MATHEMATICAL METHODS IN THE APPLIED SCIENCES
*Math. Meth. Appl. Sci.* 2003; **26**:359–374 (DOI: 10.1002/mma.341)
MOS subject classification: 49 J 40; 49 Q 10; 49 Q 12; 73 M 25

# Shape sensitivity of a plane crack front

Victor A. Kovtunenko[1,2,*,†]

[1] *Lavrentyev Institute of Hydrodynamics, 630090 Novosibirsk, Russia*
[2] *Mathematical Institute A, Stuttgart University, 70569 Stuttgart, Germany*

Communicated by W. Wendland

SUMMARY

The 3D-elasticity model of a solid with a plane crack under the stress-free boundary conditions at the crack is considered. We investigate variations of a solution and of energy functionals with respect to perturbations of the crack front in the plane. The corresponding expansions at least up to the second-order terms are obtained. The strong derivatives of the solution are constructed as an iterative solution of the same elasticity problem with specified right-hand sides. Using the expansion of the potential and surface energy, we consider an approximate quadratic form for local shape optimization of the crack front defined by the Griffith criterion. To specify its properties, a procedure of discrete optimization is proposed, which reduces to a matrix variational inequality. At least for a small load we prove its solvability and find a quasi-static model of the crack growth depending on the loading parameter.


KEY WORDS: crack; fracture; shape sensitivity; shape optimization

## 1. INTRODUCTION

In the mathematical fracture mechanics, methods of the singular perturbation theory are usually adopted, see References [1,2]. Within this approach, local asymptotic expansions of a solution are matched to construct asymptotic formulas for stress intensity factors and the corresponding potential energy [3–5]. We work within an energetic approach and adapt methods of the shape sensitivity analysis to problems with cracks.

The main feature of the model considered lies in the fact that the crack front is described by some given function. Therefore, we need to define the shape variations with respect to this function. In smooth domains, the shape sensitivity of a boundary was studied in References [6,7]. The first-order variations by a crack for linear and non-linear crack problems were investigated in References [8–11], the high-order variations by the crack shape for 2D-solids in References [12,13].

* Correspondence to: Victor A. Kovtunenko, Lavrentyev Institute of Hydrodynamics, 630090 Novosibirsk, Russia.
† E-mail: kovtunenko@hydro.nsc.ru; victork@mathematik.uni-stuttgart.de


*Received 6 June 2001*

The second subject of our consideration is the local optimization of the crack shape and the quasi-static propagation of the crack front. We assume that the crack propagates along its own plane and applies the energy expansion due to the Griffith criterion. The approach reformulating fracture criterion as a variational inequality was developed in References [14–19] and realized for the penny-shaped crack by the Irwin and the Griffith criterion in References [20–22].

## 2. VARIATION OF A PLANE CRACK

Let a crack occupy the simply connected domain $\Gamma_0 \subset \mathbb{R}^2$ on a plane in $\mathbb{R}^3$ bounded by the contour $\gamma_0 \in C^{0,1}$. We assume that $\{0\}$ lies inside the crack $\Gamma_0$ and the crack front $\gamma_0$ is described in the polar co-ordinates $(r, \phi)$ on $\mathbb{R}^2$ as follows:

$$\gamma_0 = \{r = R(\phi),\ \phi \in [0, 2\pi],\ R(0) = R(2\pi),\ R > 0\}\ R \in C^{0,1}$$

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the boundary $\Gamma \in C^{0,1}$ such that $\bar{\Gamma}_0$ lies strictly inside $\Omega$. We say that a body with a plane crack occupies the domain $\Omega_0 = \Omega \setminus \bar{\Gamma}_0$, where $\bar{\Gamma}_0 = \Gamma_0 \cup \gamma_0$.

We use the linear Hooke law for the stress and strain tensors, namely

$$\sigma_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u), \quad \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3$$

with constant, elliptic and symmetric elasticity coefficients, which satisfy

$$c_1 \xi_{ij}\xi_{ij} \leqslant c_{ijkl}\xi_{kl}\xi_{ij} \leqslant c_2 \xi_{ij}\xi_{ij} \quad \forall \xi \in \mathbb{R}^3 \times \mathbb{R}^3$$

$$c_{ijkl} = c_{jikl} = c_{klij}, \quad i, j, k, l = 1, 2, 3$$

with two constants $c_1, c_2 > 0$. For the displacement vector $u = (u_1, u_2, u_3)$ let us define the admissible functional space by

$$\tilde{H}^1(\Omega_0) = \{u \in [H^1(\Omega_0)]^3,\ u = 0 \text{ on } \Gamma\}$$

which obeys the zero displacement condition at the external boundary $\Gamma$.

We suppose the smooth volume load $f = (f_1, f_2, f_3)$, $f_i \in C^2(\bar{\Omega})$, $i = 1, 2, 3$, is given and define the potential energy functional depending on the domain by

$$\Pi(u; \Omega_0) = \tfrac{1}{2} \int_{\Omega_0} \sigma_{ij}(u)\varepsilon_{ij}(u) - \int_{\Omega_0} f_i u_i$$

In view of the well-known properties of $\Pi$, which is coercive when $u = 0$ at $\Gamma$, its minimization on $\tilde{H}^1(\Omega_0)$ possesses the unique solution $u^0 \in \tilde{H}^1(\Omega_0)$ of an equilibrium problem in the usual weak sense:

$$\int_{\Omega_0} \sigma_{ij}(u^0)\varepsilon_{ij}(v) = \int_{\Omega_0} f_i v_i \quad \forall v \in \tilde{H}^1(\Omega_0) \tag{1}$$

The boundary value problem for $u^0$ corresponding to (1) has the form

$$-\sigma_{ij,j}(u^0) = f_i, \quad i = 1, 2, 3, \quad \text{in } \Omega_0$$

$$\sigma_{i3}(u^0) = 0, \quad i = 1, 2, 3, \quad \text{on } \Gamma_0; \quad u^0 = 0 \quad \text{on } \Gamma$$

 

Let us take a small perturbation of the crack front as the function $h \in C^1([0,2\pi])$ with $h(0)=h(2\pi)$, $h'(0)=h'(2\pi)$. Then the perturbed crack $\Gamma_h$ will be bounded by the perturbed crack front

$$\gamma_h = \{r = R(\phi) + h(\phi),\ \ \phi \in [0,2\pi]\}$$

and will lie again on the plane $\mathbb{R}^2$. Note that the smooth perturbation $h$ does not add new singularities to the initial crack front $\gamma_0$, but retains its $C^{0,1}$-property. We assume that $\gamma_h$ is close to $\gamma_0$ and that both are located inside some smooth sub-domain $\omega \subset \mathbb{R}^3$ of $\Omega$ excluding the origin $\{0\}$, i.e. $\bar{\omega} \subset \Omega$, $\{0\} \notin \omega$, for example $\omega$ is a tor surrounding $\gamma_0$. For the perturbed domain $\Omega_h = \Omega \backslash \bar{\Gamma}_h$ with the perturbed crack $\Gamma_h$, by the same reason as before, there exists a unique solution $u^h \in \tilde{H}^1(\Omega_h)$ of the equilibrium problem

$$\Pi(u^h;\Omega_h) = \inf_{u \in \tilde{H}^1(\Omega_h)} \Pi(u;\Omega_h)$$

with

$$\Pi(u;\Omega_h) = \frac{1}{2}\int_{\Omega_h} \sigma_{ij}(u)\varepsilon_{ij}(u) - \int_{\Omega_h} f_i u_i$$

$$\tilde{H}^1(\Omega_h) = \{u \in [H^1(\Omega_h)]^3,\ \ u=0 \text{ on } \Gamma\}$$

which is equivalent to the variational equation

$$\int_{\Omega_h} \sigma_{ij}(u^h)\varepsilon_{ij}(v) = \int_{\Omega_h} f_i v_i \quad \forall v \in \tilde{H}^1(\Omega_h) \tag{2}$$

## 3. VARIATION OF A SOLUTION

We are now going to transform the perturbed domain $\Omega_h$ onto $\Omega_0$ by some co-ordinate transformation $\Lambda$. To construct $\Lambda$, let us choose first a smooth cut-off function $\eta$ such that $\text{supp}\{\eta\} \subset (\Omega \backslash \{0\})$ and $\eta \equiv 1$ in $\omega$. By choosing $\omega$ and $\eta$, we can assume that every point $x=(x_1,x_2,x_3) \in \text{supp}\{\eta\}$ defines uniquely the polar radius $r(x_1,x_2)$ and the polar angle $\phi(x_1,x_2)$ in $\mathbb{R}^2$ satisfying the relations

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi$$

Construct the co-ordinate transformation

$$\Lambda^{-1} = \begin{cases} y_1 = x_1 + h(\phi(x_1,x_2))\cos\phi(x_1,x_2)\eta(x) \\ y_2 = x_2 + h(\phi(x_1,x_2))\sin\phi(x_1,x_2)\eta(x) \\ y_3 = x_3 \end{cases} \tag{3}$$

for $x \in \Omega_0$ and $y \in \Omega_h$. We use for simplicity the notation

$$\theta = \frac{1}{r}\eta, \quad \theta_\alpha = x_\alpha \theta, \quad \alpha = 1,2$$

 

Then the functional matrix takes the form

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{pmatrix} 1 + (h\theta_1)_{,1} & (h\theta_1)_{,2} & (h\theta_1)_{,3} \\ (h\theta_2)_{,1} & 1 + (h\theta_2)_{,2} & (h\theta_2)_{,3} \\ 0 & 0 & 1 \end{pmatrix}$$

Its Jacobian is

$$J = 1 + (h\theta_\alpha)_{,\alpha} + \tfrac{1}{2}h^2(\theta\theta_\alpha)_{,\alpha}$$

due to $x_\alpha\phi_{,\alpha} = 0$, and, moreover, the term $h_{,\alpha}\theta_\alpha$ in $J$ is also equal to zero. Then it is easily seen that $J > 0$ for all $h$ with $\|h\|_{C([0,2\pi])} \leqslant \delta$ and some $\delta > 0$ small enough. Therefore, the correspondence $\Lambda^{-1}$ given by (3) is one to one and its inverse $\Lambda$ maps $\Omega_h$ onto $\Omega_0$. The inverse functional matrix can be expressed as

$$\frac{\partial(x_1, x_2, x_3)}{\partial y_1} = \left(1 - \frac{1}{J}(h\theta_1)_{,1} - \frac{1}{2J}\, h^2(\theta\theta_\alpha)_{,\alpha}, \quad -\frac{1}{J}(h\theta_2)_{,1}, \quad 0\right)$$
$$\frac{\partial(x_1, x_2, x_3)}{\partial y_2} = \left(-\frac{1}{J}(h\theta_1)_{,2}, \quad 1 - \frac{1}{J}(h\theta_2)_{,2} - \frac{1}{2J}\, h^2(\theta\theta_\alpha)_{,\alpha}, \quad 0\right)$$
$$\frac{\partial(x_1, x_2, x_3)}{\partial y_3} = \left(-\frac{1}{J}(h\theta_1)_{,3} - \frac{1}{2J}\, h^2(\theta\theta_1)_{,3}, -\frac{1}{J}(h\theta_2)_{,3} - \frac{1}{2J}\, h^2(\theta\theta_2)_{,3}, 1\right)$$

In what follows, we will denote $\Lambda \circ u$ by 'hat' for simplicity, i.e. in accordance with (3)

$$u(y) = u(x_1 + h\theta_1, x_2 + h\theta_2, x_3) \equiv \hat{u}(x), \quad x \in \Omega_0, \quad y \in \Omega_h$$

From the inverse matrix one gets the transformation

$$u_{,i} = \hat{u}_{,i} - \frac{1}{J}(h\theta_\alpha)_{,i}\hat{u}_{,\alpha} - \frac{1}{2J}\, h^2[(\theta\theta_\alpha)_{,\alpha}\hat{u}_{,i}(\delta_{i1} + \delta_{i2}) + (\theta\theta_\alpha)_{,3}\hat{u}_{,\alpha}\delta_{i3}]$$

for $i = 1, 2, 3$. This yields the transformation of the strain tensor

$$\varepsilon_{ij}(u) = \varepsilon_{ij}(\hat{u}) - \frac{1}{J}\, E_{ij}(h\theta_\alpha; \hat{u}_{,\alpha}) - \frac{1}{2J}\, h^2 M_{ij}(\theta^2; \hat{u}), \quad i, j = 1, 2, 3 \tag{4}$$

with the forms

$$E_{ij}(\xi; w) = \tfrac{1}{2}(\xi_{,i}w_j + \xi_{,j}w_i), \quad w = (w_1, w_2, w_3)$$
$$M_{ij}(\theta^2; w) = \tfrac{1}{2}[(\theta\theta_\alpha)_{,\alpha}(w_{i,j}(\delta_{j1} + \delta_{j2}) + w_{j,i}(\delta_{i1} + \delta_{i2}))$$
$$+ (\theta\theta_\alpha)_{,3}(w_{i,\alpha}\delta_{j3} + w_{j,\alpha}\delta_{i3})]$$

The stress tensor admits the same representation

$$\sigma_{ij}(u) = \sigma_{ij}(\hat{u}) - \frac{1}{J}\Sigma_{ij}(h\theta_\alpha; \hat{u}_{,\alpha}) - \frac{1}{2J}\, h^2 N_{ij}(\theta^2; \hat{u}), \quad i, j = 1, 2, 3 \tag{5}$$

 

where

$$\Sigma_{ij}(\xi;w)=c_{ijkl}E_{kl}(\xi;w),\quad N_{ij}(\theta^2;w)=c_{ijkl}M_{kl}(\theta^2;w)$$

Applying transformation (3) to the integrals in (2), in view of (4) and (5) we deduce from (2) the following problem:

$$\int_{\Omega_0} J\left(\sigma_{ij}(\hat{u}^h)-\frac{1}{J}\Sigma_{ij}(h\theta_\alpha;\hat{u}^h_{,\alpha})-\frac{1}{2J}\,h^2N_{ij}(\theta^2;\hat{u}^h)\right)$$
$$\times\left(\varepsilon_{ij}(v)-\frac{1}{J}\,E_{ij}(h\theta_\beta;v_{,\beta})-\frac{1}{2J}\,h^2M_{ij}(\theta^2;v)\right)$$
$$=\int_{\Omega_0} J\hat{f}_i v_i\quad\forall v\in\tilde{H}^1(\Omega_0)\tag{6}$$

with $\hat{u}^h$ being the transformed solution $u^h$ of (2). Therefore, the one-to-one correspondence defined by $\Lambda$ provides the following theorem.

*Theorem 1*
The function $\hat{u}^h\in\tilde{H}^1(\Omega_0)$ is a unique solution of the variational equation (6).

Moreover, by the coercivity of the quadratic form $\int_{\Omega_0}\sigma_{ij}(u)\varepsilon_{ij}(u)$, it can be seen that the uniform estimate follows from (6)

$$\|\hat{u}^h\|_{\tilde{H}^1(\Omega_0)}\leqslant\text{const}\tag{7}$$

for all $h$ with $\|h\|_{C^1([0,2\pi])}\leqslant\delta$, and if $\delta>0$ is small enough.

Using (3), $\hat{f}$ can be decomposed in a standard way as

$$\hat{f}=f(x_1+h\theta_1,x_2+h\theta_2,x_3)=f+h\theta_\alpha f_{,\alpha}+\tfrac{1}{2}h^2\theta_\alpha\theta_\beta f_{,\alpha\beta}+O(h^3)\tag{8}$$

From $x_\alpha\phi_{,\alpha}=0$ it follows that $J=1+h\theta_{\alpha,\alpha}+\frac{1}{2}h^2(\theta\theta_\alpha)_{,\alpha}$ and consequently

$$\int_{\Omega_0} J\hat{f}_i v_i=\int_{\Omega_0}(f_i+h(\theta_\alpha f_i)_{,\alpha}+\tfrac{1}{2}h^2[(\theta_\alpha\theta_\beta f_{i,\beta})_{,\alpha}+(\theta\theta_\alpha)_{,\alpha}f_i])v_i+O(h^3)\tag{9}$$

Using the definition of the forms $M,N,E,\Sigma$ in dependence on $(h,h')$, we can decompose the operator in (6) with respect to $h$ as

$$\int_{\Omega_0} J\left(\sigma_{ij}(u)-\frac{1}{J}\Sigma_{ij}(h\theta_\alpha;u_{,\alpha})-\frac{1}{2J}\,h^2N_{ij}(\theta^2;u)\right)$$
$$\times\left(\varepsilon_{ij}(v)-\frac{1}{J}\,E_{ij}(h\theta_\beta;v_{,\beta})-\frac{1}{2J}\,h^2M_{ij}(\theta^2;v)\right)$$
$$=\int_{\Omega_0}\sigma_{ij}(u)\varepsilon_{ij}(v)+A^1(h;u,v)+\tfrac{1}{2}A^2(h^2;u,v)+O((h,h')^3)\tag{10}$$

 

with the new bilinear forms $A^1(h;\cdot,\cdot)$ and $A^2(h^2;\cdot,\cdot)$ given by

$$A^1(h;u,v)=\int_{\Omega_0}\big(h\theta_{\alpha,\alpha}\sigma_{ij}(u)\varepsilon_{ij}(v)-\Sigma_{ij}(h\theta_\alpha;u_{,\alpha})\varepsilon_{ij}(v)-\sigma_{ij}(u)E_{ij}(h\theta_\alpha;v_{,\alpha})\big)$$

$$A^2(h^2;u,v)=\int_{\Omega_0}\big(h^2[(\theta\theta_\alpha)_{,\alpha}\sigma_{ij}(u)\varepsilon_{ij}(v)-N_{ij}(\theta^2;u)\varepsilon_{ij}(v)$$
$$-\sigma_{ij}(u)M_{ij}(\theta^2;v)]+2\Sigma_{ij}(h\theta_\alpha;u_{,\alpha})E_{ij}(h\theta_\beta;v_{,\beta})\big)\quad u,v\in\tilde{H}^1(\Omega_0)\tag{11}$$

We seek an expansion of $\hat{u}^h$ with respect to $h$ in the form

$$\hat{u}^h=u^0+\dot{u}(h)+\tfrac{1}{2}\ddot{u}(h^2)+o(h^2)\tag{12}$$

where $\dot{u}$ and $\ddot{u}$ denote the first and the second global variations of the solution $u^0$, respectively. Substituting formally (12) into (6) and applying decompositions (9) and (10), we are able to find the derivatives $\dot{u}(h)$ and $\ddot{u}(h^2)$ from the space $\tilde{H}^1(\Omega_0)$ as the unique solutions of the corresponding problems

$$\int_{\Omega_0}\sigma_{ij}(\dot{u}(h))\varepsilon_{ij}(v)=\int_{\Omega_0}h(\theta_\alpha f_i)_{,\alpha}v_i-A^1(h;u^0,v)\quad\forall v\in\tilde{H}^1(\Omega_0)\tag{13}$$

and

$$\int_{\Omega_0}\sigma_{ij}(\ddot{u}(h^2))\varepsilon_{ij}(v)=\int_{\Omega_0}h^2[(\theta_\alpha\theta_\beta f_{i,\beta})_{,\alpha}+(\theta\theta_\alpha)_{,\alpha}f_i]v_i$$
$$-2A^1(h;\dot{u}(h),v)-A^2(h^2;u^0,v)\quad\forall v\in\tilde{H}^1(\Omega_0)\tag{14}$$

One can see that the right-hand sides of (13) and (14) define at least linear continuous functionals on $\tilde{H}^1(\Omega_0)$, so that these problems are stated correctly. Let us also note that (13) and (14) are the same elasticity problems as problem (1), but with different right-hand sides specified in correspondence with the sequence of the derivatives $u^0,\dot{u},\ddot{u}$. Of course, one can continue this procedure in the same manner for the higher-order derivatives $\dddot{u}$ and so on.

In what follows, we prove the correctness of expansion (12). After subtraction of (1) from (6), we have in view of (9) and (10)

$$\int_{\Omega_0}\sigma_{ij}(\hat{u}^h-u^0)\varepsilon_{ij}(v)=\int_{\Omega_0}h(\theta_\alpha f_i)_{,\alpha}v_i-A^1(h;\hat{u}^h,v)+O((h,h')^2)$$

Substituting $v$ with $\hat{u}^h-u^0$ and using the Korn and Hölder inequalities together with (7) yield the estimate

$$\|\hat{u}^h-u^0\|_{\tilde{H}^1(\Omega_0)}\leqslant c\|h\|_{C^1([0,2\pi])}\tag{15}$$

Subtracting (1) and (13) from (6), we get with the help of representations (9) and (10)

$$\int_{\Omega_0}\sigma_{ij}(\hat{u}^h-u^0-\dot{u}(h))\varepsilon_{ij}(v)=\frac{1}{2}\int_{\Omega_0}h^2[(\theta_\alpha\theta_\beta f_{i,\beta})_{,\alpha}+(\theta\theta_\alpha)_{,\alpha}f_i]v_i$$
$$-A^1(h;\hat{u}^h-u^0,v)-\tfrac{1}{2}A^2(h^2;\hat{u}^h,v)+O((h,h')^3)$$

Consequently, using (7) and (15), the estimate follows

$$\|\hat{u}^h - u^0 - \dot{u}(h)\|_{\tilde{H}^1(\Omega_0)} \leqslant c\|h\|^2_{C^1([0,2\pi])} \tag{16}$$

Considering analogously the next terms in expansions (9) and (10), the subtraction of (1), (13) and (14), multiplied with 1/2, from (6) provides in view of (7), (15) and (16) the estimate

$$\|\hat{u}^h - u^0 - \dot{u}(h) - \tfrac{1}{2}\ddot{u}(h^2)\|_{\tilde{H}^1(\Omega_0)} \leqslant c\|h\|^3_{C^1([0,2\pi])} \tag{17}$$

We summarize the above consideration in the following theorem.

*Theorem 2*
For $\|h\|_{C^1([0,2\pi])} \leqslant \delta$ with $\delta$ sufficiently small, the strong derivatives $\dot{u}(h)$ and $\ddot{u}(h^2)$ exist and can be found as the solutions of the elasticity problems (13) and (14), respectively. Consequently, expansion (12) holds together with estimates (15)–(17).

## 4. VARIATION OF THE POTENTIAL ENERGY

Substituting the solution $u^h$ of (2) into the potential energy functional $\Pi(\cdot\,;\Omega_h)$, for every $h \in C^1([0,2\pi])$ with $h(0)=h(2\pi)$, $h'(0)=h'(2\pi)$, we define a functional $\mathscr{P}: C^1([0,2\pi]) \to \mathbb{R}$ by

$$\mathscr{P}(h) \equiv \Pi(u^h;\Omega_h) = -\frac{1}{2}\int_{\Omega_h} f_i u_i^h \tag{18}$$

due to the property

$$\int_{\Omega_h} \sigma_{ij}(u^h)\varepsilon_{ij}(u^h) = \int_{\Omega_h} f_i u_i^h$$

obtained from (2) with the test function $v = u^h$. Analogously, substituting the solution $u^0$ of (1) into $\Pi(\cdot\,;\Omega_0)$, we have

$$\mathscr{P}(0) \equiv \Pi(u^0;\Omega_0) = -\frac{1}{2}\int_{\Omega_0} f_i u_i^0 \tag{19}$$

If we now transform the integral in (18) by using transformation (3), we obtain the relation

$$\mathscr{P}(h) = -\frac{1}{2}\int_{\Omega_0} J\hat{f}_i\hat{u}_i^h \tag{20}$$

By Theorem 2 we can substitute here expansion (12), deriving together with (9) the representation

$$\mathscr{P}(h) = \mathscr{P}(0) + \mathscr{P}_0'(h) + \tfrac{1}{2}\mathscr{P}_0''(h^2) + o((h,h')^2) \tag{21}$$

with the derivatives

$$\mathscr{P}_0'(h) = -\frac{1}{2}\int_{\Omega_0} (h(\theta_\alpha f_i)_{,\alpha} u_i^0 + f_i\dot{u}_i(h)) \tag{22}$$

 

$$\mathscr{P}_0''(h^2) = -\frac{1}{2}\int_{\Omega_0} (h^2[(\theta_\alpha\theta_\beta f_{i,\beta})_{,\alpha} + (\theta\theta_\alpha)_{,\alpha} f_i]u_i^0 + 2h(\theta_\alpha f_i)_{,\alpha}\dot{u}_i(h) + f_i\ddot{u}_i(h^2)) \tag{23}$$

Note that the integrals in (18) and (19) do not depend on the cut-off function $\eta$, and therefore, in spite of the occurrence of $\eta$ in expressions (22) and (23), all the derivatives in expansion (21) are also independent of the cut-off function. This means that if one takes two different cut-off functions $\eta^1$ and $\eta^2$ into the expression for the derivatives of the potential energy, the values of these integrals must be equal.

The estimates for the derivatives of the potential energy can be easily deduced from representation (8), the smoothness of $f$ and estimates (7), (15)–(17). Indeed, subtracting (19) from (20), it follows from (7), (8) and (15) that

$$|\mathscr{P}(h) - \mathscr{P}(0)| = \frac{1}{2}\left|\int_{\Omega_0} (f_i(\hat{u}_i^h - u_i^0) + (\hat{f}_i - f_i)\hat{u}_i^h + [h\theta_{\alpha,\alpha} + \tfrac{1}{2}h^2(\theta\theta_\alpha)_{,\alpha}]\hat{f}_i\hat{u}_i^h)\right| \leqslant c\|h\|_{C^1([0,2\pi])} \tag{24}$$

Analogously, subtracting (19) and (22) from (20), we have

$$|\mathscr{P}(h) - \mathscr{P}(0) - \mathscr{P}_0'(h)| = \frac{1}{2}\left|\int_{\Omega_0} (f_i(\hat{u}_i^h - u_i^0 - \dot{u}_i(h)) + (\hat{f}_i - f_i - h\theta_\alpha f_{i,\alpha})\hat{u}_i^h + h\theta_\alpha f_{i,\alpha}(\hat{u}_i^h - u_i^0) + \tfrac{1}{2}h^2(\theta\theta_\alpha)_{,\alpha}\hat{f}_i\hat{u}_i^h + h\theta_{\alpha,\alpha}[f_i(\hat{u}_i^h - u_i^0) + (\hat{f}_i - f_i)\hat{u}_i^h])\right| \leqslant c\|h\|^2_{C^1([0,2\pi])} \tag{25}$$

due to (7), (8), (15) and (16). The same consideration with (23) provides the next estimate

$$|\mathscr{P}(h) - \mathscr{P}(0) - \mathscr{P}_0'(h) - \tfrac{1}{2}\mathscr{P}_0''(h^2)| \leqslant c\|h\|^3_{C^1([0,2\pi])} \tag{26}$$

We can now reduce the order of the variations of $u^0$ included in formulas (22) and (23). Substituting $v = \dot{u}(h)$ in (1) and $v = u^0$ in (13), we deduce the relation

$$\int_{\Omega_0} f_i\dot{u}_i(h) = \int_{\Omega_0} \sigma_{ij}(u^0)\varepsilon_{ij}(\dot{u}(h)) = \int_{\Omega_0} \sigma_{ij}(\dot{u}(h))\varepsilon_{ij}(u^0) = \int_{\Omega_0} h(\theta_\alpha f_i)_{,\alpha}u_i^0 - A^1(h; u^0, u^0)$$

Therefore, (22) yields

$$\mathscr{P}_0'(h) = -\int_{\Omega_0} h(\theta_\alpha f_i)_{,\alpha}u_i^0 + \tfrac{1}{2}A^1(h; u^0, u^0) \tag{27}$$

 

Analogously, taking $v=\ddot{u}(h^2)$ in (1) and $v=u^0$ in (14), we have

$$\int_{\Omega_0} f_i\ddot{u}_i(h^2)=\int_{\Omega_0} h^2[(\theta_\alpha\theta_\beta f_{i,\beta})_{,\alpha}+(\theta\theta_\alpha)_{,\alpha}f_i]u_i^0 \\ -2A^1(h;\dot{u}(h),u^0)-A^2(h^2;u^0,u^0)$$

and get, due to (23), that

$$\mathscr{P}_0''(h^2)=-\frac{1}{2}\int_{\Omega_0}(h^2[(\theta_\alpha\theta_\beta f_{i,\beta})_{,\alpha}+(\theta\theta_\alpha)_{,\alpha}f_i]u_i^0+h(\theta_\alpha f_i)_{,\alpha}\dot{u}_i(h)) \\ +A^1(h;\dot{u}(h),u^0)+\tfrac{1}{2}A^2(h^2;u^0,u^0)$$

Moreover, using (13) with $v=\dot{u}(h)$, we obtain

$$\int_{\Omega_0}\sigma_{ij}(\dot{u}(h))\varepsilon_{ij}(\dot{u}(h))=\int_{\Omega_0}h(\theta_\alpha f_i)_{,\alpha}\dot{u}_i(h)-A^1(h;u^0,\dot{u}(h))$$

so that we get finally

$$\mathscr{P}_0''(h^2)=-\int_{\Omega_0}(h^2[(\theta_\alpha\theta_\beta f_{i,\beta})_{,\alpha}+(\theta\theta_\alpha)_{,\alpha}f_i]u_i^0+\sigma_{ij}(\dot{u}(h))\varepsilon_{ij}(\dot{u}(h))) \\ +\tfrac{1}{2}A^2(h^2;u^0,u^0) \qquad (28)$$

As a result, the next theorem follows.

*Theorem 3*
For $\|h\|_{C^1([0,2\pi])}\leqslant\delta$ with $\delta$ sufficiently small, the strong derivatives $\mathscr{P}_0'(h)$ and $\mathscr{P}_0''(h^2)$ of the potential energy functional $\mathscr{P}$ with respect to the front shape $h$ in 0 exist and can be found by formulas (27) and (28). Consequently, expansion (21) holds together with estimates (24)–(26).

In addition to the potential energy we introduce the surface energy $\mathscr{S}$, given by the Griffith hypothesis as

$$\mathscr{S}(h)=\gamma\,\mathrm{meas}\,\Gamma_h,\quad \gamma>0,\quad \mathrm{meas}\,\Gamma_h=\frac{1}{2}\int_0^{2\pi}(R+h)^2\,\mathrm{d}\phi$$

Its initial value, corresponding to the unperturbed crack $\Gamma_0$, is equal to

$$\mathscr{S}(0)=\gamma\mathrm{meas}\,\Gamma_0,\quad \mathrm{meas}\,\Gamma_0=\frac{1}{2}\int_0^{2\pi}R^2\,\mathrm{d}\phi$$

The sum of the potential energy $\mathscr{P}$ and the surface energy $\mathscr{S}$ gives us the total potential energy $\mathscr{T}$ as a functional $\mathscr{T}:C^1([0,2\pi])\to\mathbb{R}$ defined by $\mathscr{T}(h)=\mathscr{P}(h)+\mathscr{S}(h)$. By Theorem 3 we have the following expansion:

$$\mathscr{T}(h) = \mathscr{T}(0) + \mathscr{P}_0'(h) + \gamma \int_0^{2\pi} Rh \, \mathrm{d}\phi + \frac{1}{2}\left(\mathscr{P}_0''(h^2) + \gamma \int_0^{2\pi} h^2 \, \mathrm{d}\phi\right) + o(\|h\|^2_{C^1([0,2\pi])}) \tag{29}$$

Using the representation

$$E_{ij}(h\xi; w) = hE_{ij}(\xi; w) + \xi h' E_{ij}(\phi; w), \quad w = (w_1, w_2, w_3), \quad i,j = 1,2,3$$

together with (11), we can rewrite $\mathscr{P}_0'$ as a linear continuous functional $\mathscr{L}_1 : C^1([0,2\pi]) \to \mathbb{R}$ by

$$\mathscr{L}_1(h) = \int_{\Omega_0} ([-(\theta_\alpha f_i)_{,\alpha} u_i^0 + \tfrac{1}{2}\theta_{\alpha,\alpha}\sigma_{ij}(u^0)\varepsilon_{ij}(u^0) - \Sigma_{ij}(\theta_\alpha; u^0_{,\alpha})\varepsilon_{ij}(u^0)]h - \theta_\alpha \Sigma_{ij}(\phi; u^0_{,\alpha})\varepsilon_{ij}(u^0)h') \tag{30}$$

The second derivative $\mathscr{P}_0''$ is associated with the following bilinear, continuous, symmetric form $\mathscr{L}_2 : C^1([0,2\pi]) \times C^1([0,2\pi]) \to \mathbb{R}$,

$$\begin{aligned}\mathscr{L}_2(h_1,h_2) = \int_{\Omega_0} (&-[(\theta_\alpha\theta_\beta f_{i,\beta})_{,\alpha} + (\theta\theta_\alpha)_{,\alpha} f_i]u_i^0 + \tfrac{1}{2}(\theta\theta_\alpha)_{,\alpha}\sigma_{ij}(u^0)\varepsilon_{ij}(u^0) \\ &- N_{ij}(\theta^2; u^0)\varepsilon_{ij}(u^0) + \Sigma_{ij}(\theta_\alpha; u^0_{,\alpha})E_{ij}(\theta_\beta; u^0_{,\beta})]h_1 h_2 \\ &+ \Sigma_{ij}(\theta_\alpha; u^0_{,\alpha})\theta_\beta E_{ij}(\phi; u^0_{,\beta})(h_1 h_2)' + \theta_\alpha \Sigma_{ij}(\phi; u^0_{,\alpha})\theta_\beta E_{ij}(\phi; u^0_{,\beta})h_1' h_2' \\ &- \sigma_{ij}(\dot{u}(h_1))\varepsilon_{ij}(\dot{u}(h_2)))\end{aligned} \tag{31}$$

where $\dot{u}(h_1)$ and $\dot{u}(h_2)$ are the solutions for $h = h_1$ and $h = h_2$, respectively, of the elasticity problem

$$\begin{aligned}\int_{\Omega_0} \sigma_{ij}(\dot{u}(h))\varepsilon_{ij}(v) = \int_{\Omega_0} (&[(\theta_\alpha f_i)_{,\alpha} v_i - \theta_{\alpha,\alpha}\sigma_{ij}(u^0)\varepsilon_{ij}(v) \\ &+ \Sigma_{ij}(\theta_\alpha; u^0_{,\alpha})\varepsilon_{ij}(v) + \sigma_{ij}(u^0)E_{ij}(\theta_\alpha; v_{,\alpha})]h \\ &+ \theta_\alpha[\Sigma_{ij}(\phi; u^0_{,\alpha})\varepsilon_{ij}(v) + \sigma_{ij}(u^0)E_{ij}(\phi; v_{,\alpha})]h') \quad \forall v \in \tilde{H}^1(\Omega_0)\end{aligned} \tag{32}$$

The solution $\dot{u}(h)$ defines a linear continuous mapping from $C^1([0,2\pi])$ to $\tilde{H}^1(\Omega_0)$.

The crucial point is that the properties of positiveness and convexity of $\mathscr{L}_2$ are in general unknown.

## 5. DISCRETE SHAPE OPTIMIZATION

Let us choose some finite basis $\{h_k\}_{k=1}^N$ in $C^1([0,2\pi])$ of the positive functions $h_k(\phi) \geqslant 0$ such that $h_k(0) = h_k(2\pi)$, $h_k'(0) = h_k'(2\pi)$ and $\bar{\Gamma}_{h_k} \subset \omega$, $k = 1,\ldots,N$. We define with $h_1,\ldots,h_N$

 

some elementary perturbations of the crack front and look for the complete perturbation $h$ as their linear combination with unknown coefficients, i.e.

$$h(\phi) = H_k h_k(\phi), \quad H = (H_1, \ldots, H_N) \in \mathbb{R}_+^N \tag{33}$$

where $\mathbb{R}_+$ denotes the non-negative real numbers. Physically, this means the possibility of the crack growth only.

Substituting (33) as a test function to (30), it follows from the linearity of $\mathscr{L}_1$ that $\mathscr{L}_1(H_k h_k) = H_k \mathscr{L}_1(h_k)$. Therefore we can define the row of the coefficients $\{c_k\}_{k=1}^N$ by

$$c_k = \mathscr{L}_1(h_k), \quad k = 1, \ldots, N, \quad \mathscr{L}_1(H_k h_k) = c_k H_k$$

Solving problem (32) for $h = h_k$, the $N$ functions $\dot{u}(h_1), \ldots, \dot{u}(h_N) \in \tilde{H}^1(\Omega_0)$ can be found as its unique solutions for the corresponding right-hand sides, and by the linearity we have $\dot{u}(H_k h_k) = H_k \dot{u}(h_k)$. Therefore, from (31) one can find the matrix $\{b_{kn}\}_{k,n=1}^N$ with the symmetric coefficients

$$b_{nk} = b_{kn} = \mathscr{L}_2(h_k, h_n), \quad k, n = 1, \ldots, N, \quad \mathscr{L}_2(H_k h_k, H_n h_n) = b_{kn} H_k H_n$$

in view of the symmetry and the bilinearity of $\mathscr{L}_2$ mentioned before. Analogously, the surface energy gives us the row $\{d_k\}_{k=1}^N$ of the positive coefficients

$$d_k = \gamma \int_0^{2\pi} R h_k \, \mathrm{d}\phi, \quad d_k \geqslant 0, \quad k = 1, \ldots, N$$

and the symmetric matrix $\{a_{kn}\}_{k,n=1}^N$ by

$$a_{nk} = a_{kn} = \gamma \int_0^{2\pi} h_k h_n \, \mathrm{d}\phi, \quad k, n = 1, \ldots, N$$

In view of representation (29) we consider an approximate quadratic form of the total potential energy $T : \mathbb{R}^N \to \mathbb{R}$ as

$$\mathscr{T}(H_k h_k) \approx T(H) \equiv \mathscr{T}(0) + (c_k + d_k) H_k + \tfrac{1}{2}(a_{kn} + b_{kn}) H_k H_n$$

By the general variational principle, or by the Griffith criterion in our case, the energy turns out to be minimal on the admissible functions describing the crack front, i.e.

$$T(H) = \inf_{\bar{H} \in \mathbb{R}_+^N} T(\bar{H})$$

which implies the variational inequality

$$(c_n + d_n + (a_{kn} + b_{kn}) H_k)(\bar{H}_n - H_n) \geqslant 0 \quad \forall \bar{H} \in \mathbb{R}_+^N \tag{34}$$

due to the presence of the constraint $H \geqslant 0$. Its unique solvability is guaranteed by the positive definiteness of the corresponding matrix

$$\det \{a_{kn} + b_{kn}\}_{k,n=1}^m > 0, \quad m = 1, \ldots, N$$

respectively,

$$(a_{kn} + b_{kn}) \xi_k \xi_n \geqslant c \xi_k \xi_k \equiv c|\xi|^2, \quad c > 0, \ \forall \xi \in \mathbb{R}^N \tag{35}$$

 

In general, we cannot guarantee that inequality (35) is satisfied due to the possible non-positiveness of the form $\mathscr{L}_2$. Note also that the relation $c_n + d_n \geqslant 0$ for any $n$ leads to the fact that $H \equiv 0$ is a solution of (34). This implies a condition for the non-triviality of a solution: there exists n such that $c_n + d_n < 0$.

To provide the solvability of (34) at least in the local sense, we assume first that a basis $\{h_k\}_{k=1}^N$ exists such that the matrix $\{a_{kn}\}_{k,n=1}^N$ is positive definite, i.e.

$$\det\{a_{kn}\}_{k,n=1}^m > 0, \quad m = 1, \ldots, N \tag{36}$$

Second, let us take the zero load $f \equiv 0$, which leads to $u^0 \equiv 0$, and, consequently, implies $\{b_{kn}\} = 0$ and $\{c_n\} = 0$. It follows then from (34), assumption (36) and the positiveness of $\{d_n\}$ that $H \equiv 0$. Moreover, due to the continuous dependence of the solution $u^0$ of (1) on the right-hand side $f$, it follows from (31) that

$$\det\{a_{kn} + b_{kn}\} = \det\{a_{kn}\} + O(f)$$

Therefore, the whole matrix retains its positive definiteness property (35) at least for small $f$. We formulate this result as the following theorem.

*Theorem 4*
If condition (36) is fulfilled, then at least for $\|f\|_{[L^2(\Omega)]^3}$ small enough, the variational inequality (34) is solvable and its solution minimizes the locally approximate quadratic form $T$ of the total potential energy.

Theorem 4 provides the following quasi-static model of crack propagation. Let us re-scale the load $f(t) = tf$ by the loading parameter $t \geqslant 0$. It follows from (1) that $u^0(t) = tu^0$, and therefore from (30) and (31) that $b_{kn}(t) = t^2 b_{kn}$ and $c_n(t) = t^2 c_n$ for $k, n = 1, \ldots, N$. For each value of $t \geqslant 0$ we set from (34) the quasi-static inequality

$$(t^2 c_n + d_n + (a_{kn} + t^2 b_{kn}) H_k(t))(\bar{H}_n - H_n(t)) \geqslant 0 \quad \forall \bar{H} \in \mathbb{R}_+^N \tag{37}$$

which has a solution $H(t) \in \mathbb{R}_+^N$ at least for small $t$. Again, $H(t) \equiv 0$ when $t^2 c_n + d_n \geqslant 0$, $n = 1, \ldots, N$, i.e. up to the critical value $t_{\text{growth}}$ we have

$$t \leqslant t_{\text{growth}} = \min_{c_n < 0} \sqrt{\frac{d_n}{-c_n}} \quad \Rightarrow \quad H(t) = 0 \tag{38}$$

The value $t = t_{\text{growth}}$ implies the initiation of the possible crack growth.

*Example 1*
We assume the uniform extension of the crack. This model can be described by considering the constant perturbation

$$h(\phi) \equiv A, \quad A > 0, \quad \bar{\Gamma}_h \subset \omega$$

i.e. we have only one basis function $h_N = h$ and $N = 1$. In this case we have four coefficients instead of the corresponding matrices and rows, namely

$$a = 2\pi\gamma A^2 > 0, \quad b = A^2 \mathscr{L}_2(1,1), \quad c = A\mathscr{L}_1(1), \quad d = \gamma A \int_0^{2\pi} R \, \mathrm{d}\phi > 0$$

 

The quasi-static model (37) of the local crack propagation reduces to

$$H(t)\geqslant 0, \quad (t^2c+d+(a+t^2b)H(t))(\bar{H}-H(t))\geqslant 0 \quad \forall \bar{H}\geqslant 0 \tag{39}$$

All properties of (39) are defined by the signs of the coefficients $b$ and $c$. Indeed, if $c\geqslant 0$ (the first derivative of the potential energy is positive), then $H(t)=0$ is its solution for any $t$, and the crack cannot extend. For $c<0$, from (38) the first critical value is $t_{\text{growth}}=\sqrt{d/-c}$, and, hence, $H(t)=0$ up to $t\leqslant t_{\text{growth}}$, i.e. the crack is stationary. For $t>t_{\text{growth}}$, when $b>0$ (the second derivative of the potential energy is positive), the function

$$H(t)=-\frac{t^2c+d}{a+t^2b} \tag{40}$$

is a solution of (39) with the finite asymptotic $-c/b$ for large $t$. When $b=0$, then the solution

$$H(t)=-\frac{c}{a}t^2-\frac{d}{a}$$

has quadratic order of growth. When $b<0$, then there appears the second critical value $t_{\text{critical}}=\sqrt{a/-b}$, and solution (40) exists only up to $t<t_{\text{critical}}$ with infinite asymptotic. If $t_{\text{critical}}\leqslant t_{\text{growth}}$, then in this case we have only the zero solution, too.

*Example 2*
We consider a system of local perturbations given by

$$h_k(\phi)=\begin{cases} A\cos^2\dfrac{\pi(\phi-\phi_k)}{2\delta} & \text{as } \phi\in[\phi_k-\delta,\phi_k+\delta] \\ 0 & \text{otherwise}\end{cases}$$

with $N>1$, $\delta=2\pi/N$, $\phi_k=k\delta$ and $A>0$ being a normalization factor such that $\bar{\Gamma}_{h_k}\subset\omega$, $k=1,\dots,N$. For such a choice of elementary perturbations the matrix $\{a_{kn}\}$ has the form

$$\{a_{kn}\}=\frac{3\gamma A^2\delta}{4}\begin{pmatrix} 1 & \frac{1}{6} & 0 & \dots & 0 \\ \frac{1}{6} & 1 & \frac{1}{6} & \dots & 0 \\ 0 & \frac{1}{6} & 1 & \dots & 0 \\ & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Its determinant

$$\det\{a_{kn}\}_{k,n=1}^{m}=\left(\frac{3\gamma A^2\delta}{4}\right)^m D_m$$

satisfying the recursion relation

$$D_m=D_{m-1}-\tfrac{1}{36}D_{m-2} \quad \text{for } m\geqslant 2, \quad D_0=1, \quad D_1=1$$

is strictly positive and fulfils condition (36). Hence, this case is justified by Theorem 4.

 

Concerning the treatment of the matrix variational inequality (34) in case of positive definiteness of the matrix, we propose the following simple iterative scheme. Let us denote by

$$\xi^+ = \{\xi_n^+\}_{n=1}^N = \left\{ \begin{array}{ll} \xi_n & \text{if } \xi_n \geqslant 0 \\ 0 & \text{if } \xi_n < 0 \end{array} \right\}_{n=1}^N$$

the positive part of the vector $\xi$. It defines an orthonormal projection of $\mathbb{R}^N$ onto $\mathbb{R}_+^N$, namely

$$(\xi_n - \xi_n^+)(\xi_n^+ - \bar{\xi}_n) \geqslant 0 \quad \forall \bar{\xi} \in \mathbb{R}_+^N \tag{41}$$

Now we multiply inequality (34) by an arbitrary $\lambda > 0$ and add $\pm H_n$ to it to deduce

$$(H_n - \lambda(c_n + d_n + (a_{kn} + b_{kn})H_k) - H_n)(H_n - \bar{H}_n) \geqslant 0 \quad \forall \bar{H} \in \mathbb{R}_+^N \tag{42}$$

The comparison of (41) and (42) leads to the conclusion that

$$H_n = (H_n - \lambda(c_n + d_n + (a_{kn} + b_{kn})H_k))^+, \quad n = 1, \ldots, N \tag{43}$$

The linear system of the projection equations (43) can be realized by the iterative procedure

$$H_n^m = (H_n^{m-1} - \lambda(c_n + d_n + (a_{kn} + b_{kn})H_k^{m-1}))^+, \quad n = 1, \ldots, N \quad m = 1, 2, \ldots, \quad H^0 \in \mathbb{R}^N \tag{44}$$

In addition to assumption (35), there always exists a constant $C > 0$ such that

$$|\{(a_{kn} + b_{kn})\xi_k\}_{n=1}^N| \leqslant C|\xi| \quad \forall \xi \in \mathbb{R}^N \tag{45}$$

Then the above considerations imply the following theorem.

*Theorem 5*
Under conditions (35) and (45), the solution $H^m$ of (44) converges to the solution $H$ of (34) as $m \to \infty$ for any value of $\lambda \in (0, 2c/C^2)$, and satisfies the estimate

$$|H^m - H| \leqslant \rho(\lambda)^m |H^0 - H|, \quad \rho(\lambda) = \sqrt{1 - 2c\lambda + C^2\lambda^2} < 1$$

with the optimal value $\rho_{\text{optimal}} = \sqrt{1 - (c/C)^2}$ as $\lambda_{\text{optimal}} = c/C^2$.

*Proof*
We subtract (43) from (44) and square scalarly the difference to get

$$|H^m - H|^2 = |\{(H_n^{m-1} - \lambda(c_n + d_n + (a_{kn} + b_{kn})H_k^{m-1}))^+ \\ - (H_n - \lambda(c_n + d_n + (a_{kn} + b_{kn})H_k))^+\}_{n=1}^N|^2$$

In view of the Lipschitz-continuity of the projection operator,

$$|(\xi^1)^+ - (\xi^2)^+| \leqslant |\xi^1 - \xi^2|$$

 

the estimate

$$
\begin{aligned}
|H^m - H|^2 &\leqslant |\{H_n^{m-1} - H_n - \lambda(a_{kn} + b_{kn})(H_k^{m-1} - H_k)\}_{n=1}^N|^2 \\
&= |H^{m-1} - H|^2 - 2\lambda(a_{kn} + b_{kn})(H_k^{m-1} - H_k)(H_n^{m-1} - H_n) \\
&\quad + \lambda^2|\{(a_{kn} + b_{kn})(H_k^{m-1} - H_k)\}_{n=1}^N|^2 \\
&\leqslant (1 - 2c\lambda + C^2\lambda^2)|H^{m-1} - H|^2
\end{aligned}
$$

follows due to (35) and (45). This proves the theorem. □

ACKNOWLEDGEMENTS

The research results were attained when the author was a fellow of the Alexander von Humboldt Foundation and visiting Professor W.L. Wendland at the Mathematical Institute A of the University of Stuttgart, Germany.

REFERENCES

1. Leguillon D, Sánchez-Palensia E. *Computation of Singular Solutions in Elliptic Problems and Elasticity*. Masson: Paris, 1987.
2. Mazja WG, Nazarow SA, Plamenewski BA. *Asymptotische Theorie elliptischer Randwertaufgaben in Singulär gestörten Gebieten*. Akademie-Verlag: Berlin, 1991.
3. Amestoy M, Leblond JB. Crack path in plane situation. II. Detailed form of the expansion of the stress intensity factors. *International Journal of Solids and Structures* 1992; **29**(4):465–501.
4. Leblond J-B, Mouchrif S-E, Perrin G. The tensile tunnel-crack with a slightly wavy front. *International Journal of Solids and Structures* 1966; **33**(14):1995–2022.
5. Nazarov SA, Polyakova OR. Rupture criteria, asymptotic conditions at crack tips, and selfadjoint extensions of the Lamé operator. *Transactions of the Moscow Mathematical Society* 1996; **57**:13–66.
6. Simon J. Differentiation with respect to the domain in boundary value problems. *Numerical Functional Analysis and Optimization* 1980; **2**:649–687.
7. Sokolowski J, Zolesio J-P. *Introduction to Shape Optimization. Shape Sensitivity Analysis*. Springer: Berlin, 1992.
8. Rice JR. First-order variation in elastic fields due to variation in location of a planar crack front. *Journal of Applied Mechanics* 1985; **52**:571–579.
9. Khludnev AM, Sokolowski J. The Griffith formula and the Cherepanov–Rice integral for crack problems with unilateral conditions in nonsmooth domains. *European Journal of Applied Mathematics* 1999; **10**:379–394.
10. Khludnev AM, Kovtunenko VA. *Analysis of Cracks in Solids*. WIT-Press: Southampton, Boston, 2000.
11. Bach M, Khludnev AM, Kovtunenko VA. Derivatives of the energy functional for 2D-problems with a crack under Signorini and friction conditions. *Mathematical Methods in the Applied Sciences* 2000; **23**:515–534.
12. Kovtunenko VA. Shape sensitivity of curvilinear cracks. *Newton Institute Preprint Series* N199023-SMM, Cambridge, 1999.
13. Kovtunenko VA. Sensitivity of cracks in 2D-Lamé problem via material derivatives. *Journal of Applied Mathematical Physics* (*ZAMP*), 2001; **52**:1071–1087.
14. Kolton LH, Nazarov SA. Quasistatic propagation of a mode—I crack in an elastic space. *Comptes Rendus des Seances de l'Academie des Sciences*, *Serie 2*, 1992; **315**:1453–1457.
15. Nazarov SA. Derivation of the variational inequality for small increase of mode-one crack. *Mechanical Solids* 1989; **24**:145–152.
16. Nazarov SA, Polyakova OR. On the equivalence of fracture criteria for mode-one crack in an elastic space. *Mekhanika Tverdogo Tela* 1992; **24**:101–113.
17. Nemat-Nasser, Sumi Y, Keer LM. Unstable growth of tension cracks in brittle solids: stable and unstable bifurcations, snap-through and imperfection sensitivity. *International Journal of Solids and Structures* 1980; **16**:1017–1033.
18. Nguen Quoc Son. Stabilite at bifurcation en rupture et en plasticite. *Comptes Rendus des Seances de l'Academie des Sciences, Serie* 2 1981; **292**:817–820.

 

19. Parton VZ, Morozov EM. *Mechanics of Elasto-Plastic Fracture*. Nauka: Moscow, 1974 (in Russian).
20. Bach M. Randvariationsungleichungen zur Beschreibung des quasistatischen Rißwachstums eines ebenen 3D-Risses in isotropem Material. *Doctoral Thesis* (*Ph.D.*), Stuttgart University, 1998.
21. Bach M, Nazarov SA, Wendland WL. Propagation of a penny shaped crack under the Irwin criterion. In *Analysis*, *Numerics and Applications of Differential and Integral Equations*, Bach M, Constanda C, Hsiao GC, Sändig A-M, Werner P (eds), Pitman Research Notes in Mathematical Series, vol. 379. Pitman: London, 1998; 17–21.
22. Bach M, Nazarov SA, Wendland WL. Stable propagation of a mode-1 planar crack in an anisotropic elastic media. Comparison of the Irwin and the Griffith approaches. In *Problemi Attuali dell'Analisie della Fisica Mathematica*, Ricci PE (ed). MM ARACNE EDITRICE: Roma, 2000; 167–189.

 