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Introduction

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Non-smooth variational problems and applications in mechanics

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Mathematical methods based on a variational approach are successfully used in a broad range of applications, especially those fields that are oriented on partial differential equations (PDEs). Our problem area addresses to a wide class of nonlinear problems described by all kinds of static, quasi-static and dynamic equations, inverse and optimal control problems, including shape and topology optimization. Within these directions, we focus but are not limited to multi-scale and non-flat geometries, singular and contact problems arising in solid and fluid mechanics, which are governed by complex systems of variational equations and inequalities.

Whereas classical mathematical tools are not applicable here, we aim at non-standard existence and uniqueness analysis based on the primal and the dual variational formalism, at effective numerical methods for the solution of the underlying problems by use of approximation and asymptotic techniques including homogenization. In a broad scope, the theme issue objectives are directed towards advances that are recently attained in the mathematical theory of non-smooth variational problems. Methods of the non-smooth theory are supported by the physical consistency and reported in computer simulations for novel applications to engineering sciences.

After successful publishing in 2022 in *Philosophical Transactions of the Royal Society A* of the theme issue 2236 on *Non-smooth variational problems and applications* [1], we were facing the continuation since many relevant topics remained not included there. During the International Congress on Industrial and Applied Mathematics[1] held from 20 to 25 August 2023 at Waseda University in Tokyo, Japan, we organized a mini-symposium on *Singular problems in mechanics*,[2] where recent achievements in the field were reported. Therefore, we included six selected contributions from the mini-symposium and expanded beyond, inviting contributors outside of the meeting. This avoids the content overlapping with the previous issue and ensures wider coverage of the topic in the current theme issue *Non-smooth variational problems and applications in mechanics*. Except for the issue editors, all the lead authors are new. Whereas the original topic was directed towards the modelling of non-smooth interaction phenomena, the new collection is focused more on non-smooth geometries and related matters.

In the theme issue, we bring together multiple subject areas from the field of PDEs, constrained optimization theory, shape and topology optimization, inverse and ill-posed problems and numerical methods. The mathematical tasks share a common feature of the lack of smoothness that makes it impossible to apply standard tools for analysis. The problems of our interest are motivated by consistent modelling and computer simulation of many nonlinear physical phenomena stemming from fluid and solid mechanics, porous media, damage and fracture, inverse scattering, dynamic systems with hysteresis, interaction phenomena and others.

Our theme issue collects 15 contributions of lead authors with their co-authors composing together a broad spectrum from 12 countries (accounted alphabetically): Austria, Brazil, China, Czech Republic, France, Germany, Italy, Japan, Russia, Spain, UK and USA. New developments by use of non-standard methods treat a variety of non-smooth variational problems stemming from actual applications in the field of mechanical sciences.

From a mathematical point of view, non-smooth variational methods are developed for constrained and nonlinear optimization. Existence and uniqueness analysis of the underlying problems is based on the primal and dual optimization techniques. The problems account for non-coercive energy functionals [2], injectivity constraints [3], obstacle problems [4] and inclined obstacle geometry [5] and nonlinear differential equations including the second-order gradient term to preserve the condition of frame indifference [6]. The identification of geometry parameters [7] and object detection [8] belongs to the field of inverse and ill-posed problems. For its multi-scale analysis, asymptotic and non-smooth techniques of shape and topology optimization [9], including Clarke sub-differential and second-order methods [10], phase-field description in non-smooth domains [11] and scale parameter averaging and homogenization in layered structures [12,13], are developed. Investigation of dynamic models is presented for PDEs with respect to wave evolution [14], long-time asymptotic behaviour and hysteresis [15] and global existence of solutions for nonlinear equations of Boussinesq type [16]. Methods of the non-smooth theory are supported by numerical modelling of variational inequalities using semi-smooth Newton [5] and penalization [4] methods, by computer simulations presented in two and three spatial dimensions [8,9,11] and parallel computing on the cluster [14].

From the point of view of physics, the variational approach is applied to linear and nonlinear models of elastic and inelastic solids, inclusions and body junctions [2], Kirchhoff [10] and Timoshenko plates [5] and flexural shells [4], viscoelastic materials described by quasi-linear heredity integrals [7], as well as with the constitutive response of rate type [6]. Our contributions present multi-phase phenomena of fluid diffusion in porous media [15], nonlinear models of heat transfer with boundary heat exchange and temperature-dependent coefficients [16], velocity-dependent fracture in brittle materials [11], as well as the propagation of fracture waves in a porous blocky medium with thin compliant interlayers [14]. Variational principles are suggested to minimize the maximal von Mises stress in elastic bodies [9], to model cracks

[1]ICIAM 2023 Tokyo https://iciam2023.org/

[2]Mini-symposium [00085] https://iciam2023.org/accepted_ms

in hyperelastic materials [3] and thermoelastic composites stitched with reinforcing filaments [12]. For the physical consistency, nonlinear boundary conditions of contact [4,5] and non-penetration [2,3], also accounting for adhesive and micropolar elasticity structure [13], are imposed. The results have an impact on practical methodologies. For example, the detection of damage by non-destructive tests using electromagnetic scattering experiments when testing steel welding joints with noisy limited-view data is hinted in [8].

We are happy to share with the readers many novel research topics within a variety of subject areas in the theme issue following thereafter.

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