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ITERATIVE APPROXIMATIONS OF PENALTY OPERATORS

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Abstract

A well known approach to approximate a variational inequality consists of using a penalty operator (nonlinear in general) [3, 8]. On the other hand, it is sometimes possible to use iterative approaches [1, 4, 9]. In this work an iterative equation with linear penalty operator associated with a variational inequality is constructed. The convergence of the solutions and the error estimates are proved.

Further, primary iterative procedures based on these results are proposed to find approximate solutions of variational inequalities. Estimates of the error and the iteration numbers are obtained.

These inverstigations were suggested by a study of the contact and plastic problems in solid mechanics [5]. With the described methods, approximate solutions of the contact elastoplastic problems for a plate are obtained [6,7].

1. MAIN RESULTS ON THE CONVERGENCE

Let V be a Hilbert space and V^* be its dual space. Denote by $\langle\cdot,\cdot\rangle$, $(\cdot,\cdot)$, $\|\cdot\|$ and $\|\cdot\|_*$ the duality between V and V^*, scalar product in V, norms in V and V^*, respectively. Let us introduce the dual injection $J : V \to V^*$ meaning

$$\langle Ju, v\rangle = (u, v) \quad \text{for all} \quad u, v \in V$$

and the inverse dual injection $J^{-1} : V^* \to V$. Then we have that J and J^{-1} are linear operators in the Hilbert space V [2].

K is taken to be a closed convex subset of V. The element $f \in V^*$ and operator $A : V \to V^*$ are given. We require that

(i) A be hemicontinuous,

(ii) A be strongly monotonous, i.e.

$$\langle Au - Av, u - v\rangle \geq M\|u - v\|^2 \quad \text{for all} \quad u, v \in V, \quad \text{constant} \quad M > 0. \tag{1}$$

The following variational inequality is investigated. To find $u \in K$ such that

$$\langle Au, v - u\rangle \geq \langle f, v - u\rangle \quad \text{for any} \quad v \in K. \tag{2}$$

Because of (i) and (ii), there exists a unique solution u of (2) (see [8]).

Let $P : V \to K$ be the projection operator defined by

$$\|v - Pv\| \le \|v - w\| \quad \text{for any} \quad w \in K.$$

Then P is Lipschitz continious [9], i.e.

$$\|Pv - Pw\| \le \|v - w\| \quad \text{for all} \quad v, w \in V. \tag{3}$$

Let us construct the standard penalty operator $\beta(v) = J(v - Pv)$ and define the penalty problem depending on a small positive parameter ϵ

$$Au^\epsilon + \epsilon^{-1}\beta(u^\epsilon) = f. \tag{4}$$

It is a well known result [8] that equation (4) has a unique solution $u^\epsilon \in V$ which satisfies

$$u^\epsilon \to u \quad \text{weakly in} \quad V \quad \text{as} \quad \epsilon \to 0.$$

To linearize the penalty operator in (4) we use the following iteration scheme

$$Au^{\epsilon,n} + \epsilon^{-1}Ju^{\epsilon,n} = f + \epsilon^{-1}JPu^{\epsilon,n-1}, \quad n = 1, 2, 3, ... \tag{5}$$

where $u^{\epsilon,0} \in V$ is choosen in an arbitrary way. It follows immediately from the Browder's theorem that there exists a unique solution $u^{\epsilon,n} \in V$ of (5).

Lemma 1 *The following estimates hold*

$$\|u^{\epsilon,n} - u\|^2 \le \rho_\epsilon^n \|u^{\epsilon,0} - u\|^2 + \delta_\epsilon(1 - \rho_\epsilon^n)\|f - Au\|_*^2 \quad , \tag{6}$$

$$\|u^{\epsilon,n} - u^\epsilon\|^2 \le \rho_\epsilon^n \|u^{\epsilon,0} - u^\epsilon\|^2 \quad , \tag{7}$$

where $\rho_\epsilon = (1 + M\epsilon)^{-2} < 1, \quad \delta_\epsilon = \epsilon M^{-1}(2 + M\epsilon)^{-1}.$

Proof. Let us rewrite (5) adding $(-Au - \epsilon^{-1}Ju)$ to both parts. Since J is linear, we have

$$Au^{\epsilon,n} - Au + \epsilon^{-1}J(u^{\epsilon,n} - u) = f - Au + \epsilon^{-1}J(Pu^{\epsilon,n-1} - Pu).$$

Here, $u = Pu$ due to $u \in K$. Application of the linear injection J^{-1} to this equation gives

$$J^{-1}(Au^{\epsilon,n} - Au) + \epsilon^{-1}(u^{\epsilon,n} - u) = J^{-1}(f - Au) + \epsilon^{-1}(Pu^{\epsilon,n-1} - Pu).$$

Squaring the above equality, we get

$$\|Au^{\epsilon,n} - Au\|_*^2 + 2\epsilon^{-1}\langle Au^{\epsilon,n} - Au, u^{\epsilon,n} - u\rangle + \epsilon^{-2}\|u^{\epsilon,n} - u\|^2 = \tag{8}$$

$$\|f - Au\|_*^2 + 2\epsilon^{-1}\langle f - Au, Pu^{\epsilon,n-1} - u\rangle + \epsilon^{-2}\|Pu^{\epsilon,n-1} - Pu\|^2.$$

According to (1), the left part of (8) is bounded from below by

$$(M + \epsilon^{-1})^2\|u^{\epsilon,n} - u\|^2.$$

The second term in the right hand side of (8) is negative due to (2). Further, using the inequality (3), we get

$$\|u^{\epsilon,n} - u\|^2 \le \rho_\epsilon \left(\|u^{\epsilon,n-1} - u\|^2 + \epsilon^2\|f - Au\|_*^2\right)$$

Continuing this estimate as n goes to 1, we have

$$\|u^{\epsilon,n} - u\|^2 \le \rho_\epsilon^n \left(\|u^{\epsilon,0} - u\|^2 + \epsilon^2 \sum_{i=0}^{n-1} \rho_\epsilon^i \ \|f - Au\|_*^2\right). \tag{9}$$

With the sum of the geometrical series

$$\sum_{i=0}^{n-1} \rho_\epsilon^i = (1-\rho_\epsilon^n)(1-\rho_\epsilon)^{-1},$$

the estimate (9) gives the estimate (6).

Let us next subtract (4) from (5), then

$$Au^{\epsilon,n} - Au^\epsilon + \epsilon^{-1}J(u^{\epsilon,n} - u^\epsilon) = \epsilon^{-1}J(Pu^{\epsilon,n-1} - Pu^\epsilon).$$

Doing similarly the above (to apply J^{-1} and square) gives

$$\|u^{\epsilon,n} - u^\epsilon\|^2 \leq (1+M\epsilon)^{-2}\|u^{\epsilon,n-1} - u^\epsilon\|^2.$$

This inequality reduces to the estimate (7) and completes the proof.

It follows immediately from this Lemma that

Theorem 1

$$u^{\epsilon,n} \to u^\epsilon \quad \textit{strongly in} \quad V \quad \textit{as} \quad n \to \infty, \ \epsilon \textit{ is fixed} \quad \textit{and (7) holds},$$

$$u^\epsilon \to u \quad \textit{strongly in} \quad V \quad \textit{as} \quad \epsilon \to 0 \quad \textit{and}$$

$$\|u^\epsilon - u\|^2 \leq \delta_\epsilon \|f - Au\|_*^2 \quad ,$$

where $u^{\epsilon,n}$, u^ϵ and u are the solutions of (5),(4) and (2), respectively.

2. CONSTRUCTION OF APPROXIMATING SCHEMES

Let $u^0 \in V$ be a solution of $Au^0 = f$. Hence, (1) gives

$$\|u^0 - u\| \leq M^{-1}\|f - Au\|_*. \tag{10}$$

Choose an arbitrary positive sequence h_ϵ such that

$$h_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0.$$

1. Scheme. Assume that ϵ is fixed. We define

$$v^\epsilon = u^{\epsilon,n_\epsilon},$$

where u^{ϵ,n_ϵ} is the solution of (5) under the following conditions

$$u^{\epsilon,0} = u^0,$$

$$n_\epsilon = \left[\ln\left(h_\epsilon(M^{-2} - \delta_\epsilon)^{-1}\right) / \ln \rho_\epsilon\right] + 1. \tag{11}$$

Here the bracket $[x]$ denotes the integer part of the number x.

Theorem 2 *$v^\epsilon \to u$ strongly in V as $\epsilon \to 0$ and*

$$\|v^\epsilon - u\|^2 \leq (h_\epsilon + \delta_\epsilon)\|f - Au\|_*^2.$$

In order to prove this, it is sufficient to deduce from (11) that

$$\rho_\epsilon^{n_\epsilon} \leq h_\epsilon(M^{-2} - \delta_\epsilon)^{-1}.$$

If equations (10) and (11) are taken into account, the formula (6) gives the desirable result.

2. Scheme. Another way of finding approximate solutions of (2) consists in solving (5) as ϵ decreases. Choose a sequence ϵ_k, $k = 1, 2, 3, \ldots$ such that $\epsilon_k \to 0$ as $k \to \infty$. For simplicity, let us replace $\delta_{\epsilon_k}, \rho_{\epsilon_k}, h_{\epsilon_k}$ by δ_k, ρ_k, h_k . We define that

$$u^k = u^{\epsilon_k, n_k}, \quad k = 1, 2, 3, ...$$

is the solution of (5) with the following conditions

$$u^{\epsilon_k, 0} = u^{k-1},$$

$$n_k = \left[\ln\left(h_k(h_{k-1} + \delta_{k-1} - \delta_k)^{-1}\right) / \ln \rho_k\right] + 1. \tag{12}$$

Here,

$$h_0 + \delta_0 \geq M^{-2} \tag{13}$$

Theorem 3 *$u^k \to u$ strongly in V as $k \to \infty$ and*

$$\|u^k - u\|^2 \leq (h_k + \delta_k)\|f - Au\|_*^2. \tag{14}$$

Proof. Clearly, equation (12) gives

$$\rho_k^{n_k} \leq h_k(h_{k-1} + \delta_{k-1} - \delta_k)^{-1} \tag{15}$$

For $k = 1$, due to (6),(10),(13) and (15), we have

$$\|u^1 - u\|^2 \leq \rho_1^{n_1}\|u^0 - u\|^2 + \delta_1(1 - \rho_1^{n_1})\|f - Au\|_*^2 \leq$$
$$\left(\rho_1^{n_1}(M^{-2} - \delta_1) + \delta_1\right)\|f - Au\|_*^2 \leq$$
$$\left(h_1(M^{-2} - \delta_1)(h_0 + \delta_0 - \delta_1)^{-1} + \delta_1\right)\|f - Au\|_*^2 \leq$$
$$\left(h_1(M^{-2} - \delta_1)(M^{-2} - \delta_1)^{-1} + \delta_1\right)\|f - Au\|_*^2.$$

Hence, equation (14) holds for $k = 1$. Let (14) hold for $k - 1$, i.e.

$$\|u^{k-1} - u\|^2 \leq (h_{k-1} + \delta_{k-1})\|f - Au\|_*^2. \tag{16}$$

Then we prove (14) by induction. The estimate (6) and assumption (16) give

$$\|u^k - u\|^2 \leq \rho_k^{n_k}\|u^{k-1} - u\|^2 + \delta_k(1 - \rho_k^{n_k})\|f - Au\|_*^2 \leq$$
$$\left(\rho_k^n(h_{k-1} + \delta_{k-1} - \delta_k)^{-1} + \delta_k\right)\|f - Au\|_*^2.$$

Taking into account (15), this completes the proof.

3. Scheme. We are next going to pass to the limit as $k, n \to \infty$ simultaneously. Let $v^k = u^k$ for $n_k = 1$, i.e. v^k be a solution of

$$Av^k + \epsilon_k^{-1}Jv^k = f + \epsilon_k^{-1}JPv^{k-1}, \quad k = 1, 2, 3, ..., \quad v^0 = u^0.$$

Theorem 4 *There is a sequence $\epsilon_k \to 0$ as $k \to \infty$ such that $v^k \to u$ strongly in V as $k \to \infty$. The maximum convergence*

$$\|v^k - u\|^2 \leq M^{-2}(1 + k)^{-1}\|f - Au\|_*^2$$

is provided by $\epsilon_k = (Mk)^{-1}$.

Proof. From (6) we have that

$$\|v^k - u\|^2 \leq \rho_k\left(\|v^{k-1} - u\|^2 + \epsilon_k^2\|f - Au\|_*^2\right).$$

Denote $s_0 = M^{-2}$, then (10) gives

$$\|v^0 - u\|^2 \leq s_0 \|f - Au\|_*^2.$$

We prove the Theorem by induction. Let

$$\|v^{k-1} - u\|^2 \leq s_{k-1} \|f - Au\|_*^2.$$

Obviously, if

$$\rho_k(s_{k-1} + \epsilon_k^2) \leq s_k, \tag{17}$$

then

$$\|v^k - u\|^2 \leq s_k \|f - Au\|_*^2.$$

Rewrite condition (17) as follows

$$\epsilon_k^2(1 - M^2 s_k) - 2M s_k \epsilon_k + s_{k-1} - s_k \leq 0.$$

To fulfil this, it is necessary that the following inequality holds

$$s_k \geq s_{k-1}\left(1 + M^2 s_{k-1}\right)^{-1}.$$

The equality gives the maximum convergence

$$s_k = s_{k-1}\left(1 + M^2 s_{k-1}\right)^{-1} = \cdots = s_0\left(1 + kM^2 s_0\right)^{-1} = M^2(1+k)^{-1}.$$

Because of (17), $\epsilon_k = M s_k (1 - M^2 s_k)^{-1} = (Mk)^{-1}$. This completes the proof.

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