

High-order topological expansions for Helmholtz problems in 2d

Victor A. Kovtunenko

Abstract. Methods of topological analysis are inherently related to singular perturbations. For topology variation, a trial geometric object put in a test domain is examined by reducing the object size from a finite to an infinitesimal one. Based on the singular perturbation of the forward Helmholtz problem, a topology optimization approach, which is a direct one, is described for the inverse problem of object identification from boundary measurements.

Relying on the 2d setting in a bounded domain, the high-order asymptotic result is proved rigorously for the Neumann, Dirichlet, and Robin type conditions stated at the object boundary. In particular, this implies the first-order asymptotic term called a topological derivative. For identifying arbitrary test objects, a variable parameter of the surface impedance is successful. The necessary optimality condition of minimum of the objective function with respect to trial geometric variables is discussed and realized for finding the center of the test object.

Keywords. Inverse problem, object identification, shape and topology optimization, topological derivative, derivative-free optimality condition, singular perturbation, asymptotic analysis, variational method, Helmholtz problem.

AMS classification. 35R30, 35Q93, 49Q10, 65K10.

1 Introduction

We consider both the forward and inverse Helmholtz problems with respect to a variable geometric object put in a bounded test domain. It has numerous applications for

The results were obtained with the support of the Austrian Science Fund (FWF) in the framework of the project P26147-N26: "Object identification problems: numerical analysis" (PION) and NAWI Graz.

testing methodologies by scattering with acoustic, elastic, and electromagnetic waves. The literature on this subject is numerous, so we give selected references only.

The classic methods of analysis available for the Helmholtz problem, see [16, 20, 22, 39, 58], are based mostly on the potential operator theory which is well established in case of unbounded domains. Possible approaches to inverse problems can be distinguished to iterative, see [23, 35], as well as non-iterative ones called direct. Within direct approaches to the inverse scattering, there are well known sampling, probe, factorization, singular source, MUSIC-type, and other relevant techniques, e.g. [2, 4, 27, 32, 50].

The inverse Helmholtz problem of identification of an unknown geometric object belongs to the field of shape and topology optimization [1, 24, 25, 34, 42, 44, 55], as well as parameter estimation, see [10, 15, 40, 48] for the common methods here. Recently, the concept of topological derivatives was adapted to this field, e.g. in [6, 11, 56]. The reason is that a trial object of finite size in comparison to infinitesimal one implies variation of topology of the test domain. For the analysis of topological changes we refer to the methods of singular perturbations in [28, 33, 49].

The high-order topological expansions were considered e.g. in [5, 13, 17, 18, 30] for the Poisson equation, and in [47] for a screened Poisson equation. They were the subject of discussion in [12, 54]. Regarding the inverse problems, in [17, 18] the Kohn–Vogelius cost functional over a domain was suggested as alternative to the boundary misfit functional. Numerically, the high-order terms usually have a high computational cost because they require to solve a PDE at every point where the topological derivative is computed.

To derive even formally asymptotic representations under singular perturbations is itself a hard task requiring a huge number of fine calculations. Its rigorous mathematical justification is the challenging problem. In this chapter, we obtain the high-order asymptotic formula for solutions of the singularly perturbed state problems and the corresponding optimal value functions. The first-order term implying a topological derivative is the particular case here. All subsequent asymptotic expansions are proved by rigorous estimates of the residual error in appropriate function spaces.

Our consideration addresses the topology optimization problem for identification and reconstruction of arbitrary geometries. By “arbitrary geometry” we mean the implicitly defined set of geometric variables which is parameterized by admissible triples of shape - center - size. Such general geometric assumptions can be treated within a proper variational formulation.

We provide the underlying singularly perturbed, forward and inverse, Helmholtz problem with suitable primal and dual variational principles according to the Fenchel–Legendre duality. The asymptotic terms are expressed by auxiliary Helmholtz problems given in bounded domains as well as boundary layers described by the Laplace problems in exterior domains. They all are stated in the weak form in proper Sobolev spaces. In the exterior domains, weighted Sobolev spaces are useful.

A crucial issue of the topological analysis concerns the fact that boundary conditions of the test object should be prescribed a-priori. In the present chapter, relying on a 2d spatial setting, in a unified way we consider the Helmholtz problem under Dirichlet (sound soft), Neumann (sound hard), and Robin (impedance) conditions stated at the object boundary, respectively, in Sections 3, 4, and 5. The background problem in the reference domain is described in Section 2.

From the perspective of topology optimization, the geometric variables enter the objective in a fully implicit way through the geometry-dependent state problem. This fact does not allow to find optimality conditions based on directional derivatives of the objective. Nevertheless, in Section 5.4 we show that an unknown parameter of the boundary impedance is well suitable for the purpose of variation of trial geometries. In fact, passing its imaginary part to the limit in the asymptotic expansion of the optimal value function, this gets a necessary optimality condition determining the optimal center, and this condition is derivative-free.

We start with the description of the reference configuration.

2 Background Helmholtz problem

Let $\Omega \subset \mathbb{R}^2$ be a reference domain with the Lipschitz boundary $\partial\Omega$. With $\nu = (\nu_1, \nu_2)^\top$ the unit normal vector at the boundary $\partial\Omega$ and outward to Ω is denoted. Let $\partial\Omega = \Gamma_N \cup \Gamma_D$ consist of two nonempty, mutually disjoint, parts Γ_N and Γ_D associated to the Neumann and Dirichlet boundary conditions, respectively.

For the fixed Neumann and Dirichlet data $g \in L^2(\Gamma_N; \mathbb{C})$ and $h \in H^{1/2}(\Gamma_D; \mathbb{C})$, and for a wave number $k \in \mathbb{R}_+$, the reference (called background) Helmholtz problem for the wave potential $u^0(x)$, $x = (x_1, x_2)^\top \in \Omega$, is given by:

$$-[\Delta + k^2]u^0 = 0 \quad \text{in } \Omega, \tag{2.1a}$$

$$\frac{\partial u^0}{\partial \nu} = g \quad \text{on } \Gamma_N, \tag{2.1b}$$

$$u^0 = h \quad \text{on } \Gamma_D. \tag{2.1c}$$

In (2.1) and in what follows, Δ is the Laplace operator, the usual notation $\frac{\partial}{\partial \nu} := \nu \cdot \nabla = \nu^\top \nabla$ stands for the normal derivative, " $\cdot$ " means for the inner product of vectors, ∇ is the gradient, and the upper $^\top$ denotes transposition swapping columns and rows.

In the weak form, (2.1) is described by the following variational problem: Find $u^0 \in H^1(\Omega; \mathbb{C})$ such that

$$u^0 = h \quad \text{on } \Gamma_D, \tag{2.2a}$$

$$\int_\Omega (\nabla u^0 \cdot \nabla \overline{u} - k^2 u^0 \overline{u})\, dx = \int_{\Gamma_N} g \overline{u}\, dS_x \quad \text{for all } u \in H^1(\Omega; \mathbb{C}) : u = 0 \text{ on } \Gamma_D. \tag{2.2b}$$

Here $\overline{u} = \mathrm{Re}(u) - \imath \mathrm{Im}(u)$ is the complex conjugate of $u = \mathrm{Re}(u) + \imath \mathrm{Im}(u)$, and $\imath$ is the imaginary unit.

Proposition 2.1. *For the strong solution to* (2.1)*, the boundary value problem* (2.1) *and the variational equation* (2.2) *are equivalent. There exists the unique weak solution* u^0 *to* (2.2)*. Moreover, it implies the first-order necessary optimality condition for the Cherkaev–Gibiansky variational principle:*

$$\mathcal{P}(u^0) = \min_{\mathrm{Re}(v)} \max_{\mathrm{Im}(v)} \mathcal{P}(v) \quad \textit{over } v \in H^1(\Omega;\mathbb{C}) : v = h \textit{ on } \Gamma_D \tag{2.3}$$

with the Lagrangian $\mathcal{P} : H^1(\Omega;\mathbb{C}) \mapsto \mathbb{R}$ *of the form*

$$\mathcal{P}(v) = \mathrm{Re}\Big\{ \tfrac{1}{2} \int_\Omega (\nabla v \cdot \nabla v - k^2 v^2)\, dx - \int_{\Gamma_N} g v \, dS_x \Big\}. \tag{2.4}$$

Proof. The weak formulation can be verified by standard variational arguments. Indeed, (2.2) is derived by multiplying the equation (2.1a) with a test function $\overline{u} \in H^1(\Omega;\mathbb{C})$ and subsequent integration by parts over Ω due to boundary conditions (2.1b) and $u = 0$ on Γ_D.

In return, the following Green's formula holds for every function $\mathbf{u} \in H^1(\Omega;\mathbb{C})$ such that $\Delta \mathbf{u} \in L^2(\Omega;\mathbb{C})$:

$$\begin{gathered} \int_\Omega (\nabla \mathbf{u} \cdot \nabla \overline{u} + \overline{u}\, \Delta \mathbf{u})\, dx = \langle \tfrac{\partial \mathbf{u}}{\partial \nu}, \overline{u} \rangle_{\Gamma_N} \\ \text{for all } u \in H^1(\Omega;\mathbb{C}) : u = 0 \text{ on } \Gamma_D. \end{gathered} \tag{2.5}$$

Here $\langle \frac{\partial \mathbf{u}}{\partial \nu}, \overline{u} \rangle_{\Gamma_N}$ stands for the duality pairing between $u \in H^{1/2}_{00}(\Gamma_N;\mathbb{C})$ and $\frac{\partial \mathbf{u}}{\partial \nu} \in H^{1/2}_{00}(\Gamma_N;\mathbb{C})^\star$ in the Lions–Magenes dual space, see e.g. [36, Section 1.4] for detail. With the help of (2.5), we get from (2.2b)

$$-\int_\Omega ([\Delta + k^2] u^0) \overline{u}\, dx = \langle g - \tfrac{\partial u^0}{\partial \nu}, \overline{u} \rangle_{\Gamma_N}, \quad \langle g, \overline{u} \rangle_{\Gamma_N} = \int_{\Gamma_N} g \overline{u}\, dS_x$$

and derive (2.1a) and (2.1b) by fundamental lemma of the calculus of variations when varying the test function u such that first $u = 0$ on Γ_N and then $u \neq 0$ on Γ_N.

The unique solvability of the variational equation (2.2) can be argued as usually with a Garding inequality and injectivity by using the Fredholm alternative, see e.g. [46, Theorem 5.2, Chapter 3].

Now rewriting (2.4) component-wisely for $v = \mathrm{Re}(v) + \imath \mathrm{Im}(v)$ as

$$\begin{aligned} \mathcal{P}(v) = \tfrac{1}{2} \int_\Omega \big\{ &|\nabla(\mathrm{Re}(v))|^2 - |\nabla(\mathrm{Im}(v))|^2 \\ &- k^2 \big(\mathrm{Re}(v)^2 - \mathrm{Im}(v)^2\big) \big\}\, dx - \int_{\Gamma_N} \big(\mathrm{Re}(g)\mathrm{Re}(v) - \mathrm{Im}(g)\mathrm{Im}(v)\big)\, dS_x \end{aligned}$$

and differentiating $\mathcal{P}(v)$ with respect to $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$, the necessary optimality condition for (2.3) implies two variational inequalities

$$\big\langle \tfrac{\partial}{\partial \mathrm{Re}(v)} \mathcal{P}(u^0), \mathrm{Re}(v - u^0) \big\rangle \geq 0, \quad \big\langle \tfrac{\partial}{\partial \mathrm{Im}(v)} \mathcal{P}(u^0), \mathrm{Im}(v - u^0) \big\rangle \leq 0$$

holding for all $v \in H^1(\Omega;\mathbb{C})$ such that $v = h$ on Γ_D. Inserting here $v = u^0 \pm \mathbf{u}$ with $\mathbf{u} \in H^1(\Omega;\mathbb{C})$ such that $\mathbf{u} = 0$ on Γ_D we get the following two variational equations:

$$\int_\Omega \big(\nabla \mathrm{Re}(u^0) \cdot \nabla \mathrm{Re}(\mathbf{u}) - k^2 \mathrm{Re}(u^0)\mathrm{Re}(\mathbf{u})\big)\, dx = \int_{\Gamma_N} \mathrm{Re}(g)\mathrm{Re}(\mathbf{u})\, dS_x,$$
$$\int_\Omega \big(\nabla \mathrm{Im}(u^0) \cdot \nabla \mathrm{Im}(\mathbf{u}) - k^2 \mathrm{Im}(u^0)\mathrm{Im}(\mathbf{u})\big)\, dx = \int_{\Gamma_N} \mathrm{Im}(g)\mathrm{Im}(\mathbf{u})\, dS_x.$$

The summation of these equations for $\mathbf{u} = u$ and for $\mathbf{u} = \imath u$ constitutes respectively the real and the imaginary parts of (2.2b), see (5.5a). This completes the proof. □

In the next section we give a local representation of the solution to (2.2) in the near-field of a trial point, which is called "inner" asymptotic expansion.

2.1 Inner asymptotic expansion by Fourier series in near-field

For a fixed center $x_0 \in \Omega$, a local polar coordinate system associated to x_0 can be introduced with the help of the polar radius $\rho \in \mathbb{R}_+$ and the polar angle $\theta \in (-\pi, \pi]$ such that $x - x_0 = \rho\widehat{x}$, that is

$$\rho := |x - x_0|, \quad \widehat{x} := \tfrac{x-x_0}{|x-x_0|} = (\cos\theta, \sin\theta)^\top, \quad \widehat{x}' = (-\sin\theta, \cos\theta)^\top. \tag{2.6}$$

We set $R > 0$ such that $B_R(x_0) \subset \Omega$, where $B_R(x_0)$ denotes the ball of radius R and center x_0.

In what follows, Bessel functions of the first kind $J_n(k\rho)$ and the second kind (called the Neumann functions) $Y_n(k\rho)$, $n \in \mathbb{N}_0$, will be employed for the argument $k\rho$. These functions are two linearly independent solutions of the Bessel equation:

$$(u_n)''_\rho + \tfrac{1}{\rho}(u_n)'_\rho + \big(k^2 - \tfrac{n^2}{\rho^2}\big)u_n = 0 \quad \text{for } k\rho \mapsto u_n : \mathbb{R}_+ \mapsto \mathbb{R}. \tag{2.7}$$

In particular, J_0, J_1, and Y_0 yield the expansions for $k\rho \searrow +0$:

$$J_0(k\rho) = \sum_{m=0}^{\infty} \tfrac{(-1)^m}{(m!)^2}\big(\tfrac{k\rho}{2}\big)^{2m} = 1 + a_0(k\rho), \quad a_0(k\rho) = -\tfrac{(k\rho)^2}{4} + \mathrm{O}((k\rho)^4), \tag{2.8a}$$
$$J_1(k\rho) = -J_0'(k\rho) = \tfrac{1}{2}(k\rho + a_1(k\rho)), \quad a_1(k\rho) = -\tfrac{(k\rho)^3}{8} + \mathrm{O}((k\rho)^5), \tag{2.8b}$$
$$Y_0(k\rho) = \tfrac{2}{\pi}(\ln\tfrac{k\rho}{2} + \gamma)J_0(k\rho) + a_2(k\rho), \quad a_2(k\rho) = \mathrm{O}((k\rho)^2) \tag{2.8c}$$

with the Euler constant $\gamma > 0$. Using (2.6)–(2.8) we prove the following truncated Fourier series.

Lemma 2.2. *The solution u^0 of (2.2) admits the local asymptotic representation*

$$u^0(x) = u^0(x_0)J_0(k\rho) + U_0^0(x) \quad \textit{for } x \in B_R(x_0) \subset \Omega \tag{2.9}$$

holding in the near-field, with the residual $U_0^0 \in H^1(B_R(x_0);\mathbb{C})$ *such that*

$$\int_{-\pi}^{\pi} U_0^0 \, d\theta = 0, \tag{2.10a}$$

$$U_0^0(x_0 + \rho\widehat{x}) = \mathrm{O}(\rho) \quad \textit{for } \rho \in [0,R),\ \theta \in (-\pi,\pi]. \tag{2.10b}$$

Proof. In the ball $B_\delta(x_0)$ with $\delta \in [0,R)$ we set

$$u_0^0(\rho) := \tfrac{1}{2\pi} \int_{-\pi}^{\pi} u^0 \, d\theta \quad \text{and} \quad U_0^0 := u^0 - u_0^0, \tag{2.11}$$

thus decomposing u^0 into the radial u_0^0 and residual U_0^0 functions:

$$u^0(x) = u_0^0(\rho) + U_0^0(x) \quad \text{for } x \in B_\delta(x_0). \tag{2.12}$$

According to (2.11), the residual $U_0^0 \in H^1(B_R(x_0);\mathbb{C})$ and it fulfills (2.10a).

Using (2.12) and (2.10a), the substitution of a smooth cut-off function $\eta(\rho)$ supported in $B_\delta(x_0)$ as the test function $u = \eta$ into (2.2b) and integration by parts gives

$$\begin{aligned} 0 &= \int_{B_\delta(x_0)} (\nabla u^0 \cdot \nabla \eta - k^2 u^0 \eta) \, dx \\ &= \int_0^\delta \Big\{ \tfrac{\partial}{\partial\rho} \Big(\int_{-\pi}^{\pi} (u_0^0 + U_0^0) \, d\theta \Big) \eta' - k^2 \int_{-\pi}^{\pi} (u_0^0 + U_0^0) \, d\theta \, \eta \Big\} \rho d\rho \\ &= 2\pi \int_0^\delta \big((u_0^0)'_\rho \eta' - k^2 u_0^0 \eta \big) \, \rho d\rho = -2\pi \int_0^\delta \Big(\big(\rho (u_0^0)'_\rho\big)'_\rho + k^2 \rho u_0^0 \Big) \eta \, d\rho \end{aligned}$$

for all η. This results in the Bessel equation (2.7) for $n = 0$:

$$(u_0^0)''_\rho + \tfrac{1}{\rho}(u_0^0)'_\rho + k^2 u_0^0 = 0 \quad \text{for } \rho \in (0,\delta). \tag{2.13}$$

Its general solution has the form

$$u_0^0(\rho) = K_0^0 J_0(k\rho) + S_0^0 Y_0(k\rho), \quad K_0^0, S_0^0 \in \mathbb{C}.$$

But the Neumann function $Y_0(k\rho) = \mathrm{O}(|\ln\rho|)$ in (2.8c) disagrees the inclusion $u_0^0 \in H^1((0,\delta);\mathbb{C})$ in (2.11), hence $S_0^0 = 0$ and

$$u_0^0(\rho) = K_0^0 J_0(k\rho), \quad K_0^0 \in \mathbb{C}. \tag{2.14}$$

To justify (2.10b) we apply the Saint–Venant estimate to the residual U_0^0 in (2.12).

Equation (2.13) implies $-[\Delta + k^2] u_0^0 = 0$ which together with (2.1a) yields the Helmholtz equation for the residual

$$-[\Delta + k^2] U_0^0 = 0 \quad \text{in } B_\delta(x_0). \tag{2.15}$$

With the help of (2.15) after integration by parts we have

$$I(\delta) := \int_{B_\delta(x_0)} |\nabla U_0^0|^2\, dx = \int_{B_\delta(x_0)} k^2 |U_0^0|^2\, dx + \int_{\partial B_\delta(x_0)} \frac{\partial U_0^0}{\partial \rho} \overline{U_0^0}\, dS_x. \tag{2.16}$$

Thanks to (2.10a), the Poincare and the Wirtinger (cf. (2.29)) inequalities hold:

$$\int_{B_\delta(x_0)} |U_0^0|^2\, dx \le \tfrac{(2\delta)^2}{\pi^2} \int_{B_\delta(x_0)} |\nabla U_0^0|^2\, dx, \tag{2.17a}$$

$$\int_{-\pi}^{\pi} |U_0^0|^2\, d\theta \le \int_{-\pi}^{\pi} \big|\tfrac{\partial U_0^0}{\partial \theta}\big|^2\, d\theta. \tag{2.17b}$$

From (2.17b) we can estimate the boundary integral in (2.16) as

$$\int_{\partial B_\delta(x_0)} \frac{\partial U_0^0}{\partial \rho} \overline{U_0^0}\, dS_x = \int_{-\pi}^{\pi} \frac{\partial U_0^0}{\partial \rho} \overline{U_0^0}\, \delta d\theta \le \int_{-\pi}^{\pi} \Big(\tfrac{\delta}{2} \big|\tfrac{\partial U_0^0}{\partial \rho}\big|^2 + \tfrac{1}{2\delta} |U_0^0|^2 \Big)\, \delta d\theta$$
$$\le \int_{-\pi}^{\pi} \Big(\tfrac{\delta}{2} \big|\tfrac{\partial U_0^0}{\partial \rho}\big|^2 + \tfrac{1}{2\delta} \big|\tfrac{\partial U_0^0}{\partial \theta}\big|^2 \Big)\, \delta d\theta = \tfrac{\delta}{2} \int_{\partial B_\delta(x_0)} |\nabla U_0^0|^2\, dS_x.$$

Therefore, together with (2.17a) and the co-area formula

$$\tfrac{d}{d\delta} \int_{B_\delta(x_0)} |\nabla U_0^0|^2\, dx = \int_{\partial B_\delta(x_0)} |\nabla U_0^0|^2\, dS_x, \tag{2.18}$$

from (2.16) we get the differential inequality for $I(\delta)$:

$$\big(1 - k^2 \tfrac{4\delta^2}{\pi^2}\big) I(\delta) \le \tfrac{\delta}{2} \tfrac{d}{d\delta} I(\delta) \tag{2.19}$$

Integrating (2.19) with respect to $\delta \in (r, R)$ as

$$\ln\big(\tfrac{I(R)}{I(r)}\big) = \int_r^R \tfrac{dI}{I} \ge \int_r^R \big(\tfrac{2}{\delta} - \tfrac{8k^2}{\pi^2}\delta\big)\, d\delta = \ln\big(\tfrac{R}{r}\big)^2 - \tfrac{4k^2}{\pi^2}(R^2 - r^2)$$

we derive the resulting estimate

$$\int_{B_r(x_0)} |\nabla U_0^0|^2\, dx = I(r) \le \big(\tfrac{r}{R}\big)^2 I(R) e^{\frac{4k^2}{\pi^2}(R^2 - r^2)} = \mathrm{O}(r^2). \tag{2.20}$$

Due to the fundamental theorem of calculus and using homogeneity argument, the function oscillation in $\mathbb{R}^2$ can be estimated from above (see e.g. [38]) as

$$\sup_{x,y \in B_r(x_0)} |U_0^0(x) - U_0^0(y)|^2 \le C \int_{B_r(x_0)} (|\nabla U_0^0|^2 + r^2 |\Delta U_0^0|^2)\, dx$$
$$= C \int_{B_r(x_0)} \big(|\nabla U_0^0|^2 + r^2 k^2 |U_0^0|^2\big)\, dx, \quad (C > 0) \tag{2.21}$$

where we have used (2.15). Combining estimates (2.17a) for $\delta = \rho$ with (2.20) and (2.21) for $r = \rho$, where $\rho \in [0, R)$, we derive that

$$U_0^0(x_0 + \rho\widehat{x}) = U_0^0(x_0) + \mathrm{O}(\rho) \quad \text{in } B_R(x_0).$$

But $U_0^0(x_0) = 0$ due to (2.10a), thus following (2.10b). Now passing $\rho \searrow +0$ in (2.12), in view of (2.8a) and (2.10b) from (2.14) we find the factor $K_0^0 = u^0(x_0)$, hence (2.9) holds. This completes the proof. □

We generalize Lemma 2.2 to the high-order inner asymptotic expansion below, see the related result in [49].

Proposition 2.3. *For every $N \in \mathbb{N}$, the solution u^0 of* (2.2) *admits the following local asymptotic representation in the near-field $B_R(x_0) \subset \Omega$:*

$$u^0(x) = K_0^0 J_0(k\rho) + \sum_{n=1}^{N} J_n(k\rho) K_n^0 \cdot \widehat{x}^n + U_N^0(x), \tag{2.22}$$

where the notation $\widehat{x}^n := (\cos(n\theta), \sin(n\theta))^\top$ with convention $\widehat{x}^1 = \widehat{x}$ for $n = 1$, and $K_n^0 = \big((K_n^0)_1, (K_n^0)_2\big) \in \mathbb{C}^2$. The residual $U_N^0 \in H^1(B_R(x_0); \mathbb{C})$ is such that

$$\int_{-\pi}^{\pi} U_N^0 \, d\theta = 0, \quad \int_{-\pi}^{\pi} U_N^0 \widehat{x}^n \, d\theta = 0 \quad \textit{for } n = 1, \dots, N, \tag{2.23a}$$

$$U_N^0(x_0 + \rho\widehat{x}) = \mathrm{O}(\rho^{N+1}) \quad \textit{for } \rho \in [0, R),\ \theta \in (-\pi, \pi]. \tag{2.23b}$$

Proof. Starting with u_0^0 and U_0^0 given in (2.9) and (2.10), for every $n = 1, \dots, N$ we set recursively the radial and residual functions:

$$u_n^0(\rho) := \tfrac{1}{\pi} \int_{-\pi}^{\pi} u^0 \widehat{x}^n \, d\theta \quad \text{and} \quad U_n^0 := U_{n-1}^0 - u_n^0 \cdot \widehat{x}^n, \tag{2.24}$$

which constitute the local asymptotic representation with $\delta \in [0, R)$:

$$u^0(x) = u_0^0(\rho) + \sum_{n=1}^{N} u_n^0(\rho) \cdot \widehat{x}^n + U_N^0(x) \quad \text{for } x \in B_\delta(x_0). \tag{2.25}$$

According to (2.24), all the residuals U_n^0, $n = 1, \dots, N$, fulfill

$$\int_{-\pi}^{\pi} U_n^0 \, d\theta = 0, \quad \int_{-\pi}^{\pi} U_n^0 \widehat{x}^m \, d\theta = 0 \quad \text{for } m = 1, \dots, n, \tag{2.26}$$

and (2.26) for $n = N$ implies (2.23a).

We show that every radial function u_n^0 satisfies the respective Bessel equation (2.7).

Indeed, for every $n = 1, \dots, N$, plugging into (2.2b) the test vector-function $u = \widehat{x}^n \eta(\rho)$ supported in $B_\delta(x_0)$ and the representation $u^0 = u^0_0 + \sum_{m=1}^n u^0_m \cdot \widehat{x}^m + U^0_n$ according to (2.25), recalling trigonometric calculus for $\widehat{x}^n = (\widehat{x}^n_1, \widehat{x}^n_2)^\top$:

$$\frac{\partial \widehat{x}^n_1}{\partial\theta} = -n\widehat{x}^n_2, \quad \frac{\partial \widehat{x}^n_2}{\partial\theta} = n\widehat{x}^n_1, \quad \int_{-\pi}^{\pi} \widehat{x}^n \, d\theta = 0, \quad n, m \in \mathbb{N},$$
$$\int_{-\pi}^{\pi} \widehat{x}^m_i \widehat{x}^n_j \, d\theta = \begin{cases} \pi & \text{for } m = n \text{ and } i = j \\ 0 & \text{otherwise} \end{cases}, \quad i, j = 1, 2 \tag{2.27}$$

and using orthogonality conditions in (2.23a), it succeeds in

$$0 = \int_{B_\delta(x_0)} \big(\nabla u^0 \cdot \nabla(\widehat{x}^n \eta) - k^2 u^0 \widehat{x}^n \eta\big) \, dx = \int_0^\delta \Big\{ \tfrac{\partial}{\partial\rho} \Big(\int_{-\pi}^{\pi} (u^0_0 + \sum_{m=1}^n u^0_m \cdot \widehat{x}^m$$
$$+ U^0_n) \widehat{x}^n \, d\theta \Big) \eta' + \tfrac{\eta}{\rho^2} \int_{-\pi}^{\pi} \Big(\sum_{m=1}^n u^0_m \cdot \tfrac{\partial \widehat{x}^m}{\partial\theta} + \tfrac{\partial U^0_n}{\partial\theta} \Big) \tfrac{\partial \widehat{x}^n}{\partial\theta} \, d\theta - k^2 \eta \int_{-\pi}^{\pi} (u^0_0$$
$$+ \sum_{m=1}^n u^0_m \cdot \widehat{x}^m + U^0_n) \, d\theta \Big\} \rho d\rho = \pi \int_0^\delta \big((u^0_n)'_\rho \eta' + \tfrac{n^2}{\rho^2} u^0_n \eta - k^2 u^0_n \eta \big) \rho d\rho.$$

After integration by parts we obtain the Bessel equation (2.7) possessing the general solution

$$u^0_n(\rho) = K^0_n J_n(k\rho), \quad K^0_n \in \mathbb{C}^2 \tag{2.28}$$

since the Neumann function $Y_n(k\rho)$ is singular for $\rho \searrow +0$ and disagrees the H^1-regularity of the solution u^0.

It remains to prove (2.23b). For this task, since (2.23a) holds, we refine the Wirtinger inequality (compare with (2.17b)):

$$\int_{-\pi}^{\pi} |U^0_N|^2 \, d\theta \leq \tfrac{1}{(N+1)^2} \int_{-\pi}^{\pi} \big| \tfrac{\partial U^0_N}{\partial\theta} \big|^2 \, d\theta. \tag{2.29}$$

Indeed, employment of the Fourier series

$$U^0_N = c^N_0 + \sum_{n=1}^\infty c^N_n \cdot \widehat{x}^n, \quad c^N_0 := \tfrac{1}{2\pi} \int_{-\pi}^{\pi} U^0_N \, d\theta, \quad c^N_n := \tfrac{1}{\pi} \int_{-\pi}^{\pi} U^0_N \widehat{x}^n \, d\theta$$

with scalar $c^N_0 \in \mathbb{C}$ and vectors $c^N_n \in \mathbb{C}^2$, together with the derivative

$$\tfrac{\partial U^0_N}{\partial\theta} = \sum_{n=1}^\infty n c^N_n \cdot (\widehat{x}^n)', \quad \text{where } (\widehat{x}^n)' = (-\sin(n\theta), \cos(n\theta))^\top,$$

this leads to the following Parseval identities

$$\tfrac{1}{2\pi} \int_{-\pi}^{\pi} |U^0_N|^2 \, d\theta = |c^N_0|^2 + \tfrac{1}{2} \sum_{n=1}^\infty |c^N_n|^2, \quad \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \big| \tfrac{\partial U^0_N}{\partial\theta} \big|^2 \, d\theta = \tfrac{1}{2} \sum_{n=1}^\infty n^2 |c^N_n|^2.$$

Conditions (2.23a) imply $c_0^N = \cdots = c_N^N = 0$ that allows us to estimate

$$\int_{-\pi}^{\pi} |U_N^0|^2\, d\theta = \pi \sum_{n=N+1}^{\infty} |c_n^N|^2 \le \tfrac{\pi}{(N+1)^2} \sum_{n=N+1}^{\infty} n^2 |c_n^N|^2 = \tfrac{1}{(N+1)^2} \int_{-\pi}^{\pi} \big|\tfrac{\partial U_N^0}{\partial \theta}\big|^2\, d\theta,$$

thus concluding with (2.29).

Due to (2.29) the boundary integral in $I_N(\delta)$ introduced below can be estimated as

$$\int_{\partial B_\delta(x_0)} \tfrac{\partial U_N^0}{\partial \rho} \overline{U_N^0}\, dS_x \le \int_{-\pi}^{\pi} \Big(\tfrac{\delta}{2(N+1)} \big|\tfrac{\partial U_N^0}{\partial \rho}\big|^2 + \tfrac{N+1}{2\delta} |U_N^0|^2 \Big) \delta d\theta$$
$$\le \int_{-\pi}^{\pi} \Big(\tfrac{\delta}{2(N+1)} \big|\tfrac{\partial U_N^0}{\partial \rho}\big|^2 + \tfrac{1}{2\delta(N+1)} \big|\tfrac{\partial U_N^0}{\partial \theta}\big|^2 \Big) \delta d\theta = \tfrac{\delta}{2(N+1)} \int_{\partial B_\delta(x_0)} |\nabla U_N^0|^2\, dS_x.$$

Therefore, considering similar to (2.16) the residual in the energy norm

$$I_N(\delta) := \int_{B_\delta(x_0)} |\nabla U_N^0|^2\, dx = \int_{B_\delta(x_0)} k^2 |U_N^0|^2\, dx + \int_{\partial B_\delta(x_0)} \tfrac{\partial U_N^0}{\partial \rho} \overline{U_N^0}\, dS_x,$$

applying here the Poincare inequality from (2.17a) and the co-area formula from (2.18), we derive the corresponding differential inequality

$$\big(1 - k^2 \tfrac{4\delta^2}{\pi^2}\big) I_N(\delta) \le \tfrac{\delta}{2(N+1)} \tfrac{d}{d\delta} I_N(\delta). \tag{2.30}$$

Integration of (2.30) over $\delta \in (r, R)$ leads to the estimate

$$0 \le I_N(r) \le \big(\tfrac{r}{R}\big)^{2(N+1)} I(R) e^{\frac{4(N+1)k^2}{\pi^2}(R^2 - r^2)} = \mathrm{O}(r^{2(N+1)}). \tag{2.31}$$

Applying to $U_N^0(x_0 + \rho\widehat{x})$ the point-wise estimate in the manner of (2.21), hence

$$|U_N^0(x_0 + \rho\widehat{x})|^2 \le C I_N(\rho) \quad \text{with } C > 0, \tag{2.32}$$

from (2.31) and (2.32) it follows (2.23b). The proof is completed. □

As the consequence, below we specify Proposition 2.3 for $N = 1$, where $K_0^0 = u^0(x_0)$ and $K_1^0 = \frac{2}{k}\nabla u^0(x_0)$ are calculated due to (2.8a) and (2.8b).

Corollary 2.4. *The solution u^0 of* (2.2) *admits the first-order local asymptotic representation in the near-field $B_R(x_0) \subset \Omega$ as*

$$u^0(x) = u^0(x_0) J_0(k\rho) + \tfrac{2}{k} J_1(k\rho) \nabla u^0(x_0) \cdot \widehat{x} + U_1^0(x) \tag{2.33}$$

with the residual $U_1^0 \in H^1(B_R(x_0); \mathbb{C})$ such that

$$\int_{-\pi}^{\pi} U_1^0\, d\theta = \int_{-\pi}^{\pi} U_1^0 \widehat{x}\, d\theta = 0, \tag{2.34a}$$

$$U_1^0(x_0+\rho\widehat{x}) = \mathrm{O}(\rho^2) \quad \textit{for } \rho\in[0,R),\, \theta\in(-\pi,\pi]. \tag{2.34b}$$

Moreover, the differentiation in (2.33) *according to* (2.6) *and using the rule*

$$\tfrac{\partial}{\partial x_1} = \cos\theta\tfrac{\partial}{\partial\rho} - \tfrac{\sin\theta}{\rho}\tfrac{\partial}{\partial\theta}, \quad \tfrac{\partial}{\partial x_2} = \sin\theta\tfrac{\partial}{\partial\rho} + \tfrac{\cos\theta}{\rho}\tfrac{\partial}{\partial\theta}$$

yields the following formula for the gradient in $B_R(x_0)$*:*

$$\nabla u^0(x) = \nabla u^0(x_0) + b_u^0(x) + \nabla U_1^0(x), \quad \nabla U_1^0 = \mathrm{O}(\rho) \tag{2.35a}$$

$$\textit{with the term} \quad b_u^0(x) := \big(u^0(x_0)ka_0'(k\rho) + a_1'(k\rho)\nabla u^0(x_0)\cdot\widehat{x}\big)\widehat{x} + \tfrac{a_1(k\rho)}{k\rho}(\nabla u^0(x_0)\cdot\widehat{x}')\widehat{x}', \qquad b_u^0 = \mathrm{O}(\rho). \tag{2.35b}$$

In the next sections we proceed with the "outer" asymptotic expansion which will be given in the far-field with respect to a test geometric object put in the reference domain. We will see that it depends crucially on conditions imposed at the boundary of the test object. The Neumann, Dirichlet, and Robin boundary conditions will be considered separately in Sections 3, 4, and 5, respectively.

3 Helmholtz problems for geometric objects under Neumann (sound hard) boundary condition

We start with the geometric description of a test object (inclusion, obstacle, scatterer).

Let ω stand for a generic geometric shape implying the compact set in $\mathbb{R}^2$ with the piecewise Lipschitz boundary $\partial\omega$ and the normal vector $\nu = (\nu_1,\nu_2)^\top$ outward to ω. We require that $0\in\omega$ and the unit ball $B_1(0)$ separating the near and far fields is the minimum enclosing ball centered at origin 0 such that $\omega\subset B_1(0)$. Consequently, the shapes are invariant to translations and isotropic scaling. We call by $\mathfrak{G}_\omega$ the set of such shapes ω.

Definition 3.1. Rescaling a shape $\omega\in\mathfrak{G}_\omega$ by a size parameter $\varepsilon>0$, it produces a family of admissible geometric objects

$$\omega_\varepsilon(x_0) = \big\{x\in\mathbb{R}^2 : \tfrac{x-x_0}{\varepsilon}\in\omega\big\} \subset B_\varepsilon(x_0) \tag{3.1}$$

posed at a center $x_0\in\mathbb{R}^2$.

Such admissible geometries in (3.1) admit parametrization by the triple of geometric variables $(\omega,\varepsilon,x_0)\in\mathfrak{G}_\omega\times\mathbb{R}_+\times\mathbb{R}^2$. In particular, the shape ω itself is equal to the parametrized object $\omega_1(0)$.

Definition 3.2. We call by admissible geometries $\mathfrak{G} = \mathfrak{G}_\omega\times\mathfrak{G}_\varepsilon\times\mathfrak{G}_x$ in the reference domain Ω those triples: the shape $\omega\in\mathfrak{G}_\omega$, the size $\varepsilon\in\mathfrak{G}_\varepsilon\subset\mathbb{R}_+$, and the center $x_0\in\mathfrak{G}_x\subset\Omega$ of objects $\omega_\varepsilon(x_0)$ in (3.1) which satisfy the consistency condition:

$$\omega_\varepsilon(x_0)\subset B_\varepsilon(x_0)\subset\Omega. \tag{3.2}$$

The set $\mathfrak{G}$ of admissible geometries in (3.2) will be used further for the sake of shape variation of test objects in the reference domain Ω. Moreover, passing $\varepsilon \searrow +0$ diminishes the object and represents the topology change.

For forward problems, we fix the object $\omega_\varepsilon(x_0)$ in Ω, or, equivalently, $(\omega, \varepsilon, x_0) \in \mathfrak{G}$, and we mark the dependence of functions on the size ε for the subsequent asymptotic analysis as $\varepsilon \searrow +0$.

Given the boundary data $g \in L^2(\Gamma_N; \mathbb{C})$ and $h \in H^{1/2}(\Gamma_D; \mathbb{C})$, the Neumann (sound hard) problem for the Helmholtz equation (compare with the background problem (2.1)) is considered for the wave potential $u^\varepsilon(x)$, $x \in \Omega \setminus \omega_\varepsilon(x_0)$, satisfying:

$$-[\Delta + k^2]u^\varepsilon = 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon(x_0)}, \tag{3.3a}$$

$$\frac{\partial u^\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{3.3b}$$

$$\frac{\partial u^\varepsilon}{\partial \nu} = g \quad \text{on } \Gamma_N, \tag{3.3c}$$

$$u^\varepsilon = h \quad \text{on } \Gamma_D. \tag{3.3d}$$

The weak solution to (3.3) is described by the variational problem: Find $u^\varepsilon \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ such that

$$u^\varepsilon = h \quad \text{on } \Gamma_D, \tag{3.4a}$$

$$\begin{gathered}\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla u^\varepsilon \cdot \nabla \overline{u} - k^2 u^\varepsilon \overline{u})\, dx = \int_{\Gamma_N} g\overline{u}\, dS_x \\ \text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : u = 0 \text{ on } \Gamma_D.\end{gathered} \tag{3.4b}$$

For every wave number $k \in \mathbb{R}_+$, which can be also large, the well-posedness of problem (3.4) can be argued by using the Fredholm alternative similarly to Proposition 2.1. Moreover, (3.4) follows necessarily from the corresponding variational principle (compare with (2.3) and (2.4)):

$$\mathcal{P}_\varepsilon(u^\varepsilon) = \min_{\mathrm{Re}(v)} \max_{\mathrm{Im}(v)} \mathcal{P}_\varepsilon(v) \quad \text{over } v \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : v = h \text{ on } \Gamma_D \tag{3.5}$$

with the Lagrangian $\mathcal{P}_\varepsilon : H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) \mapsto \mathbb{R}$ given by

$$\mathcal{P}_\varepsilon(v) = \mathrm{Re}\Big\{ \frac{1}{2} \int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla v \cdot \nabla v - k^2 v^2)\, dx - \int_{\Gamma_N} gv\, dS_x \Big\}. \tag{3.6}$$

From the variational problem (3.4) we can determine the boundary traction $\frac{\partial u^\varepsilon}{\partial \nu}$ in (3.3b) and (3.3c) in the weak sense using the following Green's formula. By recalling that ν denotes both the outer normal on $\partial\Omega$ and $\partial\omega_\varepsilon(x_0)$, we have for every function $\mathbf{u} \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ such that $\Delta\mathbf{u} \in L^2(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ (cf. (2.5)):

$$\begin{gathered}\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla \mathbf{u} \cdot \nabla \overline{u} + \overline{u}\, \Delta\mathbf{u})\, dx = \langle \tfrac{\partial \mathbf{u}}{\partial \nu}, \overline{u} \rangle_{\Gamma_N} - \langle \tfrac{\partial \mathbf{u}}{\partial \nu}, \overline{u} \rangle_{\partial\omega_\varepsilon(x_0)} \\ \text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : u = 0 \text{ on } \Gamma_D.\end{gathered} \tag{3.7}$$

Here $\langle \frac{\partial \mathbf{u}}{\partial \nu}, \overline{u} \rangle_{\partial\omega_\varepsilon(x_0)}$ denotes the duality pairing between $u \in H^{1/2}(\partial\omega_\varepsilon(x_0);\mathbb{C})$ and $\frac{\partial \mathbf{u}}{\partial \nu} \in H^{-1/2}(\partial\omega_\varepsilon(x_0);\mathbb{C})$.

In order to derive residual error estimates following further we require the uniform inf-sup condition (see e.g. [41]): there exists $\beta_0 > 0$ such that

$$0 < \beta_0 \leq \inf_u \sup_v \frac{\left| \int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v})\, dx \right|}{\|u\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} \|v\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})}} \tag{3.8}$$

for all $u, v \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)};\mathbb{C}) : u = v = 0$ on Γ_D, admissible geometries $(\omega, \varepsilon, x_0) \in \mathfrak{G}$, and moderate wave numbers $k \in [0, k_0]$, $k_0 > 0$. By (3.8) the well-posedness of (3.4) follows directly from the Babuska–Lions–Necas–Lax–Milgram theorem, see [31, Section 4.4].

Proposition 3.3. *The solutions* u^0 *of* (2.2) *and* u^ε *of* (3.4) *satisfy the residual estimate*

$$\|u^\varepsilon - u^0\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}(\varepsilon). \tag{3.9}$$

Proof. With the help of strong formulation (2.1) we derive from Green's formula (3.7) the variational equation for u^0 over the perturbed domain $\Omega \setminus \overline{\omega_\varepsilon(x_0)}$ in the form

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla u^0 \cdot \nabla \overline{u} - k^2 u^0 \overline{u})\, dx = \int_{\Gamma_N} g \overline{u}\, dS_x - \int_{\partial\omega_\varepsilon(x_0)} \frac{\partial u^0}{\partial \nu} \overline{u}\, dS_x \tag{3.10}$$
for all $u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)};\mathbb{C}) : u = 0$ on Γ_D.

Here we have used the fact that $\frac{\partial u^0}{\partial \nu} \in L^2(\partial\omega_\varepsilon(x_0);\mathbb{C})$ since the solution u^0 is locally H^2-smooth, see e.g. [46, Theorem 10.1, Chapter 3].

Subtracting (3.10) from (3.4) and inserting the test function $u = u^\varepsilon - u^0$ we have

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} \big(|\nabla(u^\varepsilon - u^0)|^2 - k^2 |u^\varepsilon - u^0|^2\big)\, dx = \int_{\partial\omega_\varepsilon(x_0)} \frac{\partial u^0}{\partial \nu} (\overline{u^\varepsilon - u^0})\, dS_x.$$

Applying here the inf-sup condition (3.8) and the Cauchy–Schwarz inequality, it results in the following estimate

$$\|u^\varepsilon - u^0\|^2_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} \leq \tfrac{1}{\beta_0} \|\tfrac{\partial u^0}{\partial \nu}\|_{L^2(\partial\omega_\varepsilon(x_0);\mathbb{C})} \|u^\varepsilon - u^0\|_{L^2(\partial\omega_\varepsilon(x_0);\mathbb{C})}. \tag{3.11}$$

For $\mathbf{u} \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)};\mathbb{C})$, the boundary trace theorem provides with $0 < \underline{c} < \overline{c}$:

$$\underline{c} \|\mathbf{u}\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} \leq \|\mathbf{u}\|_{H^{1/2}(\partial\Omega;\mathbb{C})} + \|\mathbf{u}\|_{H^{1/2}(\partial\omega_\varepsilon(x_0);\mathbb{C})} \leq \overline{c} \|\mathbf{u}\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})}, \tag{3.12}$$

where, by homogeneity argument, the $H^{1/2}$-norm at $\partial\omega_\varepsilon(x_0)$ implies

$$\|\mathbf{u}\|^2_{H^{1/2}(\partial\omega_\varepsilon(x_0);\mathbb{C})} = \tfrac{1}{\varepsilon} \|\mathbf{u}\|^2_{L^2(\partial\omega_\varepsilon(x_0);\mathbb{C})} + \iint_{\partial\omega_\varepsilon(x_0)} \tfrac{|\mathbf{u}(x) - \mathbf{u}(y)|^2}{|x-y|^2}\, dS_x dS_y. \tag{3.13}$$

Therefore, applying (3.12) and (3.13) with $\mathbf{u} = u^\varepsilon - u^0$ to (3.11) we get

$$\|u^\varepsilon - u^0\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} \le \tfrac{\bar{c}}{\beta_0}\sqrt{\varepsilon}\|\tfrac{\partial u^0}{\partial \nu}\|_{L^2(\partial\omega_\varepsilon(x_0);\mathbb{C})}.$$

From (2.35) in Corollary 2.4 it follows $\frac{\partial u^0}{\partial \nu} = \mathrm{O}(1)$ for $\rho = \mathrm{O}(\varepsilon)$ at $\partial\omega_\varepsilon(x_0)$, hence

$$\|\tfrac{\partial u^0}{\partial \nu}\|_{L^2(\partial\omega_\varepsilon(x_0);\mathbb{C})} = \Big(\int_{\partial\omega_\varepsilon(x_0)} \big|\tfrac{\partial u^0}{\partial \nu}\big|^2\, dS_x\Big)^{1/2} = \mathrm{O}(\sqrt{\varepsilon}) \tag{3.14}$$

and we conclude with (3.9). The proof is completed. □

We observe that the last term on the right hand side of (3.10) expresses the residual error near $\partial\omega_\varepsilon(x_0)$. This boundary integral constitutes the leading order $\mathrm{O}(\varepsilon)$ in the residual error estimate (3.9). To refine this estimate in Proposition 3.3 to the order $\mathrm{o}(\varepsilon)$, we construct a corrector for $\frac{\partial u^0}{\partial \nu} = \nabla u^0(x_0)\cdot\nu + \mathrm{O}(\varepsilon)$ (see (3.44)) in form of the boundary layer, see e.g. [59]. In will be expressed via outer asymptotic expansion in the far-field with respect to the reference geometric object ω.

3.1 Outer asymptotic expansion by Fourier series in far-field

In this section we state an auxiliary problem in the exterior domain for $y \in \mathbb{R}^2 \setminus \overline{\omega}$ with respect to the stretched variable $y = \frac{x-x_0}{\varepsilon}$ according to Definition 3.1.

For this reason we introduce the weighted Sobolev spaces (see [7, 51, 52]):

$$W^{1,p}_\mu(\mathbb{R}^2\setminus\overline{\omega};\mathbb{C}) = \{v:\quad \tfrac{v}{\mu}, \nabla v \in L^p(\mathbb{R}^2\setminus\overline{\omega};\mathbb{C})\},\quad p\in(1,\infty),$$
$$\mu(y) = \mathrm{O}(|y|\ln|y|)\quad \text{in } \mathbb{R}^2\setminus\overline{B_2(0)},\quad \mu(y) = \mathrm{O}(1)\quad \text{in } B_2(0)\setminus\overline{\omega},$$

with the weight μ suggested by the weighted Poincare inequality in exterior domains

$$\int_{\mathbb{R}^2\setminus B_2(0)} \big(\tfrac{v}{|y|\ln|y|}\big)^2 dy \le 4\int_{\mathbb{R}^2\setminus B_2(0)} |\nabla v|^2 dy \quad \text{if } \int_{\partial B_2(0)} v\, dS_x = 0. \tag{3.15}$$

We note that constant function is allowed for $p \ge 2$ and logarithm for $p > 2$ in $W^{1,p}_\mu$.

For $p = 2$ we consider the real-valued exterior Neumann problem: Find vector-function $w_\nu(y) = ((w_\nu)_1, (w_\nu)_2)^\top \in \big(W^{1,2}_\mu(\mathbb{R}^2\setminus\overline{\omega};\mathbb{R})\setminus\mathbb{P}_0\big)^2$ such that

$$\int_{\mathbb{R}^2\setminus\omega} Dw_\nu \nabla v\, dy = \int_{\partial\omega} \nu v\, dS_y \quad \text{for all } v \in W^{1,2}_\mu(\mathbb{R}^2\setminus\overline{\omega};\mathbb{R}), \tag{3.16}$$

where the derivative matrix $(Dw_\nu)_{ij} = (w_\nu)_{i,j}$ for $i,j = 1,2$, the normal vector $\nu = (\nu_1,\nu_2)^\top$, and $\mathbb{P}_0$ stands for polynomials of degree zero, i.e. constant. Excluding constant solutions, this implies the boundary value problem:

$$-\Delta w_\nu = 0 \quad \text{in } \mathbb{R}^2\setminus\overline{\omega}, \tag{3.17a}$$

$$Dw_\nu \nu = -\nu \quad \text{on } \partial\omega, \tag{3.17b}$$

$$w_\nu = \mathrm{O}\big(\tfrac{1}{|y|}\big) \quad \text{as } |y| \nearrow \infty. \tag{3.17c}$$

The existence of a solution to (3.16) follows from the result of [7]. After rescaling $y = \frac{x-x_0}{\varepsilon}$ we reduce the problem to the bounded domain $\Omega \setminus \overline{\omega_\varepsilon(x_0)}$ as follows.

Lemma 3.4. *The rescaled solution* $w^\varepsilon_\nu(x) := w_\nu(\frac{x-x_0}{\varepsilon})$ *to* (3.16) *implies the function* $w^\varepsilon_\nu \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R})^2$ *which fulfills the following relation:*

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} Dw^\varepsilon_\nu \nabla u\, dx = \int_{\Gamma_N} (Dw^\varepsilon_\nu \nu) u\, dS_x + \frac{1}{\varepsilon}\int_{\partial\omega_\varepsilon(x_0)} \nu u\, dS_x \tag{3.18}$$
for all $u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R}) : u = 0$ *on* Γ_D.

It admits the far-field representation in the Fourier series

$$w^\varepsilon_\nu(x) = \frac{\varepsilon}{\rho}\frac{1}{2\pi} M_\omega\, \widehat{x} + W^\varepsilon_\nu(x) \quad \text{for } x \in \mathbb{R}^2 \setminus \overline{B_\varepsilon(x_0)}, \tag{3.19}$$

with the 2-by-2 real matrix M_ω *and the residual function* $W^\varepsilon_\nu = ((W^\varepsilon_\nu)_1, (W^\varepsilon_\nu)_2)^\top \in H^1(\Omega \setminus \overline{B_\varepsilon(x_0)}; \mathbb{R})^2$ *such that*

$$\int_{-\pi}^{\pi} W^\varepsilon_\nu\, d\theta = \int_{-\pi}^{\pi} W^\varepsilon_\nu\, \widehat{x}\, d\theta = 0, \tag{3.20a}$$

$$W^\varepsilon_\nu(x_0 + \rho\widehat{x}) = \mathrm{O}\big((\tfrac{\varepsilon}{\rho})^2\big) \quad \text{for } \rho > \varepsilon,\ \theta \in (-\pi, \pi]. \tag{3.20b}$$

Moreover, the uniform estimates hold

$$\|Dw^\varepsilon_\nu\|_{L^2(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{R})^{2\times 2}} = \mathrm{O}(1), \quad \|w^\varepsilon_\nu\|_{L^2(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{R})^2} = \mathrm{O}(\varepsilon\sqrt{|\ln\varepsilon|}), \tag{3.21a}$$

$$\|w^\varepsilon_\nu\|_{L^2(\partial\omega_\varepsilon(x_0);\mathbb{R})^2} = \mathrm{O}(\sqrt{\varepsilon}), \quad \|Dw^\varepsilon_\nu \nu\|_{L^2(\partial\omega_\varepsilon(x_0);\mathbb{R})^2} = \mathrm{O}(\tfrac{1}{\sqrt{\varepsilon}}). \tag{3.21b}$$

Proof. The local coordinate system (2.6) after stretching implies the polar radius $|y| \in \mathbb{R}_+$ and the polar angle $\theta \in (-\pi, \pi]$ such that

$$y = \frac{x-x_0}{\varepsilon} = |y|\widehat{x}, \quad |y| = \frac{\rho}{\varepsilon}, \quad \widehat{x} = (\cos\theta, \sin\theta)^\top. \tag{3.22}$$

By using the radial vector-functions $\widehat{x}^n$ from Proposition 2.3, the harmonic vector-valued function in (3.17) admits the Fourier series in the far-field as

$$w_\nu(y) = \sum_{n=1}^{\infty} \frac{1}{|y|^n} C^\nu_n \widehat{x}^n, \quad C^\nu_n \in \mathbb{R}^{2\times 2}, \quad \text{for } y \in \mathbb{R}^2 \setminus \overline{B_1(0)} \tag{3.23}$$

with unknown coefficient matrices C^ν_n, $n \in \mathbb{N}$. Formula (3.23) implies

$$w_\nu(y) = \frac{1}{|y|}\frac{1}{2\pi} M_\omega \widehat{x} + W_\nu(y), \quad \frac{1}{2\pi} M_\omega := C^\nu_1, \quad y \in \mathbb{R}^2 \setminus \overline{B_1(0)} \tag{3.24}$$

with the square matrix $M_\omega \in \mathbb{R}^{2\times 2}$ and the vector-valued residual function $W_\nu = ((W_\nu)_1, (W_\nu)_2)^\top \in W^{1,2}_\mu(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R})^2$ such that

$$\int_{-\pi}^{\pi} W_\nu \, d\theta = \int_{-\pi}^{\pi} W_\nu \, \widehat{x} \, d\theta = 0, \tag{3.25a}$$

$$W_\nu(y) = \mathrm{O}(|y|^2) \quad \text{for } |y| > 1, \theta \in (-\pi, \pi], \tag{3.25b}$$

where (3.25a) is obtained with the help of trigonometric calculus in (2.27). Applying the coordinate transformation $y = \frac{x-x_0}{\varepsilon}$ to (3.24) and (3.25), due to (3.22) it follows straightforwardly (3.19) and (3.20) for $w^\varepsilon_\nu(x) := w_\nu(\frac{x-x_0}{\varepsilon})$ and $W^\varepsilon_\nu(x) := W_\nu(\frac{x-x_0}{\varepsilon})$.

Next we apply the coordinate transformation $y = \frac{x-x_0}{\varepsilon}$ to the problem (3.17) and employ the differential calculus according to (3.22)

$$\tfrac{\partial}{\partial y} = \varepsilon \tfrac{\partial}{\partial x}, \quad dy = \tfrac{1}{\varepsilon^2} dx, \quad dS_y = \tfrac{1}{\varepsilon} dS_x \tag{3.26}$$

to derive the following relations

$$-\Delta w^\varepsilon_\nu = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\omega_\varepsilon(x_0)}, \tag{3.27a}$$

$$D w^\varepsilon_\nu \nu = -\tfrac{1}{\varepsilon} \nu \quad \text{on } \partial\omega_\varepsilon(x_0). \tag{3.27b}$$

Since $w^\varepsilon_\nu \in W^{1,2}_\mu(\mathbb{R}^2 \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R})^2$ follows $w^\varepsilon_\nu \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R})^2$ and in view of (3.27a), the Green formula (3.7) can be applied to the vector-function $\mathbf{u} = w^\varepsilon_\nu$:

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} (D w^\varepsilon_\nu \nabla u + u \, \Delta w^\varepsilon_\nu) \, dx = \langle D w^\varepsilon_\nu \nu, u \rangle_{\Gamma_N} - \langle D w^\varepsilon_\nu \nu, u \rangle_{\partial\omega_\varepsilon(x_0)}$$
$$\text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R}) : u = 0 \text{ on } \Gamma_D.$$

As the result, using (3.27) and the fact that the solution w^ε_ν is locally smooth near Γ_N, hence $D w^\varepsilon_\nu \nu \in L^2(\Gamma_N; \mathbb{R})^2$, we arrive at formulation (3.18) in the bounded domain.

It remains to justify estimates (3.21). The first inequality in (3.21a) follows from

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} |D w^\varepsilon_\nu|^2 \, dx \le \int_{\mathbb{R}^2\setminus\omega_\varepsilon(x_0)} |D w^\varepsilon_\nu|^2 \, dx = \int_{\mathbb{R}^2\setminus\omega} |D w_\nu|^2 \, dy = \mathrm{O}(1)$$

which is calculated according to (3.26). To get the second inequality in (3.21a) we inscribe Ω in a ball $B_R(x_0)$ of radius $R > 0$ sufficiently large and decompose it in $B_R(x_0) \setminus \overline{B_\varepsilon(x_0)}$ and $B_\varepsilon(x_0) \setminus \overline{\omega_\varepsilon(x_0)}$ such that

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} |w^\varepsilon_\nu|^2 \, dx \le \int_{B_R(x_0)\setminus B_\varepsilon(x_0)} |w^\varepsilon_\nu|^2 \, dx + \int_{B_\varepsilon(x_0)\setminus\omega_\varepsilon(x_0)} |w^\varepsilon_\nu|^2 \, dx$$
$$= \int_{-\pi}^{\pi} \int_{\varepsilon}^{R} \big((\tfrac{\varepsilon}{\rho})^2 |\tfrac{1}{2\pi} M_\omega \, \widehat{x}|^2 + |W^\varepsilon_\nu|^2 \big) \, \rho d\rho d\theta + \varepsilon^2 \int_{B_1(0)\setminus\omega} |w_\nu|^2 \, dy = \mathrm{O}(\varepsilon^2 |\ln\varepsilon|)$$

due to (3.19), (3.20), and (3.26).

Finally, by homogeneity and the trace theorem from (3.12) and (3.13) it follows

$$\tfrac{1}{\sqrt{\varepsilon}}\|w^\varepsilon_\nu\|_{L^2(\partial\omega_\varepsilon(x_0);\mathbb{R})^2} \le \overline{c}\|w^\varepsilon_\nu\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{R})^2} = \mathrm{O}(1)$$

and the former inequality in (3.21b), while the latter one is confirmed from (3.27b) by

$$\|Dw^\varepsilon_\nu \nu\|^2_{L^2(\partial\omega_\varepsilon(x_0);\mathbb{R})^2} = \int_{\partial\omega_\varepsilon(x_0)} \left|\tfrac{\nu}{\varepsilon}\right|^2 dS_x = \tfrac{1}{\varepsilon^2}\mathrm{meas}_2(\partial\omega_\varepsilon(x_0)) = \tfrac{1}{\varepsilon}\mathrm{meas}_2(\partial\omega)$$

where $\mathrm{meas}_2(\partial\omega_\varepsilon(x_0))$ and $\mathrm{meas}_2(\partial\omega)$ mean the Hausdorff measure of the sets in $\mathbb{R}^2$. This completes the proof. □

We remark that the matrix M_ω in Lemma 3.4 is called added or virtual mass tensor in [53, Note G]. Its properties are given below in Lemma 3.5 following [6, 14, 26, 45].

Lemma 3.5. *The entries of M_ω have the implicit expression:*

$$(M_\omega)_{ij} = \delta_{ij}\,\mathrm{meas}_2(\omega) + \int_{\partial\omega} (w_\nu)_i \nu_j \, dS_y, \quad i, j = 1, 2. \tag{3.28}$$

The matrix $M_\omega \in \mathrm{Spsd}(\mathbb{R}^{2\times 2})$, i.e. symmetric positive semi-definite, and positive definite if $\mathrm{meas}_2(\omega) > 0$. *For ellipsoidal shapes ω it has the explicit expression*

$$M_\omega = \Theta(\alpha) M_{\omega'} \Theta(\alpha)^\top, \quad \Theta(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{3.29a}$$

$$M_{\omega'} = \pi(a+b) \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \tag{3.29b}$$

with the ellipse major $a = 1$ and minor $b \in (0, 1]$ semi-axes, where the major axis has an angle of $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with the y_1-axis counted in the anti-clockwise direction.

Proof. We split the exterior domain in the far-field $\mathbb{R}^2 \setminus \overline{B_1(0)}$ and the near-field $B_1(0) \setminus \overline{\omega}$. In the far-field, the truncated Fourier series (3.24) holds. In the near-field, for $i, j = 1, 2$ we have the second Green formula

$$\begin{aligned} 0 &= \int_{B_1(0)\setminus\omega} \{\Delta(w_\nu)_i y_j - (w_\nu)_i \Delta y_j\}\, dy \\ &= \int_{\partial B_1(0)} \left\{\tfrac{\partial(w_\nu)_i}{\partial|y|} y_j - (w_\nu)_i \tfrac{\partial y_j}{\partial|y|}\right\} dS_y - \int_{\partial\omega} \left\{\tfrac{\partial(w_\nu)_i}{\partial\nu} y_j - (w_\nu)_i \tfrac{\partial y_j}{\partial\nu}\right\} dS_y. \end{aligned}$$

Using here the Neumann condition (3.17b), $\frac{\partial y_j}{\partial|y|} = \widehat{x}_j$ according to (3.22), $y_j = \widehat{x}_j$ since $|y| = 1$ at $\partial B_1(0)$, and $\frac{\partial y_j}{\partial\nu} = \nu_j$ it follows

$$-\int_{\partial B_1(0)} \left\{\tfrac{\partial(w_\nu)_i}{\partial|y|} - (w_\nu)_i\right\} \widehat{x}_j \, dS_y = \int_{\partial\omega} \{\nu_i y_j + (w_\nu)_i \nu_j\}\, dS_y. \tag{3.30}$$

We employ (3.24), (3.25a), and the trigonometric calculus (2.27) to calculate the integral over $\partial B_1(0)$ on the left-hand side of (3.30) as

$$\int_{\partial B_1(0)} \{-\tfrac{\partial(w_\nu)_i}{\partial|y|} + (w_\nu)_i\}\widehat{x}_j\,dS_y = \int_{-\pi}^{\pi} \{\tfrac{1}{2\pi}\sum_{l=1}^{2}(M_\omega)_{il}\widehat{x}_l - \tfrac{\partial(W_\nu)_i}{\partial|y|}$$
$$+ \tfrac{1}{2\pi}\sum_{l=1}^{2}(M_\omega)_{il}\widehat{x}_l + (W_\nu)_i\}\widehat{x}_j\,d\theta = \tfrac{1}{\pi}\int_{-\pi}^{\pi}\sum_{l=1}^{2}(M_\omega)_{il}\widehat{x}_l\widehat{x}_j\,d\theta = (M_\omega)_{ij}.$$

Applying to the right-hand side of (3.30) the divergence theorem

$$\int_{\partial\omega} \nu_i y_j\,dS_y = \int_\omega y_{j,i}\,dy = \delta_{ij}\,\mathrm{meas}_2(\omega), \quad \delta_{ij} = \begin{cases} 1\,, & \text{if } i=j,\\ 0\,, & \text{if } i\neq j,\end{cases} \tag{3.31}$$

this together results in expression (3.28).

To prove the symmetry of M_ω, we insert $v = (w_\nu)_j$ as the test-function in (3.16):

$$\int_{\mathbb{R}^2\setminus\omega} \nabla(w_\nu)_i\cdot\nabla(w_\nu)_j\,dy = \int_{\partial\omega}\nu_i(w_\nu)_j\,dS_y = \int_{\partial\omega}\nu_j(w_\nu)_i\,dS_y,$$

written component-wisely for $i,j=1,2$, which follows $(M_\omega)_{ij} = (M_\omega)_{ji}$ in (3.28). For arbitrary $\xi\in\mathbb{R}^2$, from (3.16) we have the non-negative linear combinations

$$0 \le \int_{\mathbb{R}^2\setminus\omega} |\nabla\big(\xi_1(w_\nu)_1 + \xi_2(w_\nu)_2\big)|^2\,dy = \sum_{i,j=1}^{2}\int_{\partial\omega}(w_\nu)_i\xi_i\nu_j\xi_j\,dS_y.$$

Therefore, multiplying (3.28) with $\xi_i\xi_j$ and summing the result over $i,j=1,2$, the positive semi-definiteness, which is strict if $\mathrm{meas}_2(\omega)>0$, follows:

$$\sum_{i,j=1}^{2}(M_\omega)_{ij}\xi_i\xi_j = |\xi|^2\,\mathrm{meas}_2(\omega) + \sum_{i,j=1}^{2}\int_\omega (w_\nu)_i\xi_i n_j\xi_j\,dS_y \ge |\xi|^2\,\mathrm{meas}_2(\omega).$$

Finally, we derive the explicit representation of M_ω for ellipsoidal shapes ω.

Let the canonical ellipse ω' enclosed in the ball $B_1(0)$ have the major $a=1$ and the minor $b\in(0,1]$ semi-axes with respect to y'-coordinates, i.e.

$$\omega' = \{y'\in\mathbb{R}^2:\ (\tfrac{y_1'}{a})^2 + (\tfrac{y_2'}{b})^2 < 1\}, \quad a=1.$$

Let the major axis of the reference ellipse ω written in y-coordinates have an angle of $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$ with the y_1-axis counted in the anti-clockwise direction, i.e.

$$\omega = \{y\in\mathbb{R}^2:\ (\tfrac{y_1\cos\alpha + y_2\sin\alpha}{a})^2 + (\tfrac{-y_1\sin\alpha + y_2\cos\alpha}{b})^2 < 1\}, \quad a=1.$$

The y-coordinates are after rotation of y'-coordinates with the angle of $-\alpha$, and $y' \in \omega'$ when $\Theta(-\alpha)y = \Theta^\top(\alpha)y \in \omega$ with the orthogonal matrix $\Theta(\alpha)$ given in (3.29a). Therefore, we will prove formula (3.29b) for ω' and then transform $y' = \Theta^\top y$.

We introduce the elliptic coordinates $r \in \mathbb{R}_+$ and $\psi \in (-\pi, \pi]$ such that

$$y_1' = c\cosh(r)\cos\psi, \quad y_2' = c\sinh(r)\sin\psi, \quad c = \sqrt{a^2 - b^2}, \quad a = 1, \tag{3.32}$$

where c is called the linear eccentricity. By setting the distance $r_0 \in \mathbb{R}_+$ such that

$$a = c\cosh(r_0) \quad b = c\sinh(r_0), \tag{3.33a}$$

the geometry ω' can be restated as

$$\partial\omega' = \{r = r_0, \psi \in (-\pi, \pi]\}, \quad \mathbb{R}^2 \setminus \overline{\omega'} = \{r > r_0, \psi \in (-\pi, \pi]\}. \tag{3.33b}$$

From (3.32) it follows the differential calculus in elliptic coordinates:

$$\begin{gathered}\begin{cases} \frac{\partial}{\partial y_1'} = \frac{1}{\varkappa^2(r,\psi)}\big(c\cosh(r)\cos\psi\frac{\partial}{\partial r} - c\cosh(r)\sin\psi\frac{\partial}{\partial\psi}\big), \\ \frac{\partial}{\partial y_2'} = \frac{1}{\varkappa^2(r,\psi)}\big(c\cosh(r)\sin\psi\frac{\partial}{\partial r} + c\sinh(r)\cos\psi\frac{\partial}{\partial\psi}\big), \end{cases} \\ dy' = \varkappa^2(r,\psi)drd\psi, \quad \varkappa(r,\psi) = c\sqrt{\sinh^2(r) + \sin^2\psi}\end{gathered} \tag{3.34}$$

with the scale factor $\varkappa(r, \psi)$. In particular, at the ellipse boundary as $r = r_0$ in (3.33b) with the normal vector ν', using (3.32) for constant ψ and (3.33a) we have

$$\begin{gathered} \nu' = \frac{1}{\varkappa(r_0,\psi)}(b\cos\psi, a\sin\psi)^\top, \quad \frac{\partial}{\partial\nu'} = \frac{1}{\varkappa(r_0,\psi)}\frac{\partial}{\partial r}, \\ dS_{y'} = \varkappa(r_0,\psi)d\psi, \quad \varkappa(r_0,\psi) = \sqrt{a^2\cos^2\psi + b^2\sin^2\psi}. \end{gathered} \tag{3.35}$$

Applying the coordinate transformation (3.32) and calculus in (3.34) and (3.35), the exterior problem (3.17), formulated for $w_{\nu'}$ in $\mathbb{R}^2 \setminus \overline{\omega'}$ in y'-coordinates, can be rewritten in elliptic coordinates. In fact, similarly to Proposition 2.3, the harmonic vector-function $w_{\nu'}$ admits the following Fourier series in $\mathbb{R}^2 \setminus \overline{\omega'}$ (cf. (3.23)):

$$w_{\nu'} = \sum_{n=1}^{\infty} e^{-nr} C_n^{\nu'} (\cos(n\psi), \sin(n\psi))^\top \quad \text{for } r > r_0 \tag{3.36}$$

according to (3.17c) for $r \nearrow \infty$. Coefficient matrices $C_n^{\nu'} \in \mathbb{R}^{2\times 2}$ in (3.36) can be found from the boundary condition (3.17b) written due to (3.35) and (3.36) as

$$\frac{1}{\varkappa(r_0,\psi)}\frac{\partial w_{\nu'}}{\partial r}\Big|_{r=r_0} = -\sum_{n=1}^{\infty} \frac{ne^{-nr_0}}{\varkappa(r_0,\psi)} C_n^{\nu'} (\cos(n\psi), \sin(n\psi))^\top = -\frac{(b\cos\psi, a\sin\psi)^\top}{\varkappa(r_0,\psi)}.$$

Henceforth, $e^{-r_0}C_1^{\nu'}(\cos\psi,\sin\psi)^\top = (b\cos\psi, a\sin\psi)^\top$, $C_n^{\nu'} = 0$ for all $n \geq 2$, and we obtain the following analytic expression for the solution

$$w_{\nu'} = e^{r_0 - r}(b\cos\psi, a\sin\psi)^\top \quad \text{for } r \geq r_0. \tag{3.37}$$

Now the matrix $M_{\omega'}$ can be calculated analytically. After substitution of (3.37) in the representation formula (3.28) it implies the following two vectors for $j = 1, 2$:

$$(M_{\omega'})_{(\,\cdot\,,j)} = \delta_{(\,\cdot\,,j)}\mathrm{meas}_2(\omega') + \int_{\partial\omega'} (b\cos(\psi)\nu'_j, a\sin(\psi)\nu'_j)^\top\, dS_{y'}. \tag{3.38}$$

We extend $(b\cos\psi, a\sin\psi)^\top$ from the boundary inside ω' with the smooth function $(\frac{b}{a}y'_1, \frac{a}{b}y'_2)^\top$ and use the divergence theorem to calculate the elliptic integral in (3.38)

$$\int_{\partial\omega'} (\tfrac{b}{a}y'_1\nu'_j, \tfrac{a}{b}y'_2\nu'_j)^\top\, dS_{y'} = \int_{\omega'} (\tfrac{b}{a}y'_{1,j}\, \tfrac{a}{b}y'_{2,j})^\top\, dy' = \mathrm{meas}_2(\omega') \begin{cases} (\frac{b}{a}, 0)^\top, & j = 1, \\ (0, \frac{a}{b})^\top, & j = 2. \end{cases}$$

Together with $\mathrm{meas}_2(\omega') = \pi ab$ from (3.38) we arrive at (3.29b).

The transformation formula in (3.29a) can be justified by rotation $y' = \Theta^\top y$ applied to the variational equation in the manner of (3.16):

$$\int_{\mathbb{R}^2\setminus\omega'} Dw_\nu(\Theta y')\nabla v\, dy' = \int_{\partial\omega'} \nu v\, dS_{y'} \quad \text{for all } v \in W_\mu^{1,2}(\mathbb{R}^2 \setminus \overline{\omega'}; \mathbb{R}),$$

which, after the left multiplication with $\Theta^\top$, results in

$$\int_{\mathbb{R}^2\setminus\omega'} D(\Theta^\top w_\nu(\Theta y'))\nabla v\, dy' = \int_{\partial\omega'} \Theta^\top \nu v\, dS_{y'}. \tag{3.39}$$

Since $\Theta^\top\nu = \nu'$ in (3.39) and using the representation (3.24) this proves the identity

$$\begin{aligned} w_{\nu'}(y') &= \Theta^\top w_\nu(\Theta y') = \tfrac{1}{|\Theta y'|}\tfrac{1}{2\pi}\Theta^\top M_\omega \tfrac{\Theta y'}{|\Theta y'|} + \Theta^\top W_\nu(\Theta y') \\ &= \tfrac{1}{|y'|}\tfrac{1}{2\pi}\Theta^\top M_\omega \Theta \tfrac{y'}{|y'|} + \Theta^\top W_\nu(\Theta y') = \tfrac{1}{|y'|}\tfrac{1}{2\pi} M_{\omega'} \tfrac{y'}{|y'|} + W_{\nu'}(y'), \end{aligned}$$

which implies $\Theta^\top M_\omega \Theta = M_{\omega'}$, thus (3.29a), and completes the proof. □

We remark that, in the limit case when $b \searrow +0$, formulas in Lemma 3.5 describe the singular matrix of virtual mass for the straight crack, see [9, 45].

3.2 Uniform asymptotic expansion of solution of the Neumann problem

With the help of the boundary layer w_ν^ε described in Lemma 3.4 we can improve the residual error estimate (3.9) according to (3.43) in Theorem 3.6:

$$\|u^\varepsilon - u^0 - \varepsilon\nabla u^0(x_0)\cdot w_\nu^\varepsilon\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}(\varepsilon^2\sqrt{|\ln\varepsilon|}). \tag{3.40}$$

The leading order here is contributed by $\varepsilon w_\nu^\varepsilon$ over the domain $\Omega \setminus \overline{\omega_\varepsilon(x_0)}$ due to the second inequality in (3.21a). Further we construct a refined asymptotic term of order ε^2 using the auxiliary Helmholtz problem: Find $u^1 \in H^1(\Omega;\mathbb{C})^2$ such that

$$u^1 = 0 \quad \text{on } \Gamma_D, \tag{3.41a}$$

$$\begin{aligned} &\int_\Omega (Du^1 \nabla \overline{u} - k^2 u^1 \overline{u})\, dx = \frac{k^2}{\varepsilon\sqrt{|\ln\varepsilon|}} \int_{\Omega\setminus\omega_\varepsilon(x_0)} w_\nu^\varepsilon \overline{u}\, dx \\ &\text{for all } u \in H^1(\Omega;\mathbb{C}) : u = 0 \text{ on } \Gamma_D. \end{aligned} \tag{3.41b}$$

We stress that the solution u^1 to problem (3.41) is estimated uniformly with respect to ε since the right-hand side of (3.41b) is bounded by (3.21a)

$$\Big|\frac{1}{\varepsilon\sqrt{|\ln\varepsilon|}} \int_{\Omega\setminus\omega_\varepsilon(x_0)} w_\nu^\varepsilon \overline{u}\, dS_x\Big| \le \frac{\|w_\nu^\varepsilon\|_{L^2(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{R})^2}}{\varepsilon\sqrt{|\ln\varepsilon|}} \|u\|_{L^2(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} \le C\|u\|_{L^2(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})}$$

with $C > 0$. Using Green's formula (3.7) and the local H^2-regularity of the solution u^1 in $B_\varepsilon(x_0) \supset \omega_\varepsilon(x_0)$ we restate (3.41b) over the perturbed domain

$$\begin{aligned} &\int_{\Omega\setminus\omega_\varepsilon(x_0)} (Du^1 \nabla \overline{u} - k^2 u^1 \overline{u})\, dx = \frac{k^2}{\varepsilon\sqrt{|\ln\varepsilon|}} \int_{\Omega\setminus\omega_\varepsilon(x_0)} w_\nu^\varepsilon \overline{u}\, dx \\ &- \int_{\partial\omega_\varepsilon(x_0)} (Du^1\nu)\overline{u}\, dS_x \quad \text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)};\mathbb{C}) : u = 0 \text{ on } \Gamma_D. \end{aligned} \tag{3.42}$$

Theorem 3.6. *The solutions u^0 of* (2.2)*, u^ε of* (3.4)*, w_ν of* (3.16)*, and u^1 of* (3.41) *satisfy the following residual error estimate*

$$\|q_1^\varepsilon\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}(\varepsilon^2), \tag{3.43a}$$

$$q_1^\varepsilon := u^\varepsilon - u^0 - \varepsilon\nabla u^0(x_0)\cdot(w_\nu^\varepsilon + \varepsilon\sqrt{|\ln\varepsilon|}u^1). \tag{3.43b}$$

Proof. Subtracting from (3.4b) equations (3.10), (3.18) multiplied with $\varepsilon\nabla u^0(x_0)$ and (3.42) multiplied with $\varepsilon^2\sqrt{|\ln\varepsilon|}\nabla u^0(x_0)$, using

$$\frac{\partial u^0}{\partial\nu} - \nabla u^0(x_0)\cdot\nu = b_u^0\cdot\nu + \frac{\partial U_1^0}{\partial\nu} = \mathrm{O}(\varepsilon) \quad \text{on } \partial\omega_\varepsilon(x_0) \tag{3.44}$$

according to (2.35), the differential identity

$$\nabla\big(\nabla u^0(x_0)\cdot(w_\nu^\varepsilon + \varepsilon\sqrt{|\ln\varepsilon|}u^1)\big) = \nabla u^0(x_0)\cdot(Dw_\nu^\varepsilon + \varepsilon\sqrt{|\ln\varepsilon|}Du^1),$$

and the notation of q_1^ε introduced in (3.43b), we obtain the variational equation

$$\begin{aligned} &\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla q_1^\varepsilon\cdot\nabla\overline{u} - k^2 q_1^\varepsilon\overline{u})\, dx = -\varepsilon\int_{\Gamma_N} (\nabla u^0(x_0)\cdot Dw_\nu^\varepsilon\nu)\overline{u}\, dS_x \\ &+ \int_{\partial\omega_\varepsilon(x_0)} \big(b_u^0\cdot\nu + \frac{\partial U_1^0}{\partial\nu} + \varepsilon^2\sqrt{|\ln\varepsilon|}\nabla u^0(x_0)\cdot Du^1\nu\big)\overline{u}\, dS_x \\ &\text{for all } u \in H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C}) : u = 0 \text{ on } \Gamma_D. \end{aligned} \tag{3.45}$$

One difficulty is that q_1^ε is inhomogeneous, namely $q_1^\varepsilon = -\varepsilon\nabla u^0(x_0)\cdot w_\nu^\varepsilon = \mathrm{O}(\varepsilon^2)$ at Γ_D due to (3.19). For its lifting, we take a smooth cut-off function η_{Γ_D} supported in a neighborhood of Γ_D such that $\eta_{\Gamma_D} = 1$ at Γ_D. Henceforth, for

$$Q_1^\varepsilon := q_1^\varepsilon + R_1^\varepsilon, \quad R_1^\varepsilon := \varepsilon(\nabla u^0(x_0)\cdot w_\nu^\varepsilon)\eta_{\Gamma_D} = \mathrm{O}(\varepsilon^2), \quad Q_1^\varepsilon = 0 \text{ on } \Gamma_D,$$

applying to (3.45) with $q_1^\varepsilon = Q_1^\varepsilon - R_1^\varepsilon$ the Cauchy–Schwarz inequality, the asymptotic estimates in (3.19), (3.44), and the trace theorems (3.12) and (3.13), we get

$$\Big|\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla Q_1^\varepsilon\cdot\nabla\overline{u} - k^2 Q_1^\varepsilon\overline{u})\,dx\Big| \le \Big|\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla R_1^\varepsilon\cdot\nabla\overline{u} - k^2 R_1^\varepsilon\overline{u})\,dx\Big|$$

$$+ C_1\varepsilon^2\|u\|_{L^2(\Gamma_N;\mathbb{C})} + C_2\varepsilon\int_{\partial\omega_\varepsilon(x_0)} |\overline{u}|\,dS_x \le C\varepsilon^2\|u\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})}, \quad C, C_1, C_2 > 0.$$

This upper bound together with the inf-sup condition (3.8) proves $\|Q_1^\varepsilon\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}(\varepsilon^2)$, hence (3.43a) and the assertion of the theorem. □

As the consequence, from (3.45) we infer the boundary value problem for q_1^ε:

$$-[\Delta + k^2]q_1^\varepsilon = 0 \quad \text{in } \Omega\setminus\overline{\omega_\varepsilon(x_0)}, \tag{3.46a}$$

$$\tfrac{\partial q_1^\varepsilon}{\partial\nu} = -\nu\cdot\big(b_u^0 + \nabla U_1^0 + \varepsilon^2\sqrt{|\ln\varepsilon|}(Du^1)\nabla u^0(x_0)\big) \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{3.46b}$$

$$\tfrac{\partial q_1^\varepsilon}{\partial\nu} = -\varepsilon\nabla u^0(x_0)\cdot Dw_\nu^\varepsilon\nu \quad \text{on } \Gamma_N, \tag{3.46c}$$

$$q_1^\varepsilon = -\varepsilon\nabla u^0(x_0)\cdot w_\nu^\varepsilon \quad \text{on } \Gamma_D. \tag{3.46d}$$

Theorem 3.6 is useful for the asymptotic expansion with respect to $\varepsilon \searrow +0$ of the state-constrained objective function as suggested in the following section.

3.3 Inverse Helmholtz problem under Neumann boundary condition

In the inverse setting of the problem, the shape $\omega^\star \in \mathfrak{G}_\omega$, the size $\varepsilon^\star \in \mathfrak{G}_\varepsilon$, and the center $x^\star \in \mathfrak{G}_x$ for an unknown geometric object $\omega_{\varepsilon^\star}^\star(x^\star)$ being tested are to be identified and reconstructed from the known boundary measurement $u^\star \in L^2(\Gamma_N;\mathbb{C})$. The admissible set $\mathfrak{G} = \mathfrak{G}_\omega\times\mathfrak{G}_\varepsilon\times\mathfrak{G}_x$ is introduced in Definitions 3.1 and 3.2.

For this purpose, a trial geometric object $\omega_\varepsilon(x_0)$ with admissible $(\omega,\varepsilon,x_0)\in\mathfrak{G}$ is put in Ω. For such trial variables we find a family of solutions u^ε to problem (3.4) and determine the square function of the misfit at the boundary

$$J : \mathfrak{G}\mapsto\mathbb{R}_+, \quad J(\omega,\varepsilon,x_0) := \tfrac{1}{2}\int_{\Gamma_N} |u^\varepsilon - u^\star|^2\,dS_x. \tag{3.47}$$

The objective function (3.47) serves for the state-constrained, topology optimization problem: Find $(\omega^\star,\varepsilon^\star,x^\star)\in\mathfrak{G}$ which is the argument of the trivial minimum

$$0 = J(\omega^\star,\varepsilon^\star,x^\star) = \min_{(\omega,\varepsilon,x_0)\in\mathfrak{G}} J(\omega,\varepsilon,x_0) \quad \text{subject to (3.4).} \tag{3.48}$$

Since the test geometry $(\omega^\star, \varepsilon^\star, x^\star) \in \mathfrak{G}$ is feasible, the trivial minimum in (3.48) is attained at the solution $u^{\varepsilon^\star}$ of (3.4) for the test object $\omega^\star_{\varepsilon^\star}(x^\star)$ when $u^{\varepsilon^\star} = u^\star$ at Γ_N in (3.47). Uniqueness of the minimum is open.

To bring (3.48) in a form suitable for analysis, we give primal-dual arguments.

While u^ε in (3.47) implies the primal state variable, a dual state variable v^ε can be obtained by a Fenchel–Legendre duality corresponding to the variational principle:

$$\begin{aligned} &\mathcal{L}_\varepsilon(u^\varepsilon, v^\varepsilon) = \min_{\mathrm{Re}(u),\mathrm{Re}(v)} \max_{\mathrm{Im}(u),\mathrm{Im}(v)} \mathcal{L}_\varepsilon(u, v) \\ &\text{over } u, v \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : u = h,\, v = 0 \text{ on } \Gamma_D, \end{aligned} \tag{3.49}$$

where the Lagrangian $\mathcal{L}_\varepsilon$ has the form (compare with (3.5) and (3.6)):

$$\begin{aligned} \mathcal{L}_\varepsilon(u, v) :=&\mathrm{Re}\Big\{ \int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla u \cdot \nabla v - k^2 uv)\, dx - \int_{\Gamma_N} gv\, dS_x \\ &+ \tfrac{1}{2} \int_{\Gamma_N} (u - u^\star)^2\, dS_x \Big\}. \end{aligned} \tag{3.50}$$

Lemma 3.7. *The first-order necessary optimality conditions for* (3.49) *imply the primal problem* (3.4) *together with the dual variational problem: Find* $v^\varepsilon \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ *such that*

$$v^\varepsilon = 0 \quad \text{on } \Gamma_D, \tag{3.51a}$$

$$\begin{aligned} &\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla v^\varepsilon \cdot \nabla \overline{u} - k^2 v^\varepsilon \overline{u})\, dx = -\int_{\Gamma_N} (u^\varepsilon - u^\star)\overline{u}\, dS_x \\ &\text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : u = 0 \text{ on } \Gamma_D. \end{aligned} \tag{3.51b}$$

Proof. Indeed, using variational calculus from Proposition 2.1, the first order optimality condition for (3.49) necessitates four variational inequalities:

$$\begin{aligned} &\big\langle \tfrac{\partial}{\partial \mathrm{Re}(v)} \mathcal{L}_\varepsilon(u^\varepsilon, v^\varepsilon), \mathrm{Re}(v - v^\varepsilon) \big\rangle \ge 0, \quad \big\langle \tfrac{\partial}{\partial \mathrm{Im}(v)} \mathcal{L}_\varepsilon(u^\varepsilon, v^\varepsilon), \mathrm{Im}(v - v^\varepsilon) \big\rangle \le 0, \\ &\big\langle \tfrac{\partial}{\partial \mathrm{Re}(u)} \mathcal{L}_\varepsilon(u^\varepsilon, v^\varepsilon), \mathrm{Re}(u - u^\varepsilon) \big\rangle \ge 0, \quad \big\langle \tfrac{\partial}{\partial \mathrm{Im}(u)} \mathcal{L}_\varepsilon(u^\varepsilon, v^\varepsilon), \mathrm{Im}(u - u^\varepsilon) \big\rangle \le 0 \end{aligned}$$

holding for all $u, v \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ such that $u = h$, $v = 0$ on Γ_D. Inserting here $v = v^\varepsilon \pm \mathbf{v}$ and $u = u^\varepsilon \pm \mathbf{u}$ with $\mathbf{v}, \mathbf{u} \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ such that $\mathbf{v} = \mathbf{u} = 0$ on Γ_D we get the following four variational equations:

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} \big(\nabla \mathrm{Re}(u^\varepsilon) \cdot \nabla \mathrm{Re}(\mathbf{v}) - k^2 \mathrm{Re}(u^\varepsilon)\mathrm{Re}(\mathbf{v})\big)\, dx = \int_{\Gamma_N} \mathrm{Re}(g)\mathrm{Re}(\mathbf{v})\, dS_x,$$

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} \big(\nabla \mathrm{Im}(u^\varepsilon) \cdot \nabla \mathrm{Im}(\mathbf{v}) - k^2 \mathrm{Im}(u^\varepsilon)\mathrm{Im}(\mathbf{v})\big)\, dx = \int_{\Gamma_N} \mathrm{Im}(g)\mathrm{Im}(\mathbf{v})\, dS_x,$$

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} \big(\nabla \mathrm{Re}(v^\varepsilon) \cdot \nabla \mathrm{Re}(\mathbf{u}) - k^2 \mathrm{Re}(v^\varepsilon)\mathrm{Re}(\mathbf{u})\big)\, dx = \int_{\Gamma_N} \mathrm{Re}(u^\star - u^\varepsilon)\mathrm{Re}(\mathbf{u})\, dS_x,$$

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} \big(\nabla \mathrm{Im}(v^\varepsilon) \cdot \nabla \mathrm{Im}(\mathbf{u}) - k^2 \mathrm{Im}(v^\varepsilon)\mathrm{Im}(\mathbf{u})\big)\, dx = \int_{\Gamma_N} \mathrm{Im}(u^\star - u^\varepsilon)\mathrm{Im}(\mathbf{u})\, dS_x.$$

The summation of the first and the second equations for $\mathbf{v} = u$ and $\mathbf{v} = \imath u$ constitutes the real and imaginary parts of (3.4b), while the third and the fourth equations for $\mathbf{u} = u$ and $\mathbf{u} = \imath u$ contribute to (3.51b), respectively. This completes the proof. □

We emphasize that the Helmholtz problem (3.51) is analogous to (3.4) and differs from it by the boundary data at $\partial\Omega$. Therefore, the well-posedness result stated for u^ε remains true also for v^ε. It implies the weak solution to (cf. (3.3))

$$-[\Delta + k^2]v^\varepsilon = 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon(x_0)}, \tag{3.52a}$$

$$\tfrac{\partial v^\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{3.52b}$$

$$\tfrac{\partial v^\varepsilon}{\partial \nu} = -(u^\varepsilon - u^\star) \quad \text{on } \Gamma_N, \tag{3.52c}$$

$$v^\varepsilon = 0 \quad \text{on } \Gamma_D. \tag{3.52d}$$

Plugging the representation (3.52c) in (3.47) we establish the following equivalence.

Proposition 3.8. *The topology optimization problem* (3.48) *can by equivalently stated with respect to the primal and dual state variables: Find* $(\omega^\star, \varepsilon^\star, x^\star) \in \mathfrak{G}$ *such that*

$$0 = \min_{(\omega,\varepsilon,x_0)\in\mathfrak{G}} \mathrm{Re}\Big\{-\tfrac{1}{2}\int_{\Gamma_N} (u^\varepsilon - u^\star)\tfrac{\partial \overline{v^\varepsilon}}{\partial \nu}\, dS_x\Big\} \quad \textit{subject to } (3.4) \textit{ and } (3.51). \tag{3.53}$$

Further we provide asymptotic analysis of the objective as $\varepsilon \searrow +0$.

As $\varepsilon = 0$, problem (3.51) turns into the dual background problem stated in the reference domain Ω: Find $v^0 \in H^1(\Omega;\mathbb{C})$ such that

$$v^0 = 0 \quad \text{on } \Gamma_D, \tag{3.54a}$$

$$\begin{aligned}&\int_\Omega (\nabla v^0 \cdot \nabla \overline{u} - k^2 v^0 \overline{u})\, dx = -\int_{\Gamma_N} (u^0 - u^\star)\overline{u}\, dS_x\\ &\text{for all } u \in H^1(\Omega;\mathbb{C}) : u = 0 \text{ on } \Gamma_D,\end{aligned} \tag{3.54b}$$

which implies the weak solution to (cf. (2.1))

$$-[\Delta + k^2]v^0 = 0 \quad \text{in } \Omega, \tag{3.55a}$$

$$\tfrac{\partial v^0}{\partial \nu} = -(u^0 - u^\star) \quad \text{on } \Gamma_N, \tag{3.55b}$$

$$v^0 = 0 \quad \text{on } \Gamma_D. \tag{3.55c}$$

Problem (3.54) is similar to the primal background problem (2.2), henceforth, all the results of Section 2 hold true for (3.54), too. In particular, the inner asymptotic expansion holds in the near-field $B_R(x_0) \subset \Omega$ in the form

$$v^0(x) = v^0(x_0)J_0(k\rho) + V_0^0(x), \quad V_0^0(x) = \tfrac{2}{k}J_1(k\rho)\nabla v^0(x_0)\cdot\widehat{x} + V_1^0(x) \tag{3.56}$$

with the residuals $V_0^0, V_1^0 \in H^1(B_R(x_0);\mathbb{C})$ such that

$$\int_{-\pi}^{\pi} V_0^0\, d\theta = \int_{-\pi}^{\pi} V_1^0\, d\theta = \int_{-\pi}^{\pi} V_1^0 \widehat{x}\, d\theta = 0, \tag{3.57a}$$

$$V_0^0(x_0+\rho\widehat{x}) = \mathrm{O}(\rho), \quad V_1^0(x_0+\rho\widehat{x}) = \mathrm{O}(\rho^2) \quad \text{for } \theta \in (-\pi,\pi], \tag{3.57b}$$

and it implies similar to (2.35) representation of the gradient

$$\nabla v^0(x) = \nabla v^0(x_0) + b_v^0(x) + \nabla V_1^0(x), \quad \nabla V_1^0 = \mathrm{O}(\rho), \tag{3.58a}$$

$$\begin{aligned} b_v^0(x) &:= \big(v^0(x_0) k a_0'(k\rho) + a_1'(k\rho)\nabla v^0(x_0)\cdot\widehat{x}\big)\widehat{x} \\ &+ \tfrac{a_1(k\rho)}{k\rho}(\nabla v^0(x_0)\cdot\widehat{x}')\widehat{x}', \quad b_v^0 = \mathrm{O}(\rho). \end{aligned} \tag{3.58b}$$

We note that, expansion of $v^\varepsilon - v^0$ for $\varepsilon \searrow +0$ in the manner of Theorem 3.6 would be a hard task since the right-hand side of problem (3.51) itself depends on u^ε. In this respect, Proposition 3.8 will be not helpful. Instead, decomposing $u^\varepsilon - u^\star = u^\varepsilon - u^0 + u^0 - u^\star$ and using (3.55b) we express the objective in (3.47) equivalently

$$\begin{aligned} J(\omega,\varepsilon,x_0) &= J_0 - \mathrm{Re}\Big\{\int_{\Gamma_N} (u^\varepsilon - u^0)\tfrac{\partial\overline{v^0}}{\partial\nu}\, dS_x\Big\} + \tfrac12\int_{\Gamma_N} |u^\varepsilon - u^0|^2\, dS_x, \\ J_0 &:= \tfrac12\int_{\Gamma_N} |u^0 - u^\star|^2\, dS_x = \mathrm{O}(1), \quad \tfrac12\int_{\Gamma_N} |u^\varepsilon - u^0|^2\, dS_x = \mathrm{O}(\varepsilon^4|\ln\varepsilon|) \end{aligned} \tag{3.59}$$

for $\varepsilon \searrow +0$ due to Theorem 3.6. From (3.59) we infer the asymptotic result below.

Theorem 3.9. *The objective in* (3.47) *admits the high-order asymptotic expansion*

$$J(\omega,\varepsilon,x_0) = J_0 + \mathrm{Re}\big\{\varepsilon^2 J_1^N(\omega,x_0) + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon\big\} + \mathrm{O}(\varepsilon^4|\ln\varepsilon|), \tag{3.60}$$

where the asymptotic terms are expressed by formulas:

$$J_1^N(\omega,x_0) := -\nabla u^0(x_0)^\top M_\omega \nabla\overline{v^0}(x_0) + k^2 \mathrm{meas}_2(\omega) u^0(x_0)\overline{v^0}(x_0), \tag{3.61a}$$

$$\begin{aligned} J_2^\varepsilon &:= \varepsilon^3\sqrt{|\ln\varepsilon|}\int_{-\pi}^{\pi} \nabla u^0(x_0)\cdot\big(\tfrac{\partial u^1}{\partial\rho}\overline{v^0}(x_0) - u^1\nabla\overline{v^0}(x_0)\cdot\widehat{x}\big)\, d\theta \\ &= \mathrm{O}(\varepsilon^3\sqrt{|\ln\varepsilon|}), \end{aligned} \tag{3.61b}$$

$$\begin{aligned} J_3^\varepsilon &:= \varepsilon^2\int_{-\pi}^{\pi} \nabla u^0(x_0)\cdot\big(\tfrac{\partial W_\nu^\varepsilon}{\partial\rho}\overline{V_1^0} - W_\nu^\varepsilon\tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)\, d\theta \\ &- \int_{\partial\omega_\varepsilon(x_0)} \nu\cdot\big(q_1^\varepsilon\nabla v^0(x_0) + (b_u^0 + \nabla U_1^0)\overline{V_0^0}\big)\, dS_x = \mathrm{O}(\varepsilon^3), \end{aligned} \tag{3.61c}$$

$$\begin{aligned} J_4^\varepsilon &:= \varepsilon^2\sqrt{|\ln\varepsilon|}\Big(\varepsilon\int_{-\pi}^{\pi} \nabla u^0(x_0)\cdot\big\{\tfrac{\partial u^1}{\partial\rho}\varepsilon\nabla\overline{v^0}(x_0)\cdot\widehat{x} - u^1\big(\overline{v^0}(x_0)ka_0' \\ &+ \tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)\big\}\, d\theta - \int_{\partial\omega_\varepsilon(x_0)} \nu\cdot(Du^1)\nabla u^0(x_0)\overline{V_0^0}\, dS_x\Big) = \mathrm{O}(\varepsilon^4\sqrt{|\ln\varepsilon|}). \end{aligned} \tag{3.61d}$$

Proof. To verify (3.60) it needs to expand up to $\mathrm{O}(\varepsilon^4|\ln\varepsilon|)$-order asymptotic terms the boundary integral in (3.59)

$$\mathcal{I}(u^\varepsilon - u^0, v^0) := -\int_{\Gamma_N} (u^\varepsilon - u^0)\tfrac{\partial \overline{v^0}}{\partial\nu}\, dS_x.$$

For this task we employ the second Green formula in $\Omega\setminus\overline{B_\varepsilon(x_0)}$ and rewrite $\mathcal{I}$ as

$$\begin{aligned}\mathcal{I}(u^\varepsilon - u^0, v^0) &= \int_{\Omega\setminus B_\varepsilon(x_0)} \big(\Delta(u^\varepsilon - u^0)\overline{v^0} - (u^\varepsilon - u^0)\Delta\overline{v^0}\big)\, dx \\ &+ \int_{\partial B_\varepsilon(x_0)} \big(\tfrac{\partial(u^\varepsilon - u^0)}{\partial\rho}\overline{v^0} - (u^\varepsilon - u^0)\tfrac{\partial\overline{v^0}}{\partial\rho}\big)\, dS_x,\end{aligned} \tag{3.62}$$

where the domain integral disappears due to the Helmholtz equations (2.1a), (3.3a), and (3.55a). On the one hand, we note that $\mathcal{I}$ is an invariant integral which can be written over arbitrary Lipschitz-smooth boundary $\partial\mathcal{O}$ of a domain $\mathcal{O}$ such that $\omega_\varepsilon(x_0)\subseteq\mathcal{O}\subset\Omega$. On the other hand, the integral over the circle $\partial B_\varepsilon(x_0)$ is advantageous while it can be calculated by substitution of the uniform expansion (3.43) for $u^\varepsilon - u^0$ and the inner expansion (3.56) and (3.58) for v^0.

By doing so, we decompose $\mathcal{I}(u^\varepsilon - u^0, v^0) = \mathcal{I}(q_1^\varepsilon, v^0) + \mathcal{I}(u^\varepsilon - u^0 - q_1^\varepsilon, v^0)$ with the residual q_1^ε from Theorem 3.6 and calculate these two integrals separately.

First, applying to $\mathcal{I}(q_1^\varepsilon, v^0)$ the second Green formula in $B_\varepsilon(x_0)\setminus\overline{\omega_\varepsilon(x_0)}$ we get

$$\begin{aligned}\mathcal{I}(q_1^\varepsilon, v^0) &:= \int_{\partial B_\varepsilon(x_0)} \big(\tfrac{\partial q_1^\varepsilon}{\partial\rho}\overline{v^0} - q_1^\varepsilon\tfrac{\partial\overline{v^0}}{\partial\rho}\big)\, dS_x = \int_{\partial\omega_\varepsilon(x_0)} \big(\tfrac{\partial q_1^\varepsilon}{\partial\nu}\overline{v^0} - q_1^\varepsilon\tfrac{\partial\overline{v^0}}{\partial\nu}\big)\, dS_x \\ &= -\int_{\partial\omega_\varepsilon(x_0)} \nu\cdot\big\{q_1^\varepsilon(\nabla v^0(x_0) + b_v^0 + \nabla V_1^0) \\ &+ \big(b_u^0 + \nabla U_1^0 + \varepsilon^2\sqrt{|\ln\varepsilon|}(Du^1)\nabla u^0(x_0)\big)\big(\overline{v^0}(x_0)(1+a_0) + \overline{V_0^0}\big)\big\}\, dS_x\end{aligned}$$

after substitution of (3.46b), (3.56), and (3.58a). The divergence theorem provides

$$\begin{aligned}&-\int_{\partial\omega_\varepsilon(x_0)} \nu\cdot\big(b_u^0 + \nabla U_1^0 + \varepsilon^2\sqrt{|\ln\varepsilon|}(Du^1)\nabla u^0(x_0)\big)\overline{v^0}(x_0)\, dS_x \\ &= -\int_{\omega_\varepsilon(x_0)} \mathrm{div}\big(b_u^0 + \nabla U_1^0 + \varepsilon^2\sqrt{|\ln\varepsilon|}(Du^1)\nabla u^0(x_0)\big)\overline{v^0}(x_0)\, dx.\end{aligned} \tag{3.63}$$

Here $-\mathrm{div}(\nabla U_1^0) = -\Delta U_1^0 = k^2 U_1^0$ due to (2.1a) and (2.33), $b_u^0 = -u^0(x_0)\frac{k^2\rho}{2}\widehat{x} + \mathrm{O}(\rho^2)$ according to (2.8) and (2.35b), and since $\rho\widehat{x} = x - x_0$ then

$$I_1^q := \int_{\omega_\varepsilon(x_0)} \mathrm{div}\big(u^0(x_0)\tfrac{k^2(x-x_0)}{2}\overline{v^0}(x_0)\big)\, dx = k^2 u^0(x_0)\overline{v^0}(x_0)\int_{\omega_\varepsilon(x_0)} dx. \tag{3.64}$$

With the help of (3.63) and (3.64) we collect the asymptotic terms of the same order

$$\mathcal{I}(q_1^\varepsilon, v^0) = I_1^q + I_3^q + I_4^q + I_5^q \quad \text{(the term } I_2^q \text{ of order } \mathrm{O}(\varepsilon^3\sqrt{|\ln\varepsilon|}) \text{ is zero)},$$

$$I_3^q := -\int_{\partial\omega_\varepsilon(x_0)} \nu\cdot\big(q_1^\varepsilon \nabla v^0(x_0) + (b_u^0 + \nabla U_1^0)\overline{V_0^0}\big)\, dS_x = \mathrm{O}(\varepsilon^3),$$

$$I_4^q := -\varepsilon^2\sqrt{|\ln\varepsilon|}\int_{\partial\omega_\varepsilon(x_0)} \nu\cdot(Du^1)\nabla u^0(x_0)\overline{V_0^0}\, dS_x = \mathrm{O}(\varepsilon^4\sqrt{|\ln\varepsilon|}),$$

$$I_5^q := -\int_{\omega_\varepsilon(x_0)} \mathrm{div}\big(b_u^0 + u^0(x_0)\tfrac{k^2(x-x_0)}{2} + \varepsilon^2\sqrt{|\ln\varepsilon|}(Du^1)\nabla u^0(x_0)\big)\overline{v^0}(x_0)\, dx$$
$$-\int_{\partial\omega_\varepsilon(x_0)} \{\nu\cdot\big(b_u^0 + \nabla U_1^0 + \varepsilon^2\sqrt{|\ln\varepsilon|}(Du^1)\nabla u^0(x_0)\big)\overline{v^0}(x_0)a_0$$
$$+ q_1^\varepsilon(b_v^0 + \nabla V_1^0)\}\, dS_x + \int_{\omega_\varepsilon(x_0)} k^2 U_1^0\overline{v^0}(x_0)\, dx = \mathrm{O}(\varepsilon^4)$$

in view of the following asymptotic relations:

$$\begin{aligned} &q_1^\varepsilon = U_1^0 = V_1^0 = \mathrm{O}(\varepsilon^2), \quad \tfrac{\partial q_1^\varepsilon}{\partial\rho} = \tfrac{\partial U_1^0}{\partial\rho} = \tfrac{\partial V_1^0}{\partial\rho} = \mathrm{O}(\varepsilon), \quad u^1 = \tfrac{\partial u^1}{\partial\rho} = \mathrm{O}(1), \\ &V_0^0 = b_u^0 = b_v^0 = \mathrm{O}(\varepsilon), \int_{\omega_\varepsilon(x_0)} dx = \varepsilon^2\mathrm{meas}_2(\omega), \int_{\partial\omega_\varepsilon(x_0)} dS_x = \varepsilon\,\mathrm{meas}_1(\partial\omega), \end{aligned} \tag{3.65}$$

which hold in $B_\varepsilon(x_0)$ due to (2.34b), (2.35b), (3.43), and (3.57b).

Second, inserting in $\mathcal{I}$ the representations (3.19), (3.43), (3.56), and (3.58a) we have

$$\begin{aligned} \mathcal{I}(u^\varepsilon - u^0 - q_1^\varepsilon, v^0) &:= \int_{\partial B_\varepsilon(x_0)} \big(\tfrac{\partial(u^\varepsilon - u^0 - q_1^\varepsilon)}{\partial\rho}\overline{v^0} - (u^\varepsilon - u^0 - q_1^\varepsilon)\tfrac{\partial\overline{v^0}}{\partial\rho}\big)\, dS_x \\ &= \varepsilon\int_{-\pi}^{\pi} \nabla u^0(x_0)\cdot\{\big(-\tfrac{1}{2\pi\varepsilon}M_\omega\,\widehat{x} + \tfrac{\partial W_\nu^\varepsilon}{\partial\rho} + \varepsilon\sqrt{|\ln\varepsilon|}\tfrac{\partial u^1}{\partial\rho}\big)\big(\overline{v^0}(x_0)(1+a_0) \\ &+ (\varepsilon + \tfrac{a_1}{k})\nabla\overline{v^0}(x_0)\cdot\widehat{x} + \overline{V_1^0}\big) - \big(\tfrac{1}{2\pi}M_\omega\,\widehat{x} + W_\nu^\varepsilon + \varepsilon\sqrt{|\ln\varepsilon|}u^1\big) \\ &\times\big(\nabla\overline{v^0}(x_0)\cdot\widehat{x} + \overline{b_v^0}\cdot\widehat{x} + \tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)\}\,\varepsilon d\theta = I_1^w + I_2^w + I_3^w + I_4^w + I_5^w. \end{aligned} \tag{3.66}$$

We calculate the asymptotic terms $I_1^w, I_2^w, I_3^w, I_4^w, I_5^w$ in (3.66) by using the orthogonality (3.20a) and (3.57a), calculus (2.27) providing

$$\int_{-\pi}^{\pi} \nabla u^0(x_0)\cdot\tfrac{1}{2\pi}(M_\omega\,\widehat{x})(\nabla\overline{v^0}(x_0)\cdot\widehat{x})\, d\theta = \tfrac{1}{2}\nabla u^0(x_0)^\top M_\omega\nabla\overline{v^0}(x_0), \tag{3.67}$$

and the following relations holding at $\partial B_\varepsilon(x_0)$ due to (2.8), (3.20b), and (3.58b):

$$\begin{aligned} &\overline{b_v^0}\cdot\widehat{x} = \overline{v^0}(x_0)ka_0' + a_1'\nabla\overline{v^0}(x_0)\cdot\widehat{x}, \quad W_\nu^\varepsilon = \mathrm{O}(1), \quad \tfrac{\partial W_\nu^\varepsilon}{\partial\rho} = \mathrm{O}(\tfrac{1}{\varepsilon}), \\ &a_0 = \mathrm{O}(\varepsilon^2), \quad a_0' = \mathrm{O}(\varepsilon), \quad a_1 = \mathrm{O}(\varepsilon^3), \quad a_1' = \mathrm{O}(\varepsilon^2). \end{aligned} \tag{3.68}$$

With the help of (3.65), (3.67), and (3.68) the calculation results in

$$I_1^w = -\varepsilon^2\nabla u^0(x_0)^\top M_\omega\nabla\overline{v^0}(x_0),$$

$$I_2^w := \varepsilon^3\sqrt{|\ln\varepsilon|}\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big(\tfrac{\partial u^1}{\partial\rho}\overline{v^0}(x_0) - u^1\nabla\overline{v^0}(x_0)\cdot\widehat{x}\big)\,d\theta = \mathrm{O}(\varepsilon^3\sqrt{|\ln\varepsilon|}),$$

$$I_3^w := \varepsilon^2\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big(\tfrac{\partial W_\nu^\varepsilon}{\partial\rho}\overline{V_1^0} - W_\nu^\varepsilon\tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)d\theta = \mathrm{O}(\varepsilon^3),$$

$$I_4^w := \varepsilon^3\sqrt{|\ln\varepsilon|}\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big\{\tfrac{\partial u^1}{\partial\rho}\varepsilon\nabla\overline{v^0}(x_0)\cdot\widehat{x} - u^1\big(\overline{v^0}(x_0)ka_0' + \tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)\big\}\,d\theta$$
$$= \mathrm{O}(\varepsilon^4\sqrt{|\ln\varepsilon|}),$$

$$I_5^w := \varepsilon^3\sqrt{|\ln\varepsilon|}\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big\{\tfrac{\partial u^1}{\partial\rho}\big(\overline{v^0}(x_0)a_0 + \tfrac{a_1}{k\varepsilon}\nabla\overline{v^0}(x_0)\cdot\widehat{x} + \overline{V_1^0}\big)$$
$$- u^1(a_1'\nabla\overline{v^0}(x_0)\cdot\widehat{x})\big\}\,d\theta - \tfrac{\varepsilon}{2}(\tfrac{a_1}{k} + \varepsilon a_1')\nabla u^0(x_0)^\top M_\omega\nabla\overline{v^0}(x_0) = \mathrm{O}(\varepsilon^4).$$

Finally, the summation $I_1^q + I_1^w = \varepsilon^2 J_1^N$, $I_2^w = J_2^\varepsilon$, $I_3^q + I_3^w = J_3^\varepsilon$, and $I_4^q + I_4^w = J_4^\varepsilon$ gathers the asymptotic terms in (3.61) and finishes the proof. □

We make a few remarks on corollaries following from Theorem 3.9.

The first-order asymptotic term $\mathrm{Re}(J_1^N)$ given in formula (3.61a) is called the topological derivative of the objective J following the terminology of [57] since

$$\mathrm{Re}(J_1^N(\omega,x_0)) = \lim_{\varepsilon\searrow+0}\tfrac{1}{\varepsilon^2}\big(J(\omega,\varepsilon,x_0) - J_0\big). \tag{3.69}$$

It was found e.g. in [3, 8].

After approximation by (3.60), the optimization problem (3.48) forces the shape-topological control problem: For fixed $\varepsilon\in\mathfrak{G}_\varepsilon$, find $(\omega^\star,x^\star)\in\mathfrak{G}_\omega\times\mathfrak{G}_x$ such that

$$\mathrm{Re}(J_1^N(\omega^\star,x^\star)) = \min_{(\omega,x_0)\in\mathfrak{G}_\omega\times\mathfrak{G}_x}\mathrm{Re}(J_1^N(\omega,x_0)). \tag{3.70}$$

The approximated objective in (3.70) does not depend on the perturbed state. It is expressed by the reference solutions u^0 and v^0 of the primal (2.2) and the dual (3.54) background Helmholtz problems, as well as the solution w_ν of the exterior Neumann problem for the Laplace operator (3.16).

The boundary layer w_ν enters formula (3.61a) via the virtual mass tensor M_ω. Therefore, employing the explicit description of M_ω given for ellipsoidal shapes in Lemma 3.5, the control problem (3.70) can be relaxed by reducing the set of admissible shapes $\mathfrak{G}_\omega$ to a family of ellipses depending on rotation and compression.

The problem of identification of the center $x^\star$ of the test object will be discussed further in Section 5.4.

In the next Section 4 we modify our methods to treat forward and inverse problems for the Helmholtz equation under Dirichlet boundary conditions at $\partial\omega_\varepsilon(x_0)$.

4 Helmholtz problems for geometric objects under Dirichlet (sound soft) boundary condition

Given $g \in L^2(\Gamma_N; \mathbb{C})$ and $h \in H^{1/2}(\Gamma_D; \mathbb{C})$, the (forward) Dirichlet problem for the Helmholtz equation consists in finding the wave potential $u^\varepsilon(x)$ fulfilling:

$$-[\Delta + k^2]u^\varepsilon = 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon(x_0)}, \tag{4.1a}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{4.1b}$$

$$\tfrac{\partial u^\varepsilon}{\partial \nu} = g \quad \text{on } \Gamma_N, \tag{4.1c}$$

$$u^\varepsilon = h \quad \text{on } \Gamma_D, \tag{4.1d}$$

which is described by the variational problem: Find $u^\varepsilon \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ such that

$$u^\varepsilon = h \quad \text{on } \Gamma_D, \quad u^\varepsilon = 0 \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{4.2a}$$

$$\begin{gathered} \int_{\Omega \setminus \omega_\varepsilon(x_0)} (\nabla u^\varepsilon \cdot \nabla \overline{u} - k^2 u^\varepsilon \overline{u})\, dx = \int_{\Gamma_N} g\overline{u}\, dS_x \\ \text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : u = 0 \text{ on } \Gamma_D \cup \partial\omega_\varepsilon(x_0). \end{gathered} \tag{4.2b}$$

Well-posedness of the Dirichlet problem (4.2) is argued similarly to the Neumann problem (3.4). It implies necessary condition for the variational principle with $\mathcal{P}_\varepsilon$ from (3.6):

$$\begin{gathered} \mathcal{P}_\varepsilon(u^\varepsilon) = \min_{\mathrm{Re}(v)} \max_{\mathrm{Im}(v)} \mathcal{P}_\varepsilon(v) \\ \text{over } v \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : v = h \text{ on } \Gamma_D,\, v = 0 \text{ on } \partial\omega_\varepsilon(x_0). \end{gathered} \tag{4.3}$$

To evaluate the difference of u^ε from the background solution u^0, the outer asymptotic expansion in the far-field is needed, see [33, Section 3.3].

4.1 Outer and inner asymptotic expansions by Fourier series

We construct three auxiliary problems: two boundary layers in the exterior domain $\mathbb{R}^2 \setminus \omega$ and a regularized Helmholtz problem in Ω for the logarithm.

First, we define the kernel of the Laplace operator in $\mathbb{R}^2$ by means of the logarithmic capacity, see e.g. [21]. We consider the following homogeneous Dirichlet problem:

$$-\Delta w_{00} = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\omega}, \tag{4.4a}$$

$$w_{00} = 0 \quad \text{on } \partial\omega, \tag{4.4b}$$

$$w_{00} = \mathrm{O}(\ln|y|) \quad \text{as } |y| \nearrow \infty. \tag{4.4c}$$

The weak variational formulation to (4.4) can be given in the weighted Sobolev spaces introduced in Section 3.1: For $p > 2$ and $p' < 2$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, find $w_{00} \in W^{1,p}_{\mu}(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R})$ such that

$$w_{00} = 0 \quad \text{on } \partial\omega, \tag{4.5a}$$

$$\int_{\mathbb{R}^2 \setminus \omega} \nabla w_{00} \cdot \nabla v \, dy = 0 \quad \text{for all } v \in W^{1,p'}_{\mu}(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R}) : v = 0 \text{ on } \partial\omega. \tag{4.5b}$$

The existence of a nontrivial solution to (4.5) is argued as follows. Following [7], there exists a unique solution of the inhomogeneous Dirichlet problem: Find $\widetilde{w} \in W^{1,2}_{\mu}(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R})$ such that

$$\widetilde{w} = \ln|y| \quad \text{on } \partial\omega,$$

$$\int_{\mathbb{R}^2 \setminus \omega} \nabla \widetilde{w} \cdot \nabla v \, dy = 0 \quad \text{for all } v \in W^{1,2}_{\mu}(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R}) : v = 0 \text{ on } \partial\omega.$$

Since $\ln|y| \in W^{1,p}_{\mu}(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R})$ for $p > 2$, then $w_{00} = \ln|y| - \widetilde{w}$ solves (4.5). It admits the Fourier series (cf. (3.23)) in $\mathbb{R}^2 \setminus \overline{B_1(0)}$:

$$w_{00}(y) = -\ln(\mathrm{cap}(\omega)) + \ln|y| + W_{00}, \quad W_{00} = \sum_{n=1}^{\infty} \tfrac{1}{|y|^n} C^{00}_n \widehat{x}^n, \tag{4.6}$$

with $C^{00}_n \in \mathbb{R}^{2\times 2}$, $n \in \mathbb{N}$, and $\mathrm{cap}(\omega) \in \mathbb{R}_+$ called logarithmic capacity of the set ω.

After rescaling $y = \frac{x - x_0}{\varepsilon}$, the exterior problem (4.5) can be reduced to the bounded domain $\Omega \setminus \overline{\omega_\varepsilon(x_0)}$ with the help of the following Green formula holding for every function $\mathbf{u} \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ such that $\Delta \mathbf{u} \in L^2(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ (cf. (3.7)):

$$\begin{aligned} &\int_{\Omega \setminus \omega_\varepsilon(x_0)} (\nabla \mathbf{u} \cdot \nabla \overline{u} + \overline{u}\, \Delta \mathbf{u}) \, dx = \langle \tfrac{\partial \mathbf{u}}{\partial \nu}, \overline{u} \rangle_{\Gamma_N} \\ &\text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : u = 0 \text{ on } \Gamma_D \cup \partial\omega_\varepsilon(x_0). \end{aligned} \tag{4.7}$$

Then relations (4.5)–(4.7) prove the assertion of the Lemma 4.1 below.

Lemma 4.1. *The rescaled solution* $w^\varepsilon_{00}(x) := w_{00}(\frac{x - x_0}{\varepsilon})$ *to* (4.5) *fulfills*

$$w^\varepsilon_{00} = 0 \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{4.8a}$$

$$\begin{aligned} &\int_{\Omega \setminus \omega_\varepsilon(x_0)} \nabla w^\varepsilon_{00} \cdot \nabla u \, dx = \int_{\Gamma_N} \tfrac{\partial w^\varepsilon_{00}}{\partial \nu} u \, dS_x \\ &\text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R}) : u = 0 \text{ on } \Gamma_D \cup \partial\omega_\varepsilon(x_0). \end{aligned} \tag{4.8b}$$

It admits the far-field representation in the Fourier series

$$w^\varepsilon_{00}(x) = -\ln(\varepsilon \mathrm{cap}(\omega)) + \ln\rho + W^\varepsilon_{00}(x) \quad \text{for } x \in \mathbb{R}^2 \setminus \overline{B_\varepsilon(x_0)}, \tag{4.9}$$

with the residual function $W^\varepsilon_{00} \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R})$ *such that*

$$\int_{-\pi}^{\pi} W^\varepsilon_{00} \, d\theta = 0, \quad W^\varepsilon_{00} = \mathrm{O}(\tfrac{\varepsilon}{\rho}) \quad \text{for } \rho > \varepsilon,\ \theta \in (-\pi, \pi]. \tag{4.10}$$

Second, to compensate the logarithm in (4.9) which is unbounded as $\rho \searrow +0$, we construct the regularized Helmholtz problem in Ω: Find $u^{\ln} \in H^1(\Omega;\mathbb{R})$ such that

$$u^{\ln} = \ln\rho \quad \text{on } \Gamma_D, \tag{4.11a}$$

$$\int_\Omega (\nabla u^{\ln}\cdot\nabla u - k^2 u^{\ln} u)\,dx = \int_{\Gamma_N} \tfrac{\partial(\ln\rho)}{\partial\nu} u\,dS_x - \int_\Omega k^2 u \ln\rho\,dx \tag{4.11b}$$

for all $u \in H^1(\Omega;\mathbb{R}) : u = 0$ on Γ_D.

Since $\Delta[\ln\rho] = 0$, the solution of (4.11) describes the boundary value problem:

$$-[\Delta + k^2]u^{\ln} = -[\Delta + k^2]\ln\rho \quad \text{in } \Omega, \tag{4.12a}$$

$$\tfrac{\partial u^{\ln}}{\partial\nu} = \tfrac{\partial(\ln\rho)}{\partial\nu} \quad \text{on } \Gamma_N, \tag{4.12b}$$

$$u^{\ln} = \ln\rho \quad \text{on } \Gamma_D. \tag{4.12c}$$

Using Green's formula (4.7) we restate (4.11b) over the perturbed domain as

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla u^{\ln}\cdot\nabla u - k^2 u^{\ln} u)\,dx = \int_{\Gamma_N} \tfrac{\partial(\ln\rho)}{\partial\nu} u\,dS_x - \int_{\Omega\setminus\omega_\varepsilon(x_0)} k^2 u\ln\rho\,dx \tag{4.13}$$

for all $u \in H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{R}) : u = 0$ on $\Gamma_D \cup \partial\omega_\varepsilon(x_0)$.

Similarly to Lemma 2.2, below we establish the inner asymptotic expansion for $u^{\ln}$.

Lemma 4.2. *The solution $u^{\ln}$ of* (4.11) *admits the representation in the near-field*

$$u^{\ln}(x) = u^{\ln}(x_0) + (u^{\ln}(x_0) - \ln\rho)a_0 - \tfrac{\pi}{2}a_2 + U_0^{\ln}(x) \quad \textit{in } B_R(x_0) \subset \Omega, \tag{4.14}$$

with a_0 and a_2 given in (2.8) *and the residual $U_0^{\ln} \in H^1(B_R(x_0);\mathbb{R})$ such that*

$$\int_{-\pi}^{\pi} U_0^{\ln}\,d\theta = 0, \quad U_0^{\ln} = \mathrm{O}(\rho) \quad \textit{for } \rho \in [0,R),\ \theta \in (-\pi,\pi]. \tag{4.15}$$

Proof. For $B_R(x_0) \subset \Omega$, we decompose $u^{\ln}$ into the radial and residual functions:

$$\begin{aligned} &u^{\ln}(x) = u_0^{\ln}(\rho) + U_0^{\ln}(x) \quad \text{in } B_\delta(x_0),\ \delta \in [0,R), \text{ where} \\ &u_0^{\ln}(\rho) := \tfrac{1}{2\pi}\int_{-\pi}^{\pi} u^{\ln}\,d\theta, \quad U_0^{\ln} := u^{\ln} - u_0^{\ln}, \quad \text{hence } \int_{-\pi}^{\pi} U_0^{\ln}\,d\theta = 0. \end{aligned} \tag{4.16}$$

Using (4.16) we substitute a smooth cut-off function $\eta(\rho)$ supported in $B_\delta(x_0)$ as the test function $u = \eta$ into (4.11b) and integrate it by parts to derive that

$$0 = \int_{B_\delta(x_0)} \big(\nabla u^{\ln}\cdot\nabla\eta - k^2(u^{\ln} - \ln\rho)\eta\big)\,dx = 2\pi\int_0^\delta \big((u_0^{\ln})'_\rho\eta'$$

$$- k^2(u_0^{\ln} - \ln\rho)\eta\big)\,\rho d\rho = -2\pi\int_0^\delta \Big(\big(\rho(u_0^{\ln})'_\rho\big)'_\rho + k^2\rho(u_0^{\ln} - \ln\rho)\Big)\eta\,d\rho$$

which implies the inhomogeneous Bessel equation

$$(u_0^{\ln})''_\rho + \tfrac{1}{\rho}(u_0^{\ln})'_\rho + k^2 u_0^{\ln} = k^2 \ln\rho \quad \text{for } \rho \in (0,\delta). \tag{4.17}$$

Together with the particular integral $\ln\rho$, its general solution has the form

$$u_0^{\ln}(\rho) = K_0^{\ln}(1+a_0) + S_0^{\ln}\big(\tfrac{2}{\pi}(\ln\tfrac{k}{2} + \ln\rho + \gamma)(1+a_0) + a_2\big) + \ln\rho \tag{4.18}$$

with the Bessel and Neumann functions written according to (2.8a) and (2.8c), and two unknown coefficients $K_0^{\ln}, S_0^{\ln} \in \mathbb{R}$. The factor $S_0^{\ln} = -\frac{\pi}{2}$ avoids the leading logarithmic term, thus providing the function regularity $u_0^{\ln} \in H^1((0,\delta);\mathbb{R})$ in (4.16).

The inhomogeneous Helmholtz equation (4.12a) and (4.17) imply $-[\Delta+k^2]U_0^{\ln} = 0$ which argues a Fourier series representation of $U_0^{\ln}$ for $\rho \searrow +0$. From $\int_{-\pi}^{\pi} U_0^{\ln}\, d\theta = 0$ in (4.16) we get the Wirtinger inequality (2.17b) for the residual $U_0^{\ln}$. This follows the asymptotic order $U_0^{\ln} = \mathrm{O}(\rho)$ in (4.15), see for detail the proof of Lemma 2.2. Now passing $\rho \searrow +0$ in (4.16) and (4.18) we find $K_0^{\ln} = u^{\ln}(x_0) + (\ln\frac{k}{2} + \gamma)$ and, consequently, we arrive at formula (4.14). This completes the proof. □

With the help of Lemma 4.1 and Lemma 4.2 we construct the first-order correction function $w_{\ln}^{\varepsilon}$ to $u^{\varepsilon} - u^0$ which will be used further in Theorem 4.5.

Lemma 4.3. *Combining the rescaled solution w_{00}^{ε} to* (4.5) *and the solution $u^{\ln}$ to* (4.11) *it forms the first-order correction function in $H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)};\mathbb{R})$*

$$w_{\ln}^{\varepsilon} := w_{00}^{\varepsilon} + \ln(\varepsilon \mathrm{cap}(\omega)) - u^{\ln} \tag{4.19}$$

which fulfills the following relations:

$$w_{\ln}^{\varepsilon} = W_{00}^{\varepsilon} \quad \text{on } \Gamma_D, \tag{4.20a}$$

$$w_{\ln}^{\varepsilon} = \ln(\varepsilon \mathrm{cap}(\omega)) - u^{\ln}(x_0) + (\ln\rho - u^{\ln}(x_0))a_0 + \tfrac{\pi}{2}a_2 - U_0^{\ln} \ \text{on } \partial\omega_\varepsilon(x_0), \tag{4.20b}$$

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla w_{\ln}^{\varepsilon}\cdot\nabla u - k^2 w_{\ln}^{\varepsilon} u)\, dx = \int_{\Gamma_N} \frac{\partial W_{00}^{\varepsilon}}{\partial\nu} u\, dS_x - \int_{\Omega\setminus\omega_\varepsilon(x_0)} k^2 W_{00}^{\varepsilon} u\, dx \tag{4.20c}$$

for all $u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)};\mathbb{R}) : u = 0$ on $\Gamma_D \cup \partial\omega_\varepsilon(x_0)$

and admits the representation in the Fourier series

$$w_{\ln}^{\varepsilon} = \big(\ln\rho - u^{\ln}(x_0)\big)(1+a_0) + W_{00}^{\varepsilon} + \tfrac{\pi}{2}a_2 - U_0^{\ln} \quad \text{in } B_R(x_0)\setminus\overline{B_\varepsilon(x_0)}. \tag{4.21}$$

Proof. Indeed, formulas (4.20) and (4.21) are obtained by substitution of the representations (4.8) and (4.9) holding for w_{00}^{ε} as well as the representations (4.11a), (4.13), and (4.14) for $u^{\ln}$ in the combination of functions w_{00}^{ε} and $u^{\ln}$ defined in (4.19). □

Third, we construct a boundary layer which realizes the second-order correction to $u^\varepsilon - u^0$ as it will be proved further in Theorem 4.5.

For this task, we consider the vector-valued exterior Dirichlet problem:

$$-\Delta w_y = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\omega}, \tag{4.22a}$$

$$w_y = -y \quad \text{on } \partial\omega, \tag{4.22b}$$

$$w_y = \mathrm{O}\big(\tfrac{1}{|y|}\big) \quad \text{as } |y| \nearrow \infty. \tag{4.22c}$$

The boundary value problem (4.22) admits the following weak formulation: Find $w_y = ((w_y)_1, (w_y)_2)^\top \in W^{1,2}_\mu(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R})^2$ such that

$$w_y = -y \quad \text{on } \partial\omega, \tag{4.23a}$$

$$\int_{\mathbb{R}^2\setminus\omega} Dw_y \nabla v \, dy = 0 \quad \text{for all } v \in W^{1,2}_\mu(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R}) : v = 0 \text{ on } \partial\omega. \tag{4.23b}$$

The unique solution to (4.23) is guaranteed by existence theorems in [7].

Moreover, $Dw_y\nu \in H^{-1/2}(\partial\omega; \mathbb{R})^2$ is well defined at the boundary $\partial\omega$ by Green's formula holding for harmonic functions $\mathbf{u} \in W^{1,2}_\mu(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{C})$ with $\Delta\mathbf{u} = 0$ (see [7]):

$$\int_{\mathbb{R}^2\setminus\omega} \nabla\mathbf{u} \cdot \nabla\overline{u} \, dy = -\langle \tfrac{\partial\mathbf{u}}{\partial\nu}, \overline{u} \rangle_{\partial\omega} \quad \text{for all } u \in W^{1,2}_\mu(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{C}) \tag{4.24}$$

with the paring $\langle \frac{\partial\mathbf{u}}{\partial\nu}, \overline{u} \rangle_{\partial\omega}$ between $u \in H^{1/2}(\partial\omega; \mathbb{C})$ and $\frac{\partial\mathbf{u}}{\partial\nu} \in H^{-1/2}(\partial\omega; \mathbb{C})$.

After rescaling $y = \frac{x-x_0}{\varepsilon}$ we reduce the exterior Dirichlet problem to the bounded domain $\Omega \setminus \overline{\omega_\varepsilon(x_0)}$ which is described in the following lemma.

Lemma 4.4. *The rescaled solution $w^\varepsilon_y(x) := w_y(\frac{x-x_0}{\varepsilon})$ to* (4.23) *implies the vector-function $w^\varepsilon_y \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R})^2$ which fulfills the following relations:*

$$w^\varepsilon_y = -\tfrac{x-x_0}{\varepsilon} \quad \textit{on } \partial\omega_\varepsilon(x_0), \tag{4.25a}$$

$$\begin{gathered}\int_{\Omega\setminus\omega_\varepsilon(x_0)} Dw^\varepsilon_y \nabla u \, dx = \int_{\Gamma_N} (Dw^\varepsilon_y \nu) u \, dS_x \\ \textit{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R}) : u = 0 \textit{ on } \Gamma_D \cup \partial\omega_\varepsilon(x_0).\end{gathered} \tag{4.25b}$$

It admits the far-field representation in the Fourier series

$$w^\varepsilon_y(x) = \tfrac{\varepsilon}{\rho} \tfrac{1}{2\pi} P_\omega \,\widehat{x} + W^\varepsilon_y(x) \quad \textit{for } x \in \mathbb{R}^2 \setminus \overline{B_\varepsilon(x_0)}, \tag{4.26}$$

with P_ω called polarization matrix in [53, Note G] and the residual function $W^\varepsilon_y = ((W^\varepsilon_y)_1, (W^\varepsilon_y)_2)^\top \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R})^2$ such that for $\rho > \varepsilon$, $\theta \in (-\pi, \pi]$ it holds

$$\int_{-\pi}^{\pi} W^\varepsilon_y \, d\theta = \int_{-\pi}^{\pi} W^\varepsilon_y \,\widehat{x} \, d\theta = 0, \quad W^\varepsilon_y = \mathrm{O}\big((\tfrac{\varepsilon}{\rho})^2\big). \tag{4.27}$$

The entries of the 2-by-2 matrix P_ω have the implicit expression (cf. (3.28)):

$$(P_\omega)_{ij} = -\delta_{ij}\,\mathrm{meas}_2(\omega) - \langle \tfrac{\partial (w_y)_i}{\partial \nu}, y_j \rangle_{\partial\omega}, \quad i, j = 1, 2. \tag{4.28}$$

$-P_\omega$ is symmetric positive semi-definite (Spsd), and symmetric positive definite (Spd) if $\mathrm{meas}_2(\omega) > 0$. *For ellipsoidal shapes ω it has the explicit expression (cf. (3.29))*

$$P_\omega = \Theta(\alpha) P_{\omega'} \Theta(\alpha)^\top, \quad P_{\omega'} = -\pi(a+b)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{4.29}$$

with the ellipse major $a = 1$ and minor $b \in (0, 1]$ semi-axes, where the major axis has an angle of $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with the y_1-axis counted in the anti-clockwise direction.

Proof. Following the proof of Lemma 3.4 and employing the radial functions $\widehat{x}^n$ from Proposition 2.3, the harmonic function w_y in (4.22) obeys the Fourier series

$$w_y(y) = \sum_{n=1}^{\infty} \tfrac{1}{|y|^n} C_n^y \widehat{x}^n \quad \text{for } y \in \mathbb{R}^2 \setminus \overline{B_1(0)}$$

with unknown coefficient matrices $C_n^y \in \mathbb{R}^{2\times 2}$, $n \in \mathbb{N}$. This implies

$$w_y(y) = \tfrac{1}{|y|}\tfrac{1}{2\pi} P_\omega \widehat{x} + W_y(y), \quad \tfrac{1}{2\pi} P_\omega := C_1^y, \quad y \in \mathbb{R}^2 \setminus \overline{B_1(0)} \tag{4.30}$$

with the residual $W_y = ((W_y)_1, (W_y)_2)^\top \in W_\mu^{1,2}(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R})^2$ such that

$$\int_{-\pi}^{\pi} W_y\, d\theta = \int_{-\pi}^{\pi} W_y\, \widehat{x}\, d\theta = 0, \; W_y = \mathrm{O}(|y|^2) \quad \text{for } |y| > 1, \theta \in (-\pi, \pi]. \tag{4.31}$$

After the transformation $y = \frac{x - x_0}{\varepsilon}$ using the calculus (3.22), from (4.30) and (4.31) we get (4.26) and (4.27) for $w_y^\varepsilon(x) := w_y(\frac{x-x_0}{\varepsilon})$ and $W_y^\varepsilon(x) := W_y(\frac{x-x_0}{\varepsilon})$.

The coordinate transformation $y = \frac{x-x_0}{\varepsilon}$ applied to the boundary value problem (4.22) and supported by the differential calculus in (3.26) leads to the relations :

$$-\Delta w_y^\varepsilon = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\omega_\varepsilon(x_0)}, \tag{4.32a}$$

$$w_y^\varepsilon = -\tfrac{x-x_0}{\varepsilon} \quad \text{on } \partial\omega_\varepsilon(x_0). \tag{4.32b}$$

The solution to (4.32a) is locally H^2-smooth, hence $Dw_y^\varepsilon \nu \in L^2(\Gamma_N; \mathbb{R})^2$, and the inclusion $w_y^\varepsilon \in W_\mu^{1,2}(\mathbb{R}^2 \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R})^2$ implies $w_y^\varepsilon \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R})^2$. Therefore, applying Green's formula (4.7) to w_y^ε, from (4.32) we derive the weak formulation (4.25) of the transformed problem in the bounded domain.

To get the expressions of matrix P_ω, we follow the lines in the proof of Lemma 3.5.

In the near-field $B_1(0) \setminus \overline{\omega}$, due to (4.22a) from the second Green formula we have

$$\int_{\partial B_1(0)} \big\{ \tfrac{\partial (w_y)_i}{\partial |y|} y_j - (w_y)_i \tfrac{\partial y_j}{\partial |y|} \big\}\, dS_y = \langle \tfrac{\partial (w_y)_i}{\partial \nu}, y_j \rangle_{\partial\omega} - \int_{\partial\omega} (w_y)_i \tfrac{\partial y_j}{\partial \nu}\, dS_y, \quad i, j = 1, 2,$$

with the duality pairing defined in (4.24), and the Dirichlet condition (4.22b) follows

$$-\int_{\partial B_1(0)} \{\tfrac{\partial (w_y)_i}{\partial |y|} - (w_y)_i\} \widehat{x}_j \, dS_y = -\langle \tfrac{\partial (w_y)_i}{\partial \nu}, y_j \rangle_{\partial\omega} - \int_{\partial\omega} y_i \nu_j \, dS_y. \tag{4.33}$$

Using (4.30) and (4.31) we calculate the integral on the left-hand side of (4.33) as

$$\int_{\partial B_1(0)} \{-\tfrac{\partial (w_y)_i}{\partial |y|} + (w_y)_i\} \widehat{x}_j \, dS_y = \tfrac{1}{\pi} \int_{-\pi}^{\pi} \sum_{l=1}^{2} (P_\omega)_{il} \widehat{x}_l \widehat{x}_j \, d\theta = (P_\omega)_{ij}$$

due to (2.27). Applying to the right-hand side of (4.33) the divergence theorem (3.31) this results in expression (4.28).

With the help of the Green formula (4.24) and relations (4.22) we derive that

$$\int_{\mathbb{R}^2 \setminus \omega} \nabla (w_y)_i \cdot \nabla (w_y)_j \, dy = \langle \tfrac{\partial (w_y)_i}{\partial \nu}, y_j \rangle_{\partial\omega} = \langle \tfrac{\partial (w_y)_j}{\partial \nu}, y_i \rangle_{\partial\omega}, \tag{4.34}$$

which proves the symmetry $(P_\omega)_{ij} = (P_\omega)_{ji}$ in (4.28) as well as the non-negativeness

$$0 \le \int_{\mathbb{R}^2 \setminus \omega} |\nabla\big(\xi_1 (w_y)_1 + \xi_2 (w_y)_2\big)|^2 \, dy = \sum_{i,j=1}^{2} \langle \tfrac{\partial (w_y)_i}{\partial \nu} \xi_i, y_j \xi_j \rangle_{\partial\omega}$$

for arbitrary $\xi = (\xi_1, \xi_2)^\top \in \mathbb{R}^2$. Therefore, multiplying (4.28) with $-\xi_i \xi_j$ we get

$$-\sum_{i,j=1}^{2} (P_\omega)_{ij} \xi_i \xi_j = |\xi|^2 \operatorname{meas}_2(\omega) + \sum_{i,j=1}^{2} \langle \tfrac{\partial (w_y)_i}{\partial \nu} \xi_i, y_j \xi_j \rangle_{\partial\omega} \ge |\xi|^2 \operatorname{meas}_2(\omega).$$

This implies that $-P_\omega \in \operatorname{Spsd}(\mathbb{R}^{2\times 2})$, and $-P_\omega \in \operatorname{Spd}(\mathbb{R}^{2\times 2})$ if $\operatorname{meas}_2(\omega) > 0$.

Finally, let ω' be the canonical ellipsoidal shape with the major $a = 1$ and the minor $b \in (0, 1]$ semi-axes in respect to Cartesian coordinates $(y'_1, y'_2)^\top$. The ellipse can be written in the elliptic coordinates (3.32) in the form (3.33).

Composing the Fourier series for the solution $w_{y'}$ of (4.22) in $\mathbb{R}^2 \setminus \overline{\omega'}$ (cf. (3.36)):

$$w_{y'} = \sum_{n=1}^{\infty} e^{-nr} C_n^{y'} (\cos(n\psi), \sin(n\psi))^\top \quad \text{for } r > r_0, \tag{4.35}$$

the unknown coefficient matrices $C_n^{y'} \in \mathbb{R}^{2\times 2}$ in (4.35) are determined from the boundary condition (4.22b)

$$w_{y'}\big|_{r=r_0} = \sum_{n=1}^{\infty} e^{-nr_0} C_n^{y'} (\cos(n\psi), \sin(n\psi))^\top = -(a\cos\psi, b\sin\psi)^\top \quad \text{on } \partial\omega'$$

as $e^{-r_0} C_1^{y'} (\cos\psi, \sin\psi)^\top = -(a\cos\psi, b\sin\psi)^\top$ and $C_n^{y'} = 0$ for all $n \geq 2$. Henceforth, we obtain the solution analytically

$$w_{y'} = -e^{r_0 - r}(a\cos\psi, b\sin\psi)^\top \quad \text{for } r \geq r_0. \tag{4.36}$$

Now we calculate the matrix $P_{\omega'}$ explicitly by substituting (4.36) in the representation formula (4.28). Indeed, using (3.35) the normal derivative of $w_{y'}$ at $\partial\omega'$ is found

$$\tfrac{\partial w_{y'}}{\partial \nu'} = \tfrac{1}{\varkappa(r_0,\psi)} \tfrac{\partial w_{y'}}{\partial r}\Big|_{r=r_0} = \tfrac{1}{\varkappa(r_0,\psi)}(a\cos\psi, b\sin\psi)^\top = (\tfrac{a}{b}\nu_1', \tfrac{b}{a}\nu_2')^\top,$$

and it leads to the following two vectors in (4.28) for $j = 1, 2$:

$$(P_{\omega'})_{(\,\cdot\,,j)} = -\delta_{(\,\cdot\,,j)} \mathrm{meas}_2(\omega') - I_{(\,\cdot\,,j)}, \quad I_{(\,\cdot\,,j)} := \int_{\partial\omega'} \big(\tfrac{a}{b}\nu_1' y_j', \tfrac{b}{a}\nu_2' y_j'\big)^\top dS_y. \tag{4.37}$$

Using the divergence theorem we calculate two elliptic integrals $I_{(\,\cdot\,,j)}$ in (4.37)

$$I_{(\,\cdot\,,j)} = \int_{\omega'} (\tfrac{a}{b} y'_{j,1} \tfrac{b}{a} y'_{j,2})^\top \, dy' = \mathrm{meas}_2(\omega') \begin{cases} (\tfrac{a}{b}, 0)^\top & \text{for } j = 1, \\ (0, \tfrac{b}{a})^\top & \text{for } j = 2, \end{cases}$$

and arrive at the second formula in (4.29).

The first formula in (4.29) is justified by rotation of $y' \in \omega'$ to $\Theta^\top(\alpha) y \in \omega$ using the orthogonal matrix $\Theta(\alpha)$ given in (3.29a) with the angle of $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Indeed, the coordinate transformation $y = \Theta y'$ applied to the Dirichlet problem (4.23) reads

$$w_y(\Theta y') = -\Theta y' \quad \text{on } \partial\omega',$$

$$\int_{\mathbb{R}^2 \setminus \omega'} D w_y(\Theta y') \nabla v \, dy' = 0 \quad \text{for all } v \in W_\mu^{1,2}(\mathbb{R}^2 \setminus \overline{\omega'}; \mathbb{R}) : v = 0 \text{ on } \partial\omega'.$$

Henceforth, $w_{y'}(y') = \Theta^\top w_y(\Theta y')$ and the representation (4.30) provides $\Theta^\top P_\omega \Theta = P_{\omega'}$, for detail see the proof of Lemma 3.5. Our proof is finished. □

We remark that, in the limit case when $b \searrow +0$, explicit formulas (4.29) in Lemma 4.4 describe the singular polarization matrix P_ω for the straight crack, see [45].

4.2 High-order uniform asymptotic expansion of the Dirichlet problem

With the help of the first and second order correction terms $w_{\ln}^\varepsilon$ and w_y^ε given in Lemmas 4.3 and 4.4 we decompose the residual $u^\varepsilon - u^0$ for the Dirichlet problem (4.2).

Theorem 4.5. *The solutions* u^0 *of* (2.2), u^ε *of* (4.2), w_{00} *of* (4.5) *and* $u^{\ln}$ *of* (4.11) *composed together in the function* $w_{\ln}^\varepsilon$ *given in* (4.19), *and the solution* w_y *of* (4.23) *satisfy the following residual error estimate*

$$\|q_2^\varepsilon\|_{H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})} = \mathrm{O}\big(\tfrac{\varepsilon}{\sqrt{|\ln\varepsilon|}}\big), \tag{4.38a}$$

$$q_2^\varepsilon := u^\varepsilon - u^0 - \tfrac{u^0(x_0) w_{\ln}^\varepsilon}{u^{\ln}(x_0) - \ln(\varepsilon \mathrm{cap}(\omega))} - \varepsilon \nabla u^0(x_0) \cdot w_y^\varepsilon. \tag{4.38b}$$

Proof. For q_2^ε introduced in (4.38b), we use the boundary conditions (2.2a), (4.2a), and (4.20a) at Γ_D to get (4.41a). We apply the local representations (4.9), (4.21), and

$$u^0(x) = u^0(x_0)(1+a_0) + (\rho + \tfrac{a_1}{k})\nabla u^0(x_0)\cdot \widehat{x} + U_1^0 \quad \text{in } B_\varepsilon(x_0) \tag{4.39}$$

holding due to (2.8) and (2.33), the boundary conditions (4.32b) and (4.20b) implying

$$\frac{w_{\ln}^\varepsilon}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} = -1 + \frac{(\ln\rho - u^{\ln}(x_0))a_0 + \frac{\pi}{2}a_2 - U_0^{\ln}}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{4.40}$$

and $\rho\widehat{x} = x - x_0$ to calculate the residual at $\partial\omega_\varepsilon(x_0)$:

$$q_2^\varepsilon = -u^0(x_0)a_0 - \tfrac{a_1}{k}\nabla u^0(x_0)\cdot\widehat{x} - U_1^0 - u^0(x_0)\frac{(\ln\rho - u^{\ln}(x_0))a_0 + \frac{\pi}{2}a_2 - U_0^{\ln}}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))},$$

hence (4.41b). We subtract from (4.2b) the equations (3.10), (4.20c) multiplied with $\frac{u^0(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))}$, and (4.25b) multiplied with $\varepsilon\nabla u^0(x_0)$, thus obtaining the relations:

$$q_2^\varepsilon = -\frac{u^0(x_0)W_{00}^\varepsilon}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} - \varepsilon\nabla u^0(x_0)\cdot w_y^\varepsilon \quad \text{on } \Gamma_D, \tag{4.41a}$$

$$q_2^\varepsilon = -u^0(x_0)\frac{(\ln\rho - \ln(\varepsilon \mathrm{cap}(\omega)))a_0 + \frac{\pi}{2}a_2 - U_0^{\ln}}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} - \tfrac{a_1}{k}\nabla u^0(x_0)\cdot\widehat{x} - U_1^0 \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{4.41b}$$

$$\begin{aligned}\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla q_2^\varepsilon\cdot\nabla\overline{u} - k^2 q_2^\varepsilon\overline{u})\,dx &= \int_{\Omega\setminus\omega_\varepsilon(x_0)} k^2\Big(\frac{u^0(x_0)W_{00}^\varepsilon}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))}\\ &+ \varepsilon\nabla u^0(x_0)\cdot w_y^\varepsilon\Big)\overline{u}\,dx - \int_{\Gamma_N} \overline{u}\frac{\partial}{\partial\nu}\Big(\frac{u^0(x_0)W_{00}^\varepsilon}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} + \varepsilon\nabla u^0(x_0)\cdot w_y^\varepsilon\Big)dS_x\end{aligned} \tag{4.41c}$$

for all $u \in H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C}) : u = 0$ on $\Gamma_D\cup\partial\omega_\varepsilon(x_0)$.

For lifting in the boundary conditions (4.41a) and (4.41b), respectively, the cut-off functions η_{Γ_D} supported in a neighborhood of Γ_D such that $\eta_{\Gamma_D} = 1$ at Γ_D and $\eta_{x_0}^\varepsilon$ with a local support in $B_{2\varepsilon}(x_0)$ such that $\eta_{x_0}^\varepsilon = 1$ in $B_\varepsilon(x_0)$ are taken. We define

$$\begin{aligned} R_2^\varepsilon &:= \Big(u^0(x_0)\frac{(\ln\rho - \ln(\varepsilon \mathrm{cap}(\omega)))a_0 + \frac{\pi}{2}a_2 - U_0^{\ln}}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} + \tfrac{a_1}{k}\nabla u^0(x_0)\cdot\widehat{x} + U_1^0\Big)\eta_{x_0}^\varepsilon\\ &+ \Big(\frac{u^0(x_0)W_{00}^\varepsilon}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} + \varepsilon\nabla u^0(x_0)\cdot w_y^\varepsilon\Big)\eta_{\Gamma_D} = \mathrm{O}(\tfrac{\varepsilon}{|\ln\varepsilon|}), \quad Q_2^\varepsilon := q_2^\varepsilon + R_2^\varepsilon.\end{aligned} \tag{4.42}$$

Since $\frac{1}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} = \mathrm{O}(\frac{1}{|\ln\varepsilon|})$, the asymptotic order in (4.42) is provided by $a_0 = a_2 = U_1^0 = \mathrm{O}(\varepsilon^2)$, $a_1 = \mathrm{O}(\varepsilon^3)$, $U_0^{\ln} = \mathrm{O}(\varepsilon)$ holding due to the representations (2.8), (2.34b), and (4.15) in $B_{2\varepsilon}(x_0)$. At Γ_D it is argued by $w_y^\varepsilon = W_{00}^\varepsilon = \mathrm{O}(\varepsilon)$ due to (4.10), (4.26), and (4.27). Applying the Cauchy–Schwarz inequality to (4.41c) and using (4.42), where $\|\eta_{x_0}^\varepsilon\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{R})} = \mathrm{O}(1)$, we estimate with $C, C_3, C_4 > 0$:

$$\begin{aligned}\Big|\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla Q_2^\varepsilon\cdot\nabla\overline{u} - k^2 Q_2^\varepsilon\overline{u})\,dx\Big| &\le \Big|\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla R_2^\varepsilon\cdot\nabla\overline{u} - k^2 R_2^\varepsilon\overline{u})\,dx\Big|\\ &+ \frac{C_3\varepsilon}{\sqrt{|\ln\varepsilon|}}\|u\|_{L^2(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} + \frac{C_4\varepsilon}{|\ln\varepsilon|}\|u\|_{L^2(\Gamma_N;\mathbb{C})} \le \frac{C\varepsilon}{\sqrt{|\ln\varepsilon|}}\|u\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})}.\end{aligned} \tag{4.43}$$

Here we have utilized $w_y^\varepsilon = W_{00}^\varepsilon = \mathrm{O}(\varepsilon)$ at Γ_N due to (4.10), (4.26), and (4.27).

The asymptotic order $\frac{\varepsilon}{\sqrt{|\ln\varepsilon|}}$ in $\Omega\setminus\overline{\omega_\varepsilon(x_0)}$ was calculated as follows.

Inscribing Ω in a ball $B_R(x_0)$ of radius $R>0$ sufficiently large, we decompose it in the far-field $B_R(x_0)\setminus\overline{B_\varepsilon(x_0)}$ and the near-field $B_\varepsilon(x_0)\setminus\overline{\omega_\varepsilon(x_0)}$. In the far-field,

$$\int_{\Omega\setminus B_\varepsilon(x_0)} |W_{00}^\varepsilon|^2\,dx \le \int_{B_R(x_0)\setminus B_\varepsilon(x_0)} |W_{00}^\varepsilon|^2\,dx \le C\int_\varepsilon^R (\tfrac{\varepsilon}{\rho})^2\,\rho d\rho = \mathrm{O}(\varepsilon^2|\ln\varepsilon|) \quad (4.44)$$

due to (4.10), hence $\|\frac{W_{00}^\varepsilon}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}\|_{L^2(B_R(x_0)\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}\big(\frac{\varepsilon}{\sqrt{|\ln\varepsilon|}}\big)$, and analog

$$\int_{\Omega\setminus B_\varepsilon(x_0)} |w_y^\varepsilon|^2\,dx \le \int_{B_R(x_0)\setminus B_\varepsilon(x_0)} |w_y^\varepsilon|^2\,dx \le C\int_\varepsilon^R (\tfrac{\varepsilon}{\rho})^2\,\rho d\rho = \mathrm{O}(\varepsilon^2|\ln\varepsilon|) \quad (4.45)$$

due to (4.26) and (4.27), with $C>0$, hence $\|\varepsilon w_y^\varepsilon\|_{L^2(B_R(x_0)\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}(\varepsilon^2\sqrt{|\ln\varepsilon|})$. In the near-field, the transformation $y=\frac{x-x_0}{\varepsilon}$ with the calculus (3.26) provides

$$\int_{B_\varepsilon(x_0)\setminus\omega_\varepsilon(x_0)} |W_{00}^\varepsilon|^2\,dx = \varepsilon^2\int_{B_1(0)\setminus\omega} |W_{00}|^2\,dy = \mathrm{O}(\varepsilon^2) \quad (4.46)$$

since W_{00} in (4.6) does not depend on ε. Analogously for w_y from (4.23) we get

$$\int_{B_\varepsilon(x_0)\setminus\omega_\varepsilon(x_0)} |w_y^\varepsilon|^2\,dx = \varepsilon^2\int_{B_1(0)\setminus\omega} |w_y|^2\,dy = \mathrm{O}(\varepsilon^2). \quad (4.47)$$

The estimates (4.42) and (4.43) together with the inf-sup condition (3.8) holding for all $u,v\in H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C}) : u=v=0$ on $\Gamma_D\cup\partial\omega_\varepsilon(x_0)$ prove the asymptotic order in (4.38a) and the assertion of the theorem. □

As a corollary of Theorem 4.5 we state two asymptotic expansions of the low order in two propositions below.

Proposition 4.6. *The solutions* u^0 *of* (2.2)*,* u^ε *of* (4.2)*,* w_{00} *of* (4.5) *and* $u^{\ln}$ *of* (4.11) *composed together in the function* $w_{\ln}^\varepsilon$ *given in* (4.19)*, satisfy the error estimate*

$$\|q_1^\varepsilon\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}(\varepsilon), \quad q_1^\varepsilon := u^\varepsilon - u^0 - \frac{u^0(x_0)w_{\ln}^\varepsilon}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}. \quad (4.48)$$

Indeed, using the coordinate transformation $y=\frac{x-x_0}{\varepsilon}$ and calculus (3.26) we have

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} |Dw_y^\varepsilon|^2\,dx \le \int_{\mathbb{R}^2\setminus\omega_\varepsilon(x_0)} |Dw_y^\varepsilon|^2\,dx = \int_{\mathbb{R}^2\setminus\omega} |Dw_y|^2\,dy = \mathrm{O}(1),$$

hence $\|w_y^\varepsilon\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})^2} = \mathrm{O}(1)$ due to (4.45) and (4.47), and from (4.38) it follows

$$\|q_1^\varepsilon\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \|q_\varepsilon^2 + \varepsilon\nabla u^0(x_0)\cdot w_y^\varepsilon\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}(\varepsilon).$$

Proposition 4.7. *The solutions* u^0 *of* (2.2) *and* u^ε *of* (4.2) *satisfy the error estimate*

$$\|u^\varepsilon - u^0\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}\big(\tfrac{1}{|\ln\varepsilon|}\big). \tag{4.49}$$

Moreover, at Γ_N *the estimate* (4.49) *can be improved as*

$$\|u^\varepsilon - u^0\|_{L^2(\Gamma_N;\mathbb{C})} = \mathrm{O}\big(\tfrac{\varepsilon}{\sqrt{|\ln\varepsilon|}}\big). \tag{4.50}$$

Proof. The homogeneity argument and Lemma 4.3 supported by estimates (4.44) and (4.46) provides similarly $\|w^\varepsilon_{\ln}\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}(1)$ and leads to (4.49) for $u^\varepsilon - u^0 = q^\varepsilon_1 + \frac{u^0(x_0)w^\varepsilon_{\ln}}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}$ in view of Proposition 4.6.

The estimate (4.50) is argued by the representation (4.38) and $w^\varepsilon_y = W^\varepsilon_{00} = \mathrm{O}(\varepsilon)$ at Γ_N holding due to (4.10), (4.26), and (4.27). The proof is completed. □

We note that the residual estimate (4.50) will be needed for (4.53) in the next section.

4.3 Inverse Helmholtz problem under Dirichlet boundary condition

For the objective function J defined in (3.47) and characterizing the misfit of the solution u^ε to (4.2) from the known measurement $u^\star$ at the boundary Γ_N, we consider a state-constrained topology optimization problem (cf. (3.48)): Find the feasible test geometry $(\omega^\star, \varepsilon^\star, x^\star) \in \mathfrak{G}$ such that

$$0 = J(\omega^\star, \varepsilon^\star, x^\star) = \min_{(\omega,\varepsilon,x_0)\in\mathfrak{G}} J(\omega, \varepsilon, x_0) \quad \text{subject to (4.2)}. \tag{4.51}$$

For the primal state variable u^ε entering the objective J, the Fenchel–Legendre duality argues a dual state variable v^ε resulting necessarily from the variational principle:

$$\begin{gathered}\mathcal{L}_\varepsilon(u^\varepsilon, v^\varepsilon) = \min_{\mathrm{Re}(u),\mathrm{Re}(v)} \max_{\mathrm{Im}(u),\mathrm{Im}(v)} \mathcal{L}_\varepsilon(u, v) \ \text{ over } u, v \in H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C}) \\ \text{such that } u = h,\ v = 0 \text{ on } \Gamma_D \text{ and } u = v = 0 \text{ on } \partial\omega_\varepsilon(x_0),\end{gathered} \tag{4.52}$$

where the Lagrangian $\mathcal{L}_\varepsilon$ has the form (3.50).

Using the primal background solution u^0 to (2.2) and the dual background solution v^0 to (3.54), similarly to (3.59) we decompose the objective

$$\begin{gathered}J(\omega, \varepsilon, x_0) = J_0 - \mathrm{Re}\Big\{\int_{\Gamma_N} (u^\varepsilon - u^0)\tfrac{\partial\overline{v^0}}{\partial\nu}\, dS_x\Big\} + \tfrac{1}{2}\int_{\Gamma_N} |u^\varepsilon - u^0|^2\, dS_x, \\ J_0 := \tfrac{1}{2}\int_{\Gamma_N} |u^0 - u^\star|^2\, dS_x = \mathrm{O}(1), \quad \tfrac{1}{2}\int_{\Gamma_N} |u^\varepsilon - u^0|^2\, dS_x = \mathrm{O}\big(\tfrac{\varepsilon^2}{|\ln\varepsilon|}\big)\end{gathered} \tag{4.53}$$

where the asymptotic order in (4.53) is justified by (4.50) in Proposition 4.7. By analogy with Theorem 3.9 we prove the high-order asymptotic expansion.

Theorem 4.8. *The objective in* (4.53) *admits the high-order asymptotic expansion*

$$J(\omega,\varepsilon,x_0)=J_0+\mathrm{Re}\{\tfrac{J_1^D(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}+J_6^\varepsilon+J_7^\varepsilon+J_8^\varepsilon+J_9^\varepsilon\}+\mathrm{O}(\tfrac{\varepsilon^2}{|\ln\varepsilon|}) \quad (4.54)$$

for $\varepsilon\searrow+0$*, where the asymptotic terms are expressed by formulas:*

$$J_1^D(x_0):=2\pi u^0(x_0)\overline{v^0}(x_0), \quad (4.55a)$$

$$J_6^\varepsilon:=\varepsilon\int_{-\pi}^{\pi}\tfrac{\partial q_2^\varepsilon}{\partial\rho}\overline{v^0}(x_0)\,d\theta=\mathrm{O}(\tfrac{\varepsilon}{\sqrt{|\ln\varepsilon|}}), \quad (4.55b)$$

$$J_7^\varepsilon:=\varepsilon\int_{-\pi}^{\pi}\tfrac{u^0(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}\big(\tfrac{\partial W_{00}^\varepsilon}{\partial\rho}\overline{V_0^0}-W_{00}^\varepsilon\tfrac{\partial\overline{V_0^0}}{\partial\rho}\big)\,d\theta=\mathrm{O}(\tfrac{\varepsilon}{|\ln\varepsilon|}), \quad (4.55c)$$

$$\begin{aligned}J_8^\varepsilon:=\varepsilon^2\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big(\tfrac{\partial W_y^\varepsilon}{\partial\rho}\overline{V_0^0}-W_y^\varepsilon\tfrac{\partial\overline{V_0^0}}{\partial\rho}\big)\,d\theta\\ +\varepsilon^2\tfrac{\ln\varepsilon}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}\tfrac{k^2}{2}J_1^D(x_0)=\mathrm{O}(\varepsilon^2),\end{aligned} \quad (4.55d)$$

$$J_9^\varepsilon:=\varepsilon\int_{-\pi}^{\pi}\big(\tfrac{\partial q_2^\varepsilon}{\partial\rho}\overline{V_0^0}-q_2^\varepsilon\tfrac{\partial\overline{V_0^0}}{\partial\rho}\big)\overline{v^0}(x_0)\,d\theta=\mathrm{O}(\tfrac{\varepsilon^2}{\sqrt{|\ln\varepsilon|}}). \quad (4.55e)$$

Proof. The proof is based on the asymptotic expansions obtained in Section 4.2.

We express the invariant integral in (4.53) according to (3.62) equivalently

$$\begin{aligned}\mathcal{I}(u^\varepsilon-u^0,v^0)&:=-\int_{\Gamma_N}(u^\varepsilon-u^0)\tfrac{\partial\overline{v^0}}{\partial\nu}\,dS_x\\ &=\int_{\partial B_\varepsilon(x_0)}\big(\tfrac{\partial(u^\varepsilon-u^0)}{\partial\rho}\overline{v^0}-(u^\varepsilon-u^0)\tfrac{\partial\overline{v^0}}{\partial\rho}\big)\,dS_x.\end{aligned} \quad (4.56)$$

Substituting here the representations (3.56), with J_0 from (2.8a), and (4.38b) we have

$$\begin{aligned}\mathcal{I}(u^\varepsilon-u^0,v^0)&=\int_{-\pi}^{\pi}\big(\tfrac{\partial(u^\varepsilon-u^0)}{\partial\rho}\overline{v^0}-(u^\varepsilon-u^0)\tfrac{\partial\overline{v^0}}{\partial\rho}\big)\,\varepsilon d\theta\\ &=\int_{-\pi}^{\pi}\{\big(\tfrac{u^0(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}\tfrac{\partial w_{\ln}^\varepsilon}{\partial\rho}+\varepsilon\nabla u^0(x_0)\cdot\tfrac{\partial w_y^\varepsilon}{\partial\rho}+\tfrac{\partial q_2^\varepsilon}{\partial\rho}\big)\big(\overline{v^0}(x_0)(1+a_0)+\overline{V_0^0}\big)\\ &-\big(\tfrac{u^0(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}w_{\ln}^\varepsilon+\varepsilon\nabla u^0(x_0)\cdot w_y^\varepsilon+q_2^\varepsilon\big)\big(\overline{v^0}(x_0)ka_0'+\tfrac{\partial\overline{V_0^0}}{\partial\rho}\big)\}\,\varepsilon d\theta\\ &=I_1^y+I_2^y+I_3^y+I_4^y+I_5^y+I_6^y\end{aligned}$$

with the integrals $I_1^y,I_2^y,I_3^y,I_4^y,I_5^y,I_6^y$ which are defined and calculated as follows.

Using the formula (4.21) of $w_{\ln}^\varepsilon$ and the orthogonality in (4.10), (4.16), we find

$$\begin{aligned}I_1^y&:=\int_{-\pi}^{\pi}\tfrac{u^0(x_0)\overline{v^0}(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}\big(\tfrac{\partial w_{\ln}^\varepsilon}{\partial\rho}(1+a_0)-w_{\ln}^\varepsilon ka_0'\big)\,\varepsilon d\theta=2\pi\varepsilon\tfrac{u^0(x_0)\overline{v^0}(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}\\ &\times\{\big(\tfrac{1+a_0}{\varepsilon}-u^{\ln}(x_0)ka_0'+\tfrac{\pi ka_2'}{2}\big)(1+a_0)-\big((\ln\varepsilon-u^{\ln}(x_0))(1+a_0)+\tfrac{\pi a_2}{2}\big)ka_0'\}\\ &=\tfrac{J_1^D(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon\mathrm{cap}(\omega))}(1+\tfrac{k^2}{2}\varepsilon^2\ln\varepsilon)+\mathrm{O}(\tfrac{\varepsilon^2}{|\ln\varepsilon|})\qquad\big(\text{since } a_0'(k\varepsilon)=-\tfrac{k\varepsilon}{2}+\mathrm{O}(\varepsilon^3)\big)\end{aligned}$$

with the notation introduced in (4.55a), while the orthogonality in (3.57a) provides

$$
\begin{aligned}
I_2^y &:= \int_{-\pi}^{\pi} \tfrac{u^0(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} \big(\tfrac{\partial w_{\ln}^\varepsilon}{\partial \rho} \overline{V_0^0} - w_{\ln}^\varepsilon \tfrac{\partial \overline{V_0^0}}{\partial \rho} \big)\, \varepsilon d\theta \\
&= \int_{-\pi}^{\pi} \tfrac{u^0(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} \big(\tfrac{\partial (W_{00}^\varepsilon - U_0^{\ln})}{\partial \rho} \overline{V_0^0} - (W_{00}^\varepsilon - U_0^{\ln}) \tfrac{\partial \overline{V_0^0}}{\partial \rho} \big)\, \varepsilon d\theta \\
&= \varepsilon \int_{-\pi}^{\pi} \tfrac{u^0(x_0)}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} \big(\tfrac{\partial W_{00}^\varepsilon}{\partial \rho} \overline{V_0^0} - W_{00}^\varepsilon \tfrac{\partial \overline{V_0^0}}{\partial \rho} \big)\, d\theta + \mathrm{O}(\tfrac{\varepsilon^2}{|\ln \varepsilon|})
\end{aligned}
$$

due to $U_0^{\ln} = V_0^0 = \mathrm{O}(\varepsilon)$, $\frac{\partial U_0^{\ln}}{\partial \rho} = \frac{\partial V_0^0}{\partial \rho} = \mathrm{O}(1)$ in (4.15) and (3.57b), and $W_{00}^\varepsilon = \mathrm{O}(1)$, $\frac{\partial W_{00}^\varepsilon}{\partial \rho} = \mathrm{O}(\frac{1}{\varepsilon})$ in (4.10) for $\rho = \varepsilon$. The expressions (4.26) and (4.27) for w_y^ε imply that

$$
I_3^y := \int_{-\pi}^{\pi} \varepsilon \nabla u^0(x_0) \cdot \big(\tfrac{\partial w_y^\varepsilon}{\partial \rho} (1 + a_0) - w_y^\varepsilon k a_0' \big) \overline{v^0}(x_0)\, \varepsilon d\theta = 0,
$$

and with the asymptotic order $W_y^\varepsilon = \mathrm{O}(1)$, $\frac{\partial W_y^\varepsilon}{\partial \rho} = \mathrm{O}(\frac{1}{\varepsilon})$ for $\rho = \varepsilon$ it yields

$$
\begin{aligned}
I_4^y &:= \int_{-\pi}^{\pi} \varepsilon \nabla u^0(x_0) \cdot \big(\tfrac{\partial w_y^\varepsilon}{\partial \rho} \overline{V_0^0} - w_y^\varepsilon \tfrac{\partial \overline{V_0^0}}{\partial \rho} \big)\, \varepsilon d\theta \\
&= \varepsilon^2 \int_{-\pi}^{\pi} \nabla u^0(x_0) \cdot \big(\tfrac{\partial W_y^\varepsilon}{\partial \rho} \overline{V_0^0} - W_y^\varepsilon \tfrac{\partial \overline{V_0^0}}{\partial \rho} \big)\, d\theta = \mathrm{O}(\varepsilon^2).
\end{aligned}
$$

Finally, the asymptotic order given in (4.38a) in Theorem 4.5 allows us to estimate

$$
I_5^y := \int_{-\pi}^{\pi} \big(\tfrac{\partial q_2^\varepsilon}{\partial \rho} (1 + a_0) - q_2^\varepsilon k a_0' \big) \overline{v^0}(x_0)\, \varepsilon d\theta = \varepsilon \int_{-\pi}^{\pi} \tfrac{\partial q_2^\varepsilon}{\partial \rho} \overline{v^0}(x_0)\, d\theta + \mathrm{O}(\tfrac{\varepsilon^3}{\sqrt{|\ln \varepsilon|}}),
$$

$$
I_6^y := \int_{-\pi}^{\pi} \big(\tfrac{\partial q_2^\varepsilon}{\partial \rho} \overline{V_0^0} - q_2^\varepsilon \tfrac{\partial \overline{V_0^0}}{\partial \rho} \big) \overline{v^0}(x_0)\, \varepsilon d\theta = \mathrm{O}(\tfrac{\varepsilon^2}{\sqrt{|\ln \varepsilon|}})
$$

since $q_2^\varepsilon = \mathrm{O}\big(\frac{\varepsilon}{\sqrt{|\ln \varepsilon|}}\big)$ and $\frac{\partial q_2^\varepsilon}{\partial \rho} = \mathrm{O}\big(\frac{1}{\sqrt{|\ln \varepsilon|}}\big)$ at $\partial B_\varepsilon(x_0)$ by the trace theorem.

Collecting the terms of the same order in $I_1^y, I_2^y, I_3^y, I_4^y, I_5^y, I_6^y$ we arrive at (4.55), thus proving the assertion of the theorem. □

We remark the asymptotic decomposition of the ε-dependent factor in (4.54)

$$
\tfrac{1}{u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega))} = \tfrac{1}{-\ln \varepsilon} + \tfrac{u^{\ln}(x_0)}{\ln \varepsilon (u^{\ln}(x_0)-\ln(\varepsilon \mathrm{cap}(\omega)))} = \tfrac{1}{-\ln \varepsilon} + \mathrm{O}\big(\tfrac{1}{|\ln \varepsilon|^2}\big). \tag{4.57}
$$

With the help of (4.57), from Theorem 4.8 we infer the corollary (cf. (3.69)):

$$
\mathrm{Re}(J_1^D(x_0)) = \lim_{\varepsilon \searrow +0} \tfrac{1}{|\ln \varepsilon|^{-1}} \big(J(\omega, \varepsilon, x_0) - J_0 \big). \tag{4.58}
$$

Here the first-order asymptotic term $\mathrm{Re}(J_1^D)$ is given by formula (4.55a). It is called the topological derivative of the objective J for the Dirichlet problem (4.2), and in the form (4.58) was found e.g. in [56].

We note that, in comparison to the first asymptotic term $\mathrm{Re}(J_1^N(\omega, x_0))$ for the Neumann problem, (4.58) does not depend on the shape ω. For generalization of the concept of topological derivatives, we refer to [29, 37].

In the next section we bridge the gap between the Neumann and Dirichlet problems by employing Robin condition with unknown parameter of the boundary impedance.

5 Helmholtz problems for geometric objects under Robin (impedance) boundary condition

For the test object $\omega_\varepsilon(x_0)$ we introduce a parameter $\alpha \in \mathbb{C}$ of the boundary impedance which enters the Robin boundary condition for the forward Helmholtz problem:

$$-[\Delta + k^2]u^{(\varepsilon,\alpha)} = 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon(x_0)}, \tag{5.1a}$$

$$-\tfrac{\partial u^{(\varepsilon,\alpha)}}{\partial \nu} + \alpha u^{(\varepsilon,\alpha)} = 0 \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{5.1b}$$

$$\tfrac{\partial u^{(\varepsilon,\alpha)}}{\partial \nu} = g \quad \text{on } \Gamma_N, \tag{5.1c}$$

$$u^{(\varepsilon,\alpha)} = h \quad \text{on } \Gamma_D. \tag{5.1d}$$

We mark the dependence of (5.1) on α which, on the one hand, turns into the Neumann problem (3.3) in the case of small $|\alpha| \searrow 0$, and, on the other hand, into the Dirichlet problem (4.1) in the case of large $|\alpha| \nearrow \infty$. In this way, problem (5.1) accounts for arbitrary boundary conditions of the test object depending on the parameter α.

The weak solution to (5.1) is described by the following variational problem: Find $u^{(\varepsilon,\alpha)} \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ such that

$$u^{(\varepsilon,\alpha)} = h \quad \text{on } \Gamma_D, \tag{5.2a}$$

$$\begin{aligned} &\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla u^{(\varepsilon,\alpha)} \cdot \nabla \overline{u} - k^2 u^{(\varepsilon,\alpha)} \overline{u})\, dx + \int_{\partial\omega_\varepsilon(x_0)} \alpha u^{(\varepsilon,\alpha)} \overline{u}\, dS_x \\ &= \int_{\Gamma_N} g\overline{u}\, dS_x \quad \text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : u = 0 \text{ on } \Gamma_D. \end{aligned} \tag{5.2b}$$

Following Proposition 2.1 the corresponding variational principle is established below.

Proposition 5.1. *The variational equation* (5.2) *results necessarily from the variational principle:*

$$\begin{aligned} &\mathcal{P}_{(\varepsilon,\alpha)}(u^{(\varepsilon,\alpha)}) = \min_{\mathrm{Re}(v)} \max_{\mathrm{Im}(v)} \mathcal{P}_{(\varepsilon,\alpha)}(v) \\ &\textit{over } v \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : v = h \textit{ on } \Gamma_D, \end{aligned} \tag{5.3}$$

where the Lagrangian $\mathcal{P}_{(\varepsilon,\alpha)} : H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) \mapsto \mathbb{R}$ *is determined (cf.* (3.6)*) by*

$$\begin{aligned}\mathcal{P}_{(\varepsilon,\alpha)}(v) := \mathrm{Re}\Big\{ \tfrac{1}{2} \int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla v \cdot \nabla v - k^2 v^2)\, dx \\ + \tfrac{1}{2} \int_{\partial\omega_\varepsilon(x_0)} \alpha v^2 \, dS_x - \int_{\Gamma_N} g v \, dS_x \Big\}.\end{aligned} \tag{5.4}$$

Proof. Component-wisely for $v = \mathrm{Re}(v) + \imath \mathrm{Im}(v)$ the functional in (5.4) reads

$$\begin{aligned}\mathcal{P}_{(\varepsilon,\alpha)}(v) &= \tfrac{1}{2} \int_{\Omega\setminus\omega_\varepsilon(x_0)} \big\{ |\nabla(\mathrm{Re}(v))|^2 - |\nabla(\mathrm{Im}(v))|^2 - k^2\big(\mathrm{Re}(v)^2 - \mathrm{Im}(v)^2\big)\big\}\, dx \\ &+ \int_{\partial\omega_\varepsilon(x_0)} \big\{ \tfrac{1}{2}\mathrm{Re}(\alpha)\big(\mathrm{Re}(v)^2 - \mathrm{Im}(v)^2\big) - \mathrm{Im}(\alpha)\mathrm{Re}(v)\mathrm{Im}(v)\big\}\, dS_x \\ &- \int_{\Gamma_N} \big(\mathrm{Re}(g)\mathrm{Re}(v) - \mathrm{Im}(g)\mathrm{Im}(v)\big)\, dS_x.\end{aligned}$$

Therefore, for $v = u^{(\varepsilon,\alpha)} \pm \mathbf{u}$ in (5.3) with $\mathbf{u} \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})$ such that $\mathbf{u} = 0$ on Γ_D we get the first-order necessary optimality conditions:

$$\begin{aligned}&\int_{\Omega\setminus\omega_\varepsilon(x_0)} \big(\nabla\mathrm{Re}(u^{(\varepsilon,\alpha)}) \cdot \nabla\mathrm{Re}(\mathbf{u}) - k^2\mathrm{Re}(u^{(\varepsilon,\alpha)})\mathrm{Re}(\mathbf{u})\big)\, dx - \int_{\Gamma_N} \mathrm{Re}(g)\mathrm{Re}(\mathbf{u})\, dS_x \\ &+ \int_{\partial\omega_\varepsilon(x_0)} \big(\mathrm{Re}(\alpha)\mathrm{Re}(u^{(\varepsilon,\alpha)}) - \mathrm{Im}(\alpha)\mathrm{Im}(u^{(\varepsilon,\alpha)})\big)\mathrm{Re}(\mathbf{u})\, dS_x = 0, \\ &\int_{\Omega\setminus\omega_\varepsilon(x_0)} \big(\nabla\mathrm{Im}(u^{(\varepsilon,\alpha)}) \cdot \nabla\mathrm{Im}(\mathbf{u}) - k^2\mathrm{Im}(u^{(\varepsilon,\alpha)})\mathrm{Im}(\mathbf{u})\big)\, dx - \int_{\Gamma_N} \mathrm{Im}(g)\mathrm{Im}(\mathbf{u})\, dS_x \\ &+ \int_{\partial\omega_\varepsilon(x_0)} \big(\mathrm{Re}(\alpha)\mathrm{Im}(u^{(\varepsilon,\alpha)}) + \mathrm{Im}(\alpha)\mathrm{Re}(u^{(\varepsilon,\alpha)})\big)\mathrm{Im}(\mathbf{u})\, dS_x = 0.\end{aligned}$$

Summing these two equations first for $\mathbf{u} = u$, and then for $\mathbf{u} = \imath u$, and accounting for the identities:

$$g\overline{u} = \mathrm{Re}(g)\mathrm{Re}(u) + \mathrm{Im}(g)\mathrm{Im}(u) + \imath\big(\mathrm{Im}(g)\mathrm{Re}(u) - \mathrm{Re}(g)\mathrm{Im}(u)\big), \tag{5.5a}$$

similar ones for $u^{(\varepsilon,\alpha)}\overline{u}$ and the inner product $\nabla u^{(\varepsilon,\alpha)} \cdot \nabla\overline{u}$, and

$$\begin{aligned}\alpha u^{(\varepsilon,\alpha)}\overline{u} &= \mathrm{Re}(\alpha)\big(\mathrm{Re}(u^{(\varepsilon,\alpha)})\mathrm{Re}(u) + \mathrm{Im}(u^{(\varepsilon,\alpha)})\mathrm{Im}(u)\big) \\ &+ \mathrm{Im}(\alpha)\big(\mathrm{Im}(u^{(\varepsilon,\alpha)})\mathrm{Re}(u) - \mathrm{Re}(u^{(\varepsilon,\alpha)})\mathrm{Im}(u)\big) + \imath\big\{\mathrm{Re}(\alpha)\big(\mathrm{Im}(u^{(\varepsilon,\alpha)})\mathrm{Re}(u) \\ &- \mathrm{Re}(u^{(\varepsilon,\alpha)})\mathrm{Im}(u)\big) + \mathrm{Im}(\alpha)\big(\mathrm{Re}(u^{(\varepsilon,\alpha)})\mathrm{Re}(u) + \mathrm{Im}(u^{(\varepsilon,\alpha)})\mathrm{Im}(u)\big)\big\}\end{aligned} \tag{5.5b}$$

it constitutes the real and imaginary parts of (5.2b), thus finishing the proof. □

For fixed k, under reasonable assumptions on α, there exists a unique solution of the variational equation (5.2), see e.g. [19]. Summarizing these assumptions, α should be either not too large or it should have a definite sign such that $\mathrm{Re}(\alpha) \geq 0$ and either $\mathrm{Im}(\alpha) \geq 0$ or $\mathrm{Im}(\alpha) \leq 0$. The latter case is realized for the Dirichlet boundary condition when $|\alpha| \nearrow \infty$.

We refer to the admissible set $\mathfrak{G}_\alpha \subset \mathbb{C}$ of such α which allows solvability of (5.2). Similar to (3.8), the uniform inf-sup condition assumes $\beta_1 > 0$ such that

$$0 < \beta_1 \leq \inf_u \sup_v \frac{\left| \int_{\Omega \setminus \omega_\varepsilon(x_0)} (\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v})\, dx + \int_{\partial \omega_\varepsilon(x_0)} \alpha u \overline{v}\, dS_x \right|}{\|u\|_{H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})} \|v\|_{H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C})}} \tag{5.6}$$

for all $u, v \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : u = v = 0$ on Γ_D, admissible geometries $(\omega, \varepsilon, x_0) \in \mathfrak{G}$, impedances $\alpha \in \mathfrak{G}_\alpha$, and wave numbers $k \in [0, k_1]$, $k_1 > 0$.

5.1 Outer asymptotic expansion by Fourier series in far-field

To cancel the leading asymptotic term in (5.20) it needs the exterior Neumann problem:

$$-\Delta w_0 = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\omega}, \tag{5.7a}$$

$$\tfrac{\partial w_0}{\partial \nu} = 1 \quad \text{on } \partial\omega, \tag{5.7b}$$

$$w_0 = \mathrm{O}(\ln |y|) \quad \text{as } |y| \nearrow \infty. \tag{5.7c}$$

The unique weak solution to (5.7) can be defined in the weighted Sobolev spaces: For $p > 2$ and $p' < 2$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, find $w_0 \in W^{1,p}_\mu(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R}) \setminus \mathbb{P}_0$ such that

$$\int_{\mathbb{R}^2 \setminus \omega} \nabla w_0 \cdot \nabla v\, dy = -\int_{\partial\omega} v\, dS_y \quad \text{for all } v \in W^{1,p'}_\mu(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R}), \tag{5.8}$$

since constant test-functions v are excluded in (5.8) for $p' < 2$ and no solvability condition for the Neumann data arises. If $p = p' = 2$, then (5.8) is not solvable.

Indeed, to argue the solvability of (5.8) we employ the function w_{00} from the non-trivial kernel of the Laplace operator given in (4.4). Applying to w_{00} the Green formula in $B_1(0) \setminus \overline{\omega}$ in the manner of (4.24), for the constant test-function we calculate:

$$0 = \int_{\partial B_1(0)} \tfrac{\partial w_{00}}{\partial |y|}\, dS_y - \langle \tfrac{\partial w_{00}}{\partial \nu}, 1 \rangle_{\partial\omega} = \int_{-\pi}^{\pi} \tfrac{\partial (\ln |y|)}{\partial |y|}\Big|_{|y|=1} d\theta - \langle \tfrac{\partial w_{00}}{\partial \nu}, 1 \rangle_{\partial\omega}$$

due to the representation (4.6) and, consequently, it fulfills the solvability condition

$$0 = 2\pi - \langle \tfrac{\partial w_{00}}{\partial \nu}, 1 \rangle_{\partial\omega} = \langle \tfrac{2\pi}{|\partial\omega|} - \tfrac{\partial w_{00}}{\partial \nu}, 1 \rangle_{\partial\omega} = \tfrac{2\pi}{|\partial\omega|} \langle 1 - \tfrac{|\partial\omega|}{2\pi} \tfrac{\partial w_{00}}{\partial \nu}, 1 \rangle_{\partial\omega}$$

where $|\partial\omega| := \text{meas}_1(\partial\omega)$. Therefore, there exists a solution of the corresponding Neumann problem defined up to a constant: Find $\widetilde{w}_0 \in W^{1,2}_\mu(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R})$ such that

$$\int_{\mathbb{R}^2\setminus\omega} \nabla\widetilde{w}_0 \cdot \nabla v \, dy = \langle \tfrac{|\partial\omega|}{2\pi} \tfrac{\partial w_{00}}{\partial\nu} - 1, v\rangle_{\partial\omega} \quad \text{for all } v \in W^{1,2}_\mu(\mathbb{R}^2 \setminus \overline{\omega}; \mathbb{R}). \tag{5.9}$$

From (5.9) and Green's formula (4.24) it follows that $w_0 = \widetilde{w}_0 - \frac{|\partial\omega|}{2\pi} w_{00}$ solves (5.8).

The constant-free solution w_0 admits logarithm in the Fourier series in $\mathbb{R}^2 \setminus \overline{B_1(0)}$:

$$\begin{aligned} &w_0(y) = c^0_0 \ln|y| + W_0, \quad W_0 = \tfrac{1}{|y|} c^0_1 \cdot \widehat{x} + W_1, \\ &W_1 = \sum_{n=2}^{\infty} \tfrac{1}{|y|^n} c^0_n \cdot \widehat{x}^n, \quad c^0_n \in \mathbb{R}^2, \quad c^0_1 =: \tfrac{1}{2\pi} m_\omega. \end{aligned} \tag{5.10}$$

To determine unknown $c^0_0 \in \mathbb{R}$ in (5.10) we apply the divergence theorem in $B_1(0)\setminus\overline{\omega}$:

$$\int_{B_1(0)\setminus\omega} \Delta w_0 \, dy = \int_{\partial B_1(0)} \tfrac{\partial w_0}{\partial|y|} \, dS_y - \int_{\partial\omega} \tfrac{\partial w_0}{\partial\nu} \, dS_y = \int_{-\pi}^{\pi} c^0_0 \tfrac{\partial(\ln|y|)}{\partial|y|}\Big|_{|y|=1} d\theta - \int_{\partial\omega} dS_y$$

which implies $0 = 2\pi c^0_0 - |\partial\omega|$, hence $c^0_0 = \frac{|\partial\omega|}{2\pi}$.

After rescaling $y = \frac{x-x_0}{\varepsilon}$ with the help of calculus (3.26) and using the Green formula (4.7) we reduce the exterior problem to the bounded domain $\Omega \setminus \overline{\omega_\varepsilon(x_0)}$, and from relations (5.7)–(5.10) we derive the following result below.

Lemma 5.2. *The rescaled solution $w^\varepsilon_0(x) := w_0(\frac{x-x_0}{\varepsilon})$ to* (5.8) *fulfills*

$$\begin{aligned} &\int_{\Omega\setminus\omega_\varepsilon(x_0)} \nabla w^\varepsilon_0 \cdot \nabla u \, dx = -\tfrac{1}{\varepsilon} \int_{\partial\omega_\varepsilon(x_0)} u \, dS_x + \int_{\Gamma_N} \tfrac{\partial w^\varepsilon_0}{\partial\nu} u \, dS_x \\ &\text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{R}) : u = 0 \text{ on } \Gamma_D. \end{aligned} \tag{5.11}$$

It admits the far-field representation in the Fourier series for $x \in \mathbb{R}^2 \setminus \overline{B_\varepsilon(x_0)}$

$$w^\varepsilon_0(x) = \tfrac{|\partial\omega|}{2\pi}(\ln\rho - \ln\varepsilon) + W^\varepsilon_0(x), \quad W^\varepsilon_0(x) = \tfrac{\varepsilon}{\rho}\tfrac{1}{2\pi}(m_\omega \cdot \widehat{x}) + W^\varepsilon_1(x) \tag{5.12}$$

with the residual functions $W^\varepsilon_0(x) := W_0(\frac{x-x_0}{\varepsilon})$ and $W^\varepsilon_1 := W_1(\frac{x-x_0}{\varepsilon})$ such that

$$\int_{-\pi}^{\pi} W^\varepsilon_0 \, d\theta = \int_{-\pi}^{\pi} W^\varepsilon_1 \, d\theta = \int_{-\pi}^{\pi} W^\varepsilon_1 \widehat{x} \, d\theta = 0, \tag{5.13a}$$

$$W^\varepsilon_0 = \mathrm{O}(\tfrac{\varepsilon}{\rho}), \quad W^\varepsilon_1 = \mathrm{O}\big((\tfrac{\varepsilon}{\rho})\big)^2 \quad \text{for } \rho > \varepsilon,\ \theta \in (-\pi, \pi]. \tag{5.13b}$$

Entries of the coefficient vector $m_\omega \in \mathbb{R}^2$ have the implicit expression:

$$(m_\omega)_i = \int_{\partial\omega} (w_0 \nu_i - y_i) \, dS_y, \quad i = 1, 2, \tag{5.14}$$

and $m_\omega = 0$ is zero for ellipsoidal shapes ω.

Proof. It suffices to prove (5.14). Indeed, the second Green formula (cf. (3.30)) gives

$$-\int_{\partial B_1(0)}\big(\tfrac{\partial w_0}{\partial|y|}-w_0\big)\widehat{x}\,dS_y=-\int_{\partial\omega}\big(\tfrac{\partial w_0}{\partial\nu}y-w_0\nu\big)\,dS_y=\int_{\partial\omega}(w_0\nu-y)\,dS_y.$$

Using (5.10) we calculate the integral on the left hand side here as

$$-\int_{\partial B_1(0)}\big(\tfrac{\partial w_0}{\partial|y|}-w_0\big)\widehat{x}\,dS_y=\int_{-\pi}^{\pi}\big(-\tfrac{|\partial\omega|}{2\pi}+\tfrac{1}{2\pi}m_\omega\cdot\widehat{x}+\tfrac{\partial W_1}{\partial|y|}$$
$$+\tfrac{1}{2\pi}m_\omega\cdot\widehat{x}+W_1\big)\widehat{x}\,d\theta=\int_{-\pi}^{\pi}\tfrac{1}{\pi}(m_\omega\cdot\widehat{x})\widehat{x}\,d\theta=m_\omega$$

which succeeds in (5.14).

Resetting (5.8) in the Cartesian coordinates $y'=\Theta^\top y$, for the ellipse ω' written in the elliptic coordinates (3.32) in the form (3.33), we have the Fourier series

$$w_0'=c_{00}'+c_0'r+\sum_{n=1}^{\infty}e^{-nr}c_n'\cdot(\cos(n\psi),\sin(n\psi))\quad\text{for } r>r_0$$

for the solution, and due to (3.35) the Neumann boundary condition (5.7b) implies

$$\tfrac{1}{\varkappa(r_0,\psi)}\tfrac{\partial w_0'}{\partial r}\big|_{r=r_0}=\tfrac{1}{\varkappa(r_0,\psi)}\Big(c_0'-\sum_{n=1}^{\infty}ne^{-nr_0}c_n'\cdot(\cos(n\psi),\sin(n\psi))\Big)=1.$$

Therefore, thanks to orthogonality of the Fourier basis functions we calculate explicitly

$$\int_{\partial\omega'}y'\,dS_{y'}=\int_{\partial\omega'}\tfrac{\partial w_0'}{\partial\nu'}y'\,dS_{y'}=\int_{-\pi}^{\pi}\tfrac{1}{\varkappa(r_0,\psi)}\tfrac{\partial w_0'}{\partial r}\big|_{r=r_0}(a\cos\psi,b\sin\psi)^\top\varkappa(r_0,\psi)\,d\psi$$
$$=-\int_{-\pi}^{\pi}e^{-r_0}\big((c_1')_1\cos\psi+(c_1')_2\sin\psi\big)(a\cos\psi,b\sin\psi)^\top d\psi$$
$$=-\pi e^{-r_0}\big((c_1')_1a,(c_1')_2b\big)^\top.$$

On the other hand, due to the symmetry of ω' when changing $\psi=\phi-\pi$ it follows

$$\int_{\partial\omega'}y'\,dS_{y'}=\int_{-\pi}^{0}(a\cos\psi,b\sin\psi)^\top\varkappa(r_0,\psi)\,d\psi+\int_{0}^{\pi}(a\cos\psi,b\sin\psi)^\top\varkappa(r_0,\psi)\,d\psi$$
$$=\int_0^{\pi}(a\cos(\phi-\pi),b\sin(\phi-\pi))^\top\varkappa(r_0,\phi)d\phi+\int_0^{\pi}(a\cos\psi,b\sin\psi)^\top\varkappa(r_0,\psi)d\psi=0$$

and leads to $c_1'=0$. Inserting these expressions in (5.14) we conclude

$$m_{\omega'}=\int_{\partial\omega'}(w_0'\nu'-y')\,dS_{y'}=\int_{-\pi}^{\pi}\Big(c_{00}'+c_0'r_0+\sum_{n=2}^{\infty}e^{-nr_0}c_n'\cdot(\cos(n\psi),\sin(n\psi))\Big)$$
$$\times\tfrac{1}{\varkappa(r_0,\psi)}(b\cos\psi,a\sin\psi)^\top\varkappa(r_0,\psi)\,d\psi=0.$$

After rotation of the Cartesian coordinates $y=\Theta y'$ in (5.8) we get $w_0'(y')=w_0(\Theta y')$, hence $m_\omega=\Theta m_{\omega'}=0$ is zero as well. This proves the assertion of the lemma. □

To compensate the logarithm in (5.12) we employ the result of Section 4.1 for the regularized problem (4.11), and using Lemma 4.2 we state the corrector w_1^ε as follows.

Lemma 5.3. *The combination of w_0^ε from Lemma 5.2 and the solution $u^{\ln}$ to* (4.11)

$$w_1^\varepsilon := w_0^\varepsilon + \tfrac{|\partial\omega|}{2\pi}(\ln\varepsilon - u^{\ln}) \tag{5.15}$$

fulfills the following relations:

$$-\Delta w_1^\varepsilon = k^2 \tfrac{|\partial\omega|}{2\pi}(\ln\rho - u^{\ln}) \quad \text{in } \Omega\setminus\overline{\omega_\varepsilon(x_0)}, \tag{5.16a}$$

$$\tfrac{\partial w_1^\varepsilon}{\partial\nu} = \tfrac{1}{\varepsilon} + \tfrac{|\partial\omega|}{2\pi}\big(b_{\ln}\cdot\nu - \tfrac{\partial U_0^{\ln}}{\partial\nu}\big) \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{5.16b}$$

$$\text{where} \quad b_{\ln} := \big((\ln\rho - u^{\ln}(x_0))ka_0' + \tfrac{a_0}{\rho} + \tfrac{\pi k a_2'}{2}\big)\widehat{x} = \mathrm{O}(\rho|\ln\rho|), \tag{5.16c}$$

it satisfies the variational equation

$$w_1^\varepsilon = W_0^\varepsilon \quad \text{on } \Gamma_D, \tag{5.17a}$$

$$\begin{aligned}\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla w_1^\varepsilon\cdot\nabla u - k^2 w_1^\varepsilon u)\,dx &= \int_{\Gamma_N} \frac{\partial W_0^\varepsilon}{\partial\nu} u\,dS_x \\ - \int_{\partial\omega_\varepsilon(x_0)} \frac{\partial w_1^\varepsilon}{\partial\nu} u\,dS_x &- \int_{\Omega\setminus\omega_\varepsilon(x_0)} k^2\big(w_0^\varepsilon + \tfrac{|\partial\omega|}{2\pi}(\ln\varepsilon - \ln\rho)\big)u\,dx\end{aligned} \tag{5.17b}$$

for all $u\in H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{R}) : u = 0$ on Γ_D

and admits the far-field representation for $x\in B_R(x_0)\setminus\overline{B_\varepsilon(x_0)}$:

$$w_1^\varepsilon(x) = \tfrac{|\partial\omega|}{2\pi}\big((\ln\rho - u^{\ln}(x_0))(1+a_0) + \tfrac{\pi a_2}{2} - U_0^{\ln}\big) + W_0^\varepsilon. \tag{5.18}$$

Proof. The assertion is derived directly by substitution in (5.15) the asymptotic representations (4.14) for $u^{\ln}$ and (5.12) for w_0^ε. With the notation in (5.16c), this leads to (5.18) and (5.16b), obtained after differentiation of the radial function in (5.18) by the rule $\nabla = \frac{\partial}{\partial\rho}\widehat{x}$. Relations (5.17) follow from (4.11a), (4.13), and (5.11). □

Combining the boundary layers w_ν^ε and w_1^ε constructed in Lemma 3.4 and Lemma 5.3, respectively, in the next section we expand the solution of the Robin problem (5.2).

5.2 Combined uniform asymptotic expansion of the Robin problem

Analogous to Sections 3.2 and 4.2 the following uniform asymptotic expansion holds.

Theorem 5.4. *The solutions u^0 of* (2.2)*, w_ν of* (3.16)*, $u^{(\varepsilon,\alpha)}$ of* (5.2)*, and w_1^ε of* (5.17) *satisfy the residual error estimate*

$$\|q_1^{(\varepsilon,\alpha)}\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})} = \mathrm{O}\big(\{(1+|\alpha|)\sqrt{|\ln\varepsilon|} + |\alpha|^2\}\varepsilon^2\big), \tag{5.19a}$$

$$q_1^{(\varepsilon,\alpha)} := u^{(\varepsilon,\alpha)} - u^0 - \varepsilon\big(\alpha u^0(x_0)w_1^\varepsilon + \nabla u^0(x_0)\cdot w_\nu^\varepsilon\big). \tag{5.19b}$$

Proof. If we write the residual $q_0^{(\varepsilon,\alpha)} := u^{(\varepsilon,\alpha)} - u^0$ of (5.2) and (3.10) using (3.44)

$$\begin{aligned}&\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla q_0^{(\varepsilon,\alpha)} \cdot \nabla \overline{u} - k^2 q_0^{(\varepsilon,\alpha)} \overline{u})\, dx + \int_{\partial\omega_\varepsilon(x_0)} \alpha q_0^{(\varepsilon,\alpha)} \overline{u}\, dS_x \\ &= \int_{\partial\omega_\varepsilon(x_0)} (\tfrac{\partial u^0}{\partial\nu} - \alpha u^0)\overline{u}\, dS_x = \int_{\partial\omega_\varepsilon(x_0)} (\nabla u^0(x_0)\cdot\nu - \alpha u^0(x_0))\overline{u}\, dS_x + \mathrm{o}(\varepsilon),\end{aligned} \tag{5.20}$$

then we observe that the linear combination of the boundary layers introduced in (5.19b) will cancel the leading asymptotic term due to (3.27b) and (5.16b).

In detail, subtracting from (5.2) equations (3.10), (5.17) multiplied with $\varepsilon\alpha u^0(x_0)$, and (3.18) multiplied with $\varepsilon\nabla u^0(x_0)$, for the residual $q_1^{(\varepsilon,\alpha)}$ in (5.19b) we derive

$$q_1^{(\varepsilon,\alpha)} = -\varepsilon\alpha u^0(x_0) W_0^\varepsilon - \varepsilon\nabla u^0(x_0)\cdot w_\nu^\varepsilon = \mathrm{O}\big((1+|\alpha|)\varepsilon^2\big) \quad \text{on } \Gamma_D, \tag{5.21a}$$

$$\begin{aligned}&\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla q_1^{(\varepsilon,\alpha)} \cdot \nabla \overline{u} - k^2 q_1^{(\varepsilon,\alpha)} \overline{u})\, dx + \int_{\partial\omega_\varepsilon(x_0)} \alpha q_1^{(\varepsilon,\alpha)} \overline{u}\, dS_x \\ &= I_1 + I_2 + I_3 \quad \text{for all } u \in H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C}) : u = 0 \text{ on } \Gamma_D,\end{aligned} \tag{5.21b}$$

where the integrals I_1, I_2, I_3 are defined and estimated as follows.

According to (3.19) and (5.13) we have $W_0^\varepsilon = w_\nu^\varepsilon = \mathrm{O}(\varepsilon)$ at Γ_D, hence the asymptotic order in (5.21a), and $\frac{\partial W_0^\varepsilon}{\partial\nu} = Dw_\nu^\varepsilon\nu = \mathrm{O}(\varepsilon)$ at Γ_N providing

$$I_1 := -\varepsilon\int_{\Gamma_N} \big(\alpha u^0(x_0)\tfrac{\partial W_0^\varepsilon}{\partial\nu} + \nabla u^0(x_0)\cdot(Dw_\nu^\varepsilon\nu)\big)\overline{u}\, dS_x = \mathrm{O}\big((1+|\alpha|)\varepsilon^2\big).$$

Based on the expressions (2.9) and (2.35) for u^0 and (5.16b) for $\frac{\partial w_1^\varepsilon}{\partial\nu}$ it follows

$$\begin{aligned}I_2 :=& \int_{\partial\omega_\varepsilon(x_0)} \big(\tfrac{\partial u^0}{\partial\nu} - \alpha u^0 + \varepsilon\alpha u^0(x_0)(\tfrac{\partial w_1^\varepsilon}{\partial\nu} - \alpha w_1^\varepsilon) - \varepsilon\nabla u^0(x_0)\cdot(\tfrac{\nu}{\varepsilon} + \alpha w_\nu^\varepsilon)\big)\overline{u}\, dS_x \\ =& \int_{\partial\omega_\varepsilon(x_0)} \big\{b_u^0\cdot\nu + \tfrac{\partial U_1^0}{\partial\nu} - \alpha u^0(x_0)a_0 - \alpha U_0^0 + \varepsilon\alpha u^0(x_0)\tfrac{|\partial\omega|}{2\pi}\big(b_{\ln}\cdot\nu - \tfrac{\partial U_0^{\ln}}{\partial\nu}\big) \\ &- \varepsilon\alpha^2 u^0(x_0) w_1^\varepsilon - \varepsilon\alpha\nabla u^0(x_0)\cdot w_\nu^\varepsilon\big\}\overline{u}\, dS_x = \mathrm{O}\big((1+|\alpha|+|\alpha|^2)\varepsilon^2\big)\end{aligned}$$

due to (2.8a), (3.19), (4.16), and (5.16c), where the estimate (3.21b) implying $w_\nu^\varepsilon = \mathrm{O}(1)$ and similarly $w_1^\varepsilon = \mathrm{O}(1)$ at $\partial\omega_\varepsilon(x_0)$ were used. The representation over domain

$$\begin{aligned}I_3 :=& \varepsilon\int_{\Omega\setminus\omega_\varepsilon(x_0)} k^2\big\{\alpha u^0(x_0)\big(w_0^\varepsilon + \tfrac{|\partial\omega|}{2\pi}(\ln\varepsilon - \ln\rho)\big) + \nabla u^0(x_0)\cdot w_\nu^\varepsilon\big\}\overline{u}\, dx \\ =& \mathrm{O}\big((1+|\alpha|)\varepsilon^2\sqrt{|\ln\varepsilon|}\big)\end{aligned}$$

is argued by (3.21a) and by the similar estimate resulting by homogeneity from Lemma 5.2:

$$\|w_0^\varepsilon + \tfrac{|\partial\omega|}{2\pi}(\ln\varepsilon - \ln\rho)\|_{L^2(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{R})^2} = \mathrm{O}(\varepsilon\sqrt{|\ln\varepsilon|}).$$

Next lifting with the help of a cut-off function η_{Γ_D} such that $\eta_{\Gamma_D} = 1$ at Γ_D we set

$$R_1^{(\varepsilon,\alpha)} := \varepsilon\big(\alpha u^0(x_0) W_0^\varepsilon + \nabla u^0(x_0) \cdot w_\nu^\varepsilon\big)\eta_{\Gamma_D} = \mathrm{O}\big((1+|\alpha|)\varepsilon^2\big). \tag{5.22}$$

Applying to (5.21b) the Cauchy–Schwarz inequality, the estimates of I_1, I_2, I_3 succeed

$$\Big|\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla Q_1^{(\varepsilon,\alpha)} \cdot \nabla\overline{u} - k^2 Q_1^{(\varepsilon,\alpha)}\overline{u})\, dx + \int_{\partial\omega_\varepsilon(x_0)} \alpha Q_1^{(\varepsilon,\alpha)}\overline{u}\, dS_x\Big|$$
$$\leq C\big(\{(1+|\alpha|)\sqrt{|\ln\varepsilon|} + |\alpha|^2\}\varepsilon^2\big)\|u\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)};\mathbb{C})}, \quad Q_1^{(\varepsilon,\alpha)} := q_1^{(\varepsilon,\alpha)} + R_1^{(\varepsilon,\alpha)}$$

with $C > 0$, which together with the inf-sup condition (5.6) proves the theorem. □

Based on Theorem 5.4, in the next section we treat asymptotically for $\varepsilon \searrow +0$ the optimal value function J in (3.47) depending on the boundary impedance α.

5.3 Inverse Helmholtz problem under Robin boundary condition

In contrast to Sections 3.3 and 4.3, here the test parameter of surface impedance $\alpha^\star \in \mathfrak{G}_\alpha \subset \mathbb{C}$ (see (5.6)) is unknown a-priori and has to be identified together with the test object $\omega^\star_{\varepsilon^\star}(x^\star)$. For a feasible trial parameter $\alpha \in \mathfrak{G}_\alpha$ and the solution $u^{(\varepsilon,\alpha)}$ of (5.2) we reset the objective

$$J : \mathfrak{G} \times \mathfrak{G}_\alpha \mapsto \mathbb{R}_+, \quad J(\omega, \varepsilon, x_0, \alpha) := \tfrac{1}{2}\int_{\Gamma_N} |u^{(\varepsilon,\alpha)} - u^\star|^2\, dS_x \tag{5.23}$$

and the topology optimization problem: Find $(\omega^\star, \varepsilon^\star, x^\star, \alpha^\star) \in \mathfrak{G} \times \mathfrak{G}_\alpha$ such that

$$0 = J(\omega^\star, \varepsilon^\star, x^\star, \alpha^\star) = \min_{(\omega,\varepsilon,x_0,\alpha)\in\mathfrak{G}\times\mathfrak{G}_\alpha} J(\omega, \varepsilon, x_0, \alpha) \quad \text{subject to (5.2).} \tag{5.24}$$

Analogously to Lemma 3.7 and Proposition 5.1, a Fenchel–Legendre duality provides the primal-dual variational principle:

$$\mathcal{L}_{(\varepsilon,\alpha)}(u^{(\varepsilon,\alpha)}, v^{(\varepsilon,\alpha)}) = \min_{\mathrm{Re}(u),\mathrm{Re}(v)} \max_{\mathrm{Im}(u),\mathrm{Im}(v)} \mathcal{L}_{(\varepsilon,\alpha)}(u, v)$$
$$\text{over } u, v \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)}; \mathbb{C}) : u = h,\ v = 0 \text{ on } \Gamma_D, \tag{5.25}$$

where the Lagrangian $\mathcal{L}_{(\varepsilon,\alpha)}$ has the form (compare with (3.49) and (3.50)):

$$\mathcal{L}_{(\varepsilon,\alpha)}(u, v) := \mathrm{Re}\Big\{\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla u \cdot \nabla v - k^2 uv)\, dx - \int_{\Gamma_N} gv\, dS_x$$
$$+ \int_{\partial\omega_\varepsilon(x_0)} \alpha uv\, dS_x + \tfrac{1}{2}\int_{\Gamma_N} (u - u^\star)^2\, dS_x\Big\}, \tag{5.26}$$

thus arguing necessarily the dual variational problem: Find $v^{(\varepsilon,\alpha)} \in H^1(\Omega\backslash\overline{\omega_\varepsilon(x_0)};\mathbb{C})$ such that

$$v^{(\varepsilon,\alpha)} = 0 \quad \text{on } \Gamma_D, \tag{5.27a}$$

$$\begin{aligned}&\int_{\Omega\backslash\omega_\varepsilon(x_0)} (\nabla v^{(\varepsilon,\alpha)} \cdot \nabla\overline{u} - k^2 v^{(\varepsilon,\alpha)}\overline{u})\,dx + \int_{\partial\omega_\varepsilon(x_0)} \alpha v^{(\varepsilon,\alpha)}\overline{u}\,dS_x \\ &= \int_{\Gamma_N} (u^\star - u^{(\varepsilon,\alpha)})\overline{u}\,dS_x \quad \text{for all } u \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)};\mathbb{C}) : u = 0 \text{ on } \Gamma_D.\end{aligned} \tag{5.27b}$$

Again utilizing the primal u^0 and the dual v^0 background solutions of (2.2) and (3.54), which are independent of the test object as well as its surface impedance, similarly to (3.59) and (4.53) we express the objective in (5.23) equivalently as

$$\begin{aligned}&J(\omega,\varepsilon,x_0,\alpha) = J_0 + \mathrm{Re}\big\{\mathcal{I}(u^{(\varepsilon,\alpha)} - u^0, v^0)\big\} + \tfrac{1}{2}\int_{\Gamma_N} |u^{(\varepsilon,\alpha)} - u^0|^2\,dS_x, \\ &\mathcal{I}(u^{(\varepsilon,\alpha)} - u^0, v^0) := \int_{\Gamma_N} (u^0 - u^{(\varepsilon,\alpha)})\tfrac{\partial\overline{v}^0}{\partial\nu}\,dS_x, \quad J_0 := \tfrac{1}{2}\int_{\Gamma_N} |u^0 - u^\star|^2\,dS_x \\ &= \mathrm{O}(1), \quad \tfrac{1}{2}\int_{\Gamma_N} |u^{(\varepsilon,\alpha)} - u^0|^2\,dS_x = \mathrm{O}\big(\big\{(1+|\alpha|)\sqrt{|\ln\varepsilon|} + |\alpha|^2\big\}^2\varepsilon^4\big)\end{aligned} \tag{5.28}$$

with the asymptotic order due to Theorem 5.4. The refinement of (5.28) follows.

Theorem 5.5. *The objective in* (5.28) *admits the high-order asymptotic expansion*

$$\begin{aligned}&J(\omega,\varepsilon,x_0,\alpha) = J_0 + \mathrm{Re}\big\{\varepsilon\tfrac{|\partial\omega|}{2\pi}\alpha J_1^D(x_0) + \big(\varepsilon^2 J_1^N(\omega,x_0) + \alpha J_2^{(\varepsilon,\alpha)}\big) \\ &+ J_3^{(\varepsilon,\alpha)} + J_4^{(\varepsilon,\alpha)} + J_5^{(\varepsilon,\alpha)} + J_6^{(\varepsilon,\alpha)}\big\} + \mathrm{O}\big(\big\{(1+|\alpha|)\sqrt{|\ln\varepsilon|} + |\alpha|^2\big\}^2\varepsilon^4\big),\end{aligned} \tag{5.29}$$

where J_1^N and J_1^D are expressed in (3.61a) *and* (4.55a)*, and the asymptotic terms are*

$$\begin{aligned}J_2^{(\varepsilon,\alpha)} &:= \int_{\partial\omega_\varepsilon(x_0)} \big(\varepsilon\alpha u^0(x_0)w_1^\varepsilon + \nabla u^0(x_0)\cdot(x + \varepsilon w_\nu^\varepsilon)\big)\overline{v^0}(x_0)\,dS_x \\ &- \varepsilon^2 u^0(x_0)(m_\omega \cdot \nabla\overline{v^0}(x_0)) = \mathrm{O}((1+|\alpha|)\varepsilon^2),\end{aligned} \tag{5.30a}$$

$$J_3^{(\varepsilon,\alpha)} := -\varepsilon\alpha\int_{\partial\omega_\varepsilon(x_0)} u^0(x_0)\tfrac{|\partial\omega|}{2\pi}(b_{\ln}\cdot\nu)\overline{v^0}(x_0)\,dS_x = \mathrm{O}(|\alpha|\varepsilon^3|\ln\varepsilon|), \tag{5.30b}$$

$$\begin{aligned}J_4^{(\varepsilon,\alpha)} &:= \int_{\partial\omega_\varepsilon(x_0)} \alpha q_1^{(\varepsilon,\alpha)}(\overline{v^0}(x_0) - \nabla\overline{v^0}(x_0)\cdot\nu)\,dS_x + \int_{\partial B_\varepsilon(x_0)} \frac{\partial q_1^{(\varepsilon,\alpha)}}{\partial\rho}\overline{V_0^0}\,dS_x \\ &= \mathrm{O}\big((1+|\alpha|)\big\{(1+|\alpha|)\sqrt{|\ln\varepsilon|} + |\alpha|^2\big\}\varepsilon^3\big),\end{aligned} \tag{5.30c}$$

$$J_5^{(\varepsilon,\alpha)} := \int_{\partial\omega_\varepsilon(x_0)} \big(\alpha u^0(x_0)a_0 + \alpha U_1^0 - (b_u^0 + u^0(x_0)\tfrac{k^2\rho}{2}\widehat{x})\cdot\nu\big)\overline{v^0}(x_0)\,dS_x$$
$$+\,\varepsilon^2 \int_{-\pi}^{\pi} \big\{\alpha u^0(x_0)\big[\tfrac{\partial W_1^\varepsilon}{\partial\rho}\overline{V_1^0} - W_1^\varepsilon \tfrac{\partial \overline{V_1^0}}{\partial\rho} - \tfrac{|\partial\omega|}{2\pi}\big(\tfrac{\partial U_0^{\ln}}{\partial\rho}\overline{V_0^0} - U_0^{\ln}\tfrac{\partial\overline{V_0^0}}{\partial\rho}\big)\big] \tag{5.30d}$$
$$+\,\nabla u^0(x_0)\cdot\big(\tfrac{\partial W_\nu^\varepsilon}{\partial\rho}\overline{V_1^0} - W_\nu^\varepsilon\tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)\big\}\,d\theta + \varepsilon(2a_0 + \varepsilon\tfrac{\pi k a_2'}{2})\tfrac{|\partial\omega|}{2\pi}\alpha J_1^D(x_0)$$
$$= \mathrm{O}((1+|\alpha|)\varepsilon^3),$$

$$J_6^{(\varepsilon,\alpha)} := \int_{\partial B_\varepsilon(x_0)} \big(\tfrac{\partial q_1^{(\varepsilon,\alpha)}}{\partial\rho}\overline{v^0}(x_0)a_0 - q_1^{(\varepsilon,\alpha)}(\overline{b_v^0} + \nabla\overline{V_1^0}(x_0))\cdot\widehat{x}\big)\,dS_x \tag{5.30e}$$
$$= \mathrm{O}\big(\{(1+|\alpha|)\sqrt{|\ln\varepsilon|} + |\alpha|^2\}\varepsilon^4\big).$$

Proof. Employing the equivalent expression (4.56) over the circle $\partial B_\varepsilon(x_0)$ for the invariant integral $\mathcal{I}(u^{(\varepsilon,\alpha)} - u^0, v^0)$ in (5.28) and substituting here the asymptotic representation (5.19b) for $u^{(\varepsilon,\alpha)} - u^0$ we have

$$\mathcal{I}(u^\varepsilon - u^0, v^0) = \int_{\partial B_\varepsilon(x_0)} \big\{\big(\varepsilon\alpha u^0(x_0)\tfrac{\partial w_1^\varepsilon}{\partial\rho} + \varepsilon\nabla u^0(x_0)\cdot\tfrac{\partial w_\nu^\varepsilon}{\partial\rho} + \tfrac{\partial q_1^{(\varepsilon,\alpha)}}{\partial\rho}\big)\overline{v^0}$$
$$-\big(\varepsilon\alpha u^0(x_0)w_1^\varepsilon + \varepsilon\nabla u^0(x_0)\cdot w_\nu^\varepsilon + q_1^{(\varepsilon,\alpha)}\big)\tfrac{\partial\overline{v^0}}{\partial\rho}\big\}\,dS_x = I_1^\alpha + I_2^\alpha + I_3^\alpha.$$

The integrals $I_1^\alpha, I_2^\alpha, I_3^\alpha$ are set as follows.

On the one hand, the first-order representation $v_0 = v_0(x_0)(1+a_0) + V_0^0$ from (3.56) and (2.8a) provides the expression

$$I_1^\alpha := \int_{\partial B_\varepsilon(x_0)} \varepsilon\alpha u^0(x_0)\big(\tfrac{\partial w_1^\varepsilon}{\partial\rho}\overline{v^0} - w_1^\varepsilon\tfrac{\partial\overline{v^0}}{\partial\rho}\big)\,dS_x = \int_{-\pi}^{\pi}\varepsilon\alpha u^0(x_0)\overline{v^0}(x_0)\big(\tfrac{\partial w_1^\varepsilon}{\partial\rho}(1+a_0)$$
$$-\,w_1^\varepsilon k a_0'\big)\,\varepsilon d\theta + \int_{-\pi}^{\pi}\varepsilon\alpha u^0(x_0)\big(\tfrac{\partial w_1^\varepsilon}{\partial\rho}\overline{V_0^0} - w_1^\varepsilon\tfrac{\partial\overline{V_0^0}}{\partial\rho}\big)\,\varepsilon d\theta =: I_{11}^\alpha + I_{12}^\alpha.$$

Inserting in I_1^α the representations (5.12) for w_0^ε and (5.18) for w_1^ε from Lemmas 5.2 and 5.3 and using the orthogonality conditions for $U_0^{\ln}$, W_0^ε, and W_1^ε, we calculate

$$I_{11}^\alpha = \int_{-\pi}^{\pi}\varepsilon\alpha u^0(x_0)\overline{v^0}(x_0)\tfrac{|\partial\omega|}{2\pi}\big\{\big(\tfrac{1+a_0}{\varepsilon} + \tfrac{\pi k a_2'}{2}\big)(1+a_0) - \tfrac{\pi a_2}{2}k a_0'\big\}\,\varepsilon d\theta$$
$$= \varepsilon|\partial\omega|\alpha u^0(x_0)\overline{v^0}(x_0)\big\{1 + (2a_0 + \varepsilon\tfrac{\pi k a_2'}{2}) + \big(a_0^2 + \varepsilon\tfrac{\pi k}{2}(a_2' a_0 - a_2 a_0')\big)\big\},$$
$$I_{12}^\alpha = -\varepsilon^2\alpha\int_{-\pi}^{\pi} u^0(x_0)\tfrac{|\partial\omega|}{2\pi}\big(\tfrac{\partial U_0^{\ln}}{\partial\rho}\overline{V_0^0} - U_0^{\ln}\tfrac{\partial\overline{V_0^0}}{\partial\rho}\big)\,d\theta + I_{13}^\alpha,$$
$$I_{13}^\alpha := \varepsilon^2\alpha\int_{-\pi}^{\pi} u^0(x_0)\big(\tfrac{\partial W_0^\varepsilon}{\partial\rho}\overline{V_0^0} - W_0^\varepsilon\tfrac{\partial\overline{V_0^0}}{\partial\rho}\big)\,d\theta = \varepsilon^2\alpha\int_{-\pi}^{\pi} u^0(x_0)\big\{\big(-\tfrac{1}{\varepsilon}\tfrac{1}{2\pi}(m_\omega\cdot\widehat{x})$$
$$+\,\tfrac{\partial W_1^\varepsilon}{\partial\rho}\big)\big((\varepsilon + \tfrac{a_1}{k})(\nabla\overline{v^0}(x_0)\cdot\widehat{x}) + \overline{V_1^0}\big) - \big(\tfrac{1}{2\pi}(m_\omega\cdot\widehat{x}) + W_1^\varepsilon\big)\big((1+a_1')(\nabla\overline{v^0}(x_0)\cdot\widehat{x})$$
$$+\,\tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)\big\}\,d\theta = \varepsilon^2\alpha\int_{-\pi}^{\pi} u^0(x_0)\big(\tfrac{\partial W_1^\varepsilon}{\partial\rho}\overline{V_1^0} - W_1^\varepsilon\tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)\,d\theta + I_{14}^\alpha,$$

$$I^\alpha_{14} := -\varepsilon^2\alpha \int_{-\pi}^{\pi} u^0(x_0)\tfrac{1}{2\pi}(m_\omega\cdot\widehat{x})\bigl(1+\tfrac{a_1}{\varepsilon k}+1+a_1'\bigr)(\nabla\overline{v^0}(x_0)\cdot\widehat{x})\,d\theta$$
$$= -\varepsilon^2\alpha u^0(x_0)(m_\omega\cdot\nabla\overline{v^0}(x_0))\bigl(1+\tfrac12(\tfrac{a_1}{\varepsilon k}+a_1')\bigr).$$

On the other hand, the second-order expansion of $v_0 = v_0(x_0)(1+a_0)+(\rho+\frac{a_1}{k})(\nabla v^0(x_0)\cdot\widehat{x})+V_1^0$ in the manner of (4.39) and its derivative from (3.58) yields

$$I^\alpha_2 := \int_{\partial B_\varepsilon(x_0)} \varepsilon\nabla u^0(x_0)\cdot\bigl(\tfrac{\partial w^\varepsilon_\nu}{\partial\rho}\overline{v^0} - w^\varepsilon_\nu\tfrac{\partial\overline{v^0}}{\partial\rho}\bigr)\,dS_x$$
$$= \int_{-\pi}^{\pi} \varepsilon\nabla u^0(x_0)\cdot\bigl(\tfrac{\partial w^\varepsilon_\nu}{\partial\rho}(\varepsilon+\tfrac{a_1}{k})(\nabla\overline{v^0}(x_0)\cdot\widehat{x}) - w^\varepsilon_\nu(\nabla\overline{v^0}(x_0)+\overline{b^0_v})\cdot\widehat{x}\bigr)\,\varepsilon d\theta$$
$$+ \int_{-\pi}^{\pi} \varepsilon\nabla u^0(x_0)\cdot\bigl(\tfrac{\partial w^\varepsilon_\nu}{\partial\rho}\overline{V^0_1} - w^\varepsilon_\nu\tfrac{\partial\overline{V^0_1}}{\partial\rho}\bigr)\,\varepsilon d\theta =: I^\alpha_{21}+I^\alpha_{22},$$

where we have used $\int_{-\pi}^{\pi} w^\varepsilon_\nu\,d\theta = 0$ due to (3.19) and (3.20). Inserting in I^α_2 the representation (3.19) of w^ε_ν it recalls the calculation in (3.66)–(3.68) getting

$$I^\alpha_{21} = -\varepsilon^2\nabla u^0(x_0)^\top M_\omega\nabla\overline{v^0}(x_0)\bigl(1+\tfrac12(\tfrac{a_1}{\varepsilon k}+a_1')\bigr),$$
$$I^\alpha_{22} = \varepsilon^2\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\bigl(\tfrac{\partial W^\varepsilon_\nu}{\partial\rho}\overline{V^0_1} - W^\varepsilon_\nu\tfrac{\partial\overline{V^0_1}}{\partial\rho}\bigr)\,d\theta.$$

It remains to estimate I^α_3 which includes the residual $q_1^{(\varepsilon,\alpha)}$ given in Theorem 5.4:

$$I^\alpha_3 := \int_{\partial B_\varepsilon(x_0)} \bigl(\tfrac{\partial q_1^{(\varepsilon,\alpha)}}{\partial\rho}\overline{v^0} - q_1^{(\varepsilon,\alpha)}\tfrac{\partial\overline{v^0}}{\partial\rho}\bigr)\,dS_x$$
$$= \int_{\partial B_\varepsilon(x_0)} \bigl(\tfrac{\partial q_1^{(\varepsilon,\alpha)}}{\partial\rho}\overline{v^0}(x_0) - q_1^{(\varepsilon,\alpha)}\nabla\overline{v^0}(x_0)\cdot\widehat{x}\bigr)\,dS_x + \int_{\partial B_\varepsilon(x_0)}\tfrac{\partial q_1^{(\varepsilon,\alpha)}}{\partial\rho}\overline{V^0_0}\,dS_x$$
$$+ \int_{\partial B_\varepsilon(x_0)} \bigl(\tfrac{\partial q_1^{(\varepsilon,\alpha)}}{\partial\rho}\overline{v^0}(x_0)a_0 - q_1^{(\varepsilon,\alpha)}(\overline{b^0_v}+\nabla\overline{V^0_1}(x_0))\cdot\widehat{x}\bigr)\,dS_x =: I^\alpha_{31}+I^\alpha_{32}+I^\alpha_{33}.$$

By applying to I^α_{31} the second Green formula in $B_\varepsilon(x_0)\setminus\overline{\omega_\varepsilon(x_0)}$ it can be rewritten as

$$I^\alpha_{31} = \langle\tfrac{\partial q_1^{(\varepsilon,\alpha)}}{\partial\nu},\overline{v^0}(x_0)\rangle_{\partial\omega_\varepsilon(x_0)} - \int_{\partial\omega_\varepsilon(x_0)} q_1^{(\varepsilon,\alpha)}(\nabla\overline{v^0}(x_0)\cdot\nu)\,dS_x. \tag{5.31}$$

We extract the normal derivative $\frac{\partial q_1^{(\varepsilon,\alpha)}}{\partial\nu}$ at $\partial\omega_\varepsilon(x_0)$ from I_2 in (5.21b):

$$\tfrac{\partial q_1^{(\varepsilon,\alpha)}}{\partial\nu} = \alpha q_1^{(\varepsilon,\alpha)} + \alpha u^0 - \tfrac{\partial u^0}{\partial\nu} + \varepsilon\alpha u^0(x_0)(\alpha w^\varepsilon_1 - \tfrac{\partial w^\varepsilon_1}{\partial\nu})$$
$$+ \varepsilon\nabla u^0(x_0)\cdot(\alpha w^\varepsilon_\nu - Dw^\varepsilon_\nu\nu) = \alpha q_1^{(\varepsilon,\alpha)} + \alpha u^0(x_0)a_0 + \alpha U^0_0 - b^0_u\cdot\nu \tag{5.32}$$
$$- \tfrac{\partial U^0_1}{\partial\nu} + \varepsilon\alpha u^0(x_0)\bigl(\alpha w^\varepsilon_1 - \tfrac{|\partial\omega|}{2\pi}(b_{\ln}\cdot\nu - \tfrac{\partial U^{\ln}_0}{\partial\nu})\bigr) + \varepsilon\alpha\nabla u^0(x_0)\cdot w^\varepsilon_\nu$$

and substitute it in (5.31). Analog to (3.64), the divergence theorem in $\omega_\varepsilon(x_0)$ provides

$$\begin{aligned}
&\int_{\partial\omega_\varepsilon(x_0)} \nu\cdot\bigl(u^0(x_0)\tfrac{k^2\rho}{2}\widehat{x} - \nabla U_1^0 + \varepsilon\alpha u^0(x_0)\tfrac{|\partial\omega|}{2\pi}\nabla U_0^{\ln}\bigr)\overline{v^0}(x_0)\,dS_x\\
&= \int_{\omega_\varepsilon(x_0)} \operatorname{div}\bigl(u^0(x_0)\tfrac{k^2\rho}{2}\widehat{x} - \nabla U_1^0 + \varepsilon\alpha u^0(x_0)\tfrac{|\partial\omega|}{2\pi}\nabla U_0^{\ln}\bigr)\overline{v^0}(x_0)\,dx \qquad (5.33)\\
&= \varepsilon^2\mathrm{meas}_2(\omega)k^2u^0(x_0)\overline{v^0}(x_0) + \int_{\omega_\varepsilon(x_0)} k^2\bigl(U_1^0 - \varepsilon\alpha u^0(x_0)\tfrac{|\partial\omega|}{2\pi}U_0^{\ln}\bigr)\overline{v^0}(x_0)\,dx
\end{aligned}$$

since $\operatorname{div}(\rho\widehat{x}) = 2$, $\Delta U_1^0 = -k^2U_1^0$, and $\Delta U_0^{\ln} = -k^2U_0^{\ln}$. With the help of (5.32), (5.33), and due to $U_0^0 = \nabla u^0(x_0)\cdot(x + \frac{a_1}{k}\widehat{x}) + U_1^0$, from (5.31) we infer

$$\begin{aligned}
I_{31}^\alpha &= \alpha\int_{\partial\omega_\varepsilon(x_0)} \bigl(\varepsilon\alpha u^0(x_0)w_1^\varepsilon + \nabla u^0(x_0)\cdot(x+\varepsilon w_\nu^\varepsilon)\bigr)\overline{v^0}(x_0)\,dS_x\\
&+ \varepsilon^2\mathrm{meas}_2(\omega)k^2u^0(x_0)\overline{v^0}(x_0) + \int_{\partial\omega_\varepsilon(x_0)} \bigl\{\bigl(\alpha q_1^{(\varepsilon,\alpha)} + \alpha u^0(x_0)a_0\\
&+ \alpha(\tfrac{a_1}{k}\nabla u^0(x_0)\cdot\widehat{x} + U_1^0) - (b_u^0 + u^0(x_0)\tfrac{k^2\rho}{2}\widehat{x})\cdot\nu - \varepsilon\alpha u^0(x_0)\tfrac{|\partial\omega|}{2\pi}b_{\ln}\cdot\nu\bigr)\overline{v^0}(x_0)\\
&- q_1^{(\varepsilon,\alpha)}(\nabla\overline{v^0}(x_0)\cdot\nu)\bigr\}\,dS_x + \int_{\omega_\varepsilon(x_0)} k^2\bigl(U_1^0 - \varepsilon\alpha u^0(x_0)\tfrac{|\partial\omega|}{2\pi}U_0^{\ln}\bigr)\overline{v^0}(x_0)\,dx.
\end{aligned}$$

After collection of the asymptotic terms of the same order in view of the asymptotic relations (3.65) and (3.68) in the proof of Theorem 3.9, (4.15) in Lemma 4.2, (5.13) in Lemma 5.2, (5.16c) in Lemma 5.3, and (5.19) in Theorem 5.4, it follows the expansion

$$\begin{aligned}
\mathcal{I}(u^\varepsilon - u^0, v^0) &= I_{11}^\alpha + I_{12}^\alpha + I_{21}^\alpha + I_{22}^\alpha + I_{31}^\alpha + I_{32}^\alpha + I_{33}^\alpha = \varepsilon\tfrac{|\partial\omega|}{2\pi}\alpha J_1^D(x_0)\\
&+ \bigl(\varepsilon^2J_1^N(\omega,x_0) + \alpha J_2^{(\varepsilon,\alpha)}\bigr) + J_3^{(\varepsilon,\alpha)} + J_4^{(\varepsilon,\alpha)} + J_5^{(\varepsilon,\alpha)} + J_6^{(\varepsilon,\alpha)} + J_7^{(\varepsilon,\alpha)}
\end{aligned}$$

with J_1^N and J_1^D given in (3.61a) and (4.55a), the terms from $J_2^{(\varepsilon,\alpha)}$ to $J_6^{(\varepsilon,\alpha)}$ described in (5.30), and the residual term collected by

$$\begin{aligned}
J_7^{(\varepsilon,\alpha)} &:= \varepsilon\bigl(a_0^2 + \varepsilon\tfrac{\pi k}{2}(a_2'a_0 - a_2a_0')\bigr)\tfrac{|\partial\omega|}{2\pi}\alpha J_1^D(x_0) - \tfrac{\varepsilon^2}{2}\bigl\{\alpha u^0(x_0)(m_\omega\cdot\nabla\overline{v^0}(x_0))\\
&- \nabla u^0(x_0)^\top M_\omega\nabla\overline{v^0}(x_0)\bigr\}(\tfrac{a_1}{\varepsilon k} + a_1') + \int_{\partial\omega_\varepsilon(x_0)} \alpha\tfrac{a_1}{k}(\nabla u^0(x_0)\cdot\widehat{x})\overline{v^0}(x_0)\,dS_x\\
&+ \int_{\omega_\varepsilon(x_0)} k^2\bigl(U_1^0 - \varepsilon\alpha u^0(x_0)\tfrac{|\partial\omega|}{2\pi}U_0^{\ln}\bigr)\overline{v^0}(x_0)\,dx = \mathrm{O}((1+|\alpha|)\varepsilon^4).
\end{aligned}$$

The proof of the assertion of the theorem is completed. □

We finish with the important generalizations and consequences of Theorem 5.5.

For a variable parameter of the surface impedance $\alpha \in L^\infty(\partial\omega;\mathbb{C})$, following [43] the expansion (5.29) can be generalized by replacing α with $\alpha(\frac{x-x_0}{\varepsilon})$, $\frac{|\partial\omega|}{2\pi}\alpha$ with the average $\frac{1}{2\pi}\int_{\partial\omega}\alpha(y)\,dS_y$, and the modulus $|\alpha|$ with the L^∞-norm.

Sections 3 and 4 correspond, respectively, to the case $|\alpha| \searrow +0$ and a special case of $|\alpha| \nearrow \infty$ which will be explained further. Moreover, they provide our construction with the auxiliary functions w_ν^ε, w_{00}, and $u^{\ln}$ as well as the expressions of J_1^N and J_1^D used here. Therefore, these sections are not redundant.

For fixed $\alpha \in \mathbb{C}$, the concept of the topological derivatives from (3.69) and (4.58) can be generalized to the Robin problem (5.2):

$$\tfrac{|\partial\omega|}{2\pi} \operatorname{Re}(\alpha J_1^D(x_0)) = \lim_{\varepsilon\searrow+0} \tfrac{1}{\varepsilon}\big(J(\omega,\varepsilon,x_0,\alpha) - J_0\big) \tag{5.34}$$

utilizing the leading asymptotic term in the α-dependent asymptotic expansion (5.29).

However, the leading term in (5.29) changes when varying $|\alpha| \searrow +0$ or $|\alpha| \nearrow \infty$.

In fact, on the one hand, passing $|\alpha| \searrow +0$ in (5.29) we obtain the representation

$$\lim_{|\alpha|\searrow+0} J(\omega,\varepsilon,x_0,\alpha) = J_0 + \varepsilon^2 \operatorname{Re}(J_1^N(\omega,x_0)) + \mathrm{O}\big(\sqrt{|\ln\varepsilon|}\varepsilon^3\big) \tag{5.35}$$

with the residual estimate according to (5.30). Asymptotically, the limit in (5.35) coincides with formulas (3.60) and (3.61) for the Neumann problem in Theorem 3.9.

On the one hand, taking the limit as $|\alpha| \nearrow \infty$, which corresponds to the Dirichlet problem, in the next section we derive a derivative-free necessary optimality condition following from the topology optimization problem (5.24).

5.4 Necessary optimality condition for the topology optimization

Recalling the topology optimization problem (5.24) for an unknown parameter of the test impedance $\alpha^\star \in \mathbb{C}$, it allows, in particular, the both limit cases $|\alpha^\star| \searrow +0$ as well as $|\alpha^\star| \nearrow \infty$. In the latter case we derive the necessary optimality condition which does not appear in the limit optimization problems (3.48) and (4.51), when either the Neumann or Dirichlet boundary conditions of the test object are assumed a-priori.

For $\varepsilon \searrow +0$, the first-order approximation of the objective J from (5.29) implies

$$\begin{aligned} J(\omega,\varepsilon,x_0,\alpha) &= J_0 + \varepsilon\tfrac{|\partial\omega|}{2\pi}\big\{\operatorname{Re}(\alpha)\operatorname{Re}(J_1^D(x_0)) - \operatorname{Im}(\alpha)\operatorname{Im}(J_1^D(x_0))\big\} \\ &+ \mathrm{O}((1+|\alpha|+|\alpha|^2)\varepsilon^2), \quad \text{where } |\alpha| = \sqrt{|\operatorname{Re}(\alpha)|^2 + |\operatorname{Im}(\alpha)|^2}. \end{aligned} \tag{5.36}$$

The general α-dependent representation (5.36) turns into the particular one

$$\begin{aligned} J(\omega,\varepsilon,x_0,\alpha) = J_0 + \tfrac{1}{-\ln\varepsilon}\operatorname{Re}(J_1^D(x_0)) - \varepsilon\tfrac{|\partial\omega|}{2\pi}\operatorname{Im}(\alpha)\operatorname{Im}(J_1^D(x_0)) \\ + \mathrm{O}\big(\tfrac{1}{|\ln\varepsilon|^2} + (|\operatorname{Im}(\alpha)| + |\operatorname{Im}(\alpha)|^2)\varepsilon^2\big), \end{aligned} \tag{5.37a}$$

$$\text{when} \quad \operatorname{Re}(\alpha) = \tfrac{1}{\varepsilon(-\ln\varepsilon)}\tfrac{2\pi}{|\partial\omega|} + \mathrm{O}(\tfrac{1}{\varepsilon|\ln\varepsilon|^2}). \tag{5.37b}$$

Due to (4.57), formula (5.37a) coincides with the low-order asymptotic terms of expansion (4.54) from Theorem 4.8 for the Dirichlet problem, when $|\operatorname{Im}(\alpha)| = \mathrm{O}(\frac{1}{\varepsilon|\ln\varepsilon|^2})$. Nevertheless, for fixed ε and variable α it differs.

Indeed, for the test parameters $(\omega, \varepsilon, x_0) = (\omega^\star, \varepsilon^\star, x^\star)$ and $\alpha = \alpha^\star$, the optimal objective value $J(\omega^\star, \varepsilon^\star, x^\star, \alpha^\star) = 0$ in (5.24). In particular, when $|\alpha^\star| \nearrow \infty$, too. The finite (zero) limit in (5.37a) when $|\mathrm{Im}(\alpha^\star)| \nearrow \infty$ (hence $|\alpha^\star| \nearrow \infty$) and arbitrary $\varepsilon^\star$ can be preserved only if its complement $\mathrm{Im}(J_1^D(x^\star))$ is zero.

We note that this argument holds true also for finite optima $0 \neq J(\omega^\star, \varepsilon^\star, x^\star, \alpha^\star) < \infty$ when the test parameters are infeasible.

Thus, recalling the form (4.55a) of J_1^D we have proved the following main result.

Theorem 5.6. *For arbitrary geometric parameters* $(\omega^\star, \varepsilon^\star, x^\star) \in \mathfrak{G}$ *of the test object under the Dirichlet boundary condition, the necessary optimality condition of* (4.51)

$$\mathrm{Im}\big(u^0(x^\star)\overline{v^0}(x^\star)\big) = 0 \tag{5.38}$$

is expressed by the primal and dual background solutions u^0 *and* v^0 *to* (2.2) *and* (3.54).

Based on Theorem 5.6, in [43] the imaging function is introduced

$$f : \{u^\star \in \mathfrak{G}_u\} \mapsto C(\Omega; \mathbb{R}), \quad f_{u^\star}(x) := \mathrm{Im}\big(u^0(x)\overline{v^0}(x)\big) \tag{5.39}$$

over a set of feasible boundary measurements $u^\star \in \mathfrak{G}_u \subset L^2(\Gamma_N; \mathbb{C})$. Thanks to (5.38) its zero-level set contains the test center $x^\star$:

$$x^\star \in L_{=0}(f_{u^\star}) := \{x \in \Omega : \quad f_{u^\star}(x) = 0\}. \tag{5.40}$$

The formalism (5.39) and (5.40) guarantees a high-precision numerical solution of the inverse problem of identification of the center of an unknown test object from boundary measurements. In [43] the following numerical findings are reported in detail:

- Implementation of the imaging function $f_{u^\star}$ in (5.39) has low computational costs since it needs to solve the background Helmholtz problems in the fixed domain.
- The center $x^\star$ from (5.40) can be detected as the intersection point of zero-level sets
$$x^\star = \textstyle\bigcap_{i=1}^d L_{=0}(f_{u_i^\star}) \tag{5.41}$$
from two and three different measurements $u_i^\star$ in 2d and 3d, respectively.
- For low wave numbers k, formula (5.41) holds in arbitrary spatial dimensions $d \in \{1, 2, 3\}$ and for arbitrary feasible test objects $\omega^\star_{\varepsilon^\star}(x^\star) \subset \Omega$.
- The numerical identification result by (5.41) is exact when $x^\star$ coincides with a computational mesh node, it is highly stable to discretization and noise errors.
- Although Theorem 5.6 is stated for the Dirichlet problem, (5.39) and (5.41) are applicable numerically as well to finite $\alpha^\star \in \mathbb{R}$ for the Robin problem, in particular, to $\alpha^\star = 0$ for the Neumann problem.

Acknowledgments. The author thanks the organizers, especially K. Kunisch and M. Rumpf, for the support during Workshop 1 of the Special Semester on "New Trends in Calculus of Variations" held at the RICAM (Linz, Austria), October 13-17, 2014.

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Author information

Victor A. Kovtunenko, Institute for Mathematics and Scientific Computing, Karl-Franzens University of Graz, NAWI Graz, Heinrichstraße 36, 8010 Graz, Austria; and
Lavrent'ev Institute of Hydrodynamics, Sibirian Branch of the Russian Academy of Sciences, 630090 Novosibirsk, Russia.
E-mail: `victor.kovtunenko@uni-graz.at`