State-constrained optimization for identification of small inclusions

Victor A. Kovtunenko[1,2,*]

[1] Institute for Mathematics and Scientific Computing, Karl-Franzens University of Graz, Heinrichstr. 36, 8010 Graz, Austria
[2] Lavrent'ev Institute of Hydrodynamics, Siberian Branch of the Russian Academy of Sciences, 630090 Novosibirsk, Russia

The inverse problem of identification of small geometric objects (defects, inclusions) of unknown topological properties is under the investigation. This problem is treated within the state-constrained optimization framework. Using topological sensitivity analysis and methods of singular perturbations, a proper approximation by the asymptotic model is justified rigorously. The underlying parametric optimization problem is solved semi-analytically by a variational calculus.

The field of inverse problems for identification of unknown inclusions has a long hystory of numerous engineering applications in the context of non-destructive testing. From the mathematical viewpoint, the asymptotic analysis due to singular perturbations in the direct problem is useful to restore geometric and physical parameters of a test inclusion; see [4, 6]. Newly, the concept of topological derivative was adapted in this field, e.g., [1, 3]. Further, applying variational methods we avoid the restrictions on geometric properties of the test inclusions. Our approach is based on the variational principles for singular problems [7] and its perturbation analysis [2, 5, 8–11].

Let $\omega \subset B_1(0) \subset \boldsymbol{R}^2$ (where $B_1(0)$ is the unit disk centered at 0) be a generic geometric object with the piecewise Lipschitz boundary $\partial\omega$, thus allowing cracks, multi-junks and alike. Rescaling with the 'size' $\varepsilon > 0$ it produces small inclusions $\omega_\varepsilon(x_0) = \{x \in \boldsymbol{R}^2 : \frac{x-x_0}{\varepsilon} \in \omega\} \subset \Omega$ over trial points $x_0 \in \operatorname{int}\Omega$ in the reference domain $\Omega \subset \boldsymbol{R}^2$ with the Lipschitz boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$. For the test inclusion $\omega^\star_{\varepsilon^\star}(x^\star) \subset \Omega$ associated the sound hard obstacle, we consider the model described by the scalar Helmholtz equation:

$$-(\Delta + k^2)u^\star = 0 \text{ in } \Omega \setminus \overline{\omega^\star_{\varepsilon^\star}(x^\star)}, \quad \frac{\partial u^\star}{\partial n} = 0 \text{ on } \partial\omega^\star_{\varepsilon^\star}(x^\star), \quad u^\star = 0 \text{ on } \Gamma_D, \quad \frac{\partial u^\star}{\partial n} = g \text{ on } \Gamma_N.$$

Here $g \in L^2(\Gamma_N)$ and $k \in \boldsymbol{R}$ are given, and n will denote the normal vector defined almost everywhere at the boundary and outward to the domain.

Since $\omega^\star_{\varepsilon^\star}(x^\star)$ is unknown a-priori, the measurement of the true solution $u^\star$ at Γ_N provides a proper misfit function to test obstacles $\omega_\varepsilon(x_0)$ posed trial in Ω. Thus we arrive at the state-constrained optimization problem:

$$\begin{aligned} &\text{minimize } J(u^\varepsilon, \omega_\varepsilon(x_0)) := \frac{1}{2}\int_{\Gamma_N} |u^\varepsilon - u^\star|^2 \, dS_x \text{ over admissible } \omega,\, \varepsilon,\, x_0 \text{ subject to the variational problem:} \\ &\text{find } u^\varepsilon \text{ such that } \int_{\Omega\setminus\overline{\omega_\varepsilon(x_0)}} (\nabla u^\varepsilon \cdot \nabla v - k^2 u^\varepsilon v)\, dx = \int_{\Gamma_N} gv \, dS_x \text{ for all } v \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)})\text{: } v = 0 \text{ on } \Gamma_D. \end{aligned} \tag{1}$$

Further we assume that the infsup-condition hold for the bilinear form $\int(\nabla u \cdot \nabla v - k^2 uv)dx$ and is uniform over all $\Omega \setminus \overline{\omega_\varepsilon(x_0)}$.

With the help of the Fenchel–Legendre duality we can rewrite (1) equivalently as

$$\begin{aligned} &\text{minimize } L(u^\varepsilon, v^\varepsilon, \omega_\varepsilon(x_0)) := J(u^\varepsilon, \omega_\varepsilon(x_0)) + \int_{\Omega\setminus\overline{\omega_\varepsilon(x_0)}} (\nabla u^\varepsilon \cdot \nabla v^\varepsilon - k^2 u^\varepsilon v^\varepsilon)\, dx - \int_{\Gamma_N} g v^\varepsilon \, dS_x \\ &\text{over admissible } \omega,\, \varepsilon,\, x_0 \text{ and } u^\varepsilon, v^\varepsilon \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)})\text{: } u^\varepsilon = v^\varepsilon = 0 \text{ on } \Gamma_D. \end{aligned} \tag{2}$$

The 1st order optimality condition for (2) implies the direct primal and adjoint variational problems:

$$\begin{aligned} &\text{find } u^\varepsilon \text{ such that } \int_{\Omega\setminus\overline{\omega_\varepsilon(x_0)}} (\nabla u^\varepsilon \cdot \nabla v - k^2 u^\varepsilon v)\, dx = \int_{\Gamma_N} gv \, dS_x, \text{ for all } v \in H^1(\Omega \setminus \overline{\omega_\varepsilon(x_0)})\text{: } v = 0 \text{ on } \Gamma_D, \\ &\text{and } v^\varepsilon \text{ such that } \int_{\Omega\setminus\overline{\omega_\varepsilon(x_0)}} (\nabla v^\varepsilon \cdot \nabla v - k^2 v^\varepsilon v)\, dx = -\int_{\Gamma_N} (u^\varepsilon - u^\star) v \, dS_x. \end{aligned} \tag{3}$$

The unique solvability of (3) follows from the infsup condition. Next we apply the asymptotic arguments for small $\varepsilon \to 0$.

Passing $\varepsilon \to 0$ in (3), the limit problems in the reference domain Ω read:

$$\begin{aligned} &\text{find } u^0 \text{ such that } \int_{\Omega} (\nabla u^0 \cdot \nabla v - k^2 u^0 v)\, dx = \int_{\Gamma_N} gv \, dS_x, \text{ for all } v \in H^1(\Omega)\text{: } v = 0 \text{ on } \Gamma_D, \\ &\text{and } v^0 \text{ such that } \int_{\Omega} (\nabla v^0 \cdot \nabla v - k^2 v^0 v)\, dx = -\int_{\Gamma_N} (u^0 - u^\star) v \, dS_x. \end{aligned} \tag{4}$$

* Corresponding author: Email victor.kovtunenko@uni-graz.at, phone +43 316 380 8545, fax +43 316 380 9815

Using (4) and the 2nd Green formula we decompose the optimal value function J from (1) in small ε as

$$J(u^\varepsilon,\omega_\varepsilon(x_0)) = \underbrace{J(u^0)}_{O(1)} + \underbrace{I_C}_{O(\varepsilon^2)} + \underbrace{\frac{1}{2}\int_{\Gamma_N}(u^\varepsilon-u^0)^2\,dS_x}_{O(\varepsilon^4|\ln\varepsilon|)},\quad I_C := \int_C\Big(\frac{\partial v^0}{\partial n}(u^\varepsilon-u^0) - \frac{\partial(u^\varepsilon-u^0)}{\partial n}v^0\Big)\,dS_x,$$

where integral I_C is invariant over any contour $C\subset\Omega\setminus\omega_\varepsilon(x_0)$ and admits the expansion $I_C=\varepsilon^2T(\omega,x_0)+O(\varepsilon^3\sqrt{|\ln\varepsilon|})$. Note that the 1st asymptotic term $T(\omega,x_0)$ implies the *topological derivative* of J with respect to the diminishing inclusion $\omega_\varepsilon(x_0)\to\{x_0\}$. We set $C=\partial B_\varepsilon(x_0)$ to be the circle of the radius ε from the center x_0 and calculate $T(\omega,x_0)$ from I_C.

For this reason, in the disk $B_\delta(x_0)$ of the radius δ we expand u^0 and v^0 in the Fourier series with the Bessel functions J_n:

$$\begin{aligned} u^0(x) &= u^0(x_0)J_0(k|x-x_0|)+2J_1(k|x-x_0|)\nabla u^0(x_0)\cdot\tfrac{x-x_0}{|x-x_0|}+U(x), &\int_{-\pi}^{\pi}U\,d\theta=\int_{-\pi}^{\pi}U\tfrac{x-x_0}{|x-x_0|}\,d\theta=0,\\ v^0(x) &= v^0(x_0)J_0(k|x-x_0|)+2J_1(k|x-x_0|)\nabla v^0(x_0)\cdot\tfrac{x-x_0}{|x-x_0|}+V(x), &\int_{-\pi}^{\pi}V\,d\theta=\int_{-\pi}^{\pi}V\tfrac{x-x_0}{|x-x_0|}\,d\theta=0, \end{aligned}\tag{5}$$

where $\theta\in[-\pi,\pi]$ is the polar angle to $x-x_0$. The perturbed solution u^ε admits the expansion in $\Omega\setminus\overline{\omega_\varepsilon(x_0)}$ as

$$u^\varepsilon(x)=u^0(x)-\varepsilon\nabla u^0(x_0)\cdot w(\tfrac{x-x_0}{\varepsilon})+Q(x),\quad \|Q\|_{H^1(\Omega\setminus\overline{\omega_\varepsilon(x_0)})}=O(\varepsilon^2\sqrt{|\ln\varepsilon|})\tag{6}$$

with the solution $w(y)\in[H^1_\mu(\boldsymbol{R}^2\setminus\overline{\omega})\setminus\boldsymbol{P}_0]^2$ of the limit Neumann problem in the exterior of $\overline{\omega}$:

$$\begin{aligned} &\int_{\boldsymbol{R}^2\setminus\overline{\omega}}\nabla w\cdot\nabla v\,dy=-\int_{\partial\omega}nv\,dS_y \text{ for all } v\in H^1_\mu(\boldsymbol{R}^2\setminus\overline{\omega}),\ \text{in the weighted Sobolev space}\\ &H^1_\mu(\boldsymbol{R}^2\setminus\overline{\omega}):=\{\tfrac{1}{\mu}v,\nabla v\in L^2(\boldsymbol{R}^2\setminus\overline{\omega})\}\text{ with the weight }\mu(y)\sim|y|\ln|y|\text{ in }\boldsymbol{R}^2\setminus\overline{B_1(0)}. \end{aligned}\tag{7}$$

It yields the Fourier series $w(y)=A_\omega\frac{y}{|y|^2}+O(\frac{1}{|y|^2})$ in $\boldsymbol{R}^2\setminus\overline{B_1(0)}$ with the virtual mass matrix $A_\omega\in\mathrm{spsd}(\boldsymbol{R}^{2\times2})$.

Substituting (5)–(7) into I_C we calculate $T(\omega,x_0)$ and derive the asymptotic model of the optimization problem (1):

$$\text{minimize } T(\omega,x_0)=-2\pi\nabla u^0(x_0)^\top A_\omega\nabla v^0(x_0)+k^2\mathrm{meas}_2(\omega)u^0(x_0)v^0(x_0)\text{ over }A_\omega\in\mathrm{spsd}(\boldsymbol{R}^{2\times2}),\,x_0\in\Omega.\tag{8}$$

Admissible obstacles ω in the approximation (8) have only 3 degrees of freedom and can be represented by equivalent ellipses

$$\omega=\{y:(\tfrac{y_1\cos\phi+y_2\sin\phi}{a})^2+(\tfrac{y_1\sin\phi-y_2\cos\phi}{b})^2\le1\},\ A_\omega=\tfrac{a+b}{2}R\begin{pmatrix}b&0\\0&a\end{pmatrix}R^\top,\ R=\begin{pmatrix}\cos\phi&-\sin\phi\\\sin\phi&\cos\phi\end{pmatrix}\tag{9}$$

with $\mathrm{meas}_2(\omega)=\pi ab$. Note that disks occur when $a=b$, and cracks occur when $b=0$. Firstly, for every trial point x_0 in Ω we solve (8) and (9) with respect to unknown $a\ge b\ge0$, $\phi\in(-\frac{\pi}{2},\frac{\pi}{2})$ to find the optimal ellipse $\omega(x_0)$. For this reason, we determine the eigenvalues $\mu_1(x_0)\le\mu_2(x_0)$ of the symmetric matrix $M(x_0):=\mathrm{sym}(\nabla u^0(x_0)\nabla v^0(x_0)^\top)\in\boldsymbol{R}^{2\times2}$, and

$$\text{if }\mu_2(x_0)\le0\text{ then }a(x_0)=b(x_0)=0\text{ and no inclusion, else }\tfrac{b(x_0)}{a(x_0)}=\tfrac{\max\{0,\mu_1(x_0)\}}{\mu_2(x_0)},\ \tan\phi(x_0)=\tfrac{\mu_1(x_0)-M_{11}(x_0)}{M_{12}(x_0)}.$$

Second, we fulfill the minimization with the ellipses $\omega(x_0)$ over all trial points $x_0\in\Omega$ with $\mu_2(x_0)>0$:

$$\text{minimize } T(\omega(x_0),x_0)=-\max\{0,\mu_1(x_0)\}^2-\mu_2(x_0)^2+\tfrac{\mu_2(x_0)\cdot\max\{0,\mu_1(x_0)\}}{\mu_2(x_0)+\max\{0,\mu_1(x_0)\}}\,k^2u^0(x_0)v^0(x_0),$$

thus finding the solution of (8): the optimal point $x_0^\star$, the optimal ratio $\frac{b(x_0^\star)}{a(x_0^\star)}$, and the optimal direction $\phi(x_0^\star)$, but no size ε of the optimal ellipse $\omega_\varepsilon(x_0^\star)$ can be determined in this approximate model.

Acknowledgements The research is supported by the Austrian Science Fund (FWF P21411-N13), Russian Foundation for Basic Research (project 10-01-00054) and Siberian Branch of Russian Academy of Sciences (project N 90). We thank E. Kostina for her kind invitation.

References

[1] S. Amstutz and N. Dominquez, eng. anal. boundary elem. **32**, 936–947 (2008).
[2] I. I. Argatov and V. A. Kovtunenko, 2009 ann. bull. australian inst. high energetic materials **1**, 124–130 (2010).
[3] G. R. Feijoo, inverse problems **20**, 1819–1840 (2004).
[4] A. Friedman and M. Vogelius, arch. ration. mech. anal. **105**, 299–326 (1989).
[5] M. Hintermüller and V. A. Kovtunenko, optimiz. meth. software, to appear.
[6] A. M. Il'in, Matching of Asymptotic Expansions of Solutions of Boundary Value Problems (AMS, Providence, RI, 1992).
[7] A. M. Khludnev and V. A. Kovtunenko, Analysis of Cracks in Solids (WIT-Press, Southampton, Boston, 2000).
[8] A. M. Khludnev, V. A. Kovtunenko, and A. Tani, j. math. soc. japan **60**, 1219–1253 (2008).
[9] A. M. Khludnev, V. A. Kovtunenko, and A. Tani, j. math. pures appl. **94**, 571–596 (2010).
[10] V. A. Kovtunenko and K. Kunisch, zamm **87**, 809–830 (2007).
[11] V. A. Kovtunenko, K. Kunisch, and W. Ring, comput. visual sci. **12**, 397–408 (2009).

