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# ASYMPTOTIC ANALYSIS OF THE PROBLEM OF EQUILIBRIUM OF AN INHOMOGENEOUS BODY WITH HINGED RIGID INCLUSIONS OF VARIOUS WIDTHS

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**Abstract:** Two models are considered, which describe the equilibrium state of an inhomogeneous two-dimensional body with two connected rigid inclusions. The first model corresponds to an elastic body with two-dimensional rigid inclusions located in regions with a constant width (curvilinear rectangle and trapezoid). The second model involves thin inclusions described by curves. In both models, it is assumed that there is a crack described by the same curve on the interface between the elastic matrix and rigid inclusions. The crack boundaries are subjected to a one-sided condition of non-penetration. The dependence of the solutions of equilibrium problems on the width of two-dimensional inclusions is studied. It is shown that the solutions of equilibrium problems in the presence of two-dimensional inclusions in a strong topology are reduced to the solutions of problems for thin inclusions with the width parameter tending to zero.

*Keywords:* variational problem, rigid inclusion, non-penetration condition, elastic matrix, hinged connection.

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## INTRODUCTION

The interest to studying mathematical models that describe various specific features of inhomogeneous bodies is caused by numerous applications of composite materials in industry, e.g., in mechanical engineering. The geometric characteristics of structural elements of composite bodies can affect the strength properties of the structure as a whole. In particular, variation of the geometric parameter in mathematical models can lead to a topological change in the dimension. It is known that a change in the dimension of a geometric model is responsible for difficulties associated with justification of formal asymptotic methods. Therefore, studying limiting transitions in mathematical models including geometric objects of different dimensions is an urgent problem.

Physical features of inhomogeneous bodies with inclusions, such as the difference in the elasticity moduli and thermal expansion coefficients for the matrix and inclusions can lead to the formation of cracks at the interface of different materials. Problems of the crack theory that describe inhomogeneous bodies with rigid inclusions and linear boundary conditions on the crack were studied, e.g., in [1–4]. Khludnev and Kovtunenko [5] proposed a variational theory of cracks with nonlinear conditions of non-penetration between their edges. Non-penetration conditions are applied for a class of problems on composite bodies and composites containing rigid inclusions

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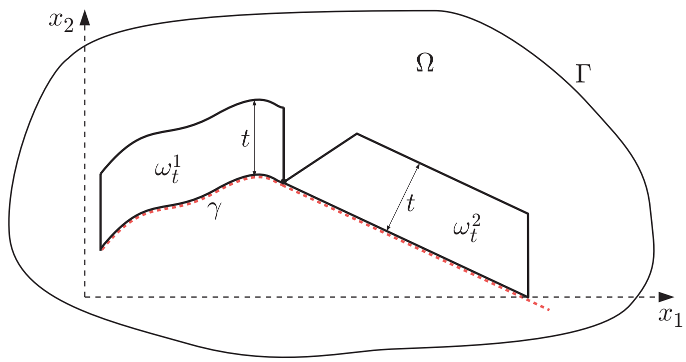


Geometry of a two-dimensional composite body with two-dimensional inclusions.

(see [6–10]). It should be noted that the analysis of nonlinear problems for cracked homogeneous and inhomogeneous bodies is complicated by the non-smooth character of their boundaries (no Lipschitz properties) [11–15].

The asymptotic analysis of variations of the shape and topology of cracked bodies for variational problems was performed in [16–22] and other publications. In particular, for models with non-penetration conditions on the crack, problems of optimal control of the shape and location of a small elastic inclusion, where the quality functional was determined in accordance with the Griffith fracture criterion, were considered in [23, 24]. The limiting transition for the elastic inclusion width tending to zero was proved in [25], and an asymptotic model of an elastic body containing a thin inclusion was developed.

Explicit formulas for the derivative of energy with respect to the parameter of changes in the shape of rigid inclusions were derived in [26, 27]. Problems of equilibrium for composite bodies with thin elastic of thin rigid inclusions under the boundary conditions of the Signorini condition type were studied, e.g., in [28–31] (a thin inclusion is understood as an inclusion whose dimension is smaller by one than the dimension of the body as a whole).

In the present study, we consider a nonlinear model that describes a two-dimensional body with two rigid inclusions connected by a hinge, i.e., it is assumed that both bodies have a common point where they are hinged. Moreover, the inclusions separate from the elastic matrix on some part of the boundary, thus, forming a crack. For problems that describe a body with two two-dimensional inclusions (the dimension of the two-dimensional inclusion coincides with the body dimension), it was proved that the limiting transition with the inclusion width parameter $t$ tending to zero yields a problem of two hinged thin rigid inclusions. The present study is a continuation of [32–35], where an asymptotic transition between the models of elastic bodies with manifolds of different dimensions was justified. In contrast to the models in [32–35], the present model contains two inclusions rather than one.

## 1. VARIATIONAL PROBLEMS

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the boundary $\Gamma$ of the class $C^{0,1}$. Let us also assume that the boundary $\Gamma$ consists of two non-intersecting segments $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\operatorname{meas}(\Gamma_0) > 0$ and $\operatorname{meas}(\Gamma_1) > 0$. Let us assume that the curve $\gamma$ is located in the domain $\Omega$, which can be divided into two subdomains $\Omega_1$ and $\Omega_2$ with the Lipschitz boundaries $\partial\Omega_1$ and $\partial\Omega_2$, so that the conditions $\gamma \subset \partial\Omega_1 \cap \partial\Omega_2$ and $\operatorname{meas}(\partial\Omega_i \cap \Gamma_0) > 0$ $(i = 1, 2)$ are satisfied. The last condition is necessary to satisfy the Korn inequality in the non-Lipschitz domain $\Omega_\gamma = \Omega \setminus \bar{\gamma}$. In what follows, the curve $\gamma$ corresponds to a crack.

### *1.1. Test Case with Two-Dimensional Inclusions*

Let us consider two families of subdomains in $\Omega$. The first family consists of the subdomains $\omega_t^1 \subset \Omega$, each shaped as a curvilinear rectangle of width $t \in (0, T]$:

$$\omega_t^1 = \{(x_1, x_2) \in \mathbb{R}^2\colon \quad g(x_1) < x_2 < g(x_1) + t, \quad x_1 \in (s_1, s_2)\}.$$

Here $g \in C^{0,1}(s_1, s_2)$ is a specified function ($s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$, and $s_1 < s_2$). Let us assume that the common lower side

$$\gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2\colon \quad x_2 = g(x_1), \quad x_1 \in (s_1, s_2)\}$$

of the curvilinear rectangles $\omega_t^1$ is part of the curve $\gamma$ (see the figure). The second family consists of trapezoids of the form $\omega_t^2 \subset \Omega$ whose width is specified by the parameter $t$. The trapezoids $\omega_t^2$ have a common fixed lower side $\gamma_2$, such that $\gamma_2 \subset \gamma$ and $\gamma_2 \cap \gamma_1 = \varnothing$. Let us assume that the choice of these two families of the domains ensures the validity of the following geometric conditions for all $t \in (0, T]$:

(1) The closures of the domains and the corresponding curves have one common intersection point

$$\bar{\omega}_t^1 \cap \bar{\omega}_t^2 = \bar{\gamma}_2 \cap \bar{\gamma}_1 = x^* = (x_1^*, x_2^*);$$

(2) The curve $\gamma$ can be continued to a certain closed curve $\Gamma$ bounding the domain $O$, such that $\overline{O} \subset \Omega$, $\partial O = \Gamma$, and the greatest inclusions are $\omega_T^i \subset O$ $(i = 1, 2)$. The boundaries of the domains $O \setminus \overline{\omega_t^1 \cup \omega_t^2}$ are the Lipschitz boundaries for all $t \in (0, T]$.

For the displacement vector $W = (w_1, w_2)$, the tensor components that describe the strain and stress of the elastic inhomogeneous body are expressed as

$$\varepsilon_{ij}(W) = (w_{i,j} + w_{j,i})/2, \qquad i, j = 1, 2,$$

$$\sigma_{ij}(W) = c_{ijkl}\varepsilon_{ij}(W), \qquad i, j = 1, 2,$$

where $c_{ijkl}$ is a specified tensor of the elasticity moduli, which is assumed to be symmetric and positively determined:

$$c_{ijkl} = c_{klij} = c_{jikl}, \quad i, j, k, l = 1, 2, \qquad c_{ijkl} = \text{const},$$

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2 \quad \forall \xi\colon \quad \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = \text{const}, \quad c_0 > 0.$$

Let us introduce the Sobolev space

$$H^1_{\Gamma_0}(\Omega_\gamma) = \{w \in H^1(\Omega_\gamma)\colon w = 0 \text{ on } \Gamma_0\}, \qquad H(\Omega_\gamma) = H^1_{\Gamma_0}(\Omega_\gamma)^2.$$

In accordance with the assumption that the domain $\Omega_\gamma$ can be divided into the Lipschitz subdomains, the Korn inequality yields the uniform estimate from below

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(W)\varepsilon_{ij}(W)\, dx \geq c\|W\|^2_{H(\Omega_\gamma)} \quad \forall\, W \in H(\Omega_\gamma) \tag{1}$$

with a constant $c > 0$ independent of $W$ (see [5, 36]).

**Remark 1.** Inequality (1) ensures the equivalence of the standard norm in $H(\Omega_\gamma)$ and the half-norm determined by its left side, as well as the coercitivity of the energy functional $\Pi(W)$ defined below.

To formulate the problem of equilibrium of an elastic body with hinged two-dimensional inclusions, we fix the parameter $t \in (0, T]$. Let us assume that the domains $\omega_t^1$ and $\omega_t^2$ correspond to the rigid inclusions, while the complement $\Omega_\gamma \setminus \overline{\omega_t^1 \cup \omega_t^2}$ corresponds to the elastic body. Following the approach [37], we describe the movable rigid inclusion corresponding to some set $Q \subset \Omega$ by setting the fields of displacements on $Q$, so that to satisfy the condition $W|_Q = \rho$, $\rho \in R(Q)$, where $R(Q)$ is the space of infinitesimal rigid displacements [37]:

$$R(Q) = \{\rho = (\rho_1, \rho_2)\colon \quad \rho(x) = b(x_2, -x_1) + (c_1, c_2); \quad b, c_1, c_2 \in \mathbb{R}, \quad x \in Q\}.$$

The presence of two two-dimensional rigid inclusions that occupy the domains $\omega_t^1$ and $\omega_t^2$ is modeled by the relations

$$W\Big|_{\omega_t^i} = \rho^i, \quad \rho^i \in R(\omega_t^i), \quad i = 1, 2.$$

**Remark 2.** The conditions $W \in H(\Omega_\gamma)$ and $W|_{\omega_t^i} = \rho^i$, $\rho^i \in R(\omega_t^i)$ $(i = 1, 2)$ mean that the functions

$$\rho^i = b^i(x_2, -x_1) + (c_1^i, c_2^i), \qquad (x_1, x_2) \in \omega_t^i$$

can be continued to the boundary of the sets $\bar{\omega}_t^1$ and $\bar{\omega}_t^2$ with the previous values

$$\rho^i(x) = b^i(x_2, -x_1) + (c_1^i, c_2^i), \qquad (x_1, x_2) \in \bar{\omega}_t^i,$$

thus, obtaining the equality at the hinge point

$$\rho^1(x^*) = \rho^2(x^*),$$

where

$$W\Big|_{\omega_t^i} = \rho^i, \qquad \rho^i \in R(\bar{\omega}_t^i), \quad i = 1, 2.$$

The condition of mutual non-penetration on two opposite edges of the crack $\gamma^+$ [above the curve $\gamma$ (see the figure)] and $\gamma^-$ [below the curve $\gamma$ (see the figure)] is imposed according to [5]:

$$[W]\nu \geq 0 \quad \text{on} \quad \gamma. \tag{2}$$

Here $\nu = (\nu_1, \nu_2)$ is the unit vector of the normal to $\gamma$ directed inward the inclusions (see the figure), and $[v] = v|_{\gamma^+} - v|_{\gamma^-}$ is the jump of the function $v$ on $\gamma$. The uniform Dirichlet condition is set on the segment $\Gamma_0$ of the external boundary $\Gamma$. Let us introduce the energy functional

$$\Pi(W) = \frac{1}{2} \int\limits_{\Omega_\gamma} \sigma_{ij}(W)\varepsilon_{ij}(W)\,dx - \int\limits_{\Omega_\gamma} FW\,dx,$$

where $F = (f_1, f_2) \in L^2(\Omega_\gamma)^2$ is the specified vector of external forces. The problem of the equilibrium state of a cracked body can be formulated in the form of the minimization problem: it is necessary to find the values of $U_t \in K_t$ such that

$$\Pi(U_t) = \inf_{W \in K_t} \Pi(W), \tag{3}$$

where $K_t$ is the set of admissible displacements:

$$K_t = \{W \in H(\Omega_\gamma)\colon \quad W\Big|_{\omega_t^i} = \rho^i, \quad \rho^i \in R(\omega_t^i), \quad i = 1, 2\}.$$

It can be easily demonstrated that the set $K_t$ is convex and closed; hence, it is weakly closed. The coercitivity of the quadratic energy functional $\Pi(W)$ (1) guarantees that problem (3) has a unique solution. Moreover, the functional $\Pi(W)$ is Gateaux-differentiable, and problem (3) is equivalent to the following variational inequality (see [5]):

$$U_t \in K_t, \qquad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_t)\varepsilon_{ij}(W - U_t)\,dx \geq \int\limits_{\Omega_\gamma} F(W - U_t)\,dx \quad \forall\, W \in K_t. \tag{4}$$

Following [6], we can demonstrate that the variational problem (3) with a sufficiently smooth solution is equivalent to the problem in the differential formulation

$$-\sigma_{ij,j}(U_t) = f_i \quad \text{in} \quad \Omega \setminus \overline{\omega_t^1 \cup \omega_t^2}, \quad i = 1, 2,$$

$$\sigma_\tau^-(U_t) = (0, 0), \quad [U_t]\nu \geq 0, \quad \sigma_\nu^-(U_t) \leq 0, \quad \sigma_\nu^-(U_t)([U_t]\nu) = 0 \quad \text{on} \quad \gamma,$$

$$U_t = 0 \quad \text{on} \quad \Gamma_0,$$

$$U_t\Big|_{\omega_t^2} = \rho_t^1, \quad U_t\Big|_{\omega_t^2} = \rho_t^2, \quad \rho_t^i \in R(\omega_t^i), \quad i = 1, 2,$$

$$-\int\limits_{\partial\omega_t^1} \sigma_{ij}(U_t)n_j^1\rho_i^1 - \int\limits_{\partial\omega_t^2} \sigma_{ij}(U_t)n_j^2\rho_i^2 = \int\limits_{\omega_t^1} F\rho^1 + \int\limits_{\omega_t^2} F\rho^2$$

for all $\rho^1 = (\rho_1^1, \rho_2^1) \in R(\omega_t^1)$ and $\rho^2 = (\rho_1^2, \rho_2^2) \in R(\omega_t^2)$. Here $n^1 = (n_1^1, n_2^1)$ is the external normal to $\omega_t^1$, $n^2 = (n_1^2, n_2^2)$ is the external normal to $\omega_t^2$, and

$$\sigma_\nu(U_t) = \sigma_{ij}(U_t)\nu_i\nu_j; \tag{5}$$

$$\sigma_\tau(U_t) = (\sigma_\tau^1(U_t), \sigma_\tau^2(U_t)) = (\sigma_{1j}(U_t)\nu_j, \sigma_{2j}(U_t)\nu_j) - \sigma_\nu(U_t)\nu. \tag{6}$$

### 1.2. Test Case of Thin Inclusions

In addition to the problems on the equilibrium state of an elastic body with two-dimensional rigid inclusions, we also formulate a problem of the equilibrium state of an elastic body with two hinged thin inclusions. These inclusions correspond to the curves $\gamma_1$ and $\gamma_2$. Similar to the test case considered above (see Section 1.1), we assume that the body contains a crack located at the interface of different materials. In the undeformed state, the crack is described by the same curve $\gamma$. For the case of thin inclusions, we determine the convex and closed set of admissible displacements

$$K_0 = \{W \in H(\Omega_\gamma)\colon \quad W\Big|_{\gamma_i^+} = \rho^i, \quad \rho^i \in R(\gamma_i), \quad i = 1, 2\}.$$

Similar to that in Remark 2, we can continue the functions $\rho^i \in R(\gamma_i)$ $(i = 1, 2)$ to the closed sets $\bar{\gamma}_i$ $(i = 1, 2)$ and write the equality at the hinge point

$$\rho^1(x^*) = \rho^2(x^*),$$

where

$$W\Big|_{\gamma_i^+} = \rho^i, \quad \rho^i \in R(\bar{\gamma}_i), \quad i = 1, 2.$$

The same properties $K_0$ and $\Pi(W)$ as those for problem (3) allow us to prove the existence and uniqueness of the solution $U_0$ of the following variational problem:

$$\Pi(U_0) = \inf_{W \in K_0} \Pi(W). \tag{7}$$

We can show that this problem is equivalent to the variational inequality (see [6])

$$U_0 \in K_0, \qquad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_0)\varepsilon_{ij}(W - U_0)\,dx \geq \int\limits_{\Omega_\gamma} F(W - U_0)\,dx \quad \forall\, W \in K_0. \tag{8}$$

The following statement is also valid: if the solution $U_0$ is sufficiently smooth, problem (5) is equivalent to the following problem in the differential formulation [6]:

$$-\sigma_{ij,j}(U_0) = f_i \quad \text{in} \quad \Omega_\gamma, \quad i = 1, 2,$$

$$\sigma_\tau^-(U_0) = (0, 0), \quad [U_0]\nu \geq 0, \quad \sigma_\nu^-(U_0) \leq 0, \quad \sigma_\nu^-(U_0)([U_0]\nu) = 0 \quad \text{on} \quad \gamma,$$

$$U_0 = 0 \quad \text{on} \quad \Gamma_0,$$

$$U_0\Big|_{\gamma_1} = \rho_0^1, \quad U_0\Big|_{\gamma_2} = \rho_0^2, \quad \rho_0^i \in R(\gamma_i), \quad i = 1, 2,$$

$$\int\limits_{\gamma_1} [\sigma_{ij}(U_0)\nu_j]\rho_i^1 + \int\limits_{\gamma_2} [\sigma_{ij}(U_0)\nu_j]\rho_i^2 = 0$$

for all $\rho^1 = (\rho_1^1, \rho_2^1) \in R(\gamma_1)$ and $\rho^2 = (\rho_1^2, \rho_2^2) \in R(\gamma_2)$. The functions $\sigma_\nu^-(U_0)$ and $\sigma_\tau^-(U_0)$ are determined in the same way as the functions $\sigma_\nu^-(U_t)$ and $\sigma_\tau^-(U_t)$ in Eqs. (5) and (6).

## 2. LIMITING TRANSITION

Let us show that problem (8) can be obtained as a limiting problem for problems with two-dimensional inclusions (4) as $t \to 0$. Let us prove that the solutions $\{U_t\}$ of problems (4) strongly converge to $U_0$ as $t \to 0$ in $H(\Omega_\gamma)$.

To justify the above-mentioned limiting transition, we construct converging sequences of the test functions from the admissible sets $K_t$ and $K_0$.

**Lemma 1.** *Let $\{t_n\} \subset (0, T]$ be a sequence of real numbers converging to zero as $n \to \infty$. Then, for an arbitrary function $W \in K_0$, there exists a subsequence $\{t_k\} = \{t_{n_k}\} \subset \{t_n\}$ and a sequence of the functions $\{W_k\}$, such that $W_k \in K_{t_k}$ $(k \in \mathbb{N})$, and $W_k \to W$ strongly in $H(\Omega_\gamma)$ as $k \to \infty$.*

**Proof.** If the subsequence is $t_{n_k} = 0$, then, Lemma 1 is valid for $W_k \equiv W$ $(k \in \mathbb{N})$. Thus, it is sufficient to consider the case with $t_n > 0$ for sufficiently large $n$. Let us use $\rho^i = (b^i x_2 + c_1^i, -b^i x_1 + c_2^i)$ to denote the rigid displacements on $\gamma_i$ $(i = 1, 2)$ and extend the domains of definition of $\rho^1$ and $\rho^2$ to the greatest domains $\omega_T^i$ $(i = 1, 2)$ with the equalities

$$\rho^i = (b^i x_2 + c_1^i, -b^i x_1 + c_2^i), \quad x = (x_1, x_2) \in \omega_T^i \quad (i = 1, 2).$$

To construct the sought subsequence $\{W_k\}$, we consider auxiliary problems with special boundary conditions and conditions for the displacements on $\omega_t^1$ and $\omega_t^2$, $t \in (0, T]$. Let us fix an arbitrary value of $t \in (0, T]$ and formulate the following auxiliary variational problems: we have to find the values of $W_t \in K_t'$, such that

$$p(W_t) = \inf_{\chi \in K_t'} p(\chi), \tag{9}$$

where

$$p(\chi) = \int\limits_{\Omega_\gamma} \sigma_{ij}(\chi - W)\varepsilon_{ij}(\chi - W)\, dx,$$

$$K_t' = \Big\{\chi \in H(\Omega_\gamma)\colon \quad \chi = W \quad \text{on} \quad \Gamma_1 \cup \gamma^+ \cup \gamma^-, \quad \chi\Big|_{\omega_t^i} = \rho^i, \quad i = 1, 2\Big\}.$$

By virtue of the Korn inequality (1), the quadratic functional $p(\chi)$ is coercive and weakly semi-continuous from below in the space $H(\Omega_\gamma)$. It can be easily shown that the set $K_t'$ is convex and closed in $H(\Omega_\gamma)$. In view of the rigorous convexity of $p(\chi)$, these properties ensure the existence of the unique solution $W_t$ of problem (9), which is equivalent to the following variational inequality (see [5]):

$$W_t \in K_t', \qquad \int\limits_{\Omega_\gamma} \sigma_{ij}(W_t - W)\varepsilon_{ij}(\chi - W_t)\, dx \geq 0 \quad \forall\, \chi \in K_t'. \tag{10}$$

Let us construct a test function $\hat{W}$ that belongs to the sets $K_t'$ for all $t \in (0, T]$. Using the lifting operators for the Lipschitz domains $O \setminus \overline{\omega_T^1 \cup \omega_T^2}$ and $\Omega \setminus \overline{O}$, we define a function $\hat{W} \in H(\Omega_\gamma)$, such that $\hat{W} = \rho^i$ in $\omega_T^i$ $(i = 1, 2)$ and $\hat{W} = W$ on $\Gamma_1 \cup \gamma^+ \cup \gamma^-$. Substituting $\hat{W}$ as test functions into Eq. (10), we obtain

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(W_t - W)\varepsilon_{ij}(\hat{W})\, dx + \int\limits_{\Omega_\gamma} \sigma_{ij}(W)\varepsilon_{ij}(W_t)\, dx \geq \int\limits_{\Omega_\gamma} \sigma_{ij}(W_t)\varepsilon_{ij}(W_t)\, dx \quad \forall\, t \in (0, T].$$

In view of the Korn inequality, this inequality yields a uniform estimate with respect to $t$:

$$\|W_t\| \leq c \quad \forall\, t \in (0, T].$$

Thus, from the sequence $\{W_{t_n}\}$, we can choose a subsequence $\{W_l\}$, which is determined by the equalities $W_l = W_{t_{n_l}}$, $l \in \mathbb{N}$ (in what follows, the subsequence elements are denoted by $t_l = t_{n_l}$) and weakly converges to a certain limit $\tilde{W}$ in $H(\Omega_\gamma)$. Let us now demonstrate that $\tilde{W} = W$ coincides with the specified function.

As $W_l - W \in H_0^1(\Omega_\gamma)^2$ due to construction, by virtue of the weak closedness of $H_0^1(\Omega_\gamma)^2$, the function $\tilde{W} - W$ belongs to the space $H_0^1(\Omega_\gamma)^2$. Let us consider combinations of the form $\chi_l^\pm = W_l \pm \varphi$, where $\varphi$ is an arbitrary function in the space of smooth finite functions $C_0^\infty(\Omega_\gamma)^2$. Note that $\chi_l^\pm \in K_{t_l}'$ for sufficiently large values of $l$. Let us substitute the elements of the sequences $\chi_l^+$ and $\chi_l^-$ as test functions in inequalities (10) corresponding to the parameters $t_l$. As a result, we obtain

$$W_l \in K_{t_l}', \qquad \int\limits_{\Omega_\gamma} \sigma_{ij}(W_l - W)\varepsilon_{ij}(\varphi)\, dx = 0. \tag{11}$$

Let us fix the function $\varphi$. Passing to the limit $l \to \infty$ in Eq. (11), we find

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(\tilde{W} - W)\varepsilon_{ij}(\varphi)\, dx = \int\limits_{\Omega_\gamma} \sigma_{ij}(\tilde{W} - W)\varepsilon_{ij}(\varphi)\, dx = 0 \quad \forall\, \varphi \in C_0^\infty(\Omega_\gamma)^2.$$

As the space $C_0^\infty(\Omega_\gamma)$ is dense in $H_0^1(\Omega_\gamma)$, we obtain $\tilde{W} - W = 0$ in $H_0^1(\Omega_\gamma)^2$. Thus, the equality $\tilde{W} = W$ is valid in $H(\Omega_\gamma)$. Therefore, there exists a subsequence $\{W_l\}$, such that $W_l \in K_{t_l}'$ $(l \in \mathbb{N})$ and $W_l \to W$ weakly in $H(\Omega_\gamma)$ as $l \to \infty$.

Let us prove strong convergence $W_l \to W$. According to the Mazur theorem, there exists a function $N\colon \mathbb{N} \to \mathbb{N}$ and a sequence of real numbers $\{\alpha(n)_i\colon i = n, \ldots, N(n)\}$ with the properties

$$\alpha(n)_i \geq 0, \quad i = n, \ldots, N(n), \qquad \sum_{i=n}^{N(n)} \alpha(n)_i = 1,$$

such that the sequence $\{W_n\}$ defined by the convex combination

$$W_n = \sum_{i=n}^{N(n)} \alpha(n)_i W_i$$

converges to $W$ strongly in $H(\Omega_\gamma)$. Let us define the sought sequence $\{W_k\}$ as follows:

$$W_1 = W_{N(1)}, \quad W_2 = W_{N(N(1))}, \quad \ldots, \quad W_k = W_{N^k(1)}, \quad \ldots .$$

The constructed function $W_k$ belongs to the set $K_{t_{N^k(1)}}$ corresponding to the subsequence $\{t_{N^k(1)}\}$ of the sequence $\{t_n\}$.

The following theorem is derived.

**Theorem 1.** *Let $U_t$ be solutions of the minimization problem with two-dimensional inclusions* (3) *corresponding to the parameters $t \in (0, T]$, and let $U_0$ be a solution of the problem with thin inclusions* (7). *Then, $U_t \to U_0$ strongly in $H(\Omega_\gamma)$ as $t \to 0$.*

**Proof.** Let us prove the theorem by means of reductio ad absurdum. Let us assume that there exists a number $\epsilon_0 > 0$ and a sequence $\{t_n\} \subset (0, T]$, such that $t_n \to 0$, $\|U_n - U_0\|_{H(\Omega_\gamma)} \geq \epsilon_0$, where $U_n = U_{t_n}$ $(n \in \mathbb{N})$ are solutions of problem (3) corresponding to $t_n$. As $W^0 \equiv 0 \in K_t$ for all $t \in (0, T]$, we substitute $W = W^0$ into the variational inequality (4) for a fixed value of $t \in (0, T]$ and obtain

$$U_t \in K_t, \qquad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_t)\varepsilon_{ij}(U_t)\, dx \leq \int\limits_{\Omega_\gamma} FU_t\, dx \quad \forall\, t \in (0, T].$$

Thus, it follows from here that, for all $t \in (0, T]$, one can use the uniform estimate

$$\|U_t\|_{H(\Omega_\gamma)} \leq c$$

with a certain constant $c > 0$ independent of $t$. Replacing the sequence $\{U_n\}$ by its subsequence, we find that $\{U_n\}$ weakly converges to a certain function $\tilde{U}$ in $H(\Omega_\gamma)$.

Let us demonstrate that $\tilde{U} \in K_0$. We have $U_n|_{\omega^i_{t_n}} = \rho^i_n \in R(\omega^i_{t_n})$, $i = 1, 2$. In accordance with Sobolev's trace theorem (see [5]), we have

$$U_n\Big|_\gamma \to \tilde{U}\Big|_\gamma \quad \text{strongly in} \quad L_2(\gamma^+)^2 \quad \text{and} \quad L_2(\gamma^-)^2 \quad \text{as} \quad n \to \infty. \tag{12}$$

Choosing the subsequence, we assume that $U_n \to \tilde{U}$ almost everywhere as $n \to \infty$ both on $\gamma^+$ and on $\gamma^-$. This fact allows us to conclude that the absolute value of each of the numerical sequences $\{b^i_n\}$, $\{c^i_{1n}\}$, and $\{c^i_{2n}\}$ determining the structure of the rigid displacements $\rho^i_n$ on $\gamma_i$ $(i = 1, 2)$ is limited. Identifying subsequences for which the previous notations with the subscript $n$ are retained, we find

$$b^i_n \to b^i, \quad c^i_{1n} \to c^i_1, \quad c^i_{2n} \to c^i_2 \quad (i = 1, 2) \quad \text{as} \quad n \to \infty.$$

In view of the discussion above and by virtue of Eq. (12), we can conclude that the trace on the upper edge of the crack is a function with a specified linear structure

$$\tilde{U}\Big|_{\gamma^+_i} = \rho^i = (b^i x_2 + c^i_1, -b^i x_1 + c^i_2) \quad \text{on} \quad \gamma_i,$$

i.e., $\rho^i \in R(\gamma_i)$, $i = 1, 2$. The validity of the non-penetration condition (2) for the limiting function $\tilde{U}$ follows from the existence of a subsequence such that $\{U_{n_k}\} \to \tilde{U}$ almost everywhere on $\gamma$ (see, e.g., [5, 32]). Thus, we have $\tilde{U} \in K_0$.

Let us prove that the limit $\tilde{U} = U_0$ coincides with the solution of problem (7). For this purpose, we demonstrate the possibility of the limiting transition in inequalities (4). According to Lemma 1, for any $W \in K_0$ there exists a subsequence $\{t_k\} = \{t_{n_k}\} \subset \{t_n\}$ and a sequence of the functions $\{W_k\}$, such that $W_k \in K_{t_k}$ and $W_k \to W$

strongly in $H(\Omega_\gamma)$ as $k \to \infty$. Because of the weak convergence $U_n \to \tilde{U}$, we can pass to the limit as $k \to \infty$ in inequality (4) for $t = t_k$ and the test functions

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(U_{n_k})\varepsilon_{ij}(W_{n_k} - U_{n_k})\,dx \geq \int\limits_{\Omega_\gamma} F(W_{n_k} - U_{n_k})\,dx.$$

In the limit, we obtain the variational inequality

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(\tilde{U})\varepsilon_{ij}(W - \tilde{U})\,dx \geq \int\limits_{\Omega_\gamma} F(W - \tilde{U})\,dx \quad \forall\, W \in K_0.$$

Unique solvability of this inequality proves the identity $\tilde{U} \equiv U_0$.

Let us prove strong convergence $U_n \to U_0$ in $H(\Omega_\gamma)$. From the variational inequality (4), for a certain $t \in (0, T]$, using the substitution $W = 2U_t$ and $W = 0$, we obtain the equality

$$U_t \in K_t, \qquad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_t)\varepsilon_{ij}(U_t)\,dx = \int\limits_{\Omega_\gamma} FU_t\,dx \quad \forall\, t \in (0, T]. \tag{13}$$

In addition to weak convergence $U_n \to U_0$ in $H(\Omega_\gamma)$ as $n \to \infty$, it follows from Eq. (13) that

$$\lim_{n\to\infty} \int\limits_{\Omega_\gamma} \sigma_{ij}(U_n)\varepsilon_{ij}(U_n)\,dx = \lim_{n\to\infty} \int\limits_{\Omega_\gamma} FU_n\,dx = \int\limits_{\Omega_\gamma} FU_0\,dx = \int\limits_{\Omega_\gamma} \sigma_{ij}(U_0)\varepsilon_{ij}(U_0)\,dx.$$

With allowance for the equivalence of the norms (see Remark 1), the last equality means strong convergence $U_n \to U_0$ in $H(\Omega_\gamma)$ as $n \to \infty$.

## CONCLUSIONS

A family of variational problems (3) that describe the equilibrium state of elastic bodies with two hinged two-dimensional inclusions of varied width $t \in (0, T]$ is analyzed. The nonlinearity of the examined problems is caused by the fact that non-penetration conditions of the inequality type are imposed on the curve that describes the crack at the interface between the inclusions. It is proved that the problem for a body with hinged thin inclusions (7) can be represented as a limiting family of problems (3) for two hinged two-dimensional inclusions. It is found that the solutions $U_t$ of problem (3) strongly converge to the solution $U_0$ of problem (7) as $t \to 0$.

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