Modified model for proportional loading and unloading of hypoplastic materials

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This work was supported by the Project No. 7AMB16AT035 within the MSMT Mobility Programme (Austria), by the Project SGS18/006/OHK1/1T/11 of the Czech Technical University, and by the Project of Excelence CZ.02.1.01/0.0/0.0/16_019/0000778 of the Ministry of Education, Youth and Sports of the Czech Republic.

Abstract: Classification of inner processes during loading and unloading tests in models of hypoplasticity developed by D. Kolymbas for the constitutive behavior of granular materials is the main aim of this work. We focus on a modified model proposed by E. Bauer. By introducing a dimensionless time parameter s we transform the constitutive equation into a rate independent form, and study the stress paths in different proportional loading regimes.

Introduction

We pursue here the study started in [**4**] of the asymptotic behavior of stress trajectories under proportional loading and unloading in granular materials under the hypoplasticity hypothesis. Note that the idea of rate-independent hypoplasticity goes back to the works by Kolymbas, see, e. g., [**10**]. In engineering literature, this concept receives a lot attention and it is appropriate to cite at least [**5, 7, 8, 14, 15, 16, 12**]. Its main purpose was to explain the phenomenon of *ratchetting* which is very strong in granular materials, and other existing rate-independent constitutive models do not manifest a satisfactory agreement with real experiments. Rachetting in granular materials is the

Research Perspectives CRM Barcelona, Summer 2018, vol. 11, in Trends in Mathematics
Springer-Birkhäuser, Basel

process of accumulation of permanent deformation during cyclic loading and unloading. This behaviour is characterised by progressively shifted loops in the strain-stress diagram.

Indeed, ratchetting is present in nonlinear kinematic hardening models of elastoplasticity of Armstrong-Frederick type, see [**2**], and the mathematical techniques developed in [**6**] for proving the well-posedness of these models have motivated the present study. Another mathematical approach to granular and multiphase media within the variational theory was proposed in [**1, 9, 11**].

An analytic identification of the asymptotic states in hypoplasticity whose existence was established in, e. g., in [**13**, Chapter 3.4], has been carried out in [**4**] for a simple one-parameter model suggested in [**3**]. Localization of the parameter domain which ensures Lyapunov stability of proportional strain paths was the main result there. Here, we obtain similar results for a modified model involving an additional physical parameter and discussed also in [**3**].

1 Description of the model

1.1 Original model

Our starting point is the model from [**4**] for inner processes in granular material under the strain-stress law

$$\dot{\boldsymbol{\sigma}}(t) = c_1\left(\dot{\boldsymbol{\varepsilon}}(t)a^2\mathrm{tr}\,\boldsymbol{\sigma} + \frac{\boldsymbol{\sigma}}{\mathrm{tr}\,\boldsymbol{\sigma}}(\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}) + a(2\boldsymbol{\sigma} - \frac{1}{3}(\mathrm{tr}\,\boldsymbol{\sigma})\mathbf{I})\|\dot{\boldsymbol{\varepsilon}}\|\right), \tag{1}$$

where $\boldsymbol{\varepsilon}$ is the strain tensor and $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{I}$ is the Kronecker tensor $\mathrm{tr}\,\boldsymbol{\sigma} = usi:\mathbf{I}$ is the trace of $\boldsymbol{\sigma}$, $a>0$ is a model parameter, and $c_1<0$ is a scaling parameter which, as we shall see, has no influence on the asymptotic behavior of the model. We consider proportional strain paths of the form

$$\boldsymbol{\varepsilon}(t) = \varepsilon(t)\mathbf{U},\ \dot{\boldsymbol{\varepsilon}}(t) = \dot{\varepsilon}(t)\mathbf{U}, \tag{2}$$

where $\varepsilon(t):[0,\infty)\to\mathbb{R}$ is a given monotone function, and $\mathbf{U}$ is a fixed symmetric tensor

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}. \tag{3}$$

This is what we call a *proportional loading*.

1.2 Modified model

The modification of equation (1) consists in including an additional physical parameter $b>0$ and considering the equation

$$\dot{\boldsymbol{\sigma}}(t) = c_1\left(\dot{\boldsymbol{\varepsilon}}(t)a^2\mathrm{tr}\,\boldsymbol{\sigma} + \frac{\boldsymbol{\sigma}}{\mathrm{tr}\,\boldsymbol{\sigma}}(\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}) + b\mathrm{tr}\,\boldsymbol{\sigma}\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}}\mathbf{I} + a(2\boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma}\mathbf{I})\|\dot{\boldsymbol{\varepsilon}}\|\right) \tag{4}$$

Let us denote by $\langle\cdot,\cdot\rangle$ the canonical scalar product in the space of tensors $\langle\boldsymbol{\varepsilon},\boldsymbol{\sigma}\rangle = \boldsymbol{\varepsilon}:\boldsymbol{\sigma}$. With this notation and Hypothesis (2), equation (4) is of the form

$$\dot{\boldsymbol{\sigma}}(t) = c_1\dot{\varepsilon}(t)\big((a^2\mathbf{U} + b\langle\mathbf{U},\mathbf{I}\rangle\mathbf{I})\langle\boldsymbol{\sigma},\mathbf{I}\rangle + \frac{\langle\boldsymbol{\sigma},\mathbf{U}\rangle}{\langle\boldsymbol{\sigma},\mathbf{I}\rangle}\boldsymbol{\sigma} + a\|\mathbf{U}\|\,\mathrm{sign}\,\dot{\varepsilon}(t)\big(2\boldsymbol{\sigma} - \frac{1}{3}\langle\boldsymbol{\sigma},\mathbf{I}\rangle\mathbf{I}\big)\big). \tag{5}$$

Our analysis of equation (5) will be carried out under the following hypotheses:

(i) The material is initially compressed, that is, $\boldsymbol{\sigma}(0)$ is a given vector such that $\langle \sigma(0), \mathbf{I} \rangle < 0$;

(ii) we investigate below the different dynamics of the model under increasing compression (or loading) corresponding to $\langle \mathbf{U}, \mathbf{I} \rangle > 0$, decreasing compression (or unloading) corresponding to $\langle \mathbf{U}, \mathbf{I} \rangle < 0$, and volume preserving compression corresponding to $\langle \mathbf{U}, \mathbf{I} \rangle = 0$;

(iii) $\varepsilon : [0, \infty) \to \mathbb{R}$ is absolutely continuous, $\dot{\varepsilon}(t) < 0$ for a. e. $t > 0$, $\lim_{t\to\infty} \varepsilon(t) = -\infty$;

By introducing a time transformation $s(t)$ through the formula

$$\dot{s}(t) = c_1 \dot{\varepsilon}(t), \quad s(0) = 0, \text{ and } \boldsymbol{\sigma}'(s) = \frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}s},$$

we are able with the above assumptions to transform equation (5) into a rate-independent form:

$$\begin{aligned} \boldsymbol{\sigma}' &= a^2 \langle \boldsymbol{\sigma}, \mathbf{I} \rangle \mathbf{U} + \frac{\langle \boldsymbol{\sigma}, \mathbf{U} \rangle}{\langle \boldsymbol{\sigma}, \mathbf{I} \rangle} \boldsymbol{\sigma} - a\|\mathbf{U}\| \left(2\boldsymbol{\sigma} - \frac{1}{3} \langle \boldsymbol{\sigma}, \mathbf{I} \rangle \mathbf{I} \right) + b \langle \boldsymbol{\sigma}, \mathbf{I} \rangle \langle \mathbf{U}, \mathbf{I} \rangle \mathbf{I} \\ &= \langle \boldsymbol{\sigma}, \mathbf{I} \rangle \left(a^2 \mathbf{U} + \frac{a}{3} \|\mathbf{U}\| \mathbf{I} + b \langle \mathbf{U}, \mathbf{I} \rangle \mathbf{I} \right) + \boldsymbol{\sigma} \left(\frac{\langle \boldsymbol{\sigma}, \mathbf{U} \rangle}{\langle \boldsymbol{\sigma}, \mathbf{I} \rangle} - 2a\|\mathbf{U}\| \right). \end{aligned} \tag{6}$$

2 Loading

2.1 Isotropic loading

First, we consider the case of the isotropic loading, that is, $\mathbf{U} = \mathbf{I}$ Then $\|\mathbf{U}\| = \sqrt{3}$ and $\langle \mathbf{U}, \mathbf{I} \rangle = 3$. In this case, equation (6) reduces to

$$\boldsymbol{\sigma}' = \langle \boldsymbol{\sigma}, \mathbf{I} \rangle \left(a^2 + \tfrac{a}{\sqrt{3}} + 3b \right) \mathbf{I} + \boldsymbol{\sigma} \left(1 - 2a\sqrt{3} \right). \tag{7}$$

The scalar product of (7) with $\mathbf{I}$ yields

$$\langle \boldsymbol{\sigma}', \mathbf{I} \rangle = \lambda \langle \boldsymbol{\sigma}, \mathbf{I} \rangle, \quad \text{with } \lambda = 3a^2 - a\sqrt{3} + 1 + 9b, \tag{8}$$

where $3a^2 - a\sqrt{3} + 1 \geq \frac{3}{4}$ and $b > 0$, therefore $\lambda > 0$. Hence

$$\langle \boldsymbol{\sigma}(s), \mathbf{I} \rangle = \langle \boldsymbol{\sigma}(0), \mathbf{I} \rangle \, \mathrm{e}^{\lambda s},$$

and equation (7) can thus be written as

$$\boldsymbol{\sigma}' = -\mu \boldsymbol{\sigma} + R \mathrm{e}^{\lambda s} \mathbf{I},$$

where

$$\mu = 2a\sqrt{3} - 1, \; R = \frac{\langle \boldsymbol{\sigma}(0), \mathbf{I} \rangle}{3} (\lambda + \mu) > 0.$$

The solution of (10) is

$$\boldsymbol{\sigma}(s) = \mathrm{e}^{-\mu s} \boldsymbol{\sigma}(0) + \frac{\langle \boldsymbol{\sigma}(0), \mathbf{I} \rangle}{3} \left(\mathrm{e}^{\lambda s} - \mathrm{e}^{-\mu s} \right) \mathbf{I}. \tag{9}$$

In other words, we have

$$\boldsymbol{\sigma}(s) - \frac{\langle \boldsymbol{\sigma}(0), \mathbf{I} \rangle}{3} \mathrm{e}^{\lambda s} \mathbf{I} = \mathrm{e}^{-\mu s} \left(\boldsymbol{\sigma}(0) - \frac{\langle \boldsymbol{\sigma}(0), \mathbf{I} \rangle}{3} \mathbf{I} \right). \tag{10}$$

The physically relevant case observed in experiments is $\mu > 0$, that is,

$$a > \frac{1}{2\sqrt{3}} \approx 0.289\,. \tag{11}$$

Then (10) means that the trajectory of $\boldsymbol{\sigma}(s)$ exponentially converges to the linear trajectory $\frac{\langle \boldsymbol{\sigma}(0),\mathbf{I}\rangle}{3}\mathrm{e}^{\lambda s}\mathbf{I}$ along the unit tensor $\mathbf{I}$ with initial condition given by the orthogonal projection of the initial condition $\boldsymbol{\sigma}(0)$ onto the line spanned by $\mathbf{I}$. The phenomenon that the influence of the initial condition is exponentially decreasing is typical for hypoplastic materials.

2.2 Anisotropic loading

Let now $\mathbf{U} \in \mathbb{R}^{3\times 3}$ be arbitrary. As mentioned above, loading corresponds to $\langle \mathbf{U},\mathbf{I}\rangle > 0$. It follows from (6) that the term $\frac{\langle \boldsymbol{\sigma},\mathbf{U}\rangle}{\langle \boldsymbol{\sigma},\mathbf{I}\rangle}$ satisfies a linear ODE

$$\left(\frac{\langle \boldsymbol{\sigma},\mathbf{U}\rangle}{\langle \boldsymbol{\sigma},\mathbf{I}\rangle}\right)' = A - \eta\frac{\langle \boldsymbol{\sigma},\mathbf{U}\rangle}{\langle \boldsymbol{\sigma},\mathbf{I}\rangle} \tag{12}$$

with

$$A = a^2\|\mathbf{U}\|^2 + \frac{a}{3}\|\mathbf{U}\|\,\langle \mathbf{U},\mathbf{I}\rangle + b\,\langle \mathbf{U},\mathbf{I}\rangle^2\,, \quad \eta = a\|\mathbf{U}\| + (a^2+3b)\,\langle \mathbf{U},\mathbf{I}\rangle\,.$$

The solution of (12) is of the form

$$\frac{\langle \boldsymbol{\sigma}(s),\mathbf{U}\rangle}{\langle \boldsymbol{\sigma}(s),\mathbf{I}\rangle} = B + C\mathrm{e}^{-\eta s},$$

with

$$B = \frac{A}{\eta}, \quad C = \frac{\langle \boldsymbol{\sigma}(0),\mathbf{U}\rangle}{\langle \boldsymbol{\sigma}(0),\mathbf{I}\rangle} - B,$$

Equation (6) is therefore equivalent to a linear equation

$$\boldsymbol{\sigma}' = \langle \boldsymbol{\sigma},\mathbf{I}\rangle\left(a^2\mathbf{U} + \frac{a}{3}\|\mathbf{U}\|\mathbf{I} + b\,\langle \mathbf{U},\mathbf{I}\rangle\,\mathbf{I}\right) + \boldsymbol{\sigma}\left(B - 2a\|\mathbf{U}\| + C\mathrm{e}^{-\eta s}\right). \tag{13}$$

To solve equation (13), we proceed as in the isotropic case taking the scalar product of (13) with $\mathbf{I}$, which yields

$$\left\langle \boldsymbol{\sigma}',\mathbf{I}\right\rangle = \left(D + C\mathrm{e}^{-\eta s}\right)\langle \boldsymbol{\sigma},\mathbf{I}\rangle\,,$$

where

$$D = (a^2+3b)\,\langle \mathbf{U},\mathbf{I}\rangle - a\|\mathbf{U}\| + B = \frac{1}{\eta}\Big((a^2+3b)^2\,\langle \mathbf{U},\mathbf{I}\rangle^2 + \frac{a}{3}\|\mathbf{U}\|\,\langle \mathbf{U},\mathbf{I}\rangle + b\,\langle \mathbf{U},\mathbf{I}\rangle^2\Big) > 0.$$

Hence,

$$\langle \boldsymbol{\sigma}(s),\mathbf{I}\rangle = \langle \boldsymbol{\sigma}(0),\mathbf{I}\rangle\,\mathrm{e}^{f(s)}, \quad f(s) = Ds + \frac{C}{\eta}(1-\mathrm{e}^{-\eta s}),$$

and we can rewrite (13) as

$$\boldsymbol{\sigma}'(s) = \langle \boldsymbol{\sigma}(0),\mathbf{I}\rangle\,\mathrm{e}^{f(s)}\mathbf{V} + g'(s)\boldsymbol{\sigma}(s), \tag{14}$$

where

$$\mathbf{V} = a^2\mathbf{U} + \frac{a}{3}\|\mathbf{U}\|\mathbf{I} + b\,\langle \mathbf{U},\mathbf{I}\rangle\,\mathbf{I}, \quad g(s) = (B - 2a\|\mathbf{U}\|)s + \frac{C}{\eta}(1-\mathrm{e}^{-\eta s}).$$

From (14) it follows that

$$\left(\mathrm{e}^{-g(s)}\boldsymbol{\sigma}(s)\right)' = \langle \boldsymbol{\sigma}(0),\mathbf{I}\rangle\,\mathrm{e}^{f(s)-g(s)}\mathbf{V} = \langle \boldsymbol{\sigma}(0),\mathbf{I}\rangle\,\mathrm{e}^{(D+2a\|\mathbf{U}\|-B)s}\mathbf{V},$$

hence

$$\mathrm{e}^{-g(s)}\boldsymbol{\sigma}(s) = \boldsymbol{\sigma}(0) + \frac{\langle \boldsymbol{\sigma}(0), \mathbf{I}\rangle}{D + 2a\|\mathbf{U}\| - B}\mathrm{e}^{(D+2a\|\mathbf{U}\|-B)s} - 1)\mathbf{V},$$

provided $D + 2a\|\mathbf{U}\| - B \neq 0$. We note that

$$B = \frac{\langle \mathbf{V}, \mathbf{U}\rangle}{\langle \mathbf{V}, \mathbf{I}\rangle}, \quad \eta = \langle \mathbf{V}, \mathbf{I}\rangle\,, \quad C = \frac{\langle \boldsymbol{\sigma}(0), \mathbf{U}\rangle}{\langle \boldsymbol{\sigma}(0), \mathbf{I}\rangle} - \frac{\langle \mathbf{V}, \mathbf{U}\rangle}{\langle \mathbf{V}, \mathbf{I}\rangle},$$

and

(15) $$D + 2a\|\mathbf{U}\| - B = a\|\mathbf{U}\| + a^2\langle \mathbf{U}, \mathbf{I}\rangle + 3b\langle \mathbf{U}, \mathbf{I}\rangle = \langle \mathbf{V}, \mathbf{I}\rangle = \eta.$$

The formula for $\boldsymbol{\sigma}(s)$ then reads

(16) $$\boldsymbol{\sigma}(s) = \mathrm{e}^{g(s)}\boldsymbol{\sigma}(0) + \frac{\langle \boldsymbol{\sigma}(0), \mathbf{I}\rangle}{\langle \mathbf{V}, \mathbf{I}\rangle}\left(\mathrm{e}^{f(s)} - \mathrm{e}^{g(s)}\right)\mathbf{V}\,.$$

We are in the same situation as in (9) provided

(17) $$\lim_{s\to\infty} g(s) = -\infty\,, \quad \lim_{s\to\infty} f(s) = +\infty.$$

This condition can be reformulated in terms of the parameters a, b, c, where

$$c = \frac{\langle \mathbf{U}, \mathbf{I}\rangle}{\sqrt{3}\|\mathbf{U}\|}$$

is the cosine of the angle between the loading direction $\mathbf{U}$ and the isotropic direction $\mathbf{I}$. It can be stated as follows.

Theorem 1 *The stability condition* (17) *is satisfied if and only if*

(18) $$3b(c^2 - 2\sqrt{3}ac) < \left(2a^2 - \frac{1}{3}\right)\sqrt{3}ac + a^2.$$

Proof. We have $D > 0$ according to (15), then the fact that $\lim_{s\to\infty} f(s) = +\infty$ follows immediately. It remains to find conditions on $\mathbf{U}$ and a under which

(19) $$B < 2a\|\mathbf{U}\|$$

in order to obtain $\lim_{s\to\infty} g(s) = -\infty$. A straightforward computation yields that (19) is fulfilled if and only if

(20) $$-a^2 + \frac{a\langle \mathbf{U}, \mathbf{I}\rangle}{3\|\mathbf{U}\|} - \frac{2a^3\langle \mathbf{U}, \mathbf{I}\rangle}{\|\mathbf{U}\|} - \frac{6ab\langle \mathbf{U}, \mathbf{I}\rangle}{\|\mathbf{U}\|} + \frac{b\langle \mathbf{U}, \mathbf{I}\rangle^2}{\|\mathbf{U}\|^2} < 0\,,$$

which we wanted to prove. □

Rewriting (16) as

(21) $$\boldsymbol{\sigma}(s) - \frac{\langle \boldsymbol{\sigma}(0), \mathbf{I}\rangle}{\langle \mathbf{V}, \mathbf{I}\rangle}\mathrm{e}^{f(s)}\mathbf{V} = \mathrm{e}^{g(s)}\left(\boldsymbol{\sigma}(0) - \frac{\langle \boldsymbol{\sigma}(0), \mathbf{I}\rangle}{\langle \mathbf{V}, \mathbf{I}\rangle}\mathbf{V}\right),$$

we see that the trajectory of $\boldsymbol{\sigma}(s)$ exponentially converges to the linear trajectory $\frac{\langle \boldsymbol{\sigma}(0), \mathbf{I}\rangle}{\langle \mathbf{V}, \mathbf{I}\rangle}\mathrm{e}^{f(s)}\mathbf{V}$ propagating along $\mathbf{V}$ with initial condition given by the projection of $\boldsymbol{\sigma}(0)$ orthogonal to $\mathbf{I}$ onto the line spanned by $\mathbf{V}$.

Remark 2 The stability condition in Theorem 1 admits a geometric intepretation in the parameter space. A more detailed discussion will be made in a forthcoming paper. Notice only that if inequality (11) holds, then (20) holds for every $\mathbf{U} \in \mathbb{R}^{3\times 3}$. Indeed, we have $c \le 1$, hence $c^2 - 2\sqrt{3}ac < 0$ for all a satisfying (11). This means in particular that the interval of the parameters a which ensure the stability of the proportional path is minimal in case of isotropic loading, while for the "almost volume preserving loading" $\langle \mathbf{U}, \mathbf{I} \rangle / \|\mathbf{U}\| \to 0$ it becomes maximal.

3 Unloading

3.1 Isotropic unloading

Isotropic loading can be describes by $\mathbf{U} = -\mathbf{I}$. In comparison with the isotropic loading case, in the equation (10) the coefficients $\lambda = -3a^2 - a\sqrt{3} - 9b - 1 < 0$ and $\mu = 1 + 2\sqrt{3}a > 0$. The condition $\mu > 0$ is satisfied for all $a > 0$.

3.2 Anisotropic unloading

The case of the anisotropic loading described by $\langle \mathbf{U}, \mathbf{I} \rangle < 0$. Asymptotic convergence to the asymptotic direction $\mathbf{V}^-$ is guaranteed by the condition $g(s) \to 0$ and $f(s) \to -\infty$ in the equation (16). It is satisfied when the constants

$$D < 0, \quad \eta > 0, \quad B - 2a\|\mathbf{U}\| < 0.$$

The last condition coincides with the case (18) of anisotropic loading. Here, however, additional conditions come into play, namely $\eta > 0$, which is equivalent to

$$D + 2a\|\mathbf{U}\| - B > 0,$$

and in terms of c it can be rewritten as

$$\sqrt{3}c > -\frac{a}{a^2 + 3b}.$$

The condition $D < 0$ holds provided $3((a^2 + 3b)^2 + b)c^2 + \frac{ac}{\sqrt{3}} < 0$.

Conclusion

The modified model for constitutive behaviour of granular materials proposed by Bauer [**3**] was studied in here for the particular case of proportional loading and unloading. We have determined the parameter range in which asymptotic stability of the proportional loading and unloading processes are guaranteed.

References

[1] B. D. Annin, V. A. Kovtunenko and V. M. Sadovskii, *Variational and hemivariational inequalities in mechanics of elastoplastic, granular media, and quasibrittle cracks*, in Analysis, Modelling, Optimization, and Numerical Techniques, G. O. Tost, O. Vasilieva, eds., Springer Proc. Math. Stat., 121 (2015), 49–56.

[2] P. J. Armstrong and C. O. Frederick, *A mathematical representation of the multiaxial Bauschinger effect*, C.E.G.B., Report RD/B/N 731, 1966.

[3] E. Bauer, *Modelling limit states within the framework of hypoplasticity*, AIP Conf. Proc., 1227, J. Goddard, P. Giovine and J. T. Jenkin, eds., AIP, 2010, pp. 290–305.

[4] E. Bauer, V.A. Kovtunenko, P. Krejčí,L. Siváková, A. Zubkova, *On Lyapunov Stability in Hypoplasticity*, Proc. Equadiff Conf.,K. Mikula, D. Ševčovič and J. Urbán, eds., Spektrum STU, 2017, pp. 107–116.

[5] E. Bauer and W. Wu, *A hypoplastic model for granular soils under cyclic loading*, Proc. Int. Workshop Modern Approaches to Plasticity, D. Kolymbas, ed., Elsevier, 2010, pp. 247–258.

[6] M. Brokate and P. Krejčí, *Wellposedness of kinematic hardening models in elastoplasticity*, RAIRO Modél. Math. Anal. Numér., (1998), 177–209.

[7] G. Gudehus, *Physical Soil Mechanics*, Springer, Berlin, Heidelberg, 2011.

[8] W. Huang and E. Bauer, *Numerical investigations of shear localization in a micro-polar hypoplastic material*, Int. J. Numer. Anal. Meth. Geomech., 27 (2003), pp. 325–352.

[9] A.M. Khludnev and V.A. Kovtunenko, *Analysis of Cracks in Solids*, WIT-Press, Southampton, Boston, 2000.

[10] D. Kolymbas, *Introduction to Hypoplasticity*, A.A. Balkema, Rotterdam, 2000.

[11] V.A. Kovtunenko and A.V. Zubkova, *Mathematical modeling of a discontinuous solution of the generalized Poisson–Nernst–Planck problem in a two-phase medium*, Kinet. Relat. Mod., 11 (2018), pp.119–135.

[12] P. Krejčí, P. O'Kane, A. Pokrovskii and D. Rachinskii, *Stability results for a soil model with singular hysteretic hydrology*, in Proc. 5th Int. Workshop Multi-Rate Processes and Hysteresis, A. Ivanyi, P. Ivanyi, D. Rachinskii and V. Sobolev, eds., J. Phys. Conf. Ser., 268 (2011), 012016.

[13] A. Niemunis, *Extended Hypoplastic Models for Soils*, Habilitation thesis, Ruhr University, Bochum, 2002.

[14] A. Niemunis and I. Herle, *Hypoplastic model for cohesionless soils with elastic strain range*, Mech. Cohes.-Frict. Mat., 2 (1997), pp. 279–299.

[15] B. Svendsen, K. Hutter and L. Laloui, *Constitutive models for granular materials including quasi-static frictional behaviour: toward a thermodynamic theory of plasticity*, Continuum Mech. Therm., 4 (1999), pp. 263–275.

[16] W. Wu, E. Bauer and D. Kolymbas, *Hypoplastic constitutive model with critical state for granular materials*, Mech. Mater., 23 (1996), pp. 45–69.