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Author for correspondence:
H. Itou
e-mail: h-itou@rs.tus.ac.jp

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Forward and inverse problems for creep models in viscoelasticity

H. Itou[1], V. A. Kovtunenko[2,3] and G. Nakamura[4,5]

[1]Department of Mathematics, Tokyo University of Science, Tokyo 162-8601, Japan
[2]Department of Mathematics and Scientific Computing, University of Graz, NAWI Graz, Heinrichstraße 36, Graz 8010, Austria
[3]Lavrentyev Institute of Hydrodynamics, Siberian Division of the Russian Academy of Sciences, Novosibirsk 630090, Russia
[4]Department of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan
[5]Research Center of Mathematics for Social Creativity, Research Institute for Electronic Science, Hokkaido University, Sapporo 060-0812, Japan

HI, 0000-0002-6224-8031; VAK, 0000-0001-5664-2625

This study examines a class of time-dependent constitutive equations used to describe viscoelastic materials under creep in solid mechanics. In nonlinear elasticity, the strain response to the applied stress is expressed via an implicit graph allowing multi-valued functions. For coercive and maximal monotone graphs, the existence of a solution to the quasi-static viscoelastic problem is proven by applying the Browder–Minty fixed point theorem. Moreover, for quasi-linear viscoelastic problems, the solution is constructed as a semi-analytic formula. The inverse viscoelastic problem is represented by identification of a design variable from non-smooth measurements. A non-empty set of optimal variables is obtained based on the compactness argument by applying Tikhonov regularization in the space of bounded measures and deformations. Furthermore, an illustrative example is given for the inverse problem of isotropic kernel identification.

This article is part of the theme issue 'Non-smooth variational problems with applications in mechanics'.

1. Introduction

This study deals with the response of a material, which can be represented with respect to the linearized strain $\boldsymbol{\varepsilon}$ and Cauchy stress $\boldsymbol{\sigma}$ in the form of a function composition,

$$\boldsymbol{\varepsilon}(t) = [\mathscr{I}(t) \circ \mathscr{F}]\boldsymbol{\sigma}, \quad t \in [0, T], \tag{1.1}$$

for the fixed final time $T > 0$, where the time-dependent function $\mathscr{I}$ represents the creep owing to the viscous phenomena, and the function $\mathscr{F}$ stands for the material response. In general, both functions are assumed to be nonlinear.

In a nonlinear elastic model, the material response $\mathscr{F}$ in (1.1) is expressed as

$$[\mathscr{F}]\boldsymbol{\sigma} = \mathbf{E}_1(p)p + \mathbf{E}_2(\|\boldsymbol{\sigma}^*\|)\boldsymbol{\sigma}^*, \tag{1.2}$$

determined by parameter-dependent tensors $\mathbf{E}_1$ and $\mathbf{E}_2$, where $\|\cdot\|$ denotes the Frobenius norm, the mechanical pressure p is defined according to the volumetric–deviatoric decomposition,

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}^*, \quad p = -\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}, \tag{1.3}$$

and $\mathbf{I}$ is the second-rank identity tensor. The constitutive relation (1.2) describes the non-constant elastic moduli, as in the study by Itou *et al.* [1–4]. It is suitable for porous and other materials such as bone, ceramics, concrete, inter-metallic alloys and rocks loaded below their yield strength. The case of constant moduli $\mathbf{E}_1 = -\mathbf{S}$ and $\mathbf{E}_2 = \mathbf{S}$ with the compliance tensor $\mathbf{S}$ reduces (1.2) to the linear Hooke's law,

$$[\mathscr{F}]\boldsymbol{\sigma} = \mathbf{S}\boldsymbol{\sigma}. \tag{1.4}$$

Other nonlinear representatives of the function $\mathscr{F}$ for the strain-limiting model were proposed by Gou & Mallikarjunaiah [5]. Itou *et al.* [6, 7] investigated the well-posedness of the constitutive model (1.1) with respect to the properties of $\mathscr{F}$ under a specific $\mathscr{I}$ described in equation (1.13). However, in this study, we investigate both the admissible functions $\mathscr{I}$ and $\mathscr{F}$.

The integral constitutive relation to describe the viscoelastic response was proposed by Boltzmann [8] based on the superposition principle. The linear theory yields the tensorial hereditary integral of

$$[\mathscr{I}(t)]\boldsymbol{\epsilon} = \mathbf{J}(t)\boldsymbol{\epsilon}(0) + \int_0^t \mathbf{J}(t-s)\frac{\partial \boldsymbol{\epsilon}(s)}{\partial s}\,ds, \tag{1.5}$$

or

$$[\mathscr{I}(t)]\boldsymbol{\epsilon} = \mathbf{J}(0)\boldsymbol{\epsilon}(t) - \int_0^t \left(\frac{\partial}{\partial s}\mathbf{J}(t-s)\right)\boldsymbol{\epsilon}(s)\,ds, \tag{1.6}$$

with a tensorial creep function $\mathbf{J}$. The fourth-order creep tensor $\mathbf{J} = (J_{ijkl})$ describes the multi-axial creep behaviour of viscoelastic materials. In isotropic materials, $\mathbf{J}$ is independent of direction. Consequently, $\mathbf{J}$ is a spherical tensor with components of the form

$$J_{ijkl} = A\delta_{ij}\delta_{kl} + \frac{B}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.7}$$

with two scalar creep functions A and B. A simpler case when $A \equiv 0$ has been studied in depth. Consequently, in the constitutive relations presented in (1.5), $\mathbf{J}$ was replaced by B; for example, equation (1.6) becomes

$$[\mathscr{I}(t)]\boldsymbol{\epsilon} = B(0)\boldsymbol{\epsilon}(t) - \int_0^t \left(\frac{\partial}{\partial s}B(t-s)\right)\boldsymbol{\epsilon}(s)\,ds. \tag{1.8}$$

According to Wineman [9], the general representative form of the creep function $\mathscr{I}$ in (1.1) can be represented by the following integral operator using the tensor $\boldsymbol{\epsilon} = [\mathscr{F}]\boldsymbol{\sigma}$:

$$[\mathscr{I}(t)]\boldsymbol{\epsilon} = \Phi_1(\boldsymbol{\epsilon}(t)) + \int_0^t \Phi_2(t-s, \boldsymbol{\epsilon}(t), \boldsymbol{\epsilon}(s))\, ds, \quad t \in [0, T], \tag{1.9}$$

which is determined by functions Φ_1 and Φ_2. Ageing viscoelasticity models that were linear with respect to $\boldsymbol{\epsilon}$ in (1.9) were introduced to describe the behaviour of concrete based on constitutive theories as presented by Grasley & Lange [10]:

$$[\mathscr{I}(t)]\boldsymbol{\epsilon} = \mathbf{A}_0\boldsymbol{\epsilon}(t) + \int_0^t \mathbf{A}(t,s)\boldsymbol{\epsilon}(s)\, ds, \tag{1.10}$$

where $\mathbf{A}_0$ and $\mathbf{A}$ are creep tensors. As is evident, equation (1.10) is a more general model than equation (1.6). As a case of scalar creep functions in (1.10), the time-shift approach implies the constitutive relation,

$$[\mathscr{I}(t)]\boldsymbol{\epsilon} = \tilde{B}_0\,\boldsymbol{\epsilon}(t) + \int_0^t \tilde{B}\left(\int_s^t \left(\frac{\tau}{s_0}\right)^{\mu} d\tau\right)\left(\frac{s}{s_0}\right)^{\mu} \boldsymbol{\epsilon}(s)\, ds, \tag{1.11}$$

where s_0 is the reference value, and parameter μ requires fitting. Here $\tilde{B}_0$ is a constant and $\tilde{B}(t,s)$ is a scalar function.

Since the time-fractional derivative facilitates a more sensitive expression of the dependence of viscous response on deformation history (cf. [11]), it has been naturally incorporated for modelling the integral viscoelastic relations. One such derivative is the Caputo fractional derivative ∂_t^{α} with respect to t, and is defined as

$$\partial_t^{\alpha} g(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} g(\tau)\, d\tau, & n-1 < \alpha < n, \quad n \in \mathbb{N}, \\ \dfrac{d^n}{dt^n} g(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

where Γ is the Gamma function, refer to [12,13]. A fractional viscoelastic model described by a spring-pot element instead of spring and dashpot elements corresponds to the case $0 < \alpha < 1$ (cf. [14]). Another example with the power-law creep is the resulting constitutive equation derived by Desch & Grimmer [15] and Mainardi [16], expressed as

$$[\mathscr{I}(t)]\boldsymbol{\epsilon} = \tilde{B}_0\boldsymbol{\epsilon}(t) + c\int_{-\infty}^t \frac{\boldsymbol{\epsilon}(s)}{(t-s)^{1-\nu}}\, ds, \tag{1.12}$$

which can be equivalently expressed using the Liouville–Weyl integral

$${}_{-\infty}J_t^{\nu}\boldsymbol{\epsilon} = \frac{1}{\Gamma(\nu)} \int_{-\infty}^t (t-s)^{\nu-1} \boldsymbol{\epsilon}(s)\, ds,$$

of fractional order $\nu \in (0,1)$. Previously, we have derived the quasi-linear viscoelastic model expressed as the Volterra equation of convolution type as in equation (1.8) in the form:

$$[\mathscr{I}(t)]\boldsymbol{\epsilon} = B_0\boldsymbol{\epsilon}(t) + \sum_{n=1}^{N} \frac{B_n}{\tau_n} \int_0^t e^{-(t-s)/\tau_n} \boldsymbol{\epsilon}(s)\, ds, \quad N \geq 1, \tag{1.13}$$

where $B_0, \ldots, B_N$ and $\tau_1, \ldots, \tau_N$ are the creep parameters. In the particular case when $N = 1$ and $B_0 = 0$, the differentiation of equation (1.13) with respect to t yields the Kelvin–Voigt model. The constitutive relation (1.13) was applied to the Boussinesq indentation problem by Itou *et al.* [17,18] and Kovtunenko [19]. In this study, we consider all representations (1.9)–(1.13).

An overview of nonlinear viscoelastic models of rate type can be found in Şengül [20]; moreover, non-classical memory kernels in linear viscoelasticity have been discussed by Carillo

& Giorgi [21]. We refer to the studies by Bulíček *et al*. [22] and de Hoop *et al*. [23] for the well-posedness of viscoelastic models; when accounting for contact in Han *et al*. [24]; the study by Itou & Tani [25] for cracks; the studies by Khludnev & Kovtunenko [26] and Khludnev & Sokolowski [27] for cracks with contact. Furthermore, the studies by Anderssen *et al* [28] and Gavioli & Krejčí [29] are referred to address the hysteresis issues in viscoelasticity.

Within forward problems described by the constitutive law (1.1), we consider a class of inverse problems stated with respect to the design variable $\mathbf{q}$. This is a model parameter, which can be a scalar, vector or tensor-valued function. For a given measurement $\mathbf{M}(t)$ for $t \in [0, T]$, the inverse problem involves the identification of the model variables that satisfy the observation:

$$[\mathscr{O}](\mathbf{q}, \varepsilon, \boldsymbol{\sigma}) = \mathbf{M}\,. \tag{1.14}$$

Measurements are conducted over the observed part of a geometric domain or its boundary. The typical objects for identification are external forces, elasticity coefficients and memory kernel. The inverse problem of kernel identification in linear viscoelasticity has been well studied and reduced to the solution of the Volterra equation of the first or second kind. Further details can be found in the studies by Grasselli *et al*. [30], Janno & von Wolfersdorf [31], Lorenzi *et al*. [32] and Romanov & Yamamoto [33].

However, practical measurements are noisy; thus, the exact observation (1.14) is substituted by minimizing the corresponding Tikhonov approximation:

$$[\mathscr{Q}](\mathbf{q}, \varepsilon, \boldsymbol{\sigma}) \colon = \| [\mathscr{O}](\mathbf{q}, \varepsilon, \boldsymbol{\sigma}) - \mathbf{M} \| + [\mathscr{R}](\mathbf{q}) \to \min, \tag{1.15}$$

where $\mathscr{R}$ implies a regularization term. The relevant discussion of the Tikhonov regularization function can be found in Hasanov *et al*. [34]. In the optimization context, we refer to Ikehata & Itou [35], and Kovtunenko & Kunisch [36] for the shape reconstruction and identification of the geometry variable $\mathbf{q}$, which may describe cavity, crack, defect and inhomogeneity. In this study, we discuss the conditions on the function $\mathscr{Q}$ in (1.15) that guarantee the existence of a solution for the inverse identification problems suitable for the viscoelastic models represented by equations (1.1)–(1.13).

2. Problem formulation

We start with the description of the geometry. Let the solid body occupy a bounded domain Ω in the Euclidean space $\mathbb{R}^d$, where $d = 1, 2, 3$. Let the boundary $\partial\Omega$ be Lipschitz continuous with the unit normal vector $\mathbf{n} = (n_1, \ldots, n_d)^\top$ pointing outward Ω for $d \geq 2$, where the superscript $^\top$ denotes the transposition. For $d = 1$, we assume that $\partial\Omega$ comprises two points. We consider a disjoint union $\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}}$ of the Dirichlet boundary Γ_{D} and the Neumann boundary Γ_{N}, where the overbar denotes set closure and $\Gamma_{\mathrm{D}} \neq \varnothing$, Γ_{N} are open subsets of $\partial\Omega$. Also, for $d = 3$, the boundaries of Γ_{D}, Γ_{N} are Lipschitz if $\Gamma_{\mathrm{N}} \neq \varnothing$. For points $\mathbf{x} = (x_1, \ldots, x_d)^\top \in \overline{\Omega}$ and $t \in [0, T]$ with final time $T > 0$, the corresponding time-space cylinder is denoted by $\Omega^T = (0, T) \times \Omega$, and the cylindrical surface by $\Gamma_{\mathrm{D}}^T = (0, T) \times \Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{N}}^T = (0, T) \times \Gamma_{\mathrm{N}}$.

Let the body force $\mathbf{f}(t, \mathbf{x}) = (f_1, \ldots, f_d)^\top$, and the boundary force $\mathbf{g}(t, \mathbf{x}) = (g_1, \ldots, g_d)^\top$ be given. The creep $\mathscr{I}(t) \colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \mapsto \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and response $\mathscr{F} \colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \mapsto \mathbb{R}^{d\times d}_{\mathrm{sym}}$ functions are assumed to be described in the space of second-order symmetric tensors $\mathbb{R}^{d\times d}_{\mathrm{sym}}$. We look for the symmetric *stress tensor* $\boldsymbol{\sigma}(t, \mathbf{x}) = (\sigma_{ij})_{i,j=1}^d$ and *displacement vector* $\mathbf{u}(t, \mathbf{x}) = (u_1, \ldots, u_d)^\top$, which determine the *linearized strain tensor* $\boldsymbol{\varepsilon}(t, \mathbf{x}) = (\varepsilon_{ij})_{i,j=1}^d$ as the symmetric part of the displacement gradient:

$$\varepsilon(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + \nabla^{\top}\mathbf{u}\right), \tag{2.1}$$

and satisfy the equilibrium equation without an inertial term:

$$-\nabla\cdot\boldsymbol{\sigma} = \mathbf{f} \quad \text{in } \Omega^T, \tag{2.2}$$

the viscoelastic constitutive equation:

$$\varepsilon(\mathbf{u}) = [\mathscr{I}\circ\mathscr{F}]\boldsymbol{\sigma} \quad \text{in } \Omega^T, \tag{2.3}$$

and the Dirichlet and Neumann boundary conditions:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}^T, \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\mathrm{N}}^T. \tag{2.4}$$

Multiplying equation (2.2) with a smooth test function $\mathbf{v}(\mathbf{x})$ and integrating the result by parts over domain Ω using the Neumann boundary condition presented in equation (2.4), the boundary value problem (2.1)–(2.4) can be described using the variational formulation for fixed $t\in[0,T]$. Let $C([0,T];X)$ denote the space of continuous functions with values in X. For the prescribed $\mathbf{f}\in C([0,T];L^2(\Omega;\mathbb{R}^d))$ and $\mathbf{g}\in C([0,T];L^2(\Gamma_{\mathrm{N}};\mathbb{R}^d))$, we determine the weak solution,

$$\mathbf{u}\in C\left([0,T];H^1_{\Gamma_{\mathrm{D}}}\left(\Omega;\mathbb{R}^d\right)\right), \quad \boldsymbol{\sigma}\in C\left([0,T];L^2\left(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}\right)\right), \tag{2.5}$$

where the Sobolev space $H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)$ includes the Dirichlet boundary condition in equation (2.4):

$$H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d) = \left\{\mathbf{v}(\mathbf{x}) = (v_1,\ldots,v_d)^{\top}\in H^1(\Omega;\mathbb{R}^d)\;\middle|\;\mathbf{v}=\mathbf{0}\text{ on }\Gamma_{\mathrm{D}}\right\},$$

which satisfies the variational equation

$$\int_{\Omega}\boldsymbol{\sigma}\cdot\varepsilon(\mathbf{v})\,d\mathbf{x} = \int_{\Omega}\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} + \int_{\Gamma_{\mathrm{N}}}\mathbf{g}\cdot\mathbf{v}\,dS_{\mathbf{x}}, \tag{2.6}$$

for all test functions $\mathbf{v}\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)$; here, the dot indicates the scalar product of tensors, and the strain tensor $\varepsilon(\mathbf{v})$ is defined in equation (2.1). The constitutive relation (2.3) can be generalized to the selection

$$(\boldsymbol{\sigma},\varepsilon(\mathbf{u}))\in\mathfrak{G}(t), \quad t\in[0,T], \tag{2.7}$$

on the time-dependent graph between the stress and strain

$$\mathfrak{G}(t) = \left\{(\mathbf{T},\boldsymbol{\varepsilon})\in\left(\mathbb{R}^{d\times d}_{\mathrm{sym}}\right)^2\;\middle|\;\boldsymbol{\varepsilon} = [\mathscr{I}(t)\circ\mathscr{F}]\mathbf{T}\right\}. \tag{2.8}$$

The implicit description in equation (2.8) is well suited for single-valued as well as set-valued functions.

3. Solvability analysis

Next, we assert the solvability result for the viscoelastic problem.

Theorem 3.1. *Let $\mathfrak{G}(t)\subset(\mathbb{R}^{d\times d}_{\mathrm{sym}})^2$ defined in equation (2.8) be continuous in t, and the following conditions hold for all $t\in[0,T]$.*

(*i*) *The graph includes the origin*:

$$(\mathbf{0},\mathbf{0})\in\mathfrak{G}(t). \tag{3.1}$$

(*ii*) *The graph is coercive with a uniform estimate for all $(\mathbf{T},\boldsymbol{\varepsilon})\in\mathfrak{G}(t)$: there exist M_1, $M_2>0$ such that $M_1M_2\le\frac{1}{4}$ and*

$$\boldsymbol{\varepsilon}\cdot\mathbf{T}\ge M_1\|\mathbf{T}\|^2 + M_2\|\boldsymbol{\varepsilon}\|^2. \tag{3.2}$$

(*iii*) *The graph is monotone for all pairs* $(\mathbf{T}^1, \epsilon^1)$ *and* $(\mathbf{T}^2, \epsilon^2) \in \mathfrak{G}(t)$:

$$\left(\epsilon^1 - \epsilon^2\right) \cdot \left(\mathbf{T}^1 - \mathbf{T}^2\right) \geq 0. \tag{3.3}$$

(*iv*) *The graph is maximal monotone: for* $(\mathbf{T}^1, \epsilon^1) \in \left(\mathbb{R}^{d\times d}_{\text{sym}}\right)^2$,

$$\text{if } \left(\epsilon^1 - \epsilon^2\right) \cdot \left(\mathbf{T}^1 - \mathbf{T}^2\right) \geq 0 \quad \text{for all } \left(\mathbf{T}^2, \epsilon^2\right) \in \mathfrak{G}(t), \quad \text{then } \left(\mathbf{T}^1, \epsilon^1\right) \in \mathfrak{G}(t). \tag{3.4}$$

Under assumptions of conditions (3.1)–(3.4), *there exists a solution* $(\boldsymbol{\sigma}, \mathbf{u})$ *for the implicitly constituted viscoelasticity problem given in relations* (2.5)–(2.8).

Proof. We carry out the proof by applying the Galerkin approximation in finite-dimensional spaces, and then passing it to the limit.

For a dense sequence forming the basis in the separable Banach spaces:

$$\boldsymbol{\phi}^k \in C\left([0,T]; H^1_{\Gamma_\mathrm{D}}\left(\Omega; \mathbb{R}^d\right)\right), \quad \mathbf{T}^k \in C\left([0,T]; L^2\left(\Omega; \mathbb{R}^{d\times d}_{\text{sym}}\right)\right), \tag{3.5}$$

we search for the finite Galerkin series,

$$\mathbf{u}^m = \sum_{k=1}^{m} U_k \boldsymbol{\phi}^k, \quad \boldsymbol{\sigma}^m = \sum_{k=1}^{m} \Sigma_k \mathbf{T}^k, \tag{3.6}$$

with unknown coefficients $\mathbf{U}^m = (U_1, \ldots, U_m)^\top$ and $\boldsymbol{\Sigma}^m = (\Sigma_1, \ldots, \Sigma_m)^\top$, which should satisfy the finite-dimensional system stemming from relations (2.5) to (2.8):

$$\int_\Omega \boldsymbol{\sigma}^m \cdot \boldsymbol{\varepsilon}(\boldsymbol{\phi}^l)\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \boldsymbol{\phi}^l\, d\mathbf{x} + \int_{\Gamma_\mathrm{N}} \mathbf{g} \cdot \boldsymbol{\phi}^l\, dS_\mathbf{x}, \quad l = 1, \ldots, m, \tag{3.7}$$

$$\int_\Omega (\boldsymbol{\varepsilon}(\mathbf{u}^m) - [\mathscr{I}(t) \circ \mathscr{F}]\boldsymbol{\sigma}^m) \cdot \mathbf{T}^l\, d\mathbf{x} = 0, \quad l = 1, \ldots, m. \tag{3.8}$$

Under finite dimensions, monotone and maximal monotone graphs are continuous, and consequently represent single-valued functions, as presented in Showalter [37]. Therefore, by virtue of the Browder–Minty fixed point theorem, from assumptions of conditions (3.1)–(3.4) we infer the unique solution to problem (3.5)–(3.8) for each fixed $m \in \mathbb{N}$.

Multiplying equations (3.7) by U_l and summing up over $l = 1, \ldots, m$, it follows from the coercivity (3.2) that

$$M_1 \|\boldsymbol{\sigma}^m\|^2_{L^2(\Omega)} + M_2 \|\boldsymbol{\varepsilon}(\mathbf{u}^m)\|^2_{L^2(\Omega)} \leq \int_\Omega \boldsymbol{\sigma}^m \cdot \boldsymbol{\varepsilon}(\mathbf{u}^m)\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{u}^m\, d\mathbf{x} + \int_{\Gamma_\mathrm{N}} \mathbf{g} \cdot \mathbf{u}^m\, dS_\mathbf{x}.$$

Following the application of the trace theorem to the right-hand side, Poincaré–Korn and weighted Young's inequalities, we obtain the uniform in $m \in \mathbb{N}$ estimate for all $t \in [0, T]$. Namely, there exist $M_3, M_4 > 0$ such that

$$M_1 \|\boldsymbol{\sigma}^m\|^2_{L^2(\Omega)} + M_3 \|\mathbf{u}^m\|^2_{H^1(\Omega)} \leq M_4\left(\|\mathbf{f}\|^2_{L^2(\Omega)} + \|\mathbf{g}\|^2_{L^2(\Gamma_\mathrm{N})}\right). \tag{3.9}$$

The maximum over $t \in [0, T]$ in inequality (3.9) and the compactness principle provide the existence of an accumulation point $(\boldsymbol{\sigma}, \mathbf{u})$ with $\mathbf{u} = \mathbf{0}$ on Γ^T_D, and a subsequence that converges as $m \longrightarrow \infty$:

$$\mathbf{u}^m \rightharpoonup \mathbf{u} \quad \text{weakly in } H^1_{\Gamma_\mathrm{D}}\left(\Omega; \mathbb{R}^d\right), \tag{3.10}$$

$$\boldsymbol{\sigma}^m \rightharpoonup \boldsymbol{\sigma} \quad \text{weakly in } L^2\left(\Omega; \mathbb{R}^{d\times d}_{\text{sym}}\right). \tag{3.11}$$

Taking the limit in equation (3.7) based on convergence (3.11) yields the variational equation (2.6).

The limit in the nonlinear function (3.8) can be considered as follows. For $\big(\mathbf{T}^2, [\mathscr{I}(t) \circ \mathscr{F}]\mathbf{T}^2\big) \in \big(L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})\big)^2$ on the graph $\mathfrak{G}(t)$, using the monotonicity (3.3), equations (3.7) and (3.8), we obtain

$$\begin{aligned} 0 &= \int_\Omega (\boldsymbol{\varepsilon}(\mathbf{u}^m) - [\mathscr{I}(t) \circ \mathscr{F}]\boldsymbol{\sigma}^m) \cdot (\boldsymbol{\sigma}^m - \mathbf{T}^2)\, d\mathbf{x} \\ &\le \int_\Omega \Big(\boldsymbol{\varepsilon}(\mathbf{u}^m) \cdot (\boldsymbol{\sigma}^m - \mathbf{T}^2) - [\mathscr{I}(t) \circ \mathscr{F}]\mathbf{T}^2 \cdot (\boldsymbol{\sigma}^m - \mathbf{T}^2)\Big)\, d\mathbf{x} \\ &= \int_\Omega \mathbf{f} \cdot \mathbf{u}^m\, d\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{u}^m\, dS_{\mathbf{x}} - \int_\Omega \Big(\boldsymbol{\varepsilon}(\mathbf{u}^m) \cdot \mathbf{T}^2 + [\mathscr{I}(t) \circ \mathscr{F}]\mathbf{T}^2 \cdot (\boldsymbol{\sigma}^m - \mathbf{T}^2)\Big)\, d\mathbf{x}. \end{aligned}$$

The convergences (3.10) and (3.11) applied here together with equation (2.6) yield the lower limit

$$\begin{aligned} 0 &\le \int_\Omega \mathbf{f} \cdot \mathbf{u}\, d\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{u}\, dS_{\mathbf{x}} - \int_\Omega \Big(\boldsymbol{\varepsilon}(\mathbf{u}) \cdot \mathbf{T}^2 + [\mathcal{I}(t) \circ \mathcal{F}]\mathbf{T}^2 \cdot (\boldsymbol{\sigma} - \mathbf{T}^2)\Big)\, d\mathbf{x} \\ &= \int_\Omega \Big(\boldsymbol{\varepsilon}(\mathbf{u}) - [\mathcal{I}(t) \circ \mathcal{F}]\mathbf{T}^2\Big) \cdot \Big(\boldsymbol{\sigma} - \mathbf{T}^2\Big)\, d\mathbf{x}. \end{aligned}$$

According to the maximal monotone property (3.4), it follows the inclusion (2.7). For time-continuous **f** and **g** in equation (2.6), the solution $\boldsymbol{\sigma}$ and hence $\boldsymbol{\varepsilon}$ in equation (2.8) is also continuous in time. This completes the proof. ∎

Itou *et al.* [3,4] presented examples of the nonlinear function $\mathscr{F}$ in the form of equations (1.2) and (1.3), and $\mathscr{I}$ described as equation (1.13), which satisfied the assumptions given in conditions (3.1)–(3.4) in theorem 3.1. As a particular case, it is considered as a linear model $\mathscr{F} = \mathbf{S}$ with a constant compliance tensor **S** in equation (1.4) and $\mathscr{I}(t)$ defined by equation (1.5) with an anisotropic kernel **J** satisfying conditions (3.1)–(3.4). This may be applicable to the Burgers model [38], see also [39], which has applications to geophysical and seismological fields [40].

In §4, we investigate the quasi-linear functions $\mathscr{I}(t)$, which are nonlinear for $t \in [0, T]$ and can be expressed by linear integral equations.

4. Quasi-linear viscoelasticity

We introduce an auxiliary problem for nonlinear elasticity to find

$$\mathbf{u}^{\mathrm{e}} \in C\big([0,T]; H^1_{\Gamma_{\mathrm{D}}}\big(\Omega; \mathbb{R}^d\big)\big), \quad \boldsymbol{\sigma}^{\mathrm{e}} \in C\big([0,T]; L^2\big(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}\big)\big), \tag{4.1}$$

which solves the quasi-static variational equation

$$\int_\Omega \boldsymbol{\sigma}^{\mathrm{e}} \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{v}\, dS_{\mathbf{x}}, \tag{4.2}$$

for all test functions $\mathbf{v} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d)$, and the set-valued inclusion holds:

$$(\boldsymbol{\sigma}^{\mathrm{e}}, \boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{e}})) \in \mathfrak{G}^{\mathrm{e}}, \tag{4.3}$$

where the graph $\mathfrak{G}^{\mathrm{e}}$ is determined as

$$\mathfrak{G}^{\mathrm{e}} = \Big\{(\mathbf{T}, \boldsymbol{\varepsilon}) \in \big(\mathbb{R}^{d\times d}_{\mathrm{sym}}\big)^2 \;\Big|\; \boldsymbol{\varepsilon} = [\mathscr{F}]\mathbf{T}\Big\}. \tag{4.4}$$

This corresponds to the following boundary value problem:

$$-\nabla \cdot \boldsymbol{\sigma}^{\mathrm{e}} = \mathbf{f} \quad \text{in } \Omega^T, \tag{4.5}$$

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$$\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{e}}) = [\mathscr{F}]\boldsymbol{\sigma}^{\mathrm{e}} \quad \text{in } \Omega^T, \tag{4.6}$$

$$\mathbf{u}^{\mathrm{e}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}^T, \quad \boldsymbol{\sigma}^{\mathrm{e}}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\mathrm{N}}^T. \tag{4.7}$$

As a consequence of theorem 3.1, we derive the following.

Proposition 4.1. *Let the graph* $\mathfrak{G}^{\mathrm{e}} \subset (\mathbb{R}^{d\times d}_{\mathrm{sym}})^2$ *defined in equation* (4.4) *satisfy the following conditions.*

(i) The graph includes the origin:

$$(\mathbf{0}, \mathbf{0}) \in \mathfrak{G}^{\mathrm{e}}. \tag{4.8}$$

(ii) The graph is coercive with a uniform estimate for all $(\mathbf{T}, \boldsymbol{\epsilon}) \in \mathfrak{G}^{\mathrm{e}}$: *there exist* $M_1^{\mathrm{e}}, M_2^{\mathrm{e}} > 0$ *such that* $M_1^{\mathrm{e}} M_2^{\mathrm{e}} \le \frac{1}{4}$ *and*

$$\boldsymbol{\epsilon} \cdot \mathbf{T} \ge M_1^{\mathrm{e}} \|\mathbf{T}\|^2 + M_2^{\mathrm{e}} \|\boldsymbol{\epsilon}\|^2. \tag{4.9}$$

(iii) The graph is monotone for all pairs $(\mathbf{T}^1, \boldsymbol{\epsilon}^1)$ *and* $(\mathbf{T}^2, \boldsymbol{\epsilon}^2) \in \mathfrak{G}^{\mathrm{e}}$:

$$(\boldsymbol{\epsilon}^1 - \boldsymbol{\epsilon}^2) \cdot (\mathbf{T}^1 - \mathbf{T}^2) \ge 0. \tag{4.10}$$

(iv) The graph is maximal monotone: for $(\mathbf{T}^1, \boldsymbol{\epsilon}^1) \in (\mathbb{R}^{d\times d}_{\mathrm{sym}})^2$,

$$\text{if } (\boldsymbol{\epsilon}^1 - \boldsymbol{\epsilon}^2) \cdot (\mathbf{T}^1 - \mathbf{T}^2) \ge 0 \quad \text{for all } (\mathbf{T}^2, \boldsymbol{\epsilon}^2) \in \mathfrak{G}^{\mathrm{e}}, \quad \text{then } (\mathbf{T}^1, \boldsymbol{\epsilon}^1) \in \mathfrak{G}^{\mathrm{e}}. \tag{4.11}$$

Under the assumptions of conditions (4.8)–(4.11), *there exists a solution* $(\boldsymbol{\sigma}^{\mathrm{e}}, \mathbf{u}^{\mathrm{e}})$ *to the implicitly constituted nonlinear elasticity problem given in relations* (4.1)–(4.4).

Based on proposition 4.1, we construct a semi-analytic formula for the solution of viscoelastic problems (2.5)–(2.8) with isotropic kernels of the form of equation (1.7) according to equation (1.8), and its specific representations, as in equations (1.11)–(1.13).

Theorem 4.1. *Under the assumptions of conditions* (4.8)–(4.11), *let the creep function be described as*

$$[\mathscr{I}(t)]\boldsymbol{\epsilon} = J_0 \boldsymbol{\epsilon}(t) + \int_0^t J(t,s)\boldsymbol{\epsilon}(s)\, ds, \tag{4.12}$$

with a constant J_0 *and a continuous isotropic kernel* $J(t,s)$ for $t, s \in [0, T]$. *Then, there exists a solution* $(\boldsymbol{\sigma}, \mathbf{u})$ *for the viscoelastic problems* (2.5)–(2.8), *which can be constructed as follows:*

$$(\boldsymbol{\sigma}, \mathbf{u}) = (\boldsymbol{\sigma}^{\mathrm{e}}, [\mathscr{I}|_{\mathbb{R}^d}]\mathbf{u}^{\mathrm{e}}), \tag{4.13}$$

from the solution $(\boldsymbol{\sigma}^{\mathrm{e}}, \mathbf{u}^{\mathrm{e}})$ *of the nonlinear elasticity problem* (4.1)–(4.4). *Herein, the continuous restriction* $\mathscr{I}|_{\mathbb{R}^d}(t) : \mathbb{R}^d \mapsto \mathbb{R}^d$ *is determined for* $t \in [0, T]$ *as follows:*

$$[\mathscr{I}|_{\mathbb{R}^d}(t)]\mathbf{v} = J_0 \mathbf{v}(t) + \int_0^t J(t,s)\mathbf{v}(s)\, ds. \tag{4.14}$$

Proof. The linear function $\mathscr{I}(t) : \mathbb{R}^{d\times d}_{\mathrm{sym}} \mapsto \mathbb{R}^{d\times d}_{\mathrm{sym}}$ in equation (4.12) preserves tensor addition:

$$[\mathscr{I}](\boldsymbol{\epsilon}^1 + \boldsymbol{\epsilon}^2) = [\mathscr{I}]\boldsymbol{\epsilon}^1 + [\mathscr{I}]\boldsymbol{\epsilon}^2, \tag{4.15}$$

and its continuous restriction $\mathscr{I}|_{\mathbb{R}^d}(t)$, expressed as in equation (4.14), commutes with the strain from equation (2.1):

$$[\mathscr{I}]\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}([\mathscr{I}|_{\mathbb{R}^d}]\mathbf{u}). \tag{4.16}$$

Based on algebra in equations (4.15) and (4.16) applied to the elastic response (4.6), we obtain

$$\boldsymbol{\varepsilon}([\mathscr{I}|_{\mathbb{R}^d}]\mathbf{u}^{\mathrm{e}}) = [\mathscr{I}]\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{e}}) = [\mathscr{I} \circ \mathscr{F}]\boldsymbol{\sigma}^{\mathrm{e}}.$$

Hence, the viscoelastic response (2.3) holds, and vice versa. For $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{e}}$, the variational equations (2.6) and (4.2) coincide. This proves the viscoelastic solution in the form of equations (4.13) and (4.14). ∎

Remark 4.1. The result of theorem 4.1 holds true for any creep function $\mathscr{J}$ satisfying the assumptions of equations (4.15) and (4.16). However, the commutative property (4.16) fails for general anisotropic kernels presented by the fourth-order creep tensor $\mathbf{J}$ in equation (1.5).

Remark 4.2. The result of theorem 4.1 does not follow directly from theorem 3.1, since there are no assumptions considered on the isotropic kernel J of function $\mathscr{J}$ in equation (4.12).

5. Inverse problem

Let $\mathbf{q}(t, \mathbf{x})$ be a design variable, which can be a scalar, vector or tensor-valued function within the parameters of the model. The admissible candidates are the forces $\mathbf{f}$ and $\mathbf{g}$ in the governing equations (2.2) and (2.4), elastic moduli entering the response functions $\mathscr{F}$ in (1.2)–(1.4), and the scalar and matrix creep parameters constituting the creep function $\mathscr{J}$ in the viscoelastic relations (1.9)–(1.13).

For the function formulation in Banach spaces, since the L^1-space is not reflexive, we introduce the $\mathscr{M}^1$-space of bounded measures, which is dual to the C_c-space of continuous functions with compact support, and the related space of bounded deformations, see Khludnev & Kovtunenko [26]:

$$BD\left(\Omega; \mathbb{R}^d\right) = \left\{\mathbf{v} \in L^1\left(\Omega; \mathbb{R}^d\right) \;\middle|\; \boldsymbol{\varepsilon}(\mathbf{v}) \in \mathscr{M}^1\left(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}\right)\right\},$$

according to the common structure (2.1) of the linearized strain tensor in elasticity. Motivated by the Tikhonov regularization, as in equation (1.15), we propose an observation function described as

$$[\mathscr{Q}](\mathbf{q}, \mathbf{u}, \boldsymbol{\sigma})\colon\ \mathscr{M}^1\left(\Omega^T\right) \times BD\left(\Omega^T; \mathbb{R}^d\right) \times \mathscr{M}^1\left(\Omega^T; \mathbb{R}^{d\times d}_{\mathrm{sym}}\right) \mapsto \mathbb{R}\,. \tag{5.1}$$

For smooth $\mathbf{q}$, let a reflexive Banach space U be continuously embedded in $\mathscr{M}^1\left(\Omega^T\right)$. We mark a solution to the direct problem in relations (2.5)–(2.8) depending on the design variable $\mathbf{q}$ via $(\mathbf{u}(\mathbf{q}), \boldsymbol{\sigma}(\mathbf{q}))$, and represent an inverse problem by the optimization over the admissible set as

$$\min_{\mathbf{q}\in U_{\mathrm{ad}}} [\mathscr{Q}](\mathbf{q}, \mathbf{u}(\mathbf{q}), \boldsymbol{\sigma}(\mathbf{q})) \quad \text{over} \quad U_{\mathrm{ad}} \subset U\,. \tag{5.2}$$

Following the arguments of Migórski *et al.* [41], the existence theorem is established as below.

Theorem 5.1.

(i) Let the admissible set U_{ad} be weakly sequentially compact: for any sequence $\mathbf{q}^m \in U_{\mathrm{ad}}$,

$$\text{there exists a subsequence } \mathbf{q}^m \rightharpoonup \mathbf{q}^\star \text{ weakly in } U \text{ as } m \to \infty, \text{ and } \mathbf{q}^\star \in U_{\mathrm{ad}}\,. \tag{5.3}$$

(ii) The solution dependent on $\mathbf{q} \mapsto (\mathbf{u}(\mathbf{q}), \boldsymbol{\sigma}(\mathbf{q}))$ is weak-to-weak continuous:

$$\begin{gathered}\text{if } \mathbf{q}^m \rightharpoonup \mathbf{q}^\star \text{ weakly in } U \text{ as } m \to \infty, \text{ then}\\ (\mathbf{u}(\mathbf{q}^m), \boldsymbol{\sigma}(\mathbf{q}^m)) \rightharpoonup (\mathbf{u}(\mathbf{q}^\star), \boldsymbol{\sigma}(\mathbf{q}^\star)) \text{ weakly in } BD\left(\Omega^T; \mathbb{R}^d\right) \times \mathscr{M}^1\left(\Omega^T; \mathbb{R}^{d\times d}_{\mathrm{sym}}\right).\end{gathered} \tag{5.4}$$

(iii) The observation function in map (5.1) is weakly lower semi-continuous:

$$\begin{gathered}\text{if } (\mathbf{q}^m, \mathbf{u}^m, \boldsymbol{\sigma}^m) \rightharpoonup (\mathbf{q}^\star, \mathbf{u}^\star, \boldsymbol{\sigma}^\star) \text{ weakly in } U \times BD\left(\Omega^T; \mathbb{R}^d\right) \times \mathscr{M}^1\left(\Omega^T; \mathbb{R}^{d\times d}_{\mathrm{sym}}\right),\\ \text{then the lower limit } \liminf_{m\to\infty} [\mathscr{Q}](\mathbf{q}^m, \mathbf{u}^m, \boldsymbol{\sigma}^m) \geq [\mathscr{Q}](\mathbf{q}^\star, \mathbf{u}^\star, \boldsymbol{\sigma}^\star)\,.\end{gathered} \tag{5.5}$$

Under the assumptions of conditions (3.1)–(3.4) and (5.3)–(5.5), the set of solutions $\mathbf{q}^\star$ to the inverse problem (5.2) is non-empty and compact.

Proof. Let $\mathbf{q}^m \in U_{\rm ad}$ be the minimizing sequence:

$$\liminf_{m\to\infty} [\mathscr{Q}](\mathbf{q}^m, \mathbf{u}(\mathbf{q}^m), \boldsymbol{\sigma}(\mathbf{q}^m)) = \inf_{\mathbf{q}\in U_{\rm ad}} [\mathscr{Q}](\mathbf{q}, \mathbf{u}(\mathbf{q}), \boldsymbol{\sigma}(\mathbf{q})). \tag{5.6}$$

From condition (5.3), it follows that there exist $\mathbf{q}^\star \in U_{\rm ad}$ and a subsequence such that

$$\mathbf{q}^m \rightharpoonup \mathbf{q}^\star \quad \text{weakly in } U \text{ as } m\to\infty.$$

Consequently, the properties, as in conditions (5.4) and (5.5), lead to the lower bound

$$\liminf_{m\to\infty} [\mathscr{Q}](\mathbf{q}^m, \mathbf{u}(\mathbf{q}^m), \boldsymbol{\sigma}(\mathbf{q}^m)) \ge [\mathscr{Q}](\mathbf{q}^\star, \mathbf{u}(\mathbf{q}^\star), \boldsymbol{\sigma}(\mathbf{q}^\star)),$$

which together with equation (5.6) justifies that $\mathbf{q}^\star$ is the argument of minimum in problem (5.2). Repeating these arguments for a sequence of solutions $(\mathbf{q}^\star)^m$ proves the compactness of the solution set and the assertion of the theorem. ∎

As an appropriate illustration, we consider an inverse problem of memory kernel identification, see Kaltenbacher *et al*. [42], as stated in the frame of quasi-linear viscoelasticity of §4.

Let $\Gamma_{\rm O} \subset \Gamma_{\rm N}$ be the observation boundary, and $\Gamma_{\rm O}^T = (0,T)\times\Gamma_{\rm O}$. For a given non-smooth measurement $\mathbf{M} \in L^1(\Gamma_{\rm O}^T;\mathbb{R}^d)$, we describe the over-determined Dirichlet data

$$\mathbf{u} = \mathbf{M} \quad \text{on} \quad \Gamma_{\rm O}^T, \tag{5.7}$$

for the direct viscoelastic problem (2.5)–(2.8). Our task is to identify from equation (5.7) the ageing memory kernel J in the isotropic creep function $\mathscr{I}$ in equation (4.12) and its restriction $\mathscr{I}|_{\mathbb{R}^d}$ in equation (4.14):

$$[\mathscr{I}(t)]y = J_0 y(t) + \int_0^t J(t,s)y(s)\,ds \quad \text{for } y \in C([0,T]), \tag{5.8}$$

in the space of continuous and uniformly bounded functions:

$$\mathbf{q} := J \in C\big([0,T]^2\big) =: U, \quad U_{\rm ad} = \Big\{ J\in U \;\Big|\; \max_{t,s\in[0,T]} |J(t,s)| \le M_5 \Big\}, \quad M_5 > 0. \tag{5.9}$$

Since equation (5.7) cannot be satisfied exactly because of the smoothness gap between $\mathbf{u}$ and $\mathbf{M}$, applying the Tikhonov regularization yields the observation function in map (5.1):

$$[\mathscr{Q}](J,\mathbf{u}) := \|\mathbf{u}-\mathbf{M}\|_{L^1(\Gamma_{\rm O}^T;\mathbb{R}^d)} + \beta\|J\|_{C([0,T]^2)}, \tag{5.10}$$

where $\beta \ge 0$ is a regularization parameter. We denote the solution to the direct problem (2.5)–(2.8) depending on the model parameter J as

$$\mathbf{u}(J) \in C\big([0,T]; H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^d)\big), \quad \boldsymbol{\sigma}(J) \in C\big([0,T]; L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})\big),$$

and formulate the optimization problem (5.2) as

$$\min_{J\in U_{\rm ad}} [\mathscr{Q}](J,\mathbf{u}(J)). \tag{5.11}$$

From theorems 4.1 and 5.1, we deduce the following existence results.

Theorem 5.2. *Under the assumptions of conditions* (4.8)–(4.11) *considered for the response function* $\mathscr{F}$ *determining the graph* $\mathfrak{G}^{\rm e}$ *in inclusion* (4.4), *and the creep function* $\mathscr{I}$ *described by the isotropic ageing memory kernel* J *in equation* (5.8) *for the quasi-linear viscoelastic direct problem* (2.5)–(2.8), *there exists a solution* $J^\star \in U_{\rm ad}$ *to the inverse problem* (5.11) *with the observation function from equation* (5.10).

Proof. We will check three conditions (5.3)–(5.5) in theorem 5.1. First, the admissible set given in (5.9) is evidently compact, thus the condition (5.3) holds.

Second, under the assumptions of conditions (4.8)–(4.11), there exists a solution $(\boldsymbol{\sigma}^{\rm e}, \mathbf{u}^{\rm e})$ to the nonlinear elasticity problem (4.1)–(4.4), which is independent of J. For the creep function described by formula (5.8), according to theorem 4.1, a solution $(\boldsymbol{\sigma}(J), \mathbf{u}(J))$ to the viscoelastic problem (2.5)–(2.8) can be constructed explicitly in the form of equation (4.13):

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\rm e}, \quad [\mathbf{u}(J)](t) = J_0\mathbf{u}^{\rm e}(t) + \int_0^t J(t,s)\mathbf{u}^{\rm e}(s)\,ds \quad \text{for } t \in [0,T]\,. \tag{5.12}$$

A sequence of continuous functions converges weakly if and only if it converges point-wise and is uniformly bounded. If $J^m \in U_{\rm ad}$ and $J^m \to J^\star$ in $C([0,T]^2)$ as $m \to \infty$, then it holds in a point-wise manner:

$$\big[\mathbf{u}(J^m)\big](t,\mathbf{x}) = J_0\mathbf{u}^{\rm e}(t,\mathbf{x}) + \int_0^t J^m(t,s)\mathbf{u}^{\rm e}(s,\mathbf{x})\,ds \to \big[\mathbf{u}(J^\star)\big](t,\mathbf{x}) \quad \text{a.e. } \Omega^T.$$

Therefore, the J-independent stress $\boldsymbol{\sigma}$ and linear in J displacement $\mathbf{u}(J)$ from equation (5.12) satisfy the property of weak-to-weak continuity given in (5.4).

Third, the trace operator is continuous from $\mathbf{u} \in BD(\Omega^T;\mathbb{R}^d)$ to $\mathbf{u} \in L^1(\Gamma^T_{\rm O};\mathbb{R}^d)$, see Temam & Strang [43]. The norm is a weakly lower semi-continuous function, hence the inequality for lower limit given in (5.5) holds true:

$$\liminf_{m\to\infty}\,[\mathcal{Q}](J^m, \mathbf{u}(J^m)) \geq [\mathcal{Q}]\big(J^\star, \mathbf{u}(J^\star)\big).$$

According to theorem 5.1, this provides the assertion on the solvability of the inverse problem (5.11). Thus, the theorem is proven. ∎

Remark 5.1. Let the regularization parameter $\beta = 0$, and the measurement $\mathbf{M} \in C([0,T]; H^{1/2}(\Gamma^T_{\rm O};\mathbb{R}^d))$ be smooth. If the optimal memory kernel $J^\star$ lies inside the admissible set $U_{\rm ad}$, then the observation (5.7) describes the optimality condition for the unconstrained minimization (5.11).

6. Conclusion

In this article, we considered mathematical models of quasi-linear viscoelasticity whose constitutive law is time-dependent and represents creep phenomena. Example models are presented by Volterra convolution and ageing-type memory kernels describing isotropic and anisotropic creep tensors, and by the Liouville–Weyl integral equation stemming from fractional derivatives. First, we have established theorems 3.1 and 4.1 on the existence of a solution to the quasi-static boundary value problem under certain conditions for the constitutive relation expressed by an implicit graph. Theorems 3.1 and 4.1 are valid for a wide variety of models of quasi-linear viscoelasticity mentioned at the end of §3 and remarks 4.1 and 4.2. In future research, we will show a specific form of the anisotropic creep tensor $\mathbf{J}$.

Second, we considered the inverse problem represented by the identification of a design variable from non-smooth measurements. We have shown in theorem 5.1 that a set of optimal variables minimizing the corresponding Tikhonov approximation is non-empty and compact. In theorem 5.2, the inverse problem of isotropic memory kernel identification is provided as an appropriate illustration of theorem 5.1.

Data accessibility. This article has no additional data.

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All authors gave final approval for publication and agreed to be held accountable for the work performed therein

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