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Optimization in constrained crack problems

Victor A. Kovtunenko[1,2,*] and **Alexander M. Khludnev**[2,**]

[1] Institute for Mathematics and Scientific Computing, Karl-Franzens University of Graz, Heinrichstr. 36, 8010 Graz, Austria.
[2] Lavrent'ev Institute of Hydrodynamics, Siberian Branch of the Russian Academy of Sciences, 630090 Novosibirsk, Russia.

Following the optimization approach to brittle fracture, we consider the evolution of a crack in a domain as solution of the global problem of constrained minimization of the total potential energy with respect to both state variables (the displacement vector) and shape variables (parameters of the crack shape). This formulation describes the initiation of a crack in a homogeneous body and its stable as well as unstable propagation, which allows also a kink (topology change) of the crack during the evolution process. By this we assume that contact can occurs between the opposite crack faces.

1 Introduction

To investigate well-posedness of the optimization problem for cracks, as necessary tools we consider the following:

- Proper kinematic description of the crack evolution is given a-priori by vector of the velocity field, and, conversely, the velocity field can be reconstructed from the given evolution.
- Modeling of equilibrium of solids with cracks in the variational framework allows to account for various physical effects of interaction between the crack faces.
- Shape sensitivity analysis via regular perturbations is applied to investigate continuity and differentiability properties of the reduced energy (cost) function.
- The global optimization is realized in on the set of extremal points of the cost function by using its regularity properties.
- By numerical implementation, a primal-dual active-set strategy is proposed for efficient computing of constrained crack problems.

Solving numerically the optimization problem we present examples of crack propagation also in such situations where the classic Griffith fracture hypothesis is not applicable.

2 Problem formulation

In the framework of optimization approach to brittle fracture [1], we consider evolution in time-parameter t of crack Γ_C in domain $\Omega \subset \mathbf{R}^N$, $N = 2, 3$, as solution of the global minimization problem:

Find $\Gamma_C(t) \in \Sigma(\Omega)$ for $t \geq 0$, such that

$$u(t) \in K(\Omega \setminus \Gamma_C(t)), \quad T(u(t); \Omega \setminus \Gamma_C(t)) \leq T(v; \Omega \setminus \Gamma_C(t)) \quad \text{for all } v \in K(\Omega \setminus \Gamma_C(t));$$

$$\Gamma_C(t) \supset \bigcup_{s<t} \Gamma_C(s), \quad T(u(t); \Omega \setminus \Gamma_C(t)) \leq T(u(\Gamma_C); \Omega \setminus \Gamma_C) \quad \text{for all } \Gamma_C \supset \bigcup_{s<t} \Gamma_C(s),$$

$$\text{where } u(\Gamma_C) \in K(\Omega \setminus \Gamma_C), \quad T(u(\Gamma_C); \Omega \setminus \Gamma_C) \leq T(v; \Omega \setminus \Gamma_C) \quad \text{for all } v \in K(\Omega \setminus \Gamma_C).$$

Here $u \mapsto T(u; \Omega \setminus \Gamma_C) : K(\Omega \setminus \Gamma_C) \mapsto \mathbf{R}$ is a cost functional of the total potential energy of the solid with crack, occupying the domain $\Omega \setminus \Gamma_C$, over the set of admissible displacements $u \in K(\Omega \setminus \Gamma_C)$, and $\Sigma(\Omega)$ is the set of admissible crack paths in Ω.

The problem deals with finding optimal crack shape $\Gamma_C(t)$ presented by geometrical or topological parameters of a domain with unknown crack. For the pre-defined crack path along the curve $\Sigma \in \Sigma(\Omega)$ in $\mathbf{R}^2$, respective one-parametric optimization problem is well posed with respect to the length parameter of the crack $\Gamma_C(t)$. It describes appearance and quasi-static propagation of the crack, for instance, during the delamination process [8]. A two-parametric optimization problem stated with respect to unknown parameters of the crack length and the angle of its kink is stated in [5]. The brittle approach is generalized for quasi-brittle fracture in [10].

* Corresponding author E-mail: victor.kovtunenko@uni-graz.at, Phone: +43 316 380 5160, Fax: +43 316 380 9815
** E-mail: khlud@hydro.nsc.ru

3 Problem ingredients

The necessary ingredients of the optimization setting include kinematic description of cracks as codimensional-1 open manifolds. The crack evolution with given a-priori vector of the velocity field $V \in C([0,L]; W^{1,\infty}(\Omega))^N$ can be managed in two ways [9]: constructing homeomorphic maps $y = \Phi(t,x)$ between domains with cracks as solution of the nonlinear ordinary differential equation

$$\frac{d\Phi}{dt}(t) = V(t, \Phi(t)), \quad \Phi(0) = x,$$

or by implicit surfaces $\Gamma_C(t) = \{y : \rho(t,y) = 0\}$, where ρ solves the linear transport equation

$$\frac{\partial \rho}{\partial t}(t,y) + V(t,y) \cdot \nabla \rho(t,y) = 0, \quad \rho(0) = \rho_0.$$

A suitable modeling of equilibrium of solids with cracks is given in [4] in framework of the variational formulation, accounting conditions of contact between the crack faces in the admissible set $K(\Omega \setminus \Gamma_C)$, which results in constrained crack problems: Find $u \in K(\Omega \setminus \Gamma_C)$, such that

$$T(u; \Omega \setminus \Gamma_C) \leq T(v; \Omega \setminus \Gamma_C) \text{ for all } v \in K(\Omega \setminus \Gamma_C).$$

The nano-level interaction (cohesion) effects during contact of the crack faces is described within this formulation in [10].

To find the global minimum with respect to Γ_C, the optimization algorithm is realized in a constructive way on the set of extremal points of T by the path-following method, which uses the differentiability properties of the reduced cost functional $\Gamma_C \mapsto T(u(\Gamma_C); \Omega \setminus \Gamma_C)$. This approach requires sensitivities of the crack problems with respect to perturbations of shape parameters, in particular, it employs the shape derivative (the energy release rate at the crack tip).

The shape sensitivity technique [6] is applied to describe kinematic characteristics of the crack problems due to regular perturbations of parameters of the crack shape. Developing this approach allows us to find the energy release rate for curvilinear cracks [7] as well as for kinking cracks [5] by means of directional derivatives $-T'_V$ with respect to velocity vector field V.

For numerical implementation, based on the generalized differentiability property, a semi-smooth Newton method proposed in the form of primal-dual active-set strategy (PDAS-method) is used in [2] as the efficient numerical technique for solution of the constrained minimization problems, in particular with cracks, due to the property of its unconditional global and, moreover, monotone convergence.

Numerical examples presenting quasi-static propagation of cracks [3, 8], which is global in time, demonstrate advantages of our optimization approach as refinement of the classic Griffith fracture hypothesis for initiation of a crack, unstable crack propagation, kink of the crack path, and when contact occurs between the crack faces.

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