Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 4, 1996

SOLUTION OF THE PROBLEM OF A BEAM WITH A CUT

V. A. Kovtunenko

UDC 539.3+519.6

In this paper, we consider the Kirckhoff model for a thin elastic beam which is fixed at the edges and is in equilibrium under the action of external load. The beam has a transverse cut at whose edges the condition of mutual nonpenetration is imposed. The present model is described by the energy-functional minimization problem or by an equivalent variational inequality. The solution of the problem of a beam without a cut is used to derive an analytical solution of the equilibrium problem for a beam with a cut. The main characteristics of the state of the beam are determined using the solution obtained. Examples of solution are presented for particular functions of external forces.

Next, we consider the problem of optimal control of a cut with two criteria: minimum opening of the cut and minimum departure of stresses from given values. In both cases, we study the behavior of optimal cuts and present examples of solutions.

Formulations of problems of elastic bodies with cracks (cuts) can be found, for example, in [1, 2]. Nonpenetration boundary conditions at the crack edges were stated by Khludnev [3, 4]. Glovinsky et al. [5, 6] have solved minimization problems with a constraint and variational inequalities. Some examples of exact solutions of variational inequalities are given in [7, 8]. The problems of optimal control for plates with cracks are considered in [3, 4]. A numerical method of solving contact problems for plates is proposed in [9, 10].

The Problem of the Equilibrium of a Beam. Let the median of a beam coincide with segment $\Omega = (a, b)$. We look for the function of displacements of the beam's points $w(x)$ under the action of a given external force $f(x)$ (Fig. 1). If $f \in L^2(\Omega)$, then $w \in H^2(\Omega) \cap H_0^1(\Omega)$ is a unique solution of the boundary-value problem

$$-D^2 w = f, \qquad w(a) = w(b) = 0. \tag{1}$$

Let the beam have a transverse cut at the point $y, a < y < b$. We seek the displacement function $u(x)$ for the beam (Fig. 1). The condition of nonpenetration for the cut edges is written as in [3, 4]:

$$[u]_y = u(y+0) - u(y-0) \geqslant 0.$$

Denote $\Omega_y = \Omega \backslash \{y\}$. We define the principal Hilbert space as

$$X_y = \{u \in H^1(\Omega_y), \quad u(a) = u(b) = 0\}$$

and the closed convex set as

$$K_y = \{u \in X_y, \quad [u]_y \geqslant 0\}.$$

We introduce the scalar product $(u, v)_y = \langle Du, Dv \rangle_y$ in X_y and the equivalent norm $\|u\|_y^2 = (u, u)_y$, where D is a differentiation operator; $\langle \cdot, \cdot \rangle_y$ denotes integration over Ω_y. The potential energy of the beam is defined by $\Pi_y(v) = 0.5\|v\|_y^2 - \langle f, v \rangle_y$. Then, the problem of the equilibrium of the beam consists in minimization of the functional Π_y [3, 4]:

$$\inf_{v \in K_y} \Pi_y(v) = \Pi_y(u). \tag{2}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 4, pp. 160–166, July–August, 1996. Original article submitted June 5, 1995.

0021-8944/96/3704-0595 $15.00

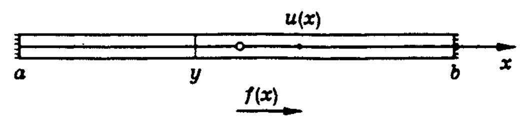

Fig. 1

Another equivalent formulation of problem (2) has the form of the variational inequality

$$u \in K_y, \quad (u, v-u)_y \geqslant \langle f, v-u\rangle_y \qquad \forall v \in K_y. \tag{3}$$

We introduce the notation

$$a^+ = \begin{cases} a, & a > 0, \\ 0, & a \leqslant 0, \end{cases} \qquad a^- = \begin{cases} 0, & a \geqslant 0, \\ -a, & a < 0, \end{cases}$$

from which we have $a = a^+ - a^-$, $a^+, a^- \geqslant 0$, $a^+a^- = 0$.

Theorem 1. *A unique solution of problem* (2) *is the function*

$$u(x) = w(x) - Dw^+(y)\alpha_y(x), \tag{4}$$

where

$$\alpha_y(x) = \begin{cases} x-a, & x \in (a, y-0), \\ x-b, & x \in (y+0, b); \end{cases}$$

w is a solution of problem (1).

Proof. By virtue of the equivalence of (2) and (3) we shall prove (3). Note the following properties of the function $\alpha_y \in C^\infty(\Omega_y) \cap X_y$ (Fig. 2):

$$[\alpha_y]_y = -(b-a), \quad D\alpha_y(x) \equiv 1 \quad (x \neq y), \qquad D^2\alpha_y(x) \equiv 0 \quad (x \neq y).$$

Then, integrating by parts, we obtain $(\alpha_y, h)_y = \langle -D^2\alpha_y, h\rangle_y - [D\alpha_y h]_y = -[h]_y$. Further, on the strength of (1), $\langle f, h\rangle_y = \langle -D^2 w, h\rangle_y = (w, h)_y + Dw(y)[h]_y$, $(u, h)_y = (w, h)_y - Dw^+(y)(\alpha_y, h)_y = (w, h)_y + Dw^+(y)[h]_y$. is valid. We calculate $[u]_y = [w]_y - Dw^+(y)[\alpha_y]_y = (b-a)Dw^+(y)$. Thus, we have $(u, v-u)_y - \langle f, v-u\rangle_y = (Dw^+(y) - Dw(y))[v-u]_y = Dw^-(y)[v-u]_y = Dw^-(y)[v]_y - (b-a)Dw^-(y)Dw^+(y) = Dw^-(y)[v]_y \geqslant 0$ $\forall v \in K_y$. Since $[u]_y \geqslant 0$, u given by formula (4) belongs to K_y and is a solution of (3). Let us demonstrate its uniqueness. Let u_1 and u_2 be two solutions of variational inequality (3), i.e., we have $(u_1, v_1 - u_1)_y \geqslant \langle f, v_1 - u_1\rangle_y$ $\forall v_1 \in K_y$ and $(u_2, v_2 - u_2)_y \geqslant \langle f, v_2 - u_2\rangle_y$ $\forall v_2 \in K_y$. We take $v_1 = u_2$ and $v_2 = u_1$ and combine the two inequalities. We then obtain $\|u_1 - u_2\|_y^2 \leqslant 0$, from which follows the uniqueness of the solution. The theorem is proved.

Remark 1. It follows from (1) that the function w and, hence, the solution u belong to the class $H^2(\Omega_y)$. Then, by virtue of the properties of w and α_y, solution (4) of problem (3) is a solution of the boundary-value problem

$$-D^2 u = f \quad \text{in} \quad \Omega_y, \qquad [u]_y = (b-a)Dw^+(y), \qquad [Du]_y = 0, \quad Du(y) = -Dw^-(y).$$

Remark 2 (smoothness of the solution). It follows from (1) and (4), that if $f \in H^n(\Omega_y)$ $(n \geqslant 0)$, we have $u \in H^{n+2}(\Omega_y)$. If $f \in C^n(\Omega_y)$ $(n \geqslant 0)$, we have $u \in C^{n+2}(\Omega_y)$.

Remark 3 (inverse problem). Let an arbitrary function u belong to the class $H^2(\Omega_y) \cap X_y$ and let the following be valid for the function: $[Du]_y = 0$, $Du(y)[u]_y = 0$, $[u]_y \geqslant 0$, $Du(y) \leqslant 0$. Then u is a solution of problem (3) for $f = -D^2 u$.

Let us prove the latter result. We have $u \in K_y$. Integrating by parts, we obtain

$$(u, v-u)_y - \langle f, v-u\rangle_y = \langle -D^2 u - f, v-u\rangle_y - [Du(v-u)]_y = -Du(y)[v]_y + Du(y)[u]_y = -Du(y)[v]_y \geqslant 0$$

for any $v \in K_y$, which proves the theorem.

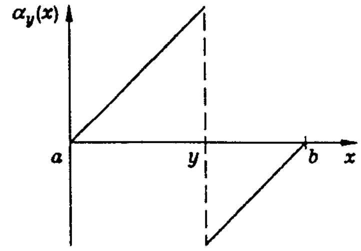

Fig. 2

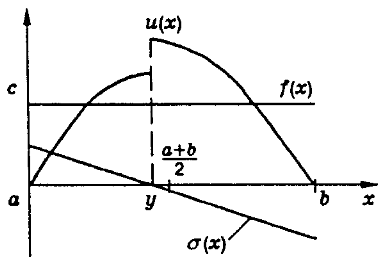

Fig. 3

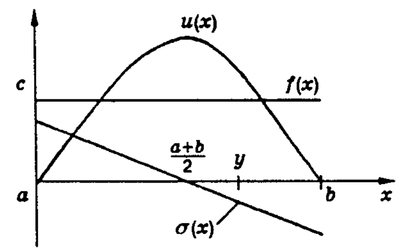

Fig. 4

Having found the displacement function $u(x) = w(x) - Dw^+(y)\alpha_y(x)$, we can calculate all physical characteristics of the solution:
the strain $\varepsilon(x)$ or stress $\sigma(x)$

$$\varepsilon(x) = \sigma(x) = Du(x) = Dw(x) - Dw^+(y)$$

[it should be noted that the stress $\sigma(x)$ is a function that is continuous on (a, b), i.e., it belongs to the class $H^1(\Omega_y) \cap C(a, b)$],
the contact force

$$p = -\sigma(y) = Dw^-(y) \geqslant 0,$$

and the potential energy of the beam

$$\Pi(u) = 0.5\|u\|_y^2 - \langle f, u\rangle_y = -0.5\|u\|_y^2 - [Duu]_y = -0.5\|Dw - Dw^+(y)\|_0^2.$$

Here $\|\cdot\|_0$ denotes the norm in $L_2(a, b)$.

Examples of Exact Solutions. Example 1. Let $f(x) \equiv c \quad (c \geqslant 0)$, then $w(x) = -0.5c(x-a)(x-b)$, $Dw(y) = c(0.5(a+b) - y)$. If $a < y \leqslant 0.5(a+b)$, we have

$$u(x) = -\frac{c}{2}\begin{cases} (x-a)(x+a-2y), & x \in (a, y-0), \\ (x-b)(x+b-2y), & x \in (y+0, b), \end{cases}$$

$$[u]_y = 0.5c(b-a)(a+b-2y) \geqslant 0, \quad \sigma(u) = c(y-x), \quad p = 0 \qquad \text{(Fig. 3)}.$$

If $0.5(a+b) \leqslant y < b$, then $u(x) = -0.5c(x-a)(x-b)$, $[u]_y = 0$, $\sigma(u) = 0.5c(a+b-2x)$, $p = 0.5c(2y-(a+b))$ (Fig. 4). The curves of $[u]_y$ versus p on y are shown in Fig. 5.

Example 2. Let $f(x) = \sin k(x-a)$ $[k = \pi/(b-a)]$, then

$$w(x) = k^{-2}\sin k(x-a), \quad Dw(x) = k^{-1}\cos k(x-a).$$

Hence, we have

$$u(x) = k^{-2}\begin{cases} \sin k(x-a), & y \geqslant 0.5(a+b), \\ \sin k(x-a) - k\cos k(y-a)\alpha_y(x), & y \leqslant 0.5(a+b) \end{cases} \qquad \text{(Fig. 6)},$$

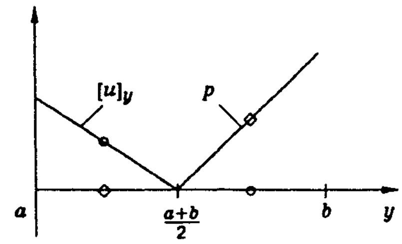

Fig. 5

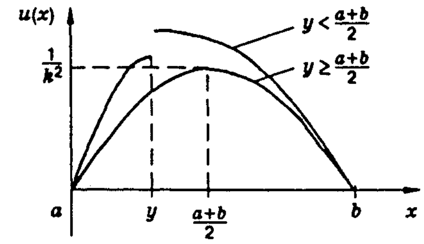

Fig. 6

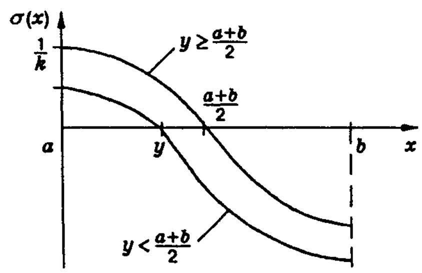

Fig. 7

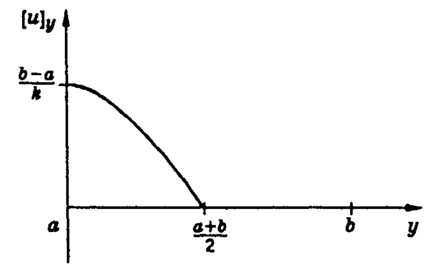

Fig. 8

$$\sigma(x) = k^{-1}\begin{cases} \cos k(x-a), & y \geqslant 0.5(a+b), \\ \cos k(x-a) - \cos k(y-a), & y \leqslant 0.5(a+b) \end{cases} \qquad \text{(Fig. 7)},$$

$$p = k^{-1}\begin{cases} -\cos k(y-a), & y \geqslant 0.5(a+b), \\ 0, & y \leqslant 0.5(a+b). \end{cases}$$

The jump

$$[u]_y = k^{-1}(b-a)\begin{cases} 0, & y \geqslant 0.5(a+b), \\ \cos k(y-a), & y \leqslant 0.5(a+b) \end{cases}$$

is shown in Fig. 8.

Optimal Control of Cut. 1) Let us consider the problem of minimization of the crack opening [2]:

$$\inf_{a<y<b} [u]_y, \tag{5}$$

where u is a solution of problem (2). By virtue of (4), problem (5) is equivalent to

$$\inf_{a<y<b} Dw^+(y). \tag{6}$$

Let us define the following sets:

$$I^+(w) = \{y \in \Omega, \quad Dw(y) > 0\}, \qquad I^-(w) = \{y \in \Omega, \quad Dw(y) \leqslant 0\}.$$

We then have $I^+(w) \cup I^-(w) = \Omega$. Since the function $w \in C^1(\Omega)$ (on the strength of embedding theorems), Dw is a continuous function. Since $w(a) = w(b) = 0$, we have $I^-(w) \neq \varnothing$. Thus, by virtue of $Dw^+(y) \geqslant 0$, we obtain $\inf_{a<y<b} [u]_y = 0$ for any $y \in I^-(w)$.

Example 1. For $f(x) \equiv c\,(c \geqslant 0)$ and for any $0.5(a+b) \leqslant y < b$, the function $u(x) = -0.5c(x-a)(x-b)$ is a solution of problem (5) (see Fig. 5).

Example 2. For $f(x) = \sin k(x-a)$ $[k = \pi/(b-a)]$ and for any $0.5(a+b) \leqslant y < b$,

$$u(x) = k^{-2} \sin k(x-a)$$

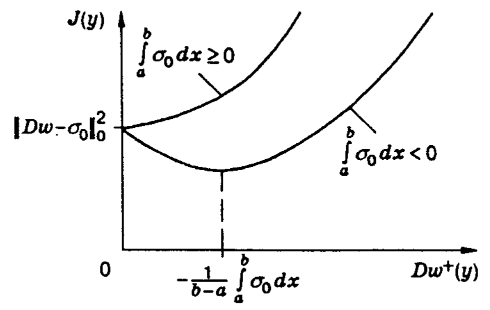

Fig. 9

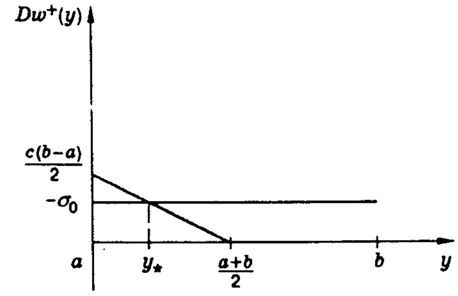

Fig. 10

is a solution of problem (5) (Fig. 8).

2) We consider the problem of optimization of stresses

$$\inf_{a<y<b}\{J(y) = \|\sigma - \sigma_0\|_0^2\}, \tag{7}$$

where $\sigma_0 \in L_2(\Omega)$ is the given stress function, and $\sigma = Du$ [u is a solution of problem (2)]. In view of the properties of the function w, we obtain

$$J(y) = \|Dw - Dw^+(y) - \sigma_0\|_0^2 = \|Dw - \sigma_0\|_0^2 + 2\int_a^b \sigma_0 dx Dw^+(y) + (b-a)(Dw^+(y))^2.$$

If $\int_a^b \sigma_0 dx \geqslant 0$, then $\inf_{a<y<b} J(y) = \|Dw - \sigma_0\|_0^2$ is obtained at any $y \in I^-(w)$ [then $Dw^+(y) = 0$]. If $\int_a^b \sigma_0 dx < 0$ and $I^+(w) \neq \varnothing$, then the infimum is obtained at y_* such that $Dw^+(y_*) + (b-a)^{-1}\int_a^b \sigma_0 dx \to \inf$. In particular, if there is $y_* \in I^+(w)$ such that $Dw^+(y_*) = -(b-a)^{-1}\int_a^b \sigma_0 dx$, we have

$$J(y_*) = \|Dw - \sigma_0\|_0^2 - (b-a)^{-1}\Big(\int_a^b \sigma_0 dx\Big)^2 \qquad \text{(Fig. 9)}.$$

We take as an example the case where $\sigma_0(x) \equiv \text{const}$ and $\sigma_0 < 0$ and, hence,

$$(b-a)^{-1}\int_a^b \sigma_0 dx = \sigma_0 < 0.$$

Example 1. Let $f(x) \equiv c$ ($c \geqslant 0$). Then

$$Dw^+(y) = \begin{cases} 0.5c(a+b-2y), & a < y \leqslant 0.5(a+b), \\ 0, & 0.5(a+b) \leqslant y < b. \end{cases}$$

If $-\sigma_0 < 0.5c(b-a)$, then at the point $y_* = 0.5(a+b) + \sigma_0/c$ we have $Dw^+(y_*) = -\sigma_0$ (Fig. 10) and the minimum of (7) is reached as

$$J(y_*) = \|0.5\,c(a+b-2x)\|_0^2 = \frac{c^2(b-a)^3}{12}.$$

If $-\sigma_0 \geqslant 0.5c(b-a)$, then $y_* = a$, and the infimum

$$J(a) = \|0.5\,c(a+b-2x) - 0.5c\,(b-a) - \sigma_0\|_0^2 = \frac{c^2(b-a)^3}{12}\Big(1 + 3\Big(1 + \frac{2\sigma_0}{c(b-a)}\Big)^2\Big)$$

is not reached. If $\sigma_0 \geqslant 0$, then for any $y_* \in I^-(w)$ the minimum of (7) is

$$J(y_*) = \|0.5\, c(a+b-2x) - \sigma_0\|_0^2 = \frac{c^2(b-a)^3}{12}\Big(1 + 3\Big(\frac{2\sigma_0}{c(b-a)}\Big)^2\Big).$$

Example 2. Let $f(x) = \sin k(x-a)$ $[k = \pi/(b-a)]$. Then

$$Dw^+(y) = k^{-1}\begin{cases} \cos k(y-a), & a < y \leqslant 0.5(a+b), \\ 0, & 0.5(a+b) \leqslant y < b. \end{cases}$$

Let $\sigma_0 < 0$. If $-\sigma_0 < k^{-1}$, we have $Dw^+(y_*) = -\sigma_0$ at the point $y_* = a + k^{-1}\arccos(-k\sigma_0)$ and the minimum

$$J(y_*) = \|Dw(x)\|_0^2 = \frac{b-a}{2k^2}$$

is reached. If $-\sigma_0 \geqslant k^{-1}$, the infimum of $J(y)$ is not reached and has the form

$$J(a) = \|k^{-1}\cos k(x-a) - k^{-1} - \sigma_0\|_0^2 = (b-a)\left(\frac{1}{2k^2} + (k^{-1}+\sigma_0)^2\right).$$

If $\sigma_0 \geqslant 0$, then for any $y_* \geqslant 0.5(a+b)$ the minimum of $J(y)$ is

$$J(y_*) = \|k^{-1}\cos k(x-a) - \sigma_0\|_0^2 = (b-a)\left(\frac{1}{2k^2} + \sigma_0^2\right).$$

This work was supported by the Russian Foundation for Fundamental Research (Grant 95-01-00886a).

REFERENCES

1. N. F. Morozov, *Mathematical Issues of the Theory of Cracks* [in Russian], Nauka, Moscow (1984).
2. V. V. Panasyuk, A. V. Andreikiv, and S. E. Kovchik, *Methods of Estimating the Fracture Toughness of Structural Materials* [in Russian], Kiev, 1977.
3. A. M. Khludnev, "On extremal shapes of cuts in a plate," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 1, 170–176 (1992).
4. A. M. Khludnev, "Existence of extreme unilateral cracks in a plate," *Contr. Cybern.*, **23**, No. 3, 453–460 (1994).
5. R. Glovinsky, J. L. Lyons, and R. Tremoller, *Numerical Analysis of Variational Inequalities* [Russian translation], Mir, Moscow (1979).
6. M. Minoux, *Mathematical Programming: Theory and Algorithms* [Russian translation], Nauka, Moscow (1990).
7. G. Cimatti, "The constrained elastic beam," *Meccanica*, **86**, No. 2, 119–124 (1973).
8. V. Barbu and P. Korman, "Approximating optimal controls for elastic obstacle problem by monotone iteration schemes," *Numer. Funct. Anal. Optimiz.*, No. 12, 429–442 (1991).
9. V. A. Kovtunenko, "Iterative method for solving variational inequalities of a contact elastoplastic problem using the penalty method," *Zh. Vychis. Mat. Mat. Fiz.*, No. 9, 1409–1415 (1993).
10. V. A. Kovtunenko, "Numerical method of solving the problem of the contact of an elastic plate with an obstacle," *Prikl. Mekh. Tekh. Fiz.*, **35**, No. 5, 142–146 (1994).