Proseminar Optimierung II

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Список литературы

[1] D.P. Bertsekas, A. Nedić, A.E. Ozdaglar, *Convex Analysis and Optimization.* Athena Scientific, Belmont, 2003, 534 pp.

[2] C. Geiger, C. Kanzow, *Numerische Verfahren zur Lösung unrestringierter Optimierungsaufgaben.* Springer-Verlag, Berlin, 1999, 487 S.

[3] C. T. Kelley, *Iterative Methods for Optimization.* SIAM, Philadelphia, PA, 1999, 180 pp.

[4] A.M. Khludnev, V.A. Kovtunenko, *Analysis of Cracks in Solids.* WIT-Press, Southampton, Boston, 2000, 408 pp.

[5] D.G. Luenberger, Y. Ye, *Linear and Nonlinear Programming.* Springer, New York, 2008, 546 pp.

[6] J. Nocedal, S.J. Wright, *Numerical Optimization.* Springer-Verlag, New York, 1999, 636 pp.

[7] В.М. Алексеев, Э.М. Галеев, В.М. Тихомиров, Сборник задач по оптимизации. Москва: Наука, 1984, 288 с.

1. Linear programs

Linear programs (LP) are stated in the canonical (standard) form:

$$\min c^\top x \quad \text{over } x \in \mathbb{R}^n \text{ subject to } Ax = b,\ x \geq 0, \tag{1.1}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$, and $m < n$. The *feasible set* is defined

$$\mathcal{F} = \{x \in \mathbb{R}^n : \quad Ax = b, \quad x \geq 0\}, \tag{1.2}$$

and its boundary corresponds to the *active set*

$$\mathcal{A} = \{x \in \mathbb{R}^n : \quad Ax = b, \quad \exists x_j = 0\}. \tag{1.3}$$

The *fundamental theorem of LP* states that optimal solutions of (1.1) occur along $\mathcal{A}$ at extreme (corner) points of $\mathcal{F}$.

Exercise 1.1. Rewrite the following LP in the standard form:

$$\min\{x_1 + 2x_2 + 3x_3\} \quad \text{over } x \in \mathbb{R}^3 \text{ subject to}$$
$$4 \leq x_1 + x_2 \leq 5, \quad 6 \leq x_1 + x_3 \leq 7, \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0,$$

and determine A, b, c in (1.1). Hint: introduce slack variables $x_4 \geq 0$, $x_5 \geq 0$, $x_6 \geq 0$, $x_7 \geq 0$ to satisfy the non-standard inequality constraints.

Exercise 1.2. Rewrite the following LP in the standard form:

$$\min\{x + y + z\} \quad \text{over } (x, y, z) \in \mathbb{R}^3 \text{ subject to}$$
$$x + y \leq 1, \quad 2x + z = 3,$$

and determine A, b, c in (1.1). Hint: introduce non-negative variables $(x_1, \dots, x_6)$ by $x = x_1 - x_2$, $y = x_3 - x_4$, $z = x_5 - x_6$.

Exercise 1.3. Convert the ℓ^1-minimization problem to LP:

$$\min |x| \quad \text{over } x \in \mathbb{R}^1 \text{ subject to } Ax = b.$$

Hint: use the inequality $-|x| \leq x \leq |x|$.

Exercise 1.4. Given the constraints:

$$x_1 + 2x_2 \leq 16, \quad 2x_1 + x_2 \leq 12, \quad x_1 + 2x_2 \geq 2, \quad x_1 \geq 0, \quad x_2 \geq 0,$$

a) plot the feasible set $\mathcal{F}$ and determine the extreme (corner) points,
b) plot the active set $\mathcal{A}$.

Exercise 1.5. Consider feasible set $\mathcal{F} \subset \mathbb{R}^2$ satisfying the following inequalities:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_2 - x_1 \leq 2, \quad x_1 + x_2 \leq 7.$$

a) Plot $\mathcal{F}$ and list its extreme points,
b) compute $\min\{x_1 - 2x_2\}$ over $x \in \mathbb{R}^2$ subject to $x \in \mathcal{F}$,
c) compute $\max\{x_1 - 2x_2\}$ over $x \in \mathbb{R}^2$ subject to $x \in \mathcal{F}$.

Hint: apply the fundamental theorem of LP.

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2. Nonlinear programs

We consider *nonlinear programs (NLP)* of the general form:

$$\min J(x) \quad \text{over } x \in \mathbb{R}^n \text{ subject to } e(x) = 0,\ g(x) \leq 0, \tag{2.1}$$

where $J : \mathbb{R}^n \mapsto \mathbb{R}$, $e : \mathbb{R}^n \mapsto \mathbb{R}^m$, $e = (e_1, \dots, e_m)$, $m < n$, and $g : \mathbb{R}^n \mapsto \mathbb{R}^p$, $g = (g_1, \dots, g_p)$. The *feasible set* is defined

$$\mathcal{F} = \{x \in \mathbb{R}^n : \quad e(x) = 0, \quad g(x) \leq 0\}, \tag{2.2}$$

and its boundary corresponds to the *active set*

$$\mathcal{A} = \{x \in \mathbb{R}^n : \quad e(x) = 0, \quad \exists g_j(x) = 0\}. \tag{2.3}$$

A vector $x^\star \in \mathcal{F}$ is called a *global solution* of (2.1) if $J(x^\star) \leq J(x)$ for all $x \in \mathcal{F}$, respectively, a *local solution* if there is a neighborhood $U(x^\star) \subset \mathcal{F}$ such that $J(x^\star) \leq J(x)$ for all $x \in U(x^\star)$.

Exercise 2.1. In $\mathbb{R}^2$ consider the constraints:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_2 - (x_1 - 1)^2 \leq 0, \quad x_1 \leq 2.$$

Plot the feasible set $\mathcal{F}$ in (2.2) and show that $x^\star = (1, 0)$ is feasible but not regular. Hint: A point $x^\star$ at the hyperplane $\mathcal{E} = \{x \in \mathbb{R}^n : e_1(x) = 0, \dots, e_k(x) = 0\}$ is regular if vectors $\nabla e_1(x^\star), \dots, \nabla e_k(x^\star)$ are linearly independent (called linear independence constrained qualification (LICQ)).

Exercise 2.2. Let $\mathcal{F}$ be a convex set in $\mathbb{R}^2$. This implies that for all $x, y \in \mathcal{F}$ and $t \in [0, 1]$ points $tx + (1 - t)y \in \mathcal{F}$. Prove that, for nonnegative weights $c^1, \dots, c^k \in \mathbb{R}$ such that $\sum_{i=1}^k c^i = 1$, if $x^1, \dots, x^k \in \mathcal{F}$ then $\sum_{i=1}^k c^i x^i \in \mathcal{F}$. Hint: use induction over $k \geq 2$.

Exercise 2.3. Determine, whether the following functions defined on $\mathbb{R}^2_+$

a) $J(x) = x_1 x_2$ b) $J(x) = \dfrac{1}{x_1 x_2}$

c) $J(x) = x_1(\ln x_1 - 1) + x_2(\ln x_2 - 1)$

are convex, concave, or neither. Hint: twice differentiable function J is convex on a convex set if its Hessian matrix $D(\nabla J)$ is positive semidefinite (spd).

Exercise 2.4. If the objective function J is convex on the convex feasible set $\mathcal{F}$, prove that local solutions of (2.1) are also global solutions.

Exercise 2.5. Consider the problem: $\min\{x_1 + x_2\}$ subject to $x_1^2 + x_2^2 = 1$.
a) Solve the problem by eliminating the variable x_2.
b) Plot $\mathcal{F}$ and find tangent plane at feasible point $x^\star = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Hint: at a regular point $x^\star$ of hyperplane $\mathcal{E}$ the tangent plane (tangent cone) is

$$\mathcal{T}(x^\star) = \{v \in \mathbb{R}^n : \ De(x^\star) \cdot v = 0\} \quad \text{with } De := (\nabla e_1^\top, \dots, \nabla e_k^\top). \tag{2.4}$$

c) Find (the Lagrange multiplier) $\lambda^\star \in \mathbb{R}$ such that $\nabla J(x^\star) = -De(x^\star)\lambda^\star$.
d) Illustrate this problem geometrically.

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3. Equality constrained optimization

For the equality constrained minimization problem

$$\min J(x) \quad \text{over } x \in \mathbb{R}^n \text{ subject to } \{e_1(x) = 0, \dots, e_m(x) = 0\} \tag{3.1}$$

the *Lagrange function (Lagrangian)* is defined by

$$L : \mathbb{R}^{n+m} \mapsto \mathbb{R}, \quad L(x, \lambda) = J(x) + e(x)^\top \lambda \tag{3.2}$$

with the *Lagrange multipliers* (dual variables) $\lambda \in \mathbb{R}^m$. The *first order necessary condition of optimality* implies a stationary point $(x^\star, \lambda^\star)$ solving

$$\nabla_{(x,\lambda)} L(x^\star, \lambda^\star) = 0 \Leftrightarrow \{\nabla J(x^\star) + De(x^\star)\lambda^\star = 0, \quad e(x^\star) = 0\}, \tag{3.3}$$

and the *second order sufficient condition* along the tangent plane $\mathcal{T}(x^\star)$ reads

$$v^\top D(\nabla L)(x^\star, \lambda^\star)\, v > 0 \quad \text{for all } v \in \mathcal{T}(x^\star), \tag{3.4}$$

where the *Hessian matrix (Hessian)* $D(\nabla L) : \mathbb{R}^{n+m} \mapsto \mathbb{R}^{n \times n}$.

Exercise 3.1. Solve using Lagrange multipliers:

$$\min\{-x^3 - y^3\} \quad \text{subject to } x + y = 1.$$

Exercise 3.2. Consider the following minimization problem:

$$\min\left\{\tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_3^2 + 2x_1x_2\right\} \text{ subject to } x_1 + x_2 = 4,\ x_3 - 9x_1 + 16 = 0.$$

a) Construct Lagrangian and find the stationary point $(x^\star, \lambda^\star)$ of this problem.
b) Check a second order sufficient condition along the tangent plane $\mathcal{T}(x^\star)$.

Exercise 3.3. If a, b, c are positive real numbers, prove that the inequality $a^3 + b^3 + c^3 \geq 3abc$ holds. Hint: it is enough to prove that, if $3abc = K$ (with arbitrarily fixed $K \geq 0$), then $\min\{a^3 + b^3 + c^3\} = K$.

Exercise 3.4. A box with sides x, y and z is to be manufactured such that its top (xy), bottom (xy), and front (xz) faces must be doubled. Find the dimensions of such a box that maximize the volume xyz for a given face area, equal to 18 dm^2. Hint: use first order and verify second order conditions.

Exercise 3.5. Find an oriented triangle of maximal area such that one vertex is $(x_3, y_3) = (1, 0)$ and the other two vertexes (x_1, y_1), (x_2, y_2) lie on the unit circle in $\mathbb{R}^2$. Hint: use the area formula $S = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$.

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4. Inequality constrained optimization

For the general, equality and inequality constrained minimization problem

$$\min J(x) \quad \text{over } x \in \mathbb{R}^n \text{ subject to } e(x) = 0,\ g(x) \leq 0, \tag{4.1}$$

the *Lagrange function (Lagrangian)* is defined by

$$L : \mathbb{R}^{n+m+p} \mapsto \mathbb{R}, \quad L(x, \lambda, \mu) = J(x) + e(x)^\top \lambda + g(x)^\top \mu \tag{4.2}$$

with the *Lagrange multipliers* $(\lambda, \mu) \in \mathbb{R}^{m+p}$. The first order necessary optimality condition implies *Karush–Kuhn–Tucker (KKT) conditions*:

$$\nabla_{(x,\lambda)} L(x^\star, \lambda^\star, \mu^\star) = 0 \Leftrightarrow \begin{cases} \nabla J(x^\star) + De(x^\star)\lambda^\star + Dg(x^\star)\mu^\star = 0, \\ e(x^\star) = 0 \end{cases} \tag{4.3a}$$

$$\mu^\star \geq 0, \quad g(x^\star) \leq 0, \quad g_i(x^\star)\mu_i^\star = 0 \quad \text{for } i = 1, \dots, p. \tag{4.3b}$$

The *second order sufficient condition* along the tangent plane $\mathcal{T}(x^\star)$ reads

$$v^\top D(\nabla L)(x^\star, \lambda^\star, \mu^\star)\, v > 0 \quad \text{for all } v \in \mathcal{T}(x^\star). \tag{4.4}$$

Exercise 4.1. Solve using KKT conditions:

$$\min\{3x - x^3\} \quad \text{over } x \in \mathbb{R} \text{ subject to } x \leq 2$$

and plot the objective function on the feasible set.

Exercise 4.2. Solve using KKT conditions:

$$\min\{e^{x_1}\} \quad \text{over } x \in \mathbb{R}^2 \text{ subject to } e^{-x_1} + x_2 = 7,\ x_1 \geq 0,\ x_2 \geq 0.$$

Exercise 4.3. Verify that *complementarity conditions* (4.3b) follows from the *variational inequality*

$$\mu^\star \geq 0, \quad \nabla_\mu L(x^\star, \lambda^\star, \mu^\star)^\top (\mu - \mu^\star) \leq 0 \quad \text{for all } \{\mu \in \mathbb{R}^p : \mu \geq 0\}. \tag{4.5}$$

Hint: plug $\mu = (\mu_1^\star, \dots, t\mu_i^\star, \dots, \mu_p^\star)$ with arbitrary $t > 0$ and $i \in \{1, \dots, p\}$.

Exercise 4.4. Let $x^\star$, $\lambda^\star$, $\mu^\star \geq 0$ be a solution of the minimax problem:

$$\min_{(x,\lambda)} \max_{\mu} L(x, \lambda, \mu) \quad \text{over } (x, \lambda, \mu) \in \mathbb{R}^{n+m+p} \text{ subject to } \mu \geq 0. \tag{4.6}$$

(a) Verify that this solution satisfy KKT conditions (4.3).
(b) From (4.6) derive that $x^\star$ is feasible and that it solves (4.1).

Exercise 4.5. Consider the following minimization problem:

$$\min\{-(x-2)^2 - 2(y-1)^2\} \quad \text{subject to } x + 4y \leq 3 \text{ and } x \geq y.$$

Solve this problem using KKT conditions.

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5. Regularity and sensitivity

Exercise 5.1. Consider the minimization problem:

$$\min\{-x_1\} \quad \text{over } x \in \mathbb{R}^2 \text{ subject to } x_1^2 \le x_2 \text{ and } x_2 \le 0.$$

Find the global solution of this problem and verify that there is no KKT points. Hint: consider the feasible set and check its regularity.

Exercise 5.2. For the problem:

$$\min\{y^2 - x\} \quad \text{over } \mathbb{R}^2 \text{ subject to } y \le (1-x)^3,\ x + y \ge 1,\ y \le 0,$$

verify that the objective function is convex, the feasible set is convex and has a non-empty interior (Slater condition). This will guarantee that the KKT point is the solution of the convex minimization problem. Find this solution. Check a second order sufficient condition as well.

Exercise 5.3. Consider the problem:

$$\min\{4x_1 + x_2\} \quad \text{over } x \in \mathbb{R}^2 \text{ subject to } x_1^2 + x_2 \ge 9,\ x_1 \ge 0,\ x_2 \ge 0.$$

(a) Verify that the feasible set $\mathcal{F}$ is non-convex (hence, a KKT point may be not the solution of the minimization problem).
(b) Since the objective function is linear, find the global solution of the problem at the boundary (the active set $\mathcal{A}$) of $\mathcal{F}$.
(c) Find all solutions of the KKT system at $\mathcal{A}$ and determine its kind of extrema.

Exercise 5.4. Consider the following minimization problem:

$$\min\Big\{\tfrac{1}{2}\alpha x_1^2 + \tfrac{1}{2}x_2^2 + 5x_1\Big\} \quad \text{over } x \in \mathbb{R}^2 \text{ subject to } x_1 \ge 0.$$

Using first and second order optimality conditions find its solution in dependence of the parameter $\alpha \in \mathbb{R}$.

Exercise 5.5. Consider the problem:

$$\min\big\{(x-1)^2 + y^2\big\} \quad \text{over } \mathbb{R}^2 \text{ subject to } x \le \tfrac{1}{\beta}y^2.$$

For what values of the parameter $\beta \in \mathbb{R}$ its KKT point is a local solution? Hint: use a second order sufficient condition.

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6. Quadratic programs

We consider *quadratic programs (QP)* as NLP with the specific (quadratic) objective function $J(x) = \frac{1}{2}x^\top Qx + d^\top x$, where the matrix $Q \in \mathbb{R}^{n\times n}$ is symmetric positive definite (spd) and $d \in \mathbb{R}^n$.

Exercise 6.1. Verify that, for $Q \in \mathrm{spd}(\mathbb{R}^{n\times n})$, the quadratic objective function J is convex by the definition: $J\big(tx+(1-t)y\big) \le tJ(x)+(1-t)J(y)$. Hint: use the Cauchy–Schwarz inequality $x^\top Qy \le \frac{1}{2}x^\top Qx + \frac{1}{2}y^\top Qy$.

Exercise 6.2. Consider the quadratic minimization problem:

$$\min\big\{\tfrac{1}{2}x^\top Qx + d^\top x\big\} \quad \text{subject to } Ax = b, \tag{6.1}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Prove that $x^\star$ is a local solution of the problem if and only if it is a global solution.

Exercise 6.3. The problem of finding the shortest distance from a point $x^0 \in \mathbb{R}^n$ to the hyperplane $\{x \in \mathbb{R}^n : Ax = b\}$ can be formulated as

$$\min\big\{\tfrac{1}{2}(x-x^0)^\top(x-x^0)\big\} \quad \text{subject to } Ax = b.$$

(a) Verify that the problem is of the form (6.1) and determine Q and d.
(b) Show that: the matrix $AA^\top$ is nonsingular if A has full rank,
(c) the stationary point is $x^\star = x^0 - A^\top\lambda^\star$ and $\lambda^\star = (AA^\top)^{-1}(Ax^0 - b)$,
(d) the stationary point is the optimal solution.

Exercise 6.4. For given data points $(x_1, y_1), \dots, (x_n, y_n)$ in $\mathbb{R}^2$, the linear regression problem consists in fitting the line $y = d+kx$ such that to minimize the residuals:

$$\min\Big\{\frac{1}{2}\sum_{i=1}^n (d+kx_i-y_i)^\top(d+kx_i-y_i)\Big\} \quad \text{over } (d,k)^\top \in \mathbb{R}^2.$$

(a) Rewrite the problem in the following form of unconstrained QP:

$$\min\big\{\tfrac{1}{2}(Az-b)^\top(Az-b)\big\} \quad \text{over } z \in \mathbb{R}^2.$$

(b) Using optimality derive the normal equations: $A^\top Az^\star = A^\top b$.
(c) Solving these two equations verify the formula: $k = \frac{S_{xy}-\bar{x}\bar{y}}{S_{xx}-\bar{x}^2}$, $d = \bar{y} - k\bar{x}$ written in statistical terms

$$S_{xx} = \tfrac{1}{n}\textstyle\sum_{i=1}^n x_i^2, \quad S_{xy} = \tfrac{1}{n}\sum_{i=1}^n x_i y_i, \quad \bar{x} = \tfrac{1}{n}\sum_{i=1}^n x_i, \quad \bar{y} = \tfrac{1}{n}\sum_{i=1}^n y_i.$$

Exercise 6.5. Consider the problem of finding the point on the parabola $y = \frac{1}{5}(x-1)^2$ that is closest to $(x^0, y^0) = (1, 2)$. This can be formulated as

$$\min\big\{\tfrac{1}{2}(x-1)^2 + \tfrac{1}{2}(y-1)^2\big\} \quad \text{subject to } (x-1)^2 = 5y.$$

Find the stationary point of the problem and show that it is the minimum point using a second order condition.

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7. Complementarity

Exercise 7.1. The Euclidean projection $P_{\mathcal{S}}x^0$ of a point $x^0 \in \mathbb{R}^n$ on simplex

$$\mathcal{S} = \{x \in \mathbb{R}^n : \quad x \geq 0, \quad x^\top e = 1\} \quad \big(e = (1, \dots, 1)^\top\big)$$

solves the problem: $\min\big\{\frac{1}{2}\|x - x^0\|_2^2\big\}$ over $x \in \mathbb{R}^n$ subject to $x \in \mathcal{S}$.
Verify the formula $P_{\mathcal{S}}x^0 = \max(0, x^0 - \lambda^\star e)$, where $\lambda^\star \in \mathbb{R}$ is a Lagrange multiplier associated to the constraint $x^\top e = 1$.

Exercise 7.2. For the maximization of entropy find the solution:

$$\max\big\{-\big(x_1 \log(x_1) + \dots + x_n \log(x_n)\big)\big\} \quad \text{over } x \in \mathbb{R}^n \text{ subject to } x \in \mathcal{S}.$$

Exercise 7.3. $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ is called a *nonlinear complementarity problem (NCP) function* if it satisfies $\phi(x, y) = 0 \Leftrightarrow x \geq 0,\ y \geq 0,\ x \cdot y = 0$.
Verify that the following are NCP functions:

$$\phi_{\min}(x, y) = \min(x, y), \quad \phi_{\max}(x, y) = y - \max(0, y - cx) \quad \text{with } c > 0,$$

$$\phi_{\mathrm{FB}}(x, y) = \sqrt{x^2 + y^2} - (x + y) \quad \text{(Fischer–Burmeister)}$$

Exercise 7.4. Consider an abstract inequality constrained minimization

$$\min J(x) \quad \text{over } x \in \mathbb{R}^n \text{ subject to } Ax = b,\ x \geq 0. \tag{7.1}$$

(a) Using KKT conditions and a NCP function ϕ derive the NCP

$$\Phi(x^\star, \lambda^\star, \mu^\star) := \begin{pmatrix} \nabla J(x^\star) + A^\top \lambda^\star - \mu^\star \\ Ax^\star - b \\ \phi(x^\star, \mu^\star) \end{pmatrix} = 0. \tag{7.2}$$

(b) If there exists index i such that both $x_i^\star = \mu_i^\star = 0$, then prove that the matrix of derivatives $D_{(x,\lambda,\mu)}\Phi(x^\star, \lambda^\star, \mu^\star)$ is singular.
(c) Using $\phi_{\max}$ rewrite $\phi_{\max}(x^\star, \mu^\star) = 0$ as the linear system

$$x_i^\star = 0 \quad \text{for } i \in A(x^\star, \mu^\star), \quad \mu_i^\star = 0 \quad \text{for } i \in I(x^\star, \mu^\star)$$

with respect to *primal-dual strictly active A and inactive I sets* of indexes

$$A(x^\star, \mu^\star) = \big\{i : \ \mu_i^\star - cx_i^\star > 0\big\}, \quad I(x^\star, \mu^\star) = \big\{i : \ \mu_i^\star - cx_i^\star \leq 0\big\}.$$

(d) On this basis suggest iterations solving nonlinear problem (7.2).

Exercise 7.5. Consider the inequality constrained optimization

$$\min \tfrac{1}{2}\big\{x_1^2 + (x_2 - 2)^2 + (x_3 + 3)^2\big\} \quad \text{subject to } x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

Initializing $A^{(-1)} = \emptyset$, $I^{(-1)} = \{1, 2, 3\}$ iterate the active and inactive sets

$$A^{(k)} = \big\{i : \ \mu_i^{(k)} - cx_i^{(k)} > 0\big\}, \quad I^{(k)} = \big\{i : \ \mu_i^{(k)} - cx_i^{(k)} \leq 0\big\}$$

together with $x_i^{(k)} = 0$ for $i \in A^{(k-1)}$ and $\mu_i^{(k)} = 0$ for $i \in I^{(k-1)}$.
From the iteration find a solution of the KKT system for this problem.

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8. Approximations

Exercise 8.1. Let $x^\star = \operatorname{argmin}\{J(x)\}$ over $x \in \mathbb{R}^n$ subject to $e(x) = 0$, $g(x) \le 0$. Verify that there exists constant $c > 0$ such that $x^\star$ is also a minimizer of the *penalty problem*

$$\min\big\{J(x) + c\big(\|e(x)\|_1 + \|\max(0, g(x))\|_1\big)\big\} \quad \text{over } x \in \mathbb{R}^n.$$

Hint: use the fact that the associated Lagrangian L satisfies $L(x^\star, \lambda^\star, \mu^\star) \le L(x, \lambda^\star, \mu^\star)$ for all $x \in \mathbb{R}^n$.

Exercise 8.2. (a) For the constrained minimization problem

$$\min\big\{J(x) = (x_1 - 6)^2 + (x_2 - 7)^2\big\} \quad \text{over } x \in \mathbb{R}^2 \text{ subject to } x_1 + x_2 \le 7$$

find the solution $x^\star$ and the associated Lagrange multiplier $\lambda^\star$.
(b) Consider the penalty problem

$$\min\big\{J^\alpha(x) = J(x) + \alpha\big(\max(0, x_1 + x_2 - 7)\big)^2\big\} \quad \text{over } x \in \mathbb{R}^2$$

and find its solution x^α in dependence of the parameter $\alpha > 0$.
Is x^α interior or exterior to the feasible set?
(c) Verify that $x^\alpha \to x^\star$ and $2\alpha \cdot \max(0, x_1^\alpha + x_2^\alpha - 7) \to \lambda^\star$ as $\alpha \to \infty$.

Exercise 8.3. (a) Find a solution $x^\star$ of the constrained minimization problem

$$\min\big\{J(x) = 2x_1^2 + 9x_2\big\} \quad \text{over } x \in \mathbb{R}^2 \text{ subject to } x_1 + x_2 \ge 4.$$

(b) Consider the associated *inverse barrier function* $J^\tau(x) = J(x) + \frac{\tau}{x_1 + x_2 - 4}$ in dependence of the parameter $\tau > 0$. Find the solution x^τ of the unconstrained problem: $\min\{J^\tau(x)\}$ over $x \in \mathbb{R}^2$, and compare x^τ with $x^\star$.

Exercise 8.4. For $\tau > 0$ consider a *log-barrier function* constrained minimization

$$\min\Big\{J(x) - \tau \sum_{i=1}^n \log(x_i)\Big\} \quad \text{over } x \in \mathbb{R}^n \text{ subject to } Ax = b,\ x > 0. \tag{8.1}$$

(a) Verify that its KKT conditions lead to the *interior point system*

$$\begin{cases} \nabla J(x) + A^\top \lambda - \nu = 0 \\ Ax - b = 0 \\ x_i \cdot \nu_i = \tau \quad \text{for } i = 1, \dots, n, \quad x > 0, \quad \nu > 0. \end{cases} \tag{8.2}$$

(b) Verify that, conversely, if the interior point system (8.2) is solvable, then it follows the KKT system for (8.1).

Exercise 8.5. For the inequality constrained optimization problem

$$\min\big\{J(x) = -5x_1^2 + 2x_2\big\} \quad \text{over } x \in \mathbb{R}^2 \text{ subject to } x_1 \le 1,\ x_2 \ge 0,$$

write the interior point system and find its solution in dependence of parameter $\tau > 0$. Passing $\tau \to 0$ find a solution of KKT system for this problem.

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9. Simplex method

For LP: $\min\{c^\top x\}$ over $x \in \mathbb{R}^n$ subject to $Ax = b$, $x \geq 0$ with $A \in \mathbb{R}^{m\times n}$, partitioning $A = (B, D)$, $x^\top = (x_B^\top, x_D^\top)$, $c^\top = (c_B^\top, c_D^\top)$ implies the system

$$Ix_b + B^{-1}Dx_D = B^{-1}b, \quad (c_D^\top - c_B^\top B^{-1}D)x_D = c^\top x - c_B^\top B^{-1}b$$

expressed by Tableau:

$$T := \left(\begin{array}{c|c|c} I & B^{-1}D & B^{-1}b \\ \hline 0 & c_D^\top - c_B^\top B^{-1}D =: r^\top & -c_B^\top B^{-1}b \end{array}\right)$$

and follows the *simplex algorithm*: Repeat until $r \geq 0$: select

$$j = \operatorname*{argmin}_{k\in\{1,\dots,n\}} \{r_k : r_k < 0\}, \quad i = \operatorname*{argmin}_{k\in\{1,\dots,m\}} \Big\{\tfrac{(B^{-1}b)_k}{(B^{-1}D)_{kj}} : (B^{-1}b)_k(B^{-1}D)_{kj} \geq 0\Big\},$$

pivot on $(B^{-1}D)_{ij}$. If all $(B^{-1}b)_k(B^{-1}D)_{kj} < 0$ then the problem is unbounded.

Exercise 9.1. Using the simplex algorithm, iterate $x = B^{-1}b$ from T for

$$\max\{5x_1 + x_2\} \quad \text{over } x \in \mathbb{R}^2 \text{ subject to}$$
$$4x_1 + 3x_2 \leq 12, \quad -2x_1 + 3x_2 \geq 6, \quad x_1 \geq 0, x_2 \geq 0.$$

Sketch the feasible set and the path of the simplex steps stopping at the solution of this LP. Are the iterations of x feasible?

Exercise 9.2. Using the simplex algorithm, solve the following LP:

$$\max\{x_1 + x_2\} \quad \text{over } x \in \mathbb{R}^2 \text{ subject to}$$
$$-2x_1 + x_2 \leq 1, \quad x_1 - x_2 \leq 1, \quad x_1 \geq 0, x_2 \geq 0.$$

Sketch the feasible set and the path of the simplex steps.

Exercise 9.3. Find all solutions of the LP:

$$\max\{x_1 + x_2 + 2x_3\} \quad \text{over } x \in \mathbb{R}^3 \text{ subject to}$$
$$x_1 + 2x_2 + 2x_3 \leq 2, \quad x_1 + 4x_2 + 2x_3 \leq 4, \quad x_1 \geq 0, \dots, x_3 \geq 0.$$

Exercise 9.4. Find an optimal solution of the problem:

$$\min\Big\{x_1 - 3x_2 - \tfrac{2}{5}x_3\Big\} \quad \text{over } x \in \mathbb{R}^3 \text{ subject to } x_1 \geq 0, \dots, x_3 \geq 0,$$
$$3x_1 - x_2 + 2x_3 \leq 7, \quad -2x_1 + 4x_2 \leq 12, \quad -4x_1 + 3x_2 + 3x_3 \leq 14.$$

Exercise 9.5. Consider the following LP:

$$\min\{5x_1 + 3x_2\} \quad \text{over } x \in \mathbb{R}^3 \text{ subject to } x_1 \geq 0, \dots, x_3 \geq 0,$$
$$2x_1 - x_2 + 4x_3 \leq 4, \quad x_1 + x_2 + 2x_3 \leq 5, \quad 2x_1 - x_2 + x_3 \geq 1.$$

Starting pivot element $(3, 3)$, solve the problem with the (dual) simplex algorithm.

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