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Quasistatic Propagation of Cracks[1]

The problem of shape perturbation for the variational inequality describing solids with cracks under the nonpenetration condition is considered [1, 4]. The obtained asymptotic results are applied to describe the quasistatic propagation of cracks [2, 3, 7].

1. Perturbation of the crack problem. Let Ω be a bounded domain in $\mathbf{R}^N$, $N = 2, 3$, with the boundary, which includes two parts $\Gamma_{\mathcal{D}}$ and $\Gamma_{\mathcal{N}}$, $\Gamma_{\mathcal{D}} \cap \Gamma_{\mathcal{N}} = \emptyset$. Consider the crack $\Gamma(0)$ inside Ω as the $(N-1)$-manifold in $\mathbf{R}^N$. We denote the domain with crack as $\Omega(0) = \Omega \setminus \overline{\Gamma}(0)$. For small enough parameter ε, we consider the function of perturbation $\Phi \in [C^2(-\varepsilon_0, \varepsilon_0; W^{1,\infty}(\mathbf{R}^N))]^N$ with $\Phi(0)(x) = x$, $x \in \mathbf{R}^N$. Let $\Phi(\varepsilon)(\Omega) = \Omega$, then the coordinate transformation $y = \Phi(\varepsilon)(x)$ transforms the initial domain $\Omega(0)$ to the perturbed domain $\Omega(\varepsilon) = \Omega \setminus \overline{\Gamma}(\varepsilon)$ with $\Gamma(\varepsilon) = \Phi(\varepsilon)(\Gamma(0))$. We assume that for each ε the domain Ω can be splitted into two subdomains with Lipschitz-continuous boundaries and common interface, which intersects $\Gamma_{\mathcal{D}}$, such that $\Gamma(\varepsilon)$ is a part of this interface.

The Jacobian of transformation has the asymptotic $J(\varepsilon) = 1 + \varepsilon \operatorname{div}(V) + o(\varepsilon)$, where $V \in [W^{1,\infty}(\mathbf{R}^N)]^N$ denotes the velocity field $\frac{\partial \Phi}{\partial \varepsilon}|_{\varepsilon=0}$. Therefore, $\Phi(\varepsilon)$ yields the one-to-one correspondence between domains $\Omega(0)$ and $\Omega(\varepsilon)$, also between spaces $\widetilde{H}^1(\Omega(0))$ and $\widetilde{H}^1(\Omega(\varepsilon))$, when we introduce the space

$$\widetilde{H}^1(\Omega(\varepsilon)) = \{u \in [H^1(\Omega(\varepsilon))]^N, \quad u = 0 \quad \text{on } \Gamma_{\mathcal{D}}\}.$$

For the displacement vector $u = (u_1, ..., u_N)$ we consider the linear elasticity model of nonhomogeneous anisotropic solid

$$\sigma_{ij}(u) = c_{ijkl}\epsilon_{kl}(u), \quad \epsilon_{ij}(u) = 0.5(u_{i,j} + u_{j,i}), \quad i, j = 1, ..., N,$$

with elasticity coefficients $c_{ijkl} \in C^2(\mathbf{R}^N)$, $i, j, k, l = 1, ..., N$, which are symmetric and elliptic as it is usually supposed. Let us choose the unit normal

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vector $\nu(\varepsilon)$ to $\Gamma(\varepsilon)$ to distinguish its two opposite faces $\Gamma^{\pm}(\varepsilon)$, which correspond to $\pm\nu(\varepsilon)$, respectively. Then we introduce the closed convex set of admissible displacements

$$K(\varepsilon)=\{u\in\widetilde{H}^1(\Omega(\varepsilon)),\quad [\![u]\!]\nu(\varepsilon)\geq 0\quad \text{on }\Gamma(\varepsilon)\},\quad [\![u]\!]=u|_{\Gamma^+(\varepsilon)}-u|_{\Gamma^-(\varepsilon)},$$

which yields the condition of mutual nonpenetration between the crack faces. For the given traction force $g=(g_1,...,g_N)\in[L^2(\Gamma_{\mathcal{N}})]^N$, there exists the unique solution $u(\varepsilon)\in K(\varepsilon)$ of the variational inequality

$$\int_{\Omega(\varepsilon)}\sigma_{ij}(u(\varepsilon))\epsilon_{ij}(v-u(\varepsilon))\geq\int_{\Gamma_{\mathcal{N}}}g_i(v-u(\varepsilon))_i\quad\forall v\in K(\varepsilon)\tag{1}$$

describing static equilibrium of the solid with crack.

2. Asymptotics of the solution and energy. When $\nu(\varepsilon)\circ\Phi(\varepsilon)=\nu(0)$, then $\Phi(\varepsilon)$ provides the one-to-one correspondence between $K(0)$ and $K(\varepsilon)$, too. In this case, the transformed solution of (1) as the function $u(\varepsilon)\circ\Phi(\varepsilon)\in K(0)$ solves the following variational inequality

$$\begin{aligned}&\textstyle\int_{\Omega(0)}J(\varepsilon)(c_{ijkl}\circ\Phi(\varepsilon))E_{kl}(\Psi(\varepsilon);u(\varepsilon)\circ\Phi(\varepsilon))E_{ij}(\Psi(\varepsilon);v-u(\varepsilon)\circ\Phi(\varepsilon))\\&\geq\textstyle\int_{\Gamma_{\mathcal{N}}}g_i(v-u(\varepsilon)\circ\Phi(\varepsilon))_i\quad\forall v\in K(0),\end{aligned}\tag{2}$$

where $E_{ij}(\Psi;u)=0.5(u_{i,k}\Psi_{kj}+u_{j,k}\Psi_{ki})$, $\Psi=(\frac{\partial\Phi}{\partial x})^{-1}$, $i,j=1,...,N$. From (2) and (1) as $\varepsilon=0$ we prove the estimates

$$\|u(\varepsilon)\circ\Phi(\varepsilon)\|_{\widetilde{H}^1(\Omega(0))}\leq c_0,\quad \|u(\varepsilon)\circ\Phi(\varepsilon)-u(0)\|_{\widetilde{H}^1(\Omega(0))}\leq c_1\varepsilon.\tag{3}$$

This provides existence of the weak material derivative $\dot{u}(\Phi)\in Z(u(0))$ in the sense

$$\varepsilon_n^{-1}(u(\varepsilon_n)\circ\Phi(\varepsilon_n)-u(0))\ \to\ \dot{u}(\Phi)\quad\text{weakly in }\widetilde{H}^1(\Omega(0))\quad\text{as }\varepsilon_n\to 0.\tag{4}$$

Here $Z(u(0))$ is a tangential hyperplane to $K(0)$ at $u(0)$, i.e. $Z(u(0))=\{u\in\widetilde{H}^1(\Omega(0)),\ [\![u]\!]\nu(0)\geq 0\ \text{ on } C(u(0)),\ \int_{\Omega(0)}\sigma_{ij}(u(0))\epsilon_{ij}(u)=\int_{\Gamma_{\mathcal{N}}}g_iu_i\}$, for $C(u(0))\subseteq\Gamma(0)$ being the coincidence set where $[\![u(0)]\!]\nu(0)=0$. Moreover, with the help of (3) and (4) we prove two orthogonality conditions

$$\int_{\Omega(0)}\sigma_{ij}(u(0))\epsilon_{ij}(\dot{u}(\Phi))=\int_{\Gamma_{\mathcal{N}}}g_i\dot{u}(\Phi)_i=-\int_{\Omega(0)}A_1(V;u(0),u(0))\tag{5}$$

with the symmetric bilinear form as follows:
$A_1(V;u,v)=\mathrm{div}(Vc_{ijkl})\epsilon_{kl}(u)\epsilon_{ij}(v)-\sigma_{ij}(u)E_{ij}(\frac{\partial V}{\partial x};v)-\sigma_{ij}(v)E_{ij}(\frac{\partial V}{\partial x};u)$.

Let us introduce the function of potential energy $P(\Phi) : (-\varepsilon_0, \varepsilon_0) \to \mathbf{R}$ by the relation

$$P(\Phi)(\varepsilon) = 0.5 \int_{\Omega(\varepsilon)} \sigma_{ij}(u(\varepsilon))\epsilon_{ij}(u(\varepsilon)) - \int_{\Gamma_{\mathcal{N}}} g_i u(\varepsilon)_i. \tag{6}$$

Applying the coordinate transformation $y = \Phi(\varepsilon)(x)$ to the integrals in (6), due to (3)–(5) the following expansion holds

$$P(\Phi)(\varepsilon) = P(\Phi)(0) + \varepsilon P'(\Phi)(0) + o(\varepsilon),$$

with the first derivative of potential energy as follows:

$$P'(\Phi)(0) = 0.5 \int_{\Omega(0)} A_1(V; u(0), u(0)). \tag{7}$$

It is independent on the choice of perturbation function Φ in the sense that, if $\Omega(\varepsilon) = \Phi^1(\varepsilon)(\Omega(0)) = \Phi^2(\varepsilon)(\Omega(0))$, then $P'(\Phi^1)(0) = P'(\Phi^2)(0)$. Similar to (7) formula for constant elasticity coefficients and V of the $W^{2,\infty}$-class was firstly obtained in [5], and for the partial case of perturbation in [6]. When there exists $D \subset \Omega(0)$ such that $u(0) \in H^2(\overline{D})$, $V = 0$ or $\{\nabla c_{ijkl}\} = 0$ in $\overline{\Omega}(0) \setminus D$, and $c_{ijkl}(x)\xi_{kl}[0.5\text{div}(V(x))\xi_{ij} - \xi_{im}V_{m,j}(x)] = 0$ for all $\{\xi_{ij}\}$ a.e. $x \in \Omega(0) \setminus \overline{D}$, then, integrating by parts, the domain integral in (7) can be represented in the equivalent form of integral over boundary ∂D,

$$I(V) = \int_{\partial D} \sigma_{ij}(u(0))[0.5(V \cdot n)\epsilon_{ij}(u(0)) - n_j(V \cdot \nabla u(0)_i)],$$

where n is the unit outward normal vector to ∂D. This implies the general form of invariant integrals of energy.

3. Numerical example on the crack propagation. Let $\Omega \in \mathbf{R}^2$ be the unit square $\{0 < x_1 < 1,\ |x_2| < 0.5\}$ with the boundary $\overline{\Gamma}_{\mathcal{N}} \cup \overline{\Gamma}_{\mathcal{D}}$, $\Gamma_{\mathcal{D}} = \{x_1 = 1,\ |x_2| < 0.5\}$. Let us take the crack $\Gamma(0) = \{0 < x_1 < l,\ x_2 = 0\}$ of the length $0 < l < 1$. In this case $\nu(0) = (0, 1)$. The Lamé model $\sigma_{ij}(u) = 2\mu\epsilon_{ij}(u) + \lambda(\epsilon_{11}(u) + \epsilon_{22}(u))\delta_{ij}$, $i, j = 1, 2$, with $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, $\nu = 0.34$, $E = 7.3 \cdot 10^4 (mPa)$ is considered. We take the traction force g given on $\Gamma_{\mathcal{N}} = \Gamma_{\mathcal{N}}^1 \cup \Gamma_{\mathcal{N}}^2$ by the following functions

$$\begin{cases} g_1(x) = -6.5 t_0 \mu |2x_2|, \quad g_2(x) = 0 \quad \text{on } \Gamma_{\mathcal{N}}^1 = \{x_1 = 0,\ 0 < \pm x_2 < 0.5\} \\ g_1(x) = 0,\ g_2(x) = \pm t_0 \mu (1 - |2x_1 - 1|) \text{ on } \Gamma_{\mathcal{N}}^2 = \{0 < x_1 < 1,\ x_2 = \pm 0.5\} \end{cases}$$

with the multiplier $t_0 = 0.9 \cdot 10^{-3}$.

The inequality (1) as $\varepsilon = 0$ is approximated by the penalty equation

$$\int_{\Omega(0)} \sigma_{ij}(u^\delta(0))\epsilon_{ij}(v) - \frac{1}{\delta}\int_{\Gamma(0)} [\![u_2^\delta(0)]\!]^- [\![v_2]\!] = \int_{\Gamma_\mathcal{N}} g_i v_i, \quad u^- = \max\{0, -u\},$$

with small parameter $\delta > 0$, and then linearized by iterations as follows:

$$\begin{aligned} &\textstyle\int_{\Omega(0)} \sigma_{ij}(u^{\delta,n}(0))\epsilon_{ij}(v) + \frac{1}{\delta}\int_{\Gamma(0)} [\![u_2^{\delta,n}(0) - u_2^{\delta,n-1}(0)]\!][\![v_2]\!] \\ &= \textstyle\int_{\Gamma_\mathcal{N}} g_i v_i + \frac{1}{\delta}\int_{\Gamma(0)} [\![u_2^{\delta,n-1}(0)]\!]^- [\![v_2]\!], \quad n = 1, 2, \ldots . \end{aligned} \tag{8}$$

We have the convergences

$$\|u^\delta(0) - u(0)\|_{\widetilde{H}^1(\Omega(0))} \to 0 \quad \text{as } \delta \to 0 \quad (\text{of the order } \sqrt{\delta}),$$

$$\|u^{\delta,n}(0) - u^\delta(0)\|_{\widetilde{H}^1(\Omega(0))} \le (1 + C\delta)^{-0.5n} \|u^{\delta,0}(0) - u^\delta(0)\|_{\widetilde{H}^1(\Omega(0))}.$$

To solve numerically the linear equation (8), we apply the finite-element method with piecewise-linear functions on uniform triangle mesh in $\Omega(0)$ of the size $h = 0.025$. It supposed the approximation of order $\sqrt{h}$ for $u(0)$ and order h for the energy. Approximate values of the potential energy $P(\Phi)(0)$ from (6), calculated by this scheme, are shown in Fig.1 in dependence on the parameter l of the crack length.

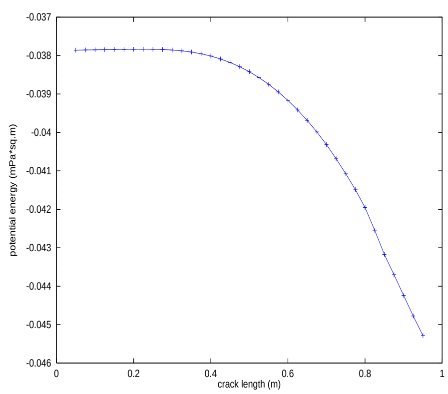

Fig.1. Potential energy $P(\Phi)(0)$

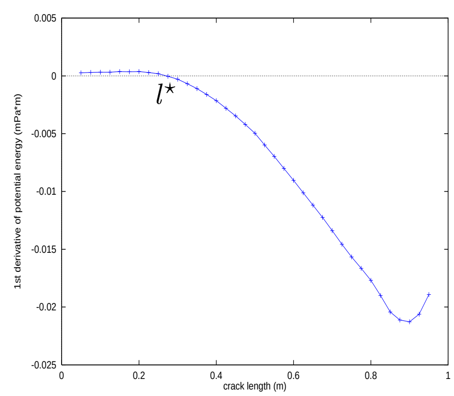

Fig.2. First derivative $P'(\Phi)(0)$

Let us take the piecewise-linear cut-off function $\chi \in W^{1,\infty}(\mathbf{R}^2)$, which has $\text{supp}(\chi) \subset \Omega$ and $\chi = 1$ in neighborhood of the crack tip $x = (l, 0)$. Applying the perturbation $\Phi(\varepsilon)(x) = (x_1 + \varepsilon\chi(x), x_2)$, it follows from formula (7) the first derivative $P'(\Phi)(0)$ with respect to perturbation of the crack length l (energy release rate), which is numerically calculated as shown in Fig.2. Here $P'(\Phi)(0) \approx 0$ for $0 < l \leq l^\star$, $l^\star \approx 0.25$. Moreover, from the well-known formula in classical fracture mechanics, $-P'(\Phi)(0) = 0.5\mu(1-\nu)|K|^2$, due to $P'(\Phi)(0) \leq 0$ we can find the modulus $|K|$ of stress intensity factors as shown in Fig.3.

For loading parameter $t \geq 0$, let us consider the linear loading $g(x)(t) = t\, g(x)$. By the cone property of $K(0)$ we obtain that $u(0)(t) = t\, u(0) \in K(0)$ is a solution of the corresponding quasistatic problem, and, therefore, $P'(\Phi)(0)(t) = t^2 P'(\Phi)(0)$. Let us fix the initial crack length l. From the Griffith fracture criterion $\gamma + P'(\Phi)(0)(t) = 0$, we deduce then the critical value of the maximal loading as

$$g_{\rm cr}(l) = 6.5 t_0 \mu\, t_{\rm cr}(l), \quad t_{\rm cr}(l) = \sqrt{\gamma\, (-P'(\Phi)(0))^{-1}},$$

and the inverse relation $l = l(g_{\rm cr})$ shown in Fig.4 as the curve L. Here constant $\gamma > 0$ is the doubled density of surface energy distributed at the crack. The Griffith fracture hypothesis implies that there is no crack growth for $0 \leq t < t_{\rm cr}(l)$, and the crack starts to grow when $t = t_{\rm cr}(l)$. In Fig.4 we see three possible cases of the crack quasistatic propagation in dependence on the initial crack length l. The crack does not grow at all for $l \leq l^\star$ with $l^\star$ from Fig.2. For $l > l^\star$ the crack propagation can be stable (with finite initial velocity), as shown in Fig.4 for $l = 0.95$, or unstable with a jump (infinite initial velocity), as shown in Fig.4 for $l = 0.825$.

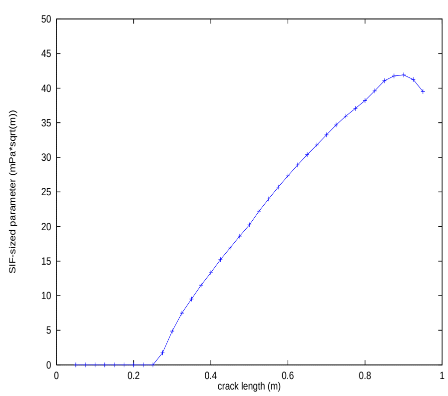

Fig.3. Modulus of SIF $|K|$

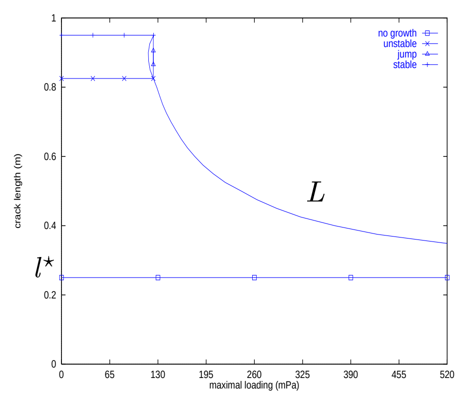

Fig.4. Critical loading $g_{\rm cr}(l)$

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