive definiteness properties of $A_{(\omega,\delta)}$. Inserting $v = w_j$, $j = 1, 2$, into (22) we have

$$\int_{\mathbb{R}^2} \chi^{\delta}_{\omega_1(0)} \nabla w_i^\top \nabla w_j \, dy = (1-\delta) \int_{\partial\omega_1(0)} n_i w_j \, dS_y = (1-\delta) \int_{\partial\omega_1(0)} n_j w_i \, dS_y,$$

and hence the symmetry $(A_{(\omega,\delta)})_{ij} = (A_{(\omega,\delta)})_{ji}$ for $i, j = 1, 2$ in (25). For arbitrary $z \in \mathbb{R}^2$, from (22) we have

$$0 \le \int_{\mathbb{R}^2} \chi^{\delta}_{\omega_1(0)} |\nabla(z_1 w_1 + z_2 w_2)|^2 \, dy = (1-\delta) \sum_{i,j=1}^{2} \int_{\partial\omega_1(0)} w_i z_i n_j z_j \, dS_y.$$

Henceforth, multiplying (25) with $z_i z_j$ and summing the result over $i, j = 1, 2$, it follows that

$$\sum_{i,j=1}^{2} (A_{(\omega,\delta)})_{ij} z_i z_j = (1-\delta) \Big\{ |z|^2 \operatorname{meas}_2(\omega_1(0)) + \sum_{i,j=1}^{2} \int_{\partial\omega_1(0)} w_i z_i n_j z_j \, dS_y \Big\}$$
$$\ge (1-\delta) |z|^2 \operatorname{meas}_2(\omega_1(0)) > 0$$

if $1 - \delta > 0$ and $\operatorname{meas}_2(\omega_1(0)) > 0$. This completes the proof. □

It is important to comment on the transmission conditions (23) in relation to the stiffness parameter $\delta > 0$. On the one hand, for $\delta \searrow +0$ implying that $\omega_1(0)$ is a hole, conditions (23) split into

$$w^- = w^+ \quad \text{on } \partial\omega_1(0)^-, \qquad \frac{\partial w^+}{\partial n} = -n \quad \text{on } \partial\omega_1(0)^+, \tag{28}$$

where the indexes $\pm$ mark the traces of the functions in (28) at $\partial\omega_1(0)^\pm$, respectively. Henceforth, to find $A_{(\omega,\delta)}$ in (25) instead of (22), it suffices to solve the exterior problem under the Neumann condition (28) as follows: Find $w \in \big(H^1_\nu(\mathbb{R}^2 \backslash \omega_1(0)) \backslash \mathbb{P}_0\big)^2$ such that for $i = 1, 2$,

$$\int_{\mathbb{R}^2 \backslash \omega_1(0)} \nabla w_i^\top \nabla v \, dy = -\int_{\partial\omega_1(0)} n_i v \, dS_y \quad \text{for all } v \in H^1_\nu(\mathbb{R}^2 \setminus \omega_1(0)).$$

In this case, $A_{(\omega,\delta)}$ is called the virtual mass or the added mass matrix according to [34].

On the other hand, for $\delta \nearrow +\infty$ implying that $\omega_1(0)$ is a rigid inclusion, conditions (23) read as

$$\frac{\partial w^-}{\partial n} = -n \quad \text{on } \partial\omega_1(0)^-, \qquad w^+ = w^- \quad \text{on } \partial\omega_1(0)^+. \tag{29}$$

In this case, (22) is split into the interior Neumann problem in $\omega_1(0)$, and the exterior Dirichlet problem in $\mathbb{R}^2 \backslash \omega_1(0)$. The respective $A_{(\omega,\delta)}$ is called the polarization matrix in [34].

Thus, we have the following.

COROLLARY 1. *The auxiliary problem* (22) *under the transmission boundary conditions* (23) *describes the general case of inclusions of varying stiffness, and it accounts for holes (hard obstacles in acoustics) under the Neumann condition* (28) *as well as rigid inclusions (soft obstacles in acoustics) under the Dirichlet condition* (29) *as the limit cases of the stiffness parameters* $\delta \searrow +0$ *and* $\delta \nearrow +\infty$*, respectively.*

With the help of the boundary layer $w$ constructed in Lemma 1, we can represent the first order asymptotic term in the expansion of the perturbed solution $u^{(\omega,\varepsilon,x^0,\delta)}$ as $\varepsilon \searrow +0$ given in the following theorem.

THEOREM 1. *The solution* $u^{(\omega,\varepsilon,x^0,\delta)} \in K(\Omega \setminus \Gamma)$ *of the heterogeneous problem* (14)*, the solution* $u^0$ *of the homogeneous problem* (17)*, and the rescaled solution* $w^\varepsilon(x) := w(\frac{x-x^0}{\varepsilon})$ *of* (22) *admit the uniform asymptotic representation for* $x \in \Omega \setminus \Gamma$ *as follows:*

$$u^{(\omega,\varepsilon,x^0,\delta)}(x) = u^0(x) + \varepsilon \nabla u^0(x^0)^\top w^\varepsilon(x) \eta_{\Gamma_D}(x) + Q(x), \tag{30}$$

*where* $\eta_{\Gamma_D}$ *is a smooth cut-off function which is equal to one except in a neighborhood of the Dirichlet boundary* $\Gamma_D$ *on which* $\eta_{\Gamma_D} = 0$*. The residual* $Q \in H(\Omega \setminus \Gamma)$ *and* $w^\varepsilon$ *exhibit the following asymptotic behavior as* $\varepsilon \searrow +0$*:*

$$\|Q\|_{H^1(\Omega\setminus\Gamma)} = \mathrm{O}(\varepsilon^2), \qquad w^\varepsilon = \mathrm{O}(\varepsilon) \quad \textit{far away from } \omega_\varepsilon(x^0). \tag{31}$$

*Proof.* Since $[\![w^\varepsilon]\!] = 0$ on $\Gamma_\infty$, we can substitute $v = u^0 + \varepsilon \nabla u^0(x^0)^\top w^\varepsilon \eta_{\Gamma_D} \in K(\Omega \setminus \Gamma)$ in (14) and $v = u^{(\omega,\varepsilon,x^0,\delta)} - \varepsilon \nabla u^0(x^0)^\top w^\varepsilon \eta_{\Gamma_D} \in K(\Omega \setminus \Gamma)$ in (19) as the test functions, which yields two inequalities. Summing them together and dividing by $\mu$, we get

$$\int_{\Omega\setminus\Gamma} \chi^\delta_{\omega_\varepsilon(x^0)} \nabla(u^{(\omega,\varepsilon,x^0,\delta)} - u^0)^\top \nabla Q \, dx \le (1-\delta) \int_{\partial\omega_\varepsilon(x^0)} \frac{\partial u^0}{\partial n} Q \, dS_x, \tag{32}$$

where $Q := u^{(\omega,\varepsilon,x^0,\delta)} - u^0(x) - \varepsilon \nabla u^0(x^0)^\top w^\varepsilon \eta_{\Gamma_D} \in H(\Omega \setminus \Gamma)$ is defined according to (30).

After rescaling $y = \frac{x-x^0}{\varepsilon}$, with the help of the Green's formula in $\Omega \setminus \Gamma$, from (22) we obtain the following variational equation written in the bounded domain for $w_i^\varepsilon(x) := w_i(\frac{x-x^0}{\varepsilon})$, $i = 1, 2$:

$$\begin{aligned} &\int_{\Omega\setminus\Gamma} \chi^\delta_{\omega_\varepsilon(x^0)} (\nabla w_i^\varepsilon)^\top \nabla v \, dx = \frac{1-\delta}{\varepsilon} \int_{\partial\omega_\varepsilon(x^0)} n_i v \, dS_x \\ &+ \int_{\Gamma_N} \frac{\partial w_i^\varepsilon}{\partial n} v \, dS_x - \int_\Gamma \frac{\partial w_i^\varepsilon}{\partial n} [\![v]\!] \, dS_x \quad \text{for all } v \in H(\Omega \setminus \Gamma). \end{aligned} \tag{33}$$

Now, inserting $v = Q$ into (33) after multiplication by the vector $\varepsilon \nabla u^0(x^0)$ and subtracting it from (32) results in the following residual estimate:

$$\begin{aligned} &\int_{\Omega\setminus\Gamma} \chi^\delta_{\omega_\varepsilon(x^0)} |\nabla Q|^2 \, dx \le (1-\delta) \int_{\partial\omega_\varepsilon(x^0)} \left( \frac{\partial u^0}{\partial n} - \nabla u^0(x^0)^\top n \right) Q \, dS_x \\ &- \varepsilon \int_{\Gamma_N} \frac{\partial}{\partial n} \big(\nabla u^0(x^0)^\top w^\varepsilon\big) Q \, dS_x + \varepsilon \int_\Gamma \frac{\partial}{\partial n} \big(\nabla u^0(x^0)^\top w^\varepsilon\big) [\![Q]\!] \, dS_x \\ &+ \varepsilon \int_{\mathrm{supp}(1-\eta_{\Gamma_D})} \chi^\delta_{\omega_\varepsilon(x^0)} \nabla\big(\nabla u^0(x^0)^\top w^\varepsilon (1-\eta_{\Gamma_D})\big)^\top \nabla Q \, dx. \end{aligned}$$

Here we apply the expansion (21) at $\partial\omega_\varepsilon(x^0)$ which implies that $\|\nabla Q\|_{L^2(\Omega\setminus\Gamma)} = \mathrm{O}(\varepsilon^2)$ and hence the first estimate in (31). The pointwise estimate $w^\varepsilon = \mathrm{O}(\varepsilon)$ holds far away from $\omega_\varepsilon(x^0)$ due to (24). The proof is complete. ∎

In the following sections we apply Theorem 1 for the topology sensitivity of the objective functions which depend on both the crack $\Gamma$ and the defect $\omega_\varepsilon(x^0)$.

**4.2. Topology sensitivity of the SIF-function.** We start with the notation of the *stress intensity factor* (SIF). At the crack tip 0, where the stress is concentrated, from (15a) and (15c) we infer the inner asymptotic expansion (compare to (20)) for $x \in B_R(0) \setminus \Gamma$ with $R = \min\{r, \mathrm{dist}(0, \partial\Omega)\}$ as follows:

$$\begin{aligned} &u^{(\omega,\varepsilon,x^0,\delta)}(x) = u^{(\omega,\varepsilon,x^0,\delta)}(0) + \frac{1}{\mu}\sqrt{\frac{2}{\pi}} c_\Gamma^{(\omega,\varepsilon,x^0,\delta)} \sqrt{\rho} \sin\frac{\theta}{2} + U(x), \\ &\int_{-\pi}^{\pi} U\, d\theta = \int_{-\pi}^{\pi} U \Big(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\Big)^\top d\theta = 0, \qquad U = \mathrm{O}(\rho),\ \nabla U = \mathrm{O}(1). \end{aligned} \tag{34}$$

In the fracture literature, the factor $c_\Gamma^{(\omega,\varepsilon,x^0,\delta)}$ in (34) is called SIF; here it is due to the mode-III crack in the antiplane setting of the spatial fracture problem. The SIF characterizes the main singularity at the crack tip. Moreover, the inequality conditions (15c) require necessarily that

$$c_\Gamma^{(\omega,\varepsilon,x^0,\delta)} \geq 0. \tag{35}$$

For the justification of (34) and (35) we refer the reader to [17, 18], where the homogeneous nonlinear model with rectilinear crack (18) was considered. Indeed, this asymptotic result is stated by the method of separation of variables locally in the neighborhood $B_R(0) \setminus \Gamma$ away from the inhomogeneity $\omega_\varepsilon(x^0)$. Here, the governing equations (15a) and (15c) for $u^0$ coincide with (18a) and (18c) for the solution $u^{(\omega,\varepsilon,x^0,\delta)}$ of the inhomogeneous problem. Therefore, the inner asymptotic expansions (34) of $u^{(\omega,\varepsilon,x^0,\delta)}$ and (48) of $u^0$ are similar and differ only by constant parameters implying the SIF. For a respective mechanical confirmation, see [27].

Below we sketch a Saint–Venant estimate proving the bound of $\nabla U$ in (34). Since $\Delta U = 0$, $U$ is a harmonic function which is infinitely differentiable in $B_R(0) \setminus \Gamma$, and integrating by parts we derive the following for $t \in (0, R)$:

$$\begin{aligned} &I(t) := \int_{B_t(0)\setminus\Gamma} |\nabla U|^2\, dx = \int_{\partial B_t(0)} \frac{\partial U}{\partial \rho} U\, dS_x - \int_{B_t(0)\cap\Gamma} \frac{\partial U}{\partial n} [\![U]\!]\, dS_x \\ &\leq \int_{-\pi}^{\pi} \frac{\partial U}{\partial \rho} U\, t d\theta \leq \int_{-\pi}^{\pi} \left( \frac{t}{2} \left(\frac{\partial U}{\partial \rho}\right)^2 + \frac{1}{2t} U^2 \right) t d\theta \leq \int_{-\pi}^{\pi} \left( \frac{t}{2} \left(\frac{\partial U}{\partial \rho}\right)^2 + \frac{1}{2t} \left(\frac{\partial U}{\partial \theta}\right)^2 \right) t d\theta \\ &= \frac{t}{2} \int_{\partial B_t(0)} |\nabla U|^2\, dS_x = \frac{t}{2} \frac{d}{dt} I(t). \end{aligned}$$

Consequently, here we have used conditions (15c), justifying that at $B_t(0) \cap \Gamma$,

$$0 = \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} [\![u^{(\omega,\varepsilon,x^0,\delta)}]\!] = \frac{\partial U}{\partial n} \left( \frac{2}{\mu} \sqrt{\frac{2}{\pi}} c_\Gamma^{(\omega,\varepsilon,x^0,\delta)} \sqrt{\rho} + [\![U]\!] \right) \leq \frac{\partial U}{\partial n} [\![U]\!]$$

due to (34) and (35), Young's and Wirtinger's inequalities, and the co-area formula. Integrating this differential inequality results in the estimate $I(t) \leq (\frac{t}{R})^2 I(R)$, which implies $I(t) = \mathrm{O}(t^2)$ and follows $\nabla U = \mathrm{O}(1)$ in (34).

From a mathematical viewpoint, the factor in (34) can be determined in the dual space of $H(\Omega \setminus \Gamma)$ through the so-called weight function, which we introduce next. While the existence of a weight function is well known for the linear crack problem (e.g., in [30, Chapter 6]), here we modify it for the underlying nonlinear problem. In fact, the modified weight function $\zeta$ provides formula (43) representing the SIF.

Let $\eta(\rho)$ be a smooth cut-off function supported in the disk $B_{2R}(0) \subset \Omega$, $\eta \equiv 1$ in $B_R(0)$, and $R > r$, where $r > 0$ always stands for the distance to the defect. With the help of the cut-off function, in $\Omega$ we extend the tangential vector $\tau$ from the crack $\Gamma$ by the vector

$$V(x) := \tau\, \eta(\rho), \quad \tau = (1, 0)^\top, \tag{36}$$

which is used further for the shape sensitivity in (54) following the velocity method commonly adopted in shape optimization [35]. Using the notation of matrices for

$$D(V) := \operatorname{div}(V)\, \mathrm{Id} - \frac{\partial V}{\partial x} - \frac{\partial V}{\partial x}^\top \in \mathrm{Sym}(\mathbb{R}^{2\times 2}), \tag{37}$$

where Id means the identity matrix, the coincidence set

$$\Xi := \{x \in \Gamma : \ [\![u^0]\!] = 0\},$$

and the "square-root" function $S(x) := \sqrt{\frac{2}{\pi}}\sqrt{\rho} \sin\frac{\theta}{2}$, we formulate the auxiliary variational problem as follows: Find $\xi \in H(\Omega \setminus \Gamma)$ such that

$$\begin{aligned} &[\![\xi]\!] = [\![V^\top \nabla S]\!] \text{ on } \Xi, \quad \int_{\Omega\setminus\Gamma} \nabla\xi^\top \nabla v\, dx = -\int_{\Omega\setminus\Gamma} \nabla S^\top D(V) \nabla v\, dx \\ &\quad \text{for all } v \in H(\Omega \setminus \Gamma) \text{ with } [\![v]\!] = 0 \text{ on } \Xi, \end{aligned} \tag{38}$$

where $V^\top \nabla S = -\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\rho}} \sin\frac{\theta}{2}\eta$ is the directional derivative of $S$ with respect to $V$, and $[\![V^\top \nabla S]\!] = -\sqrt{\frac{2}{\pi\rho}}\eta$.

*Remark* 1. Due to the inhomogeneous condition stated at $\Xi$ in (38), to provide $[\![\xi]\!] \in H^{1/2}_{00}(\Gamma)$ we assume that the coincidence set $\Xi$ where $[\![u^0]\!] = 0$ is separated from the crack tip, i.e., $0 \notin \overline{\Xi}$. For example, this assumption is guaranteed for the SIF $c^0_\Gamma > 0$ (see the definition of $c^0_\Gamma$ in (48)) when the crack is open in the vicinity. Otherwise, if the crack is closed such that $[\![u^0]\!] \equiv 0$ in a neighborhood $[-C, 0]\times\{0\} \subset \Gamma$ of the crack tip $(0, 0)$, then the crack problem can be restated for the crack tip $(-C, 0)$.

In order to get the strong formulation, we use the following identities in the right-hand side of (38):

$$\begin{aligned} &\operatorname{div}\big(\nabla S^\top D(V)\big) = \operatorname{div}(V)\Delta S - \Delta V^\top \nabla S \\ &- 2\big(\nabla V_1^\top \nabla S_{,1} + \nabla V_2^\top \nabla S_{,2}\big) = -\Delta(V^\top \nabla S), \end{aligned}$$

where we have applied $\Delta S = 0$ in $\Omega \setminus \Gamma$, and

$$\big(\nabla S^\top D(V)\big) n = 0 = -\frac{\partial}{\partial n}(V^\top \nabla S) \quad \text{on } \Gamma^\pm$$

due to $V_2 = 0$, $\frac{\partial V}{\partial n} = 0$, and $\frac{\partial S}{\partial n} = 0$, recalling that $\frac{\partial}{\partial n} = -\frac{1}{\rho}\frac{\partial}{\partial \theta}$ at $\Gamma^\pm$, as $\theta = \pm\pi$. Henceforth, after integration of (38) by parts, the unique solution of (38) satisfies the mixed Dirichlet–Neumann problem as follows:

$$-\Delta\xi = -\Delta\big(V^\top \nabla S\big) \quad \text{in } \Omega\setminus\Gamma, \tag{39a}$$

$$\xi = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial \xi}{\partial n} = 0 \quad \text{on } \Gamma_N, \tag{39b}$$

$$\begin{aligned} &[\![\xi]\!] = [\![V^\top \nabla S]\!], \qquad \left[\!\!\left[\frac{\partial \xi}{\partial n}\right]\!\!\right] = \left[\!\!\left[\frac{\partial}{\partial n}\Big(V^\top \nabla S\Big)\right]\!\!\right] = 0 \quad \text{on } \Xi,\\ &\frac{\partial \xi}{\partial n} = \frac{\partial}{\partial n}\big(V^\top \nabla S\big) = 0 \quad \text{on } (\Gamma\setminus\Xi)^\pm. \end{aligned} \tag{39c}$$

From (38) and (39) we define the *weight function* (here $t > 0$ small)

$$\zeta := \xi - V^\top \nabla S \in L^2(\Omega\setminus\Gamma) \cap H^1\big((\Omega\setminus\Gamma)\setminus B_t(0)\big), \tag{40}$$

which is a nontrivial singular solution of the homogeneous problem

$$-\Delta\zeta = 0 \quad \text{in } \Omega\setminus\Gamma, \tag{41a}$$

$$\zeta = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial \zeta}{\partial n} = 0 \quad \text{on } \Gamma_N, \tag{41b}$$

$$[\![\zeta]\!] = \left[\!\!\left[\frac{\partial \zeta}{\partial n}\right]\!\!\right] = 0 \text{ on } \Xi, \quad \frac{\partial \zeta}{\partial n} = 0 \text{ on } (\Gamma\setminus\Xi)^\pm. \tag{41c}$$

For comparison, for the linear crack problem the coincidence set $\Xi = \emptyset$ and the mixed Dirichlet–Neumann problem (41) turns into the homogeneous Neumann problem for the weight function $\zeta$. From (40) it follows that

$$\zeta(x) = \frac{1}{\sqrt{2\pi}} \tfrac{1}{\sqrt{\rho}} \sin\frac{\theta}{2} + \xi(x) \quad \text{for } x \in B_R(0)\setminus\{0\}, \tag{42}$$

which is useful in the following.

LEMMA 2. *For $0 \notin \overline{\Xi}$ providing solvability of problem* (38)*, the SIF $c_\Gamma^{(\omega,\varepsilon,x^0,\delta)}$ from* (34) *and* (35) *can be determined by the following integral formula:*

$$\begin{aligned} c_\Gamma^{(\omega,\varepsilon,x^0,\delta)} = \max\bigg\{0, &\int_{\Gamma_N} g\zeta\, dS_x - \mu \int_{\partial\omega_\varepsilon(x^0)} \left[\!\!\left[\frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n}\right]\!\!\right] \zeta\, dS_x \\ &+ \mu \int_\Xi \frac{\partial \zeta}{\partial n} [\![u^{(\omega,\varepsilon,x^0,\delta)}]\!]\, dS_x - \mu \int_{\Gamma\setminus\Xi} \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} [\![\zeta]\!]\, dS_x \bigg\}, \end{aligned} \tag{43}$$

*with the weight function $\zeta$ defined in* (38) *and* (40) *together with the properties* (41) *and* (42)*.*

*Proof.* Using the second Green's formula in $(\Omega \setminus \Gamma) \setminus B_t(0)$ with small $t \in (0, t_0)$, from (15) and (39) we derive that

$$0 = \int_{(\Omega\setminus\Gamma)\setminus B_t(0)} \{\Delta\zeta u^{(\omega,\varepsilon,x^0,\delta)} - \zeta\Delta u^{(\omega,\varepsilon,x^0,\delta)}\}\, dx = -\frac{1}{\mu}\int_{\Gamma_N} g\zeta\, dS_x$$
$$+ \int_{\partial\omega_\varepsilon(x^0)} \left[\!\left[ \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} \right]\!\right] \zeta\, dS_x - \int_{\partial B_t(0)} \left\{ \frac{\partial \zeta}{\partial \rho} u^{(\omega,\varepsilon,x^0,\delta)} - \zeta \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial \rho} \right\} dS_x$$
$$- \int_{\Gamma\setminus B_t(0)} \left\{ \frac{\partial \zeta}{\partial n} [\![ u^{(\omega,\varepsilon,x^0,\delta)} ]\!] - [\![\zeta]\!] \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} \right\} dS_x.$$

In the latter integral over $\Gamma \setminus B_t(0)$, the first summand vanishes at $(\Gamma \setminus \overline{\Xi}) \setminus B_t(0)$, and the second summand is zero at $\Xi \setminus B_t(0)$ due to (41c).

For fixed $\varepsilon$ and $t \searrow +0$, since the coincidence set is detached from the crack tip, there exists $C > 0$ such that $B_C(0) \cap \Xi = \emptyset$, and then the integral over $\Xi \setminus B_t(0)$ is uniformly bounded as follows:

$$\int_{\Xi\setminus B_t(0)} \frac{\partial \zeta}{\partial n} [\![ u^{(\omega,\varepsilon,x^0,\delta)} ]\!]\, dS_x = \int_{\Xi\setminus B_C(0)} \frac{\partial \zeta}{\partial n} [\![ u^{(\omega,\varepsilon,x^0,\delta)} ]\!]\, dS_x = \mathrm{O}(1) \quad \text{as } t \searrow +0.$$

This integral is well-defined because the solution $\zeta$ of the mixed Dirichlet–Neumann problem (38) exhibits the square-root singularity (see, e.g., [30] and references therein), and hence $\frac{\partial\zeta}{\partial n}$ has the one-over-square-root singularity which is integrable, and $[\![ u^{(\omega,\varepsilon,x^0,\delta)} ]\!]$ is $H^{3/2}$-smooth in $\Xi \setminus B_C(0)$. The $H^2$-regularity of the solution to the nonlinear crack problem up to the crack faces, except the crack vicinity, is proved rigorously, e.g., in [7] with the shift technique. Similarly, the integral over $(\Gamma \setminus \overline{\Xi}) \setminus B_t(0)$ is uniformly bounded:

$$\int_{(\Gamma\setminus\Xi)\setminus B_t(0)} \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} [\![\zeta]\!]\, dS_x = \int_{(\Gamma\setminus\Xi)\setminus B_{t_0}(0)} \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} [\![\zeta]\!]\, dS_x$$
$$+ \int_t^{t_0} \frac{1}{\rho} \frac{\partial U}{\partial \theta} \left( \sqrt{\frac{2}{\pi\rho}} + [\![\xi]\!] \right) d\rho = \mathrm{O}(1) \quad \text{as } t \searrow +0$$

due to the representations (34) and (42), and $\frac{1}{\rho}\frac{\partial U}{\partial \theta} = \mathrm{O}(1)$ according to (34).

The former integral over $\partial B_t(0)$ can be calculated by plugging the representations (34) and (42) here,

$$
\begin{aligned}
&-\int_{\partial B_t(0)} \left\{ \frac{\partial \zeta}{\partial \rho} u^{(\omega,\varepsilon,x^0,\delta)} - \zeta \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial \rho} \right\} dS_x = \frac{c_\Gamma^{(\omega,\varepsilon,x^0,\delta)}}{\mu\pi} \int_{-\pi}^{\pi} \sin^2\left(\frac{\theta}{2}\right) d\theta \\
&- t\int_{-\pi}^{\pi} \left\{ \frac{\partial \xi}{\partial \rho} \left( u^{(\omega,\varepsilon,x^0,\delta)}(0) + \frac{c_\Gamma^{(\omega,\varepsilon,x^0,\delta)}}{\mu} \sqrt{\frac{2t}{\pi}} \sin\frac{\theta}{2} + U \right) \right. \\
&\left. - \xi \left( \frac{c_\Gamma^{(\omega,\varepsilon,x^0,\delta)}}{\mu\sqrt{2\pi t}} \sin\frac{\theta}{2} + \frac{\partial U}{\partial \rho} \right) \right\} d\theta = \frac{1}{\mu} c_\Gamma^{(\omega,\varepsilon,x^0,\delta)} + \mathrm{O}(\sqrt{t}),
\end{aligned}
$$

which holds true due to $\xi = \mathrm{O}(1)$ and $\frac{\partial \xi}{\partial \rho} = \mathrm{O}(\frac{1}{\sqrt{t}})$ (similarly to (34)), using $dS_x = t d\theta$ and (34) for $U(\rho, \theta)$ as $\rho = t$ and $\theta \in (-\pi, \pi)$. Therefore, passing $t \searrow +0$ and accounting for (35), we have proven formula (43). ∎

Next, using Theorem 1 we expand the right-hand side of (43) in $\varepsilon \searrow +0$ and derive the main result of this section.

THEOREM 2. *For* $0 \notin \Xi$*, the SIF* $c_\Gamma^{(\omega,\varepsilon,x^0,\delta)}$ *of the heterogeneous problem* (14) *given in* (43) *admits the following asymptotic representation:*

$$
\begin{aligned}
&c_\Gamma^{(\omega,\varepsilon,x^0,\delta)} = \max\left\{ 0, \int_{\Gamma_N} g\zeta \, dS_x - \varepsilon^2 \mu \nabla u^0(x^0)^\top A_{(\omega,\delta)} \nabla \zeta(x^0) \right. \\
&\left. + \mu \int_\Xi \frac{\partial \zeta}{\partial n} [\![ u^{(\omega,\varepsilon,x^0,\delta)} ]\!] \, dS_x - \mu \int_{\Gamma \setminus \Xi} \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} [\![ \zeta ]\!] \, dS_x + \mathrm{Res} \right\}, \\
&\mathrm{Res} = \mathrm{O}(\varepsilon^3),
\end{aligned}
\tag{44}
$$

*where* $A_{(\omega,\delta)}$ *is the dipole matrix and* $\nabla\zeta(x^0) = \frac{1}{2\sqrt{2\pi}} r^{-3/2} (-\sin\frac{3\phi}{2}, \cos\frac{3\phi}{2})^\top + \mathrm{O}(r^{-1/2})$ *at the defect center* $x^0 = r(\cos\phi, \sin\phi)^\top$.

*Proof.* To expand the integral over $\partial\omega_\varepsilon(x^0)$ in the right-hand side of (43) as $\varepsilon \searrow +0$, here we substitute the expansion (30) of the solution $u^{(\omega,\varepsilon,x^0,\delta)}$ which implies

$$
\int_{\partial\omega_\varepsilon(x^0)} \left[\!\!\left[ \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} \right]\!\!\right] \zeta \, dS_x = \varepsilon \nabla u^0(x^0)^\top \int_{\partial\omega_\varepsilon(x^0)} \left[\!\!\left[ \frac{\partial w^\varepsilon}{\partial n} \right]\!\!\right] \zeta \, dS_x + \mathrm{O}(\varepsilon^3).
\tag{45}
$$

Below, we apply to the right-hand side of (45) the expansion (24) of the boundary layer $w^\varepsilon$ and the inner asymptotic expansion of $\zeta$, which is a $C^\infty$-function in the near-field of $x^0$, written similarly to (20) as

$$
\begin{aligned}
&\zeta(x) = \zeta(x^0) + \nabla\zeta(x^0)^\top (x - x^0) + Z(x) \quad \text{for } x \in B_R(x^0), \\
&\int_{-\pi}^{\pi} Z \, d\theta_0 = \int_{-\pi}^{\pi} Z \frac{x - x^0}{\rho_0} \, d\theta_0 = 0, \qquad Z = \mathrm{O}(\rho_0^2), \ \nabla Z = \mathrm{O}(\rho_0).
\end{aligned}
\tag{46}
$$

Next, inserting (24) and (46) into the second Green's formula in $B_\varepsilon(x^0)$, we get

$$
\int_{\partial\omega_\varepsilon(x^0)} \left\{ \left[\!\!\left[ \frac{\partial w^\varepsilon}{\partial n} \right]\!\!\right] \zeta + (1-\delta) w^\varepsilon \frac{\partial \zeta}{\partial n} \right\} dS_x = \int_{\partial B_\varepsilon(x^0)} \left\{ \frac{\partial \zeta}{\partial \rho_0} w^\varepsilon - \zeta \frac{\partial w^\varepsilon}{\partial \rho_0} \right\} dS_x
$$

and estimate its terms as follows. The divergence theorem provides

$$\int_{\partial\omega_\varepsilon(x^0)} w^\varepsilon \frac{\partial\zeta}{\partial n}\, dS_x = \int_{\partial\omega_\varepsilon(x^0)} n^\top \nabla\zeta(x^0)\, dS_x + \mathrm{O}(\varepsilon^2)$$
$$= \int_{\omega_\varepsilon(x^0)} (\nabla w^\varepsilon)^\top \nabla\zeta(x^0)\, dx + \mathrm{O}(\varepsilon^2) = \mathrm{O}(\varepsilon^2),$$

and we calculate analytically the integral over $\partial B_\varepsilon(x^0)$ as

$$\int_{\partial B_\varepsilon(x^0)} \left\{ \frac{\partial\zeta}{\partial\rho_0} w^\varepsilon - \zeta \frac{\partial w^\varepsilon}{\partial\rho_0} \right\} dS_x = \frac{\varepsilon}{\pi} \int_{-\pi}^{\pi} \nabla\zeta(x^0)^\top \frac{x-x^0}{\rho_0} A_{(\omega,\delta)} \frac{x-x^0}{\rho_0}\, d\theta_0 + \mathrm{O}(\varepsilon^2)$$
$$= \varepsilon A_{(\omega,\delta)} \nabla\zeta(x^0) + \mathrm{O}(\varepsilon^2).$$

Therefore, we obtain the asymptotic expansion

$$\int_{\partial\omega_\varepsilon(x^0)} \left[\!\!\left[ \frac{\partial w^\varepsilon}{\partial n} \right]\!\!\right] \zeta\, dS_x = \varepsilon A_{(\omega,\delta)} \nabla\zeta(x^0) + \mathrm{O}(\varepsilon^2). \tag{47}$$

Inserting (45) and (47) into (43) and using (35) yields (44). Finally, the value of $\nabla\zeta(x^0)$ can be estimated analytically from (42), while $\xi$ has the $\mathrm{O}(\rho^{1/2})$-singularity similar to (34), and hence $\nabla\xi(x^0) = \mathrm{O}(r^{-1/2})$. This completes the proof. □

As the corollary of Lemma 2 and Theorem 2, we find the SIF of the solution $u^0 \in K(\Omega \setminus \Gamma)$ of the homogeneous problem (17), which is the limit case of the heterogeneous problem as $\varepsilon \searrow +0$. Namely, similar to (34) and (35) we have the inner asymptotic expansion

$$\begin{aligned} &u^0(x) = u^0(0) + \frac{1}{\mu}\sqrt{\frac{2}{\pi}} c_\Gamma^0 \sqrt{\rho} \sin\frac{\theta}{2} + U^0(x) \quad \text{for } x \in B_R(0) \setminus \Gamma, \\ &\int_{-\pi}^{\pi} U^0\, d\theta = \int_{-\pi}^{\pi} U^0\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) d\theta = 0, \qquad U^0 = \mathrm{O}(\rho),\ \nabla U^0 = \mathrm{O}(1), \end{aligned} \tag{48}$$

with the reference SIF $c_\Gamma^0 \geq 0$ determined by the formula

$$c_\Gamma^0 = \max\left\{ 0, \int_{\Gamma_N} g\zeta\, dS_x \right\}, \tag{49}$$

where we have used the complementarity conditions $\frac{\partial\zeta}{\partial n}[\![u^0]\!] = \frac{\partial u^0}{\partial n}[\![\zeta]\!] = 0$ at $\Gamma$ due to (41c) and (18c) which provides $\frac{\partial u^0}{\partial n} = 0$ at $\Gamma \setminus \overline{\Xi}$.

In the following we derive an interpretation of Theorem 2 from the point of view of shape-topological control.

We parametrize the crack growth by means of the position of the crack tip along the fixed path $x_2 = 0$ as

$$\Gamma_\infty(t) := \{x \in \mathbb{R}^2 : x_1 < t, x_2 = 0\}, \quad \Gamma(t) := \Gamma_\infty(t) \cap \Omega,$$

such that $\Gamma = \Gamma(0)$ in this notation. Formula (43) defines the optimal value function depending on both $\Gamma(t)$ and $\omega_\varepsilon(x^0)$,

$$\begin{aligned} &J_{\mathrm{SIF}} : \mathbb{R} \times \Theta \times \mathbb{R}_+ \times (\Omega \setminus \Gamma) \times \overline{\mathbb{R}}_+ \mapsto \mathbb{R}_+, \\ &(t, \omega, \varepsilon, x^0, \delta) \mapsto J_{\mathrm{SIF}}(\Gamma(t), \omega_\varepsilon(x^0)) := c_{\Gamma(t)}^{(\omega,\varepsilon,x^0,\delta)}, \end{aligned} \tag{50}$$

which satisfies the consistency condition $\omega_\varepsilon(x^0) \subset B_\varepsilon(x^0) \subset \Omega \setminus \Gamma(t)$. From the physical point of view, the purpose of (50) is to control the SIF of the crack $\Gamma(t)$ by means of the defect $\omega_\varepsilon(x^0)$. The homogeneous reference state implies

$$J_{\mathrm{SIF}}(\Gamma(t), \emptyset) = c^0_{\Gamma(t)}. \tag{51}$$

For fixed $\Gamma(0) = \Gamma$, formula (44) proves the topology sensitivity of $J_{\mathrm{SIF}}$ from (50) and (51) with respect to diminishing the defect $\omega_\varepsilon(x^0)$ as $\varepsilon \searrow +0$.

In the following section we introduce another geometry-dependent objective function inherently related to fracture, namely, the strain energy release rate (SERR). We show its first order topology sensitivity analysis using the result of Theorem 2. The first order asymptotic term provides us with the respective *topological derivative*; see [13] for a generalized concept of topological derivatives suitable for fracture due to cracks.

**4.3. Topological derivative of the SERR-function.** The widely used Griffith criterion of fracture declares that a crack starts to grow when its SERR attains a critical value (the material parameter of fracture resistance). Therefore, decreasing the SERR would arrest the incipient crack growth, while, conversely, increasing the SERR will affect its rise. This gives us practical motivation for the topological derivative of the SERR objective function, which we construct below.

After substitution of the solution $u^{(\omega,\varepsilon,x^0,\delta)}$ of the heterogeneous problem (14), the reduced energy functional (13) implies

$$\begin{aligned} &\Pi(\Gamma(t), \omega_\varepsilon(x^0)) \\ &= \frac{1}{2}\int_{\Omega\setminus\Gamma(t)} \mu\chi^\delta_{\omega_\varepsilon(x^0)} |\nabla u^{(\omega,\varepsilon,x^0,\delta)}|^2\,dx - \int_{\Gamma_N} g u^{(\omega,\varepsilon,x^0,\delta)}\,dS_x. \end{aligned} \tag{52}$$

The derivative of $\Pi$ in (52) with respect to $t$, taken with the minus sign, is called the SERR and defines the optimal value function similar to (50) as

$$\begin{aligned} &J_{\mathrm{SERR}} : \mathbb{R} \times \Theta \times \mathbb{R}_+ \times (\Omega \setminus \Gamma) \times \overline{\mathbb{R}}_+ \mapsto \mathbb{R}_+, \\ &(t, \omega, \varepsilon, x^0, \delta) \mapsto J_{\mathrm{SERR}}(\Gamma(t), \omega_\varepsilon(x^0)) := -\frac{d}{dt}\Pi(\Gamma(t), \omega_\varepsilon(x^0)). \end{aligned} \tag{53}$$

It admits the equivalent representations (see [13, 16, 21, 22, 24] for details)

$$\begin{aligned} J_{\mathrm{SERR}} &= -\frac{1}{2}\int_{\Omega\setminus\Gamma(t)} \mu\chi^\delta_{\omega_\varepsilon(x^0)} (\nabla u^{(\omega,\varepsilon,x^0,\delta)})^\top D(V) \nabla u^{(\omega,\varepsilon,x^0,\delta)}\,dx \\ &= \lim_{R\searrow+0} I_R, \quad \text{where} \quad I_R := \mu \int_{\partial B_R((t,0))} \biggl\{ \frac{1}{2}\Bigl(V^\top \frac{x}{\rho}\Bigr) |\nabla u^{(\omega,\varepsilon,x^0,\delta)})|^2 \\ &- (V^\top \nabla u^{(\omega,\varepsilon,x^0,\delta)}) \Bigl(\frac{x^\top}{\rho} \nabla u^{(\omega,\varepsilon,x^0,\delta)}\Bigr) \biggr\}\,dS_x. \end{aligned} \tag{54}$$

The key issue is that from (54) we derive the following expression:

$$J_{\mathrm{SERR}}(\Gamma(t), \omega_\varepsilon(x^0)) = \frac{1}{2\mu}\bigl(c^{(\omega,\varepsilon,x^0,\delta)}_{\Gamma(t)}\bigr)^2 \geq 0. \tag{55}$$

Indeed, from the local asymptotic expansion (34) written at the crack tip $(t, 0)$, it follows that

$$\nabla u^{(\omega,\varepsilon,x^0,\delta)} = \frac{1}{\mu\sqrt{2\pi R}}\, c^{(\omega,\varepsilon,x^0,\delta)}_{\Gamma(t)} \Bigl(-\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\Bigr)^\top + \nabla U \quad \text{on } \partial B_R((t,0)).$$

Plugging this expression into the invariant integral $I_R$ in (54), due to $|\nabla U| = \mathrm{O}(1)$, $V = (1,0)^\top$, and $\frac{x^\top}{\rho} = (\cos\theta, \sin\theta)^\top$ at $\partial B_R((t,0))$, we calculate

$$I_R = \mu \int_{-\pi}^{\pi} \left\{ \frac{1}{2}\cos\theta \frac{1}{2\pi R\mu^2} (c_{\Gamma(t)}^{(\omega,\varepsilon,x^0,\delta)})^2 + \sin^2\left(\frac{\theta}{2}\right) \frac{1}{2\pi R\mu^2} (c_{\Gamma(t)}^{(\omega,\varepsilon,x^0,\delta)})^2 + \mathrm{O}\left(\frac{|\nabla U|}{\sqrt{R}}\right) \right\} R d\theta = \frac{1}{2\mu}\left(c_{\Gamma(t)}^{(\omega,\varepsilon,x^0,\delta)}\right)^2 + \mathrm{O}(\sqrt{R}).$$

Passing $R \searrow +0$ follows (55). Now, the substitution of expansion (44) in (55) proves directly the asymptotic model of SERR as $\varepsilon \searrow +0$ given next.

THEOREM 3. *For $0 \notin \overline{\Xi}$, the SERR at the tip of the crack $\Gamma = \Gamma(0)$ admits the following asymptotic representation when diminishing the defect $\omega_\varepsilon(x^0)$:*

(56)
$$J_{\mathrm{SERR}}(\Gamma, \omega_\varepsilon(x^0)) = \frac{1}{2\mu}\left(c_\Gamma^0\right)^2 - \varepsilon^2 c_\Gamma^0 \nabla u^0(x^0)^\top A_{(\omega,\delta)} \nabla\zeta(x^0) + c_\Gamma^0 \int_{\Xi\setminus\Xi^\varepsilon} \frac{\partial\zeta}{\partial n} [\![ u^{(\omega,\varepsilon,x^0,\delta)} - u^0 ]\!]\, dS_x - c_\Gamma^0 \int_{\Xi^\varepsilon\setminus\Xi} \frac{\partial(u^{(\omega,\varepsilon,x^0,\delta)} - u^0)}{\partial n} [\![\zeta]\!]\, dS_x + \mathrm{Res},$$
$$\mathrm{Res} = \mathrm{O}(\varepsilon^3) \quad \textit{and} \quad \mathrm{Res} \geq 0 \ \textit{if}\ c_\Gamma^0 = 0,$$

*where the perturbed coincidence set is determined by*

$$\Xi^\varepsilon := \{x \in \Gamma :\ [\![ u^{(\omega,\varepsilon,x^0,\delta)} ]\!] = 0\}.$$

*The reference $J_{\mathrm{SERR}}(\Gamma, \emptyset) = \frac{1}{2\mu}(c_\Gamma^0)^2$ implies SERR for the homogeneous state $u^0$ without defect, $A_{(\omega,\delta)}$ is the dipole matrix, and the gradient*

$$\nabla\zeta(x^0) = \frac{1}{2\sqrt{2\pi}} r^{-3/2} \left(-\sin\frac{3\phi}{2}, \cos\frac{3\phi}{2}\right)^\top + \mathrm{O}(r^{-1/2})$$

*at the defect center $x^0 = r(\cos\phi, \sin\phi)^\top$.*

*Moreover, if the coincidence sets are continuous such that* $\mathrm{meas}_1(\Xi^\varepsilon \setminus \Xi) \searrow +0$ *and* $\mathrm{meas}_1(\Xi \setminus \Xi^\varepsilon) \searrow +0$ *as $\varepsilon \searrow +0$, then the first asymptotic term in* (56) *provides the topological derivative*

$$\lim_{\varepsilon\searrow+0} \frac{J_{\mathrm{SERR}}(\Gamma,\omega_\varepsilon(x^0)) - J_{\mathrm{SERR}}(\Gamma,\emptyset)}{\varepsilon^2} = -c_\Gamma^0 \nabla u^0(x^0)^\top A_{(\omega,\delta)} \nabla\zeta(x^0). \tag{57}$$

*Proof.* To derive (56) we square (44) and (49). Then we use, first, that

$$\int_\Xi \frac{\partial\zeta}{\partial n} [\![ u^{(\omega,\varepsilon,x^0,\delta)} ]\!]\, dS_x = \int_{\Xi\setminus\Xi^\varepsilon} \frac{\partial\zeta}{\partial n} [\![ u^{(\omega,\varepsilon,x^0,\delta)} ]\!]\, dS_x = \int_{\Xi\setminus\Xi^\varepsilon} \frac{\partial\zeta}{\partial n} [\![ u^{(\omega,\varepsilon,x^0,\delta)} - u^0 ]\!]\, dS_x \tag{58}$$

holds due to $[\![ u^{(\omega,\varepsilon,x^0,\delta)} ]\!] = 0$ at $\Xi^\varepsilon$ and $[\![ u^0 ]\!] = 0$ at $\Xi$. Second, the equality

$$\int_{\Gamma\setminus\Xi} \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} [\![\zeta]\!]\, dS_x = \int_{\Xi^\varepsilon\setminus\Xi} \frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} [\![\zeta]\!]\, dS_x = \int_{\Xi^\varepsilon\setminus\Xi} \frac{\partial(u^{(\omega,\varepsilon,x^0,\delta)} - u^0)}{\partial n} [\![\zeta]\!]\, dS_x \tag{59}$$

holds due to $\frac{\partial u^{(\omega,\varepsilon,x^0,\delta)}}{\partial n} = 0$ at $\Gamma \setminus \overline{\Xi^\varepsilon}$ and $\frac{\partial u^0}{\partial n} = 0$ at $\Gamma \setminus \overline{\Xi}$ according to the complementarity conditions (15c) and (18c) and using the identity $(\Gamma \setminus \overline{\Xi}) \setminus (\Gamma \setminus \overline{\Xi^\varepsilon}) = \overline{\Xi^\varepsilon} \setminus \overline{\Xi}$.

To justify (57), it needs to pass (58) and (59) divided by $\varepsilon^2$ to the limit as $\varepsilon \searrow +0$. For this task we employ the assumption that $0 \notin \overline{\Xi}$ and the assumption of continuity of the coincidence sets; hence $0 \notin \overline{\Xi^\varepsilon}$ for sufficiently small $\varepsilon$. Otherwise, $0 \in \overline{\Xi^\varepsilon}$ implies $c_\Gamma^{(\omega,\varepsilon,x^0,\delta)} = 0$ which contradicts the convergence $c_\Gamma^{(\omega,\varepsilon,x^0,\delta)} \to c_\Gamma^0 \neq 0$ as $\varepsilon \searrow +0$ following from (34) and (48) due to Theorem 1. This implies that the sets $\Xi \setminus \overline{\Xi^\varepsilon}$ as well as $\Xi^\varepsilon \setminus \overline{\Xi}$ are detached from the crack tip. Henceforth, the functions $[\![u^{(\omega,\varepsilon,x^0,\delta)} - u^0]\!] \in H^{3/2}(\Xi \setminus \Xi^\varepsilon)$ and $\frac{\partial (u^{(\omega,\varepsilon,x^0,\delta)} - u^0)}{\partial n}, [\![\zeta]\!] \in L^2(\Xi^\varepsilon \setminus \Xi)$ are smooth here, and the following asymptotic estimates hold:

$$\begin{aligned}
&\int_{\Xi\setminus\Xi^\varepsilon} \frac{\partial \zeta}{\partial n} [\![u^{(\omega,\varepsilon,x^0,\delta)} - u^0]\!]\, dS_x = \int_{\Xi\setminus\Xi^\varepsilon} \left( \frac{\partial(\zeta - \zeta^\varepsilon)}{\partial n} + \frac{\partial \zeta^\varepsilon}{\partial n} \right) [\![u^{(\omega,\varepsilon,x^0,\delta)} - u^0]\!]\, dS_x \\
&\le \| \frac{\partial(\zeta - \zeta^\varepsilon)}{\partial n} \|_{H^{1/2}(\Gamma)^\star} \big\| [\![u^{(\omega,\varepsilon,x^0,\delta)} - u^0]\!] \big\|_{H^{1/2}(\Xi\setminus\Xi^\varepsilon)} \\
&+ \bigg\| \frac{\partial \zeta^\varepsilon}{\partial n} \|_{L^2(\Xi\setminus\Xi^\varepsilon)} \bigg\| [\![u^{(\omega,\varepsilon,x^0,\delta)} - u^0]\!] \big\|_{L^2(\Xi\setminus\Xi^\varepsilon)} = \mathrm{o}(\varepsilon^2),
\end{aligned}$$

where $\zeta^\varepsilon$ is a smooth approximation of $\zeta$ such that $\|\frac{\partial(\zeta - \zeta^\varepsilon)}{\partial n}\|_{H^{1/2}(\Gamma)^\star} = \mathrm{o}(1)$ and $\|\frac{\partial \zeta^\varepsilon}{\partial n}\|_{L^2(\Xi\setminus\Xi^\varepsilon)} = \mathrm{o}(1)$ as $\varepsilon \searrow +0$, and

$$\int_{\Xi^\varepsilon\setminus\Xi} \frac{\partial (u^{(\omega,\varepsilon,x^0,\delta)} - u^0)}{\partial n} [\![\zeta]\!]\, dS_x \le \left\| \frac{\partial (u^{(\omega,\varepsilon,x^0,\delta)} - u^0)}{\partial n} \right\|_{L^2(\Xi^\varepsilon\setminus\Xi)} \| [\![\zeta]\!] \|_{L^2(\Xi^\varepsilon\setminus\Xi)} = \mathrm{o}(\varepsilon^2)$$

provided by Theorem 1 and the assumption of the convergence $\mathrm{meas}_1(\Xi^\varepsilon \setminus \Xi) \searrow +0$ and $\mathrm{meas}_1(\Xi \setminus \Xi^\varepsilon) \searrow +0$ as $\varepsilon \searrow +0$. This proves the limit in (57) and the assertion of the theorem. □

**5. Discussion.** In the context of fracture, from Theorem 3 we can discuss the following.

The Griffith fracture criterion suggests that the crack $\Gamma$ starts to grow when $J_{\mathrm{SERR}} = G_c$ attains the fracture resistance threshold $G_c > 0$. For incipient growth of the nonlinear crack subject to inequality $c_\Gamma^0 > 0$, its arrest necessitates the negative topological derivative to decrease $J_{\mathrm{SERR}}$, which needs positive sign of $\nabla u^0(x^0)^\top A_{(\omega,\delta)} \nabla \zeta(x^0)$ in (56).

The sign and value of the topological derivative depend in a semi-analytic implicit way on the solution $u^0$, trial center $x^0$, shape $\omega$, and stiffness $\delta$ of the defect. The latter two parameters enter the topological derivative through the dipole matrix $A_{(\omega,\delta)}$. In Appendix A we present explicit values of the dipole matrix for the specific cases of the ellipse-shaped holes and inclusions. This describes also the degenerate case of cracks and thin rigid inclusions called anticracks.

**Appendix A. Ellipse and crack shaped defects.** Let the shape $\omega$ of a defect be ellipsoidal. Namely, we consider the ellipse $\omega$ enclosed in the ball $B_1(0)$, which has the major $a = 1$ and the minor $b \in (0, 1]$ semi-axes, where the major axis has an angle of $\alpha \in [0, 2\pi)$ with the $x_1$-axis counted in the counterclockwise direction.

With the rotation matrix $Q(\alpha)$, the dipole matrix for the *elliptic defect* has the form (see [8, 33])

$$A_{(\omega,\delta)} = Q(\alpha) A_{(\omega',\delta)} Q(\alpha)^\top, \quad Q(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{60a}$$

$$A_{(\omega',\delta)} = \pi(1+b) \begin{pmatrix} \frac{(1-\delta)b}{1+\delta b} & 0 \\ 0 & \frac{(1-\delta)b}{\delta+b} \end{pmatrix}. \tag{60b}$$

Further, we consider the limit cases of (60b) when the stiffness parameters $\delta \searrow +0$ and $\delta \nearrow +\infty$, which correspond to the ellipse-shaped holes and rigid inclusions according to Corollary 1.

On the one hand, for the *elliptic hole* $\omega$, passing $\delta \searrow +0$ in (60b) we obtain the virtual mass, or added mass matrix,

$$A_{(\omega',\delta)} = \pi(1+b) \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \tag{61}$$

which is positive definite. In particular, for the *straight crack* $\omega$ as $b \searrow +0$, (61) turns into the singular matrix

$$A_{(\omega',\delta)} = \pi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{62}$$

On the other hand, for the *rigid ellipse* $\omega$, passing $\delta \nearrow +\infty$ in (60b) we obtain the polarization matrix

$$A_{(\omega',\delta)} = \pi(1+b) \begin{pmatrix} -1 & 0 \\ 0 & -b \end{pmatrix}, \tag{63}$$

which is negative definite. In particular, for the *rigid segment* $\omega$ as $b \searrow +0$, (63) turns into the singular matrix

$$A_{(\omega',\delta)} = \pi \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{64}$$

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