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Investigation of implicit constitutive relations in which both the stress and strain appear linearly, adjacent to non-penetrating cracks

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A novel class of implicit constitutive relations is studied, wherein the stress and the linearized strain appear linearly, that describe material response in elastic porous bodies like rocks, ceramics, concrete, cement, bones and metals. The constitutive relation is applied to a body with a crack subjected to non-penetration conditions between the opposite crack faces. To treat well-posedness of a corresponding variational inequality, we rely on a new approximation by thresholding dilatation and apply the Lions existence theorem on pseudo-monotone variational inequalities. An analytical solution to a specific

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example (without crack) under uniform triaxial loading is constructed, wherein blow-up can take place at a finite load, and this difficulty is overcome within a thresholding approximation so that blow-up does not occur.

Keywords: Nonlinear elasticity; implicit constitutive response; blow-up; crack; non-penetration condition; variational inequality; regularization; well-posedness analysis.

AMS Subject Classification 2020: 35Q74, 35J88, 49J52, 74A20

1. Introduction

Many porous elastic bodies such as rocks, concrete, bones and ceramics and also porous metals have material properties that depend on their density, as density varies inversely as the porosity. Such materials when they undergo small deformations are modeled by a modification of the classical linearized elastic constitutive relation with Young's modulus and Poisson's ratio depending on the density (see Refs. 33, 39 and 40 for a discussion concerning concrete, Refs. 32, 36 and 51 for ceramics, Refs. 14, 24 and 35 for metals, Refs. 13 and 50 for bone and Ref. 12 for rocks). Such a constitutive relation cannot be obtained by linearizing a Cauchy elastic solid. The reason is that by virtue of the balance of mass, any dependence on the density would lead to a dependence on the trace of the linearized strain, and the constitutive relation would no longer be linear in the linearized strain. Rajagopal[41] generalized the class of elastic bodies to include those wherein the stress and the deformation gradient are related by implicit relations (see also Refs. 42 and 43) and Rajagopal and Srinivasa[47] provided a thermodynamic basis for studying the response of such elastic bodies. Recently, Rajagopal showed that within the context of such generalized constitutive relations for elastic bodies, if one linearizes constitutive relations under the assumption that the displacement gradient is small, one can obtain approximations in which the stress and the linearized strain appear linearly, but the constitutive relation is not bilinear (see Ref. 45 for an extended discussion of the linearization and other relevant issues). The class of constitutive relations obtained by linearizing the implicit relations allows one to obtain constitutive relations where the material moduli can depend on the density, thereby allowing one's arsenal of constitutive relations to include those that can describe the response of porous elastic bodies undergoing small deformations in the sense of small displacement gradients, as we expect the material properties of porous materials to depend on the density. If one were to start with a Cauchy elastic body and linearize under the assumption of small displacement gradient, one will inexorably arrive at the classical linearized elastic body whose material moduli are constant. Using the above constitutive relation Murru and Rajagopal[37, 38] studied the stress concentration due to a hole, and Itou *et al.*[19] studied whether the equations that govern the deformation of such bodies are well-posed. The fact that the material moduli can depend on the density, even when the displacement gradient, and hence the strain, is small, can lead to significant differences in the stress concentration factor.

An interesting sub-class of implicit constitutive relations that are obtained by linearizing the general class of implicit constitutive relations to describe the behavior of elastic bodies is given by the following general relation, where both the linearized strain $\boldsymbol{\varepsilon}$ and the Cauchy stress $\boldsymbol{\sigma}$ appear linearly:

$$(1 + \lambda_3 \mathrm{tr}\boldsymbol{\sigma})\boldsymbol{\varepsilon} = E_1(1 + \lambda_1 \mathrm{tr}\boldsymbol{\varepsilon})\boldsymbol{\sigma} + E_2(1 + \lambda_2 \mathrm{tr}\boldsymbol{\varepsilon})(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I},$$

where $\mathbf{I}$ is the identity transformation, and moduli E_1, E_2, λ_1, λ_2, λ_3 are constant. While the stress and the linearized strain appear linearly in the above constitutive relation, it is not a bilinear relation. When we set $\lambda_1 = \lambda_3 = 0$, we obtain the subclass

$$\boldsymbol{\varepsilon} = E_1\boldsymbol{\sigma} + [\alpha(\mathrm{tr}\boldsymbol{\varepsilon})](\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}, \quad \alpha(\mathrm{tr}\boldsymbol{\varepsilon}) = E_2(1 + \lambda_2 \mathrm{tr}\boldsymbol{\varepsilon}). \tag{1.1}$$

Furthermore, when $\lambda_2 = 0$, we recover the equation for classical linearized elasticity and identify

$$E_1 = \frac{1+\nu}{E} = \frac{1}{2\mu} > 0, \quad E_2 = -\frac{\nu}{E}, \tag{1.2}$$

where $E > 0$ and $\nu \in (0, 0.5)$ are Young's modulus and Poisson's ratio. The factor $\alpha(\mathrm{tr}\boldsymbol{\varepsilon})$ in front of $\mathrm{tr}\boldsymbol{\sigma}$ in (1.1) can be viewed as a density-dependent material modulus because $\mathrm{tr}\boldsymbol{\varepsilon}$ can be expressed in terms of the density by virtue of the balance of mass. Thus, in (1.1) both the stress and strain appear linearly (the relation is an affine transformation in the strain and the stress) and describes the small displacement gradient response of porous materials whose material moduli would depend on the density.

On taking the trace of (1.1) we get the implicit equation

$$\mathrm{tr}\boldsymbol{\varepsilon} = 3E_3(1 + \lambda_4 \mathrm{tr}\boldsymbol{\varepsilon})\mathrm{tr}\boldsymbol{\sigma}, \tag{1.3}$$

where the new coefficients are expressed using the bulk modulus K as

$$E_3 = \frac{E_1}{3} + E_2 = \frac{1-2\nu}{3E} = \frac{1}{9K} > 0, \quad \lambda_4 = \frac{E_2}{E_3}\lambda_2 = -\frac{3\nu}{1-2\nu}\lambda_2. \tag{1.4}$$

There have been several rigorous mathematical studies concerning the equations governing the response of implicit constitutive relations. Málek,[34] Blechta *et al.*,[1] Bulíček *et al.*[2–4, 7, 10] have addressed a variety of mathematical issues such as existence of global weak solutions and large data existence theory for the flow of fluids described by implicit constitutive relations for fluids satisfying slip as well as no slip boundary conditions, while Bulíček *et al.*[6, 8, 9] considered the system of partial differential equations governing the response of elastic solids described by implicit constitutive relations, mainly within the context of a body only capable of undergoing limited strains. Bulíček *et al.*[5, 11] and Şengül[48, 49] considered mathematical issues with regard to viscoelastic solids described by implicit constitutive relations. There have also been several numerical studies devoted to the response of strain limited elastic solids described by implicit constitutive relations (see Kulvait *et al.*[27–29]).

Equation (1.3) can be solved with respect to $\mathrm{tr}\boldsymbol{\varepsilon}$, and then its substitution into (1.1) provides us with the explicit constitutive relation

$$\boldsymbol{\varepsilon} = E_1\boldsymbol{\sigma} + [\beta(\mathrm{tr}\boldsymbol{\sigma})](\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}, \quad \beta(\mathrm{tr}\boldsymbol{\sigma}) = E_2\frac{1 + 3E_3(\lambda_2 - \lambda_4)\mathrm{tr}\boldsymbol{\sigma}}{1 - 3E_3\lambda_4\mathrm{tr}\boldsymbol{\sigma}}. \tag{1.5}$$

The linear-fractional transformation β in (1.5) implies a material modulus depending now on the mechanical pressure,[44] that is, the mean normal stress $p = -\mathrm{tr}\boldsymbol{\sigma}/3$. Expressing the linearized strain in terms of the stress requires one to solve the balance equations as well as the constitutive relation simultaneously. Recently, Itou *et al.*[19] studied the corresponding mixed variational formulation for three independent fields of the displacement, the deviatoric and spherical stress. In the present study, based on the implicit equation (1.3) we express the stress in terms of the linearized strain, which makes it possible to achieve the explicit variational problem by substituting the constitutive relation for the stress into the balance of linear momentum, as commonly adopted in the classical literature on continuum mechanics.

It is worth noting that neither $[\alpha(\mathrm{tr}\boldsymbol{\varepsilon})](\mathrm{tr}\boldsymbol{\sigma})$ in the constitutive equations (1.1) nor $[\beta(\mathrm{tr}\boldsymbol{\sigma})](\mathrm{tr}\boldsymbol{\sigma})$ in (1.5) are simultaneously bounded and monotone functions of their arguments. Consequently, none of the known well-posedness theorems are applicable here. By thresholding the term $[\beta(\mathrm{tr}\boldsymbol{\sigma})](\mathrm{tr}\boldsymbol{\sigma})$, well-posedness for an approximate constitutive equation (1.5) was proved in Ref. 19. In the present work, we study the constitutive equation (1.1) when thresholding the modulus α. The thresholding ensures that the calculations are carried out when the $\mathrm{tr}\boldsymbol{\varepsilon}$ entering α is consistent with the assumption of small deformation. In this context, our approach is relevant to a limiting small strain model (see Refs. 11, 15 and 16).

Numerical simulations investigating stress concentration at a circular hole under uniaxial and biaxial loading were recently reported by Murru and Rajagopal.[37, 38] Here, in addition to the constitutive equation being nonlinear we have non-linearity that stems from the inequality type condition of non-penetration between the opposite crack faces following Khludnev and Kovtunenko,[22] Khludnev and Sokolowski.[23] The variational theory of non-penetrating cracks was developed for various contact phenomena at the crack owing to friction,[20] cohesion,[26] for cracks in plates,[30] Boussinesq indentation problem,[18] limiting small strain model[15] and viscoelastic bodies with non-invertible constitutive response.[16, 17] For suitable numerical approaches for constrained crack problems, we cite Refs. 21 and 25. Problems involving fracturing of porous rocks have ramifications to important technological problems such as enhanced oil recovery and carbon dioxide sequestration, while cracks and fractures of bone and ceramic are also very important problems to which the current study is relevant. Yet another application of relevance is the cracking of cement structures.

In Sec. 2 we construct a proper approximation of the constitutive equation (1.1) by thresholding the effect of dilatation appearing in the material moduli. In Sec. 3, the nonlinear crack problem subjected to non-penetration conditions between the opposite crack faces is formulated. Then the theory of monotone variational inequalities is applied to establish its well-posedness. Finally, in Sec. 4 we present an example problem (without crack) under uniform triaxial loading. A solution to the constitutive relation (1.1) is constructed in closed form without thresholding the dilatation, and such a constitutive relation allows a blow-up for a finite load. This is inconsistent with the assumption that the displacement gradients and the strains are small, the assumption under which the constitutive relation is derived in the first place. The same non-physical feature is exhibited by the classical linearized elastic constitutive relation. Thus, one should use (1.1) in general with a great deal of care, ensuring that the solutions obtained make sense, or use a model wherein a thresholding ensures that solutions are such that they are consistent with the derivation of the constitutive relation. We also derive an analytical solution to the approximate problem when thresholding dilatation according to Sec. 2, which overcomes the difficulty of infinite growth, and we compare the two solutions.

2. Thresholding the Dilatation in the Constitutive Relation

Let us apply the volumetric-deviatoric decomposition to the stress tensor as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \frac{1}{3}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}, \quad \mathrm{tr}\boldsymbol{\sigma}^* = 0 \tag{2.1}$$

and the linearized strain tensor

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^* + \frac{1}{3}(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I}, \quad \mathrm{tr}\boldsymbol{\varepsilon}^* = 0. \tag{2.2}$$

With the help of decomposition (2.1), (2.2) and using (1.3), the material response Eq. (1.1) can be decoupled into deviatoric and spherical parts as

$$\boldsymbol{\varepsilon}^* = E_1\boldsymbol{\sigma}^*, \quad \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon} = E_3(1+\lambda_4\mathrm{tr}\boldsymbol{\varepsilon})\mathrm{tr}\boldsymbol{\sigma}, \tag{2.3}$$

where moduli E_1, E_3, λ_4 are defined in (1.2) and (1.4).

In this study, we shall express the $\mathrm{tr}\boldsymbol{\sigma}$ in terms of the $\mathrm{tr}\boldsymbol{\varepsilon}$ in the implicit constitutive relation. When the assumption of small deformation holds (hence $\mathrm{tr}\boldsymbol{\varepsilon}$ is small), following Rajagopal[46] we could replace the latter relation in (2.3) by

$$\frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon} = \frac{E_3}{1-\lambda_4\mathrm{tr}\boldsymbol{\varepsilon}}\mathrm{tr}\boldsymbol{\sigma} \tag{2.4}$$

provided we assume that λ_4 is of order 1. The constant λ_4 could be large, however, in this study, we pick it to be of order 1. In this case, if the following bound holds:

$$|\mathrm{tr}\boldsymbol{\varepsilon}| \le M_0, \quad M_0 > 0, \tag{2.5}$$

then for small constant M_0 the error is $\mathcal{O}(M_0^2)$ using the Landau big $\mathcal{O}$ notation. Moreover, for the example problem under uniform triaxial loading in Sec. 4 we find the solution error of order $\mathcal{O}(M_0^3)$.

It is important to bear in mind that the limiting dilatation in (2.5) cannot be guaranteed without truncation neither from Eq. (2.4), nor from (2.3) (see the

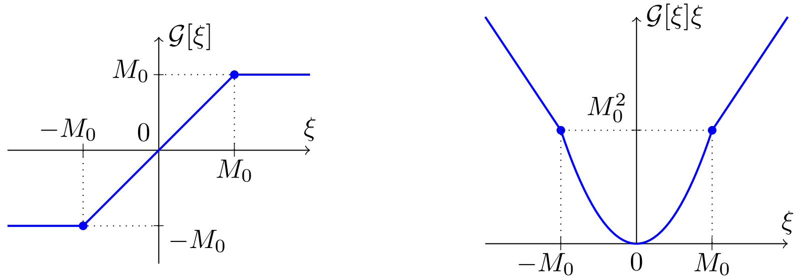

Fig. 1. (Color online) Thresholding function $\mathcal{G}[\xi]$ (left plot) and $\mathcal{G}[\xi]\xi$ (right plot).

example of blow-up in Sec. 4). Therefore, we suggest a thresholding function that is consistent with (2.5) through

$$\mathcal{G}[\xi] = \max\bigl(-M_0, \min(M_0, \xi)\bigr) = \begin{cases} -M_0 & \text{if } \xi < -M_0, \\ \xi & \text{if } |\xi| \le M_0, \\ M_0 & \text{if } \xi > M_0 \end{cases} \tag{2.6}$$

portrayed in the left plot of Fig. 1. Applying (2.6) to relation (2.4) we construct the thresholding dilatation approximation by

$$\boldsymbol{\varepsilon}^* = E_1 \boldsymbol{\sigma}^*, \quad \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon} = \frac{E_3}{1 - \lambda_4 \mathcal{G}[\mathrm{tr}\boldsymbol{\varepsilon}]}\mathrm{tr}\boldsymbol{\sigma}. \tag{2.7}$$

This implies that we fulfill (2.4) if (2.5) holds, else the linearized elastic relations

$$\begin{cases} \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon} = \frac{E_3}{1 + \lambda_4 M_0}\mathrm{tr}\boldsymbol{\sigma} & \text{if } \mathrm{tr}\boldsymbol{\varepsilon} < -M_0, \\ \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon} = \frac{E_3}{1 - \lambda_4 M_0}\mathrm{tr}\boldsymbol{\sigma} & \text{if } \mathrm{tr}\boldsymbol{\varepsilon} > M_0 \end{cases}$$

hold. Composing (2.7) together according to (2.1) and (2.2), and using relations (1.2) and (1.4) which imply that $1/E_1 = 2\mu$ and $1/(9E_3) = K$, we obtain a single equation for the stress expressed explicitly in terms of the strain:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}^* + K(1 - \lambda_4 \mathcal{G}[\mathrm{tr}\boldsymbol{\varepsilon}])(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I}. \tag{2.8}$$

We note that (2.8) cannot be viewed as a constitutive relation, since it is nonlinear in $\boldsymbol{\varepsilon}$, and thus does not conform to the small strain assumption, rather we exploit the explicit expression suitable for mathematical manipulations (see the discussion by Rajagopal[46]).

For further use, we outline the properties of thresholding.

Lemma 2.1. (Properties of $\mathcal{G}$) *The function* $\mathcal{G} : \mathbb{R} \mapsto \mathbb{R}$ *in* (2.6) *is uniformly bounded, continuous, monotone, and* $\mathcal{G}[\xi]\xi$ *has the linear growth:* $\mathcal{G}[\xi]\xi = \mathcal{O}(|\xi|)$.

Proof. Indeed, $\mathcal{G}$ is comprised of three bounded, continuous and monotone pieces, which coincide in the knots $\mathcal{G}[-M_0] = -M_0$ and $\mathcal{G}[M_0] = M_0$ as can be seen in

Fig. 1 in the left plot. In the right plot there is portrayed the function $\mathcal{G}[\xi]\xi$ which evidently satisfies the bound $|\mathcal{G}[\xi]\xi| \leq M_0|\xi|$. This proves the assertion. □

With the help of Lemma 2.1 we determine the properties of the mapping between the strain and stress (2.8) in the space $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ of symmetric 3-by-3 tensors.

Lemma 2.2. (Properties of approximate response $\boldsymbol{\varepsilon} \mapsto \boldsymbol{\sigma}$) *If in* (2.6)

$$M_0 < \frac{1}{2|\lambda_4|}, \tag{2.9}$$

then the mapping $\boldsymbol{\varepsilon} \mapsto \boldsymbol{\sigma} : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \mapsto \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ *in* (2.8) *is bounded*:

$$\|\boldsymbol{\sigma}\| \leq M_1 \|\boldsymbol{\varepsilon}\|, \quad M_1 := \max\bigl(2\mu, 3K(1 + |\lambda_4| M_0)\bigr), \tag{2.10}$$

hemi-continuous: *that implies for* $r \to 0$ *the convergence*

$$2\mu(\boldsymbol{\varepsilon} + r\bar{\boldsymbol{\varepsilon}}) + K\bigl(1 - \lambda_4 \mathcal{G}[\mathrm{tr}(\boldsymbol{\varepsilon} + r\bar{\boldsymbol{\varepsilon}})]\bigr)\mathrm{tr}(\boldsymbol{\varepsilon} + r\bar{\boldsymbol{\varepsilon}})\mathbf{I} \to 2\mu\boldsymbol{\varepsilon} + K(1 - \lambda_4 \mathcal{G}[\mathrm{tr}\boldsymbol{\varepsilon}])(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I}, \tag{2.11}$$

strongly monotone (*where dot stands for the scalar product of the tensors*):

$$(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}) \cdot (\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}) \geq M_2 \|\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}\|^2, \quad M_2 := \min\bigl(2\mu, 3K(1 - 2|\lambda_4| M_0)\bigr) \tag{2.12}$$

with respect to the Frobenius norm $\|\cdot\|$, *for arbitrary* $\boldsymbol{\varepsilon}, \bar{\boldsymbol{\varepsilon}} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ *and*

$$\bar{\boldsymbol{\sigma}} = 2\mu\bar{\boldsymbol{\varepsilon}} + K(1 - \lambda_4 \mathcal{G}[\mathrm{tr}\bar{\boldsymbol{\varepsilon}}])(\mathrm{tr}\bar{\boldsymbol{\varepsilon}})\mathbf{I} \tag{2.13}$$

and coercive:

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} \geq M_3 \|\boldsymbol{\varepsilon}\|^2, \quad M_3 := \min\bigl(2\mu, 3K(1 - |\lambda_4| M_0)\bigr). \tag{2.14}$$

Proof. The hemi-continuity in (2.11) is evident. Based on the volumetric-deviatoric decomposition (2.2), for the Frobenius norm $\|\boldsymbol{\varepsilon}\| = \sqrt{\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}}$ we have the identity

$$\|\boldsymbol{\varepsilon}\|^2 = \|\boldsymbol{\varepsilon}^*\|^2 + \frac{1}{3}|\mathrm{tr}\boldsymbol{\varepsilon}|^2. \tag{2.15}$$

On taking the square of (2.8), by virtue of the threshold $|\mathcal{G}[\mathrm{tr}\boldsymbol{\varepsilon}]| \leq M_0$ and $\|\mathbf{I}\|^2 = 3$, using (2.15) we get the upper bound in a straightforward manner as

$$\|\boldsymbol{\sigma}\|^2 \leq (2\mu\|\boldsymbol{\varepsilon}^*\|)^2 + \frac{1}{3}\bigl(3K(1 + |\lambda_4| M_0)|\mathrm{tr}\boldsymbol{\varepsilon}|\bigr)^2 \leq M_1^2 \|\boldsymbol{\varepsilon}\|^2$$

with the constant $M_1 > 0$ from (2.10). The multiplication of Eq. (2.8) with $\boldsymbol{\varepsilon}$ due to decomposition (2.2) provides the lower bound

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} \geq 2\mu\|\boldsymbol{\varepsilon}^*\|^2 + K(1 - |\lambda_4| M_0)|\mathrm{tr}\boldsymbol{\varepsilon}|^2 \geq M_3 \left(\|\boldsymbol{\varepsilon}^*\|^2 + \frac{1}{3}|\mathrm{tr}\boldsymbol{\varepsilon}|^2 \right),$$

which together with (2.15) and assumption (2.9) implies (2.14) and $M_3 > 0$.

For the property of strong monotonicity (2.12) we multiply the difference of Eqs. (2.8) and (2.13) by $\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}$ such that

$$(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}) \cdot (\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}) = 2\mu\|\boldsymbol{\varepsilon}^* - \bar{\boldsymbol{\varepsilon}}^*\|^2 + K\big\{|\mathrm{tr}(\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}})|^2 - \lambda_4\big(\mathcal{G}[\mathrm{tr}\boldsymbol{\varepsilon}](\mathrm{tr}\boldsymbol{\varepsilon}) - \mathcal{G}[\mathrm{tr}\bar{\boldsymbol{\varepsilon}}](\mathrm{tr}\bar{\boldsymbol{\varepsilon}})\big)\mathrm{tr}(\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}})\big\}. \tag{2.16}$$

Substituting into (2.16) a mean value at intermediate ξ between $\mathrm{tr}\boldsymbol{\varepsilon}$ and $\mathrm{tr}\bar{\boldsymbol{\varepsilon}}$ as

$$\mathcal{G}[\mathrm{tr}\boldsymbol{\varepsilon}](\mathrm{tr}\boldsymbol{\varepsilon}) - \mathcal{G}[\mathrm{tr}\bar{\boldsymbol{\varepsilon}}](\mathrm{tr}\bar{\boldsymbol{\varepsilon}}) = (\mathcal{G}[\xi]\xi)'\mathrm{tr}(\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}), \quad \big|(\mathcal{G}[\xi]\xi)'\big| \leq 2M_0$$

(see the right-hand side plot in Fig. 1 for the derivative of $\mathcal{G}[\xi]\xi$) the lower estimate

$$(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}) \cdot (\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}) \geq 2\mu\|\boldsymbol{\varepsilon}^* - \bar{\boldsymbol{\varepsilon}}^*\|^2 + K(1 - 2|\lambda_4|M_0)|\mathrm{tr}(\boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}})|^2$$

holds and the inequality (2.12) follows with the constant $M_2 > 0$ due to the assumption (2.9). The proof is finished. □

In the next section, we provide the full system of equations, namely the relation (2.8) and the equilibrium equation for a body with a crack and append the governing system with the appropriate boundary conditions.

3. Nonlinear Crack Problem

We start with the geometric description of the reference domain. Let Ω be a domain in the Euclidean space $\mathbb{R}^3$ with the boundary $\partial\Omega$ with the outward unit normal vector $\mathbf{n} = (n_1, n_2, n_3)$. The boundary $\partial\Omega = \overline{\Gamma_\mathrm{N}} \cup \overline{\Gamma_\mathrm{D}}$ is made up of a disjoint union of the Neumann boundary Γ_N and the nonempty Dirichlet boundary Γ_D. We assume an oriented manifold Σ with a unit normal vector $\mathbf{n} = (n_1, n_2, n_3)$, that splits Ω into two sub-domains $\Omega^\pm$ with Lipschitz continuous boundaries $\partial\Omega^\pm$ and $\mathbf{n}$ is inward to Ω^+ and thus outward to Ω^-. A nonempty intersection $\partial\Omega^\pm \cap \overline{\Gamma_\mathrm{D}} \neq \emptyset$ will be needed for a Korn–Poincaré inequality in the sub-domains. We suppose that part $\Gamma_\mathrm{c} \subseteq \Sigma$ of the interface represents a crack inside Ω, where two opposite faces $\Gamma_\mathrm{c}^\pm$ are distinguished as the corresponding parts of $\partial\Omega^\pm$, see a 2D illustration in Fig. 2. The set $\Omega_\mathrm{c} = \Omega \setminus \overline{\Gamma_\mathrm{c}}$ is called the domain with crack.

In the domain with the crack bounded by $\partial\Omega_\mathrm{c} = \partial\Omega \cup \overline{\Gamma_\mathrm{c}^+} \cup \overline{\Gamma_\mathrm{c}^-}$, a displacement vector $\mathbf{u} = (u_1, u_2, u_3)(\mathbf{x})$ is defined over spatial points $\mathbf{x} = (x_1, x_2, x_3)$. In general, the jump across the crack faces is nonzero:

$$[\![\mathbf{u}]\!](\mathbf{x}) := \mathbf{u}\big|_{\mathbf{x}\in\Gamma_\mathrm{c}^+} - \mathbf{u}\big|_{\mathbf{x}\in\Gamma_\mathrm{c}^-}. \tag{3.1}$$

In order to ensure physical consistency, Khludnev and Kovtunenko[22] (see Sec. 1.1.7) impose the normal jump across the crack faces a non-penetration condition:

$$[\![\mathbf{u} \cdot \mathbf{n}]\!](\mathbf{x}) = [\![\mathbf{u}]\!] \cdot \mathbf{n}(\mathbf{x}) \geq 0 \quad \text{for } \mathbf{x} \in \Gamma_\mathrm{c}, \tag{3.2}$$

where $\mathbf{u} \cdot \mathbf{n} = \sum_{i=1}^3 u_i n_i$ is a scalar product of vectors, and $[\![\mathbf{u} \cdot \mathbf{n}]\!] = [\![\mathbf{u}]\!] \cdot \mathbf{n}$ because the normal has no jump. The crack Γ_c is open at point $\mathbf{x}$ where the strict inequality holds in (3.2), else the crack is closed when the equality is in place.

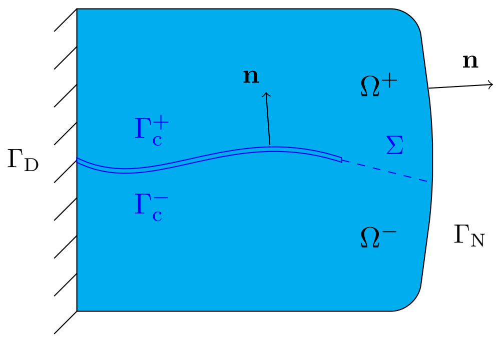

Fig. 2. (Color online) Geometry of a body with a crack.

Let the body force $\mathbf{f} = (f_1, f_2, f_3)(\mathbf{x})$ for $\mathbf{x} \in \Omega_{\mathrm{c}}$, and the boundary traction $\mathbf{g} = (g_1, g_2, g_3)(\mathbf{x})$ for $\mathbf{x} \in \Gamma_{\mathrm{N}}$ be given. In the closure $\mathbf{x} \in \overline{\Omega_{\mathrm{c}}}$, we look for a displacement vector $\mathbf{u}(\mathbf{x})$, which determines the symmetric tensor of linearized strain $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1}^3(\mathbf{x})$ according to the formula

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3 \tag{3.3}$$

and a symmetric tensor for the Cauchy stress $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^3(\mathbf{x})$, which together satisfy the equilibrium equation

$$-\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = f_i, \quad i = 1, 2, 3, \quad \text{in } \Omega_{\mathrm{c}} \tag{3.4}$$

and the approximation of constitutive equation according to (2.8):

$$\sigma_{ij} = 2\mu\varepsilon_{ij}^*(\mathbf{u}) + K\big(1 - \lambda_4 \mathcal{G}[\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u})]\big)\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u})\delta_{ij}, \quad i, j = 1, 2, 3. \tag{3.5}$$

In (3.5) the Kronecker $\delta_{ij} = 1$ if $i = j$, else zero, and due to (3.3) the trace of strain is $\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}) = \sum_{i=1}^3 \varepsilon_{ii}(\mathbf{u}) = \mathrm{div}(\mathbf{u})$ the divergence of displacement.

The governing system (3.3)–(3.5) is augmented by nonlinear boundary conditions: the Dirichlet condition for the clamp

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}; \tag{3.6}$$

the Neumann type condition for the traction

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\mathrm{N}} \tag{3.7}$$

and the complete system of conditions due to the non-penetration (3.2):

$$\boldsymbol{\sigma}\mathbf{n} - (\boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n})\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{c}}^{\pm}, \tag{3.8}$$

$$[\![\boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n}]\!] = 0 \quad \text{on } \Gamma_{\mathrm{c}}, \tag{3.9}$$

$$[\![\mathbf{u} \cdot \mathbf{n}]\!] \geq 0, \quad \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n} \leq 0, \quad (\boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n})[\![\mathbf{u} \cdot \mathbf{n}]\!] = 0 \quad \text{on } \Gamma_{\mathrm{c}}. \tag{3.10}$$

In (3.7)–(3.10) the stress at the boundary implies $\boldsymbol{\sigma}\mathbf{n} = (\sum_{j=1}^{3} \sigma_{ij} n_j)_{i=1,2,3}$, and its jump $[\![\boldsymbol{\sigma}\mathbf{n}]\!] = [\![\boldsymbol{\sigma}]\!]\mathbf{n}$ is defined akin to (3.1). The vector $\boldsymbol{\sigma}\mathbf{n}$ can be split into the normal $(\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n})\mathbf{n}$ and the tangential $\boldsymbol{\sigma}\mathbf{n} - (\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n})\mathbf{n}$ components, where the scalar

$$\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} = \sum_{i,j=1}^{3} \sigma_{ij} n_j n_i.$$

Therefore, (3.8) describes zero tangential traction and (3.9) suggests continuity of the normal traction across the crack, and the complementarity conditions (3.10) suppose that the crack is either open pointwise, when

$$[\![\mathbf{u}\cdot\mathbf{n}]\!] > 0, \quad \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} = 0$$

or closed, when

$$[\![\mathbf{u}\cdot\mathbf{n}]\!] = 0, \quad \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} \le 0.$$

Now we endow the boundary-value problem (3.3)–(3.10) with a variational formulation in an appropriate function space. Let $\mathbf{f} \in L^2(\Omega_c;\mathbb{R}^3)$ and $\mathbf{g} \in L^2(\Gamma_N;\mathbb{R}^3)$. The set of admissible displacements according to the clamping and non-penetration conditions builds the feasible cone

$$\mathcal{K} = \{\mathbf{u} \in H^1(\Omega_c;\mathbb{R}^3) \,|\, \mathbf{u} \text{ satisfies (3.6) and (3.2)}\}.$$

For feasible $\mathbf{v} = (v_1, v_2, v_3) \in \mathcal{K}$, after multiplication of the equilibrium equation (3.4) with $\mathbf{v} - \mathbf{u}$ and integration by parts over Ω_c such that

$$-\int_{\Omega_c} \sum_{i,j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}(v_i - u_i)d\mathbf{x}$$
$$= \int_{\Omega_c} \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})d\mathbf{x} - \int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n}\cdot(\mathbf{v}-\mathbf{u})dS_{\mathbf{x}} + \int_{\Gamma_c} [\![\boldsymbol{\sigma}\mathbf{n}\cdot(\mathbf{v}-\mathbf{u})]\!]dS_{\mathbf{x}},$$

with the help of boundary conditions (3.6)–(3.10) we derive

$$\int_{\Gamma_c} [\![\boldsymbol{\sigma}\mathbf{n}\cdot(\mathbf{v}-\mathbf{u})]\!]dS_{\mathbf{x}} = \int_{\Gamma_c} (\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n})[\![\mathbf{v}\cdot\mathbf{n} - \mathbf{u}\cdot\mathbf{n}]\!]dS_{\mathbf{x}}$$

and the variational inequality (for more details see Ref. 22, Sec. 1.4.4):

$$\int_{\Omega_c} \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})d\mathbf{x} \ge \int_{\Omega_c} \mathbf{f}\cdot(\mathbf{v}-\mathbf{v})d\mathbf{x} + \int_{\Gamma_N} \mathbf{g}\cdot(\mathbf{v}-\mathbf{u})dS_{\mathbf{x}}. \tag{3.11}$$

Inserting $\boldsymbol{\sigma}$ from (3.5) into (3.11) leads to the following problem: Find $\mathbf{u} \in \mathcal{K}$ such that

$$\int_{\Omega_c} \big\{2\mu\boldsymbol{\varepsilon}^*(\mathbf{u})\cdot\boldsymbol{\varepsilon}^*(\mathbf{v}-\mathbf{u}) + K\big(1 - \lambda_4\mathcal{G}[\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u})]\big)\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u})\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\big\}d\mathbf{x}$$
$$\ge \int_{\Omega_c} \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})d\mathbf{x} + \int_{\Gamma_N} \mathbf{g}\cdot(\mathbf{v}-\mathbf{u})dS_{\mathbf{x}} \tag{3.12}$$

for all test functions $\mathbf{v} \in \mathcal{K}$. Conversely, for H^2-smooth $\mathbf{u}$, pointwise relations (3.3)–(3.10) can be derived from (3.12).

In the following well-posedness analysis of problem (3.12), we apply Lemma 2.2, the Korn–Poincaré inequality

$$\|\mathbf{u}\|^2_{H^1(\Omega_c;\mathbb{R}^3)} \leq C_{\mathrm{KP}} \|\boldsymbol{\varepsilon}(\mathbf{u})\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \quad \text{if } \mathbf{u} = \mathbf{0} \text{ on } \Gamma_{\mathrm{D}} \tag{3.13}$$

and continuity of the trace operator at the boundary

$$\|\mathbf{u}\|_{L^2(\partial\Omega_c;\mathbb{R}^3)} \leq C_{\mathrm{tr}} \|\mathbf{u}\|_{H^1(\Omega_c;\mathbb{R}^3)}. \tag{3.14}$$

Theorem 3.1. (Well-posedness) *Under assumption* (2.9), *there exists a unique solution* $\mathbf{u} \in \mathcal{K}$ *to the variational inequality* (3.12). *It satisfies the a-priori estimate*:

$$M_3 \|\mathbf{u}\|_{H^1(\Omega_c;\mathbb{R}^3)} \leq C_{\mathrm{KP}} \big(\|\mathbf{f}\|_{L^2(\Omega_c;\mathbb{R}^3)} + C_{\mathrm{tr}} \|\mathbf{g}\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^3)} \big), \tag{3.15}$$

where constant $M_3 > 0$ *is given in* (2.14).

Proof. The left-hand side of the variational inequality (3.12) defines a bi-function

$$\begin{aligned} \int_{\Omega_c} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) d\mathbf{x} &= \int_{\Omega_c} \big\{ 2\mu \boldsymbol{\varepsilon}^*(\mathbf{u}) \cdot \boldsymbol{\varepsilon}^*(\mathbf{v}) + K \big(1 - \lambda_4 \mathcal{G}[\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u})]\big) \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}) \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{v}) \big\} d\mathbf{x} \\ &=: A(\mathbf{u}, \mathbf{v}) \end{aligned} \tag{3.16}$$

such that $A : H^1(\Omega_c;\mathbb{R}^3)^2 \mapsto \mathbb{R}$ is nonlinear in the first argument and linear in the second argument, whereas the right-hand side of (3.12) builds a linear function

$$F : H^1(\Omega_c;\mathbb{R}^3) \mapsto \mathbb{R}, \quad F(\mathbf{v}) := \int_{\Omega_c} \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{v}\, dS_{\mathbf{x}}. \tag{3.17}$$

Using the properties (2.10)–(2.14) of the mapping $\boldsymbol{\varepsilon}(\mathbf{u}) \mapsto \boldsymbol{\sigma}$ that hold under the assumption (2.9), we demonstrate the corresponding properties hold for the operator $A(\mathbf{u}, \mathbf{v}) - F(\mathbf{v})$ with respect to the H^1-vector norm:

$$\|\mathbf{u}\|^2_{H^1(\Omega_c;\mathbb{R}^3)} = \|\mathbf{u}\|^2_{L^2(\Omega_c;\mathbb{R}^3)} + \|\boldsymbol{\varepsilon}(\mathbf{u})\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times 3}_{\mathrm{sym}})}.$$

Applying to (3.16) and (3.17) the Cauchy–Schwarz inequality and the trace inequality (3.14), the property (2.10) for the boundedness follows:

$$\begin{aligned} |A(\mathbf{u}, \mathbf{v}) - F(\mathbf{v})| &\leq \|\boldsymbol{\sigma}\|_{H^1(\Omega_c;\mathbb{R}^3)} \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{H^1(\Omega_c;\mathbb{R}^3)} \\ &\quad + \|\mathbf{f}\|_{L^2(\Omega_c;\mathbb{R}^3)} \|\mathbf{v}\|_{L^2(\Omega_c;\mathbb{R}^3)} + \|\mathbf{g}\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^3)} \|\mathbf{v}\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^3)} \\ &\leq \big(M_1 \|\mathbf{u}\|_{H^1(\Omega_c;\mathbb{R}^3)} + \|\mathbf{f}\|_{L^2(\Omega_c;\mathbb{R}^3)} + C_{\mathrm{tr}} \|\mathbf{g}\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^3)} \big) \|\mathbf{v}\|_{H^1(\Omega_c;\mathbb{R}^3)}. \end{aligned} \tag{3.18}$$

The hemi-continuity (2.11) implies the convergence in the first argument

$$A(\mathbf{u} + r\bar{\mathbf{u}}, \mathbf{v}) \to A(\mathbf{u}, \mathbf{v}) \quad \text{as } r \to 0, \quad \mathbf{u}, \bar{\mathbf{u}}, \mathbf{v} \in H^1(\Omega_c;\mathbb{R}^3). \tag{3.19}$$

The strong monotonicity is provided by the property (2.12) using (2.13) and the Korn–Poincaré inequality (3.13) such that

$$\begin{aligned} &A(\mathbf{u}-\bar{\mathbf{u}},\mathbf{u}-\bar{\mathbf{u}}) \\ &\quad = \int_{\Omega_c} (\boldsymbol{\sigma}-\bar{\boldsymbol{\sigma}})\cdot\boldsymbol{\varepsilon}(\mathbf{u}-\bar{\mathbf{u}})d\mathbf{x} \\ &\quad \geq M_2\|\boldsymbol{\varepsilon}(\mathbf{u}-\bar{\mathbf{u}})\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \geq \frac{M_2}{C_{\mathrm{KP}}}\|\mathbf{u}-\bar{\mathbf{u}}\|^2_{H^1(\Omega_c;\mathbb{R}^3)}. \end{aligned} \tag{3.20}$$

The coercivity follows from the property (2.14) after application of the weighted Young, trace and Korn–Poincaré inequalities as

$$\begin{aligned} &A(\mathbf{u},\mathbf{u})-F(\mathbf{u}) \\ &\quad = \int_{\Omega_c}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{u})d\mathbf{x} - \int_{\Omega_c}\mathbf{f}\cdot\mathbf{u}\,d\mathbf{x} - \int_{\Gamma_N}\mathbf{g}\cdot\mathbf{u}\,dS_{\mathbf{x}} \\ &\quad \geq M_3\|\boldsymbol{\varepsilon}(\mathbf{u})\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times 3}_{\mathrm{sym}})} - \|\mathbf{f}\|_{L^2(\Omega_c;\mathbb{R}^3)}\|\mathbf{u}\|_{L^2(\Omega_c;\mathbb{R}^3)} - \|\mathbf{g}\|_{L^2(\Gamma_N;\mathbb{R}^3)}\|\mathbf{u}\|_{L^2(\Gamma_N;\mathbb{R}^3)} \\ &\quad \geq \left(\frac{M_3}{C_{\mathrm{KP}}} - \frac{C_1}{2} - \frac{C_2}{2}\right)\|\mathbf{u}\|^2_{H^1(\Omega_c;\mathbb{R}^3)} - \frac{1}{2C_1}\|\mathbf{f}\|^2_{L^2(\Omega_c;\mathbb{R}^3)} - \frac{C_{\mathrm{tr}}^2}{2C_2}\|\mathbf{g}\|^2_{L^2(\Gamma_N;\mathbb{R}^3)}, \end{aligned} \tag{3.21}$$

when the following weights are chosen, e.g. $C_1 = C_2 = M_3/(2C_{\mathrm{KP}})$ such that the factor $M_3/C_{\mathrm{KP}} - (C_1 + C_2)/2 = M_3/(2C_{\mathrm{KP}}) > 0$ in (3.21).

The properties (3.18)–(3.20) imply that the operator $A(\mathbf{u},\mathbf{v}) - F(\mathbf{v})$ is pseudo-monotone (see Lions,[31] Chap. 2, Sec. 2.4, Proposition 2.5). Together with the coercivity (3.21), solvability of the variational inequality $A(\mathbf{u},\mathbf{v}-\mathbf{u}) \geq F(\mathbf{v}-\mathbf{u})$ in the closed convex set $\mathcal{K}$ (see Lions,[31] Chap. 3, Sec. 5.3, Theorem 5.2) that is (3.12), follows. Since the monotone property (3.20) is strict, we conclude with the uniqueness of the solution. Testing (3.12) with admissible $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = 2\mathbf{u}$ we get the equation

$$\int_{\Omega_c}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{u})d\mathbf{x} = \int_{\Omega_c}\mathbf{f}\cdot\mathbf{u}\,d\mathbf{x} + \int_{\Gamma_N}\mathbf{g}\cdot\mathbf{u}\,dS_{\mathbf{x}}. \tag{3.22}$$

Applying the coercivity (2.14) and the Korn–Poincaré inequality (3.13) to the left-hand side of Eq. (3.22), then the trace (3.14) and Cauchy–Schwarz inequalities to the right-hand side of the equation, and dividing all by the norm $\|\mathbf{u}\|_{H^1(\Omega_c;\mathbb{R}^3)}$ we obtain the *a-priori* estimate (3.15). The proof is complete. □

In the following section, we construct an analytic example of the solution for the problem without crack and analyze it.

4. Example Problem Under Uniform Triaxial Loading

Let us suppose that the crack $\Gamma_c = \emptyset$ and $\Omega_c = \Omega = (0,1)^3$ is a unit cube. We consider an example of constant boundary force $g \in \mathbb{R}$ applied to the boundary

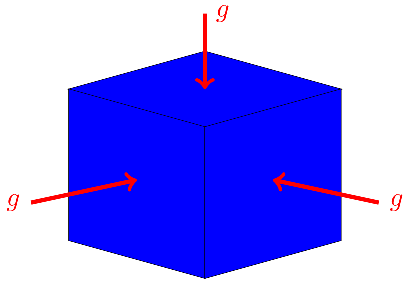

Fig. 3. (Color online) Cube under constant uniform triaxial loading g.

$\partial\Omega$ as portrayed in Fig. 3. This implies the Neumann boundary condition (3.7) written as

$$\sigma_{ii} = g, \quad \text{at } x_i \in \{0, 1\}, \quad i = 1, 2, 3. \tag{4.1}$$

We look for a solution to the equilibrium equation (3.4) in the form

$$\mathbf{u} = a\mathbf{x} + R\mathbf{x}, \quad \boldsymbol{\varepsilon}(R\mathbf{x}) = \mathbf{0} \tag{4.2}$$

with unknown parameter $a \in \mathbb{R}$ and rigid displacement $R\mathbf{x}$ expressed by the skew-symmetric matrices $R \in \mathbb{R}^{3\times3}_{\mathrm{skew}}$. From (4.2) and (3.3) we calculate the strain

$$\boldsymbol{\varepsilon}(\mathbf{u}) = a\mathbf{I}, \quad \boldsymbol{\varepsilon}^*(\mathbf{u}) = \mathbf{0}, \quad \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}) = 3a. \tag{4.3}$$

- *For the constitutive equation* (2.3), using (1.4) and (4.3) we find

$$\boldsymbol{\sigma}^* = \mathbf{0}, \quad \mathrm{tr}\boldsymbol{\sigma} = \frac{\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u})}{3E_3(1 + \lambda_4 \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}))} = 3K\frac{3a}{1 + 3a\lambda_4}$$

  and hence the stress tensor is given by

$$\boldsymbol{\sigma} = K\frac{3a}{1 + 3a\lambda_4}\mathbf{I}. \tag{4.4}$$

Inserting (4.4) into the boundary condition (4.1), the corresponding equation $K3a/(1 + 3a\lambda_4) = g$ can be solved explicitly with respect to the unknown a as

$$3a = \frac{g/K}{1 - \lambda_4 g/K}. \tag{4.5}$$

In the $3a$ versus g/K plane, the linear-fractional function in (4.5) is portrayed by a thick dashed curve in Fig. 4 for $\lambda_4 > 0$ in the left plot, and for $\lambda_4 < 0$ in the right plot.

Looking at g as a loading parameter monotone in time, we can interpret $3a \to \pm\infty$ at finite $g/K \to g_{\mathrm{cr}}$ as a blow-up. When $\lambda_4 \neq 0$ the critical value is

$$g_{\mathrm{cr}} := \frac{1}{\lambda_4}.$$

By virtue of this $\mathbf{u} \to \pm\infty$ in (4.2) and $\boldsymbol{\varepsilon}(\mathbf{u}) \to \pm\infty$ in (4.3). Such a nonphysical behavior is overcome for the thresholding dilatation approximation.

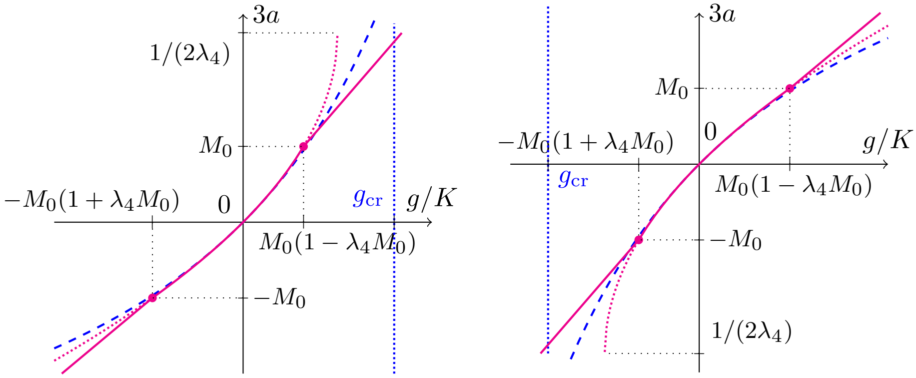

Fig. 4. (Color online) Functions of $3a$ versus g/K for $\lambda_4 > 0$ (left plot) and $\lambda_4 < 0$ (right plot).

- *For the approximate equation* (2.8), from (4.3) it follows that $\boldsymbol{\sigma}^* = \mathbf{0}$ and the stress

$$\boldsymbol{\sigma} = K(1 - \lambda_4 \mathcal{G}[3a])(3a)\mathbf{I}. \tag{4.6}$$

Inserting (4.6) into the boundary condition (4.1) and using $\mathcal{G}$ in (2.6) leads to the following cases:

$$\frac{g}{K} = \begin{cases} 3a(1 - \lambda_4 3a) & \text{if } 3|a| \le M_0, \\ 3a(1 - \lambda_4 M_0) & \text{if } 3a > M_0, \\ 3a(1 + \lambda_4 M_0) & \text{if } 3a < -M_0. \end{cases} \tag{4.7}$$

To invert (4.7), we start by considering the quadratic function

$$\psi(3a) := 3a(1 - \lambda_4 3a) \quad \text{for } -M_0 \le 3a \le M_0,$$

describing a parabola with derivative $\psi'(3a) = 1 - 2\lambda_4 3a$ and vertex at $3a = 1/(2\lambda_4)$ such that $\psi'(1/(2\lambda_4)) = 1/(4\lambda_4) < g_{\rm cr}$ as $\lambda_4 > 0$ and $\psi'(1/(2\lambda_4)) > g_{\rm cr}$ as $\lambda_4 < 0$. The restriction (2.9) assumed for parameters implies that

$$M_0 < \frac{1}{2\lambda_4} \text{ as } \lambda_4 > 0, \quad -M_0 > \frac{1}{2\lambda_4} \text{ as } \lambda_4 < 0.$$

Therefore, under the assumption (2.9) we derive that

$$\begin{cases} \text{as } \lambda_4 > 0: \psi'(3a) = 1 - 2\lambda_4 3a > 0 \quad \text{for } 3a \le M_0 < \dfrac{1}{2\lambda_4}, \\ \text{as } \lambda_4 < 0: \psi'(3a) = 1 - 2\lambda_4 3a > 0 \quad \text{for } 3a \ge -M_0 > \dfrac{1}{2\lambda_4}. \end{cases} \tag{4.8}$$

The monotone branches (4.8) of the parabola $g/K = \psi(3a)$ are portrayed by thick dashed curves in the respective left and right plots in Fig. 4.

With the help of (4.8) now we can invert (4.7) in the explicit form

$$3a = \begin{cases} \dfrac{1-\sqrt{1-4\lambda_4 g/K}}{2\lambda_4} & \text{if } -M_0(1+\lambda_4 M_0) \leq g/K \leq M_0(1-\lambda_4 M_0), \\ \dfrac{g/K}{1-\lambda_4 M_0} & \text{if } g/K > M_0(1-\lambda_4 M_0), \\ \dfrac{g/K}{1+\lambda_4 M_0} & \text{if } g/K < -M_0(1+\lambda_4 M_0), \end{cases} \tag{4.9}$$

where the square root is well-defined since the equation $g/K = \psi(3a)$ implies that $1 - 4\lambda_4 g/K = (1 - 2\lambda_4 3a)^2 > 0$. The monotone functions from (4.9) made up of three pieces are portrayed by thick solid curves in Fig. 4 for $\lambda_4 > 0$ (left plot) and $\lambda_4 < 0$ (right plot). Here, we can observe that these functions are finite at the critical values of the load $g/K = g_{\rm cr}$ in contrast to the solution (4.5).

Finally, we compare the two solutions (4.5) and (4.9) over the bounded dilatation range where (2.5) holds, then

$$g/K \in I := [-M_0(1+\lambda_4 M_0), M_0(1-\lambda_4 M_0)]$$

and the solution difference is denoted by

$$\phi(g/K) := \frac{g/K}{1-\lambda_4 g/K} - \frac{1-\sqrt{1-4\lambda_4 g/K}}{2\lambda_4}. \tag{4.10}$$

On differentiating (4.10) we find that

$$\phi'(g/K) = \frac{1}{(1-\lambda_4 g/K)^2} - \frac{1}{\sqrt{1-4\lambda_4 g/K}} \neq 0$$

if $g/K \neq 0$, because $\phi'(g/K) = 0$ implies $g/K = 0$ or the quadratic equation $(\lambda_4 g/K)^2 - 4\lambda_4 g/K + 6 = 0$ having negative discriminant. Thus, function ϕ is continuous and does not change direction on the interval I, its maxima and minima are attained at the end points, that are

$$\phi\big(-M_0(1+\lambda_4 M_0)\big) = \frac{\lambda_4^2 M_0^3}{1+\lambda_4 M_0 + (\lambda_4 M_0)^2},$$

$$\phi\big(M_0(1-\lambda_4 M_0)\big) = -\frac{\lambda_4^2 M_0^3}{1-\lambda_4 M_0 + (\lambda_4 M_0)^2}.$$

This proves the maximal difference between the solutions (4.5) and (4.9) on I,

$$\max_{g/K\in I} |\phi(g/K)| \leq \frac{\lambda_4^2 M_0^3}{1-|\lambda_4| M_0 + (\lambda_4 M_0)^2}, \tag{4.11}$$

which is of asymptotic order $\mathcal{O}(M_0^3)$.

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