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Lagrange multiplier approach to unilateral indentation problems: Well-posedness and application to linearized viscoelasticity with non-invertible constitutive response

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The Boussinesq problem describing indentation of a rigid punch of arbitrary shape into a deformable solid body is studied within the context of a linear viscoelastic model. Due to the presence of a non-local integral constraint prescribing the total contact force, the unilateral indentation problem is formulated in the general form as a quasi-variational inequality with unknown indentation depth, and the Lagrange multiplier approach is applied to establish its well-posedness. The linear viscoelastic model that is considered assumes that the linearized strain is expressed by a material response function of the

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stress involving a Volterra convolution operator, thus the constitutive relation is not invertible. Since viscoelastic indentation problems may not be solvable in general, under the assumption of monotonically non-increasing contact area, the solution for linear viscoelasticity is constructed using the convolution for an increment of solutions from linearized elasticity. For the axisymmetric indentation of the viscoelastic half-space by a cone, based on the Papkovich–Neuber representation and Fourier–Bessel transform, a closed form analytical solution is constructed, which describes indentation testing within the holding-unloading phase.

Keywords: Boussinesq problem; indentation testing; cone indenter; viscoelastic model; Volterra convolution operator; quasi-variational inequality; non-local constraint; augmented Lagrangian; variational solution.

AMS Subject Classification 2020: 35Q74, 49J40, 74D99, 74M15

1. Introduction

Indentation testing is probably one of the most often used experimental procedures and consists in pressing the tip of a rigid punch into a body whose material properties are unknown. In general the procedure is tip dependent and can be used to determine the material moduli locally in inhomogeneous materials, unlike other experimental procedures that are global providing an average value of the moduli. Nano-indentation can also be used to determine the properties of very small bodies. Thus, studying the problem of indentation is a very useful procedure in engineering in determining the variation of material properties over the body.

An indentation experiment can be expressed as an inverse problem of coefficient identification[43] from displacement-force measurements on the surface of the foundation being tested. Given the shape of the punch and the total contact force, the forward indentation problem consists in finding simultaneously the indentation depth, displacement and stress distributions over the body. This is referred to as the Boussinesq's problem.[5] The unilateral contact conditions with free boundary[10] of the contact area should be satisfied on the surface of the body. In contrast to Signorini's obstacle problem, the unilateral indentation problem has an *a-priori* unknown obstacle, which is determined only after solving the problem by adding the indentation depth to the punch shape.

There have been very few attempts in the literature to study the unilateral relations on the surface of the body as a linear complementarity problem.[27] In Sec. 2 we claim that unilateral indentation problems belong to the class of quasi-variational inequalities subject to a non-local constraint. To the best of our knowledge these problems are for first time presented here from the optimization viewpoint. The quasi-variational theory was introduced by Bensoussan and Lions[3] and characterized by a solution-dependent set of admissible states. For various applications of quasi-variational inequalities in mechanics we cite Ref. 30, in particular, inverse problems,[36] gradient constraints[17] and non-penetrating cracks under Coulomb friction.[24,29]

Prescribing the total contact force on the surface of the body implies a non-local integral constraint, which itself depends on the unknown contact stress. For the relevant volume constraint we cite Refs. 1, 7 and 32. Within the Lagrange multiplier approach[31] we introduce an augmented Lagrangian[19] combined with a penalty approximation. With its help, we provide sufficient conditions for well-posedness of the unilateral indentation problem described by general linear constitutive equations in the mixed formulation in Sec. 3. The linear elastic body is a special case. As with the study of Aguilera *et al.*,[1] we demonstrate an interesting feature, namely, that the prescribed value of total contact force is fulfilled exactly by the penalty approximation for the penalty parameter $\delta > 0$, rather than attaining it in the limit as $\delta \to 0^+$.

In our specific application, we are interested in a linear viscoelastic body when it is subject to indentation. It is natural at this juncture to ask if the correspondence principle (see, for example, Gurtin and Sternberg[14])[a] that states that the solution for a problem in linear viscoelasticity can be obtained from the corresponding solution in linearized elasticity by using the Laplace transform, provided certain conditions are met is applicable here. The applicability of the correspondence depends only on the linearity of the response function and is not restricted to linear viscoelastic response provided certain conditions are met.[40] Wineman and Rajagopal[46] discuss when the correspondence principle breaks down for linear viscoelasticity. The nonlinear contact conditions within unilateral problems is an example where one cannot appeal to the correspondence principle. For this reason, only particular solutions to the viscoelastic indentation problems were available for special choices of the contact area (see Refs. 18, 33 and 44).

Moreover, the material response that we study expresses the linearized strain in terms of the stress, keeping with causality as force/stress is the cause and the deformation/strain is the effect. A challenge with regard to this constitutive relation is that it cannot be inverted to express the stress response as a function of the strain. The model can be viewed as a linearization of a more general nonlinear model. Boltzmann[4] was the first to introduce an integral model to describe linear viscoelastic response. This was followed by several nonlinear constitutive relations to describe the response of viscoelastic bodies (see Green and Rivlin,[11] Green *et al.*,[13] Green and Rivlin,[12] Pipkin and Rogers[38] and the book by Lockett[34]). Recently, Muliana *et al.*[37] developed quasilinear viscoelastic models wherein the strain is function of the history of the stress.

In this paper we consider material response wherein the linearized strain $\boldsymbol{\varepsilon}$ is expressed with respect to the Cauchy stress $\boldsymbol{\sigma}$ by the linear mapping:

$$\boldsymbol{\varepsilon} = \mathcal{I}[\mathcal{F}\boldsymbol{\sigma}], \tag{1.1}$$

where $\mathcal{F}$ denotes the compliance tensor and $\mathcal{F}\boldsymbol{\sigma}$ is the product of two linear transformations. For instance, (1.1) reduces to the classic Hooke's law for the linear elastic body when $\mathcal{I}$ implies the identity operator such that

$$\mathcal{I}_{\mathrm{E}}[\mathcal{F}\boldsymbol{\sigma}] := \mathcal{F}\boldsymbol{\sigma}. \tag{1.2}$$

[a]In fact the result goes much farther back but was not proved rigorously.

In the case of linearized viscoelasticity (1.1) takes the integral form in time, for $t \geq 0$:

$$\mathcal{I}_{\mathrm{VE}}[\mathcal{F}\boldsymbol{\sigma}](t) := K(0)\mathcal{F}\boldsymbol{\sigma}(t) + \int_0^t K'(t-s)\mathcal{F}\boldsymbol{\sigma}(s)\,ds. \tag{1.3}$$

The non-negative kernel K that appears in the Volterra convolution operator in (1.3) is usually assumed to be given by the exponential sum

$$K(t) = K(0) + \sum_{n=1}^{N} K_n\big(1 - e^{-t/\tau_n}\big), \quad K'(t) = \sum_{n=1}^{N} \frac{K_n}{\tau_n} e^{-t/\tau_n} \tag{1.4}$$

with parameters $K(0), K_1, \dots, K_N, \tau_1, \dots \tau_N \geq 0$, that characterize generalized creep. The representation (1.3) cannot be inverted in general even though linear in $\boldsymbol{\sigma}$. In the particular case, if $N = 1$ and $K(0) = 0$ in (1.4), then

$$\mathcal{I}_{\mathrm{KV}}[\mathcal{F}\boldsymbol{\sigma}](t) := \frac{K_1}{\tau_1} \int_0^t e^{-(t-s)/\tau_1} \mathcal{F}\boldsymbol{\sigma}(s)\,ds. \tag{1.5}$$

On differentiating (1.4) with respect to t we get $K''(t) = -K'(t)/\tau_1$ and $K'(0) = K_1/\tau_1$, thus derive its inverse implying the generalized Kelvin–Voigt model[6,20]:

$$\boldsymbol{\varepsilon} + \tau_1 \dot{\boldsymbol{\varepsilon}} = K_1 \mathcal{F}\boldsymbol{\sigma}, \tag{1.6}$$

where dot stands for the time derivative.

The constitutive model (1.1) is a specific case of the more general class of nonlinear models when we substitute $\mathcal{F}\boldsymbol{\sigma}$ with a nonlinear relationship $\mathcal{F}(\boldsymbol{\sigma})$ (see related Refs. 8, 21 and 37). In the context of quasi-linear viscoelasticity, such models were investigated with respect to well-posedness in Itou *et al.*[22] A relevant subclass of contact problems adjacent to non-penetrating cracks was developed by Khludnev and Kovtunenko[28] and extended to linear viscoelastic solids.[25,39,41]

The well-posedness theory assumes the linear operator $\mathcal{I}$ in (1.1) to be coercive, bounded, weakly lower semi-continuous, self-adjoint and surjective (i.e. invertible). Since the Volterra convolution operator in (1.3) does not satisfy all these properties, indentation problems may not be solvable for arbitrary data. Under non-increasing contact area, for arbitrary punch shapes we derive a solution for a viscoelastic body using the convolution for an increment of the elastic solution subject to the linear constitutive relation (1.2) in Sec. 4. Since quasi-static indentation tests consist typically of loading-holding-unloading phases,[26] this provides a viscoelastic solution for the holding-unloading phase. Known viscoelastic solutions[18,33,44] are mostly restricted to increasing contact area that describes the loading phase.

In the literature, the relevant 3d indentation problem is usually stated for the body occupying a semi-infinite half-space $z > 0$ and a punch $\psi(x, y)$ indented into the surface of the body on the plane $z = 0$. In this setting, a collection[2,9] of analytical solutions for elastic foundations and punches presented by simple axisymmetric shapes: cylinder, cone, sphere, paraboloid, were obtained. Typically it implies an explicit function between the indentation depth $p(t)$ and the total contact force $F(t)$

applied in time, and analytic expressions in cylindrical coordinates (r, θ, z) for the displacement $\mathbf{u}(t, r, \theta, 0)$ and the stress $\boldsymbol{\sigma}(t, r, \theta, 0)$ derived from integral relations on the surface $z = 0$. Whereas the full distribution of $\mathbf{u}$ and $\boldsymbol{\sigma}$ over the half-space is challenging being often expressed by singular, discontinuous, and divergent type integrals and series.

In our previous work Itou *et al.*,[23] based on the Fourier–Bessel transform[45] (also called Hankel transform) and the Papkovich–Neuber representation[16] from potential theory, an analytical solution for flat-ended cylindrical punch pressed into the viscoelastic half-space was constructed using the convolution of the solution from linearized elasticity, since the contact area does not change in this specific configuration. In the current contribution, using the analytical solution for the cone obtained by Love[35] and Sneddon,[42] we derive in Sec. 4.1 explicit formulae for the solution describing the viscoelastic response (1.3) for the unilateral problem of cone indentation when the contact area does not increase.

2. Quasi-Variational Formulation of Indentation Problems

Let the body occupy a bounded domain Ω in an Euclidean space $\mathbb{R}^d$, $d = 2, 3$, and the boundary $\partial\Omega$ be Lipschitz continuous with the normal vector $\mathbf{n} = (n_1, \ldots, n_d)$ outward to Ω. We split $\partial\Omega = \overline{\Gamma_{\mathrm{N}}} \cup \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Sigma}$ into three parts of non-zero measure. By this we assume that the contact boundary Σ lies strictly inside $\partial\Omega \backslash \overline{\Gamma_{\mathrm{D}}}$ such that the Neumann boundary Γ_{N} separates it from the Dirichlet boundary Γ_{D}. For points in the Euclidean space $\mathbf{x} = (x_1, \ldots, x_d) \in \Omega$ and time $t \in (0, T)$, $T > 0$, the time-space cylinder is denoted by $Q^T = (0, T) \times \Omega$.

Let the total contact force $F \in L^\infty(0, T)$ be non-negative, i.e. $F(t) \geq 0$, and the punch shape ψ be given by a function bounded from above

$$\psi(\mathbf{x}) \leq \max_{\mathbf{x} \in \overline{\Sigma}} \psi(\mathbf{x}) =: p_0 \quad \text{for } \mathbf{x} \in \overline{\Sigma}, \quad p_0 \geq 0. \tag{2.1}$$

In the space of second-order symmetric d-by-d tensors $\mathbb{R}^{d \times d}_{\mathrm{sym}}$, for the given compliance $\mathcal{F}$ and the response $\mathcal{I}$, we look for the stress tensor $\boldsymbol{\sigma} = \{\sigma_{ij}\}_{i,j=1}^d(t, \mathbf{x})$, the linearized strain tensor $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}_{i,j=1}^d(t, \mathbf{x})$ determined by the displacement vector $\mathbf{u} = (u_1, \ldots, u_d)(t, \mathbf{x})$, and the indentation depth $p(t)$ that satisfy

$$-\sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad i = 1 \ldots, d, \quad \text{in } Q^T; \tag{2.2a}$$

$$\varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \ldots, d, \quad \text{in } Q^T; \tag{2.2b}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{I}[\mathcal{F}\boldsymbol{\sigma}] \quad \text{in } Q^T; \tag{2.2c}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } (0, T) \times \Gamma_{\mathrm{D}}; \tag{2.2d}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \quad \text{on } (0, T) \times \Gamma_{\mathrm{N}}; \tag{2.2e}$$

$$\boldsymbol{\sigma}\mathbf{n} - (\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n})\mathbf{n} = \mathbf{0} \quad \text{on } (0,T)\times\Sigma; \tag{2.2f}$$

$$\mathbf{u}\cdot\mathbf{n} + p + \psi \le 0, \quad \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} \le 0, \quad (\mathbf{u}\cdot\mathbf{n} + p + \psi)\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} = 0 \quad \text{on } (0,T)\times\Sigma; \tag{2.2g}$$

$$F + \int_\Sigma \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n}\, dS_{\mathbf{x}} = 0 \quad \text{for } t \in (0,T). \tag{2.2h}$$

The governing system consists of the equilibrium equation (2.2a), expression for the linearized strain (2.2b), and the constitutive equation (2.2c) under the standard Dirichlet (2.2d) and Neumann type (2.2e) boundary conditions. At the contact boundary, (2.2f) implies zero tangential stresses, and the unilateral contact conditions (2.2g) describe non-penetration of the punch into the surface of the body in the normal direction. Here dot implies the inner product of vectors, and $\boldsymbol{\sigma}\mathbf{n}$ is the linear transformation-vector multiplication. The integral condition (2.2h) prescribes the total contact force.

The complementarity relations (2.2g) and (2.2h) can be represented equivalently by splitting Σ into a coincidence set $\mathcal{C}(t)$ and its complement $\Sigma\backslash\mathcal{C}(t)$ as follows:

$$\mathbf{u}\cdot\mathbf{n} + p + \psi = 0, \quad \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} < 0 \quad \text{on } \mathcal{C}; \tag{2.3a}$$

$$\mathbf{u}\cdot\mathbf{n} + p + \psi \le 0, \quad \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma\backslash\mathcal{C}; \tag{2.3b}$$

$$F + \int_{\mathcal{C}} \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n}\, dS_{\mathbf{x}} = 0. \tag{2.3c}$$

Since $\mathcal{C}$ is unknown *a-priori*, relations (2.2a)–(2.2f) and (2.3) imply a free boundary problem.[10] On the other hand, (2.3a) and (2.3b) can be expressed with the help of nonlinear complementarity problem (NLCP) functions, e.g. min-based function:

$$\min\big(0, \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} - c(\mathbf{u}\cdot\mathbf{n} + p + \psi)\big) - \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} = 0 \quad \text{on } (0,T)\times\Sigma \tag{2.4}$$

for arbitrary constant $c > 0$. The nonlinear equation (2.4) is equivalent to

$$\mathbf{u}\cdot\mathbf{n} + p + \psi = 0 \quad \text{on } \mathcal{A} = \{\mathbf{x}\in\Sigma \,|\, \big(\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} - c(\mathbf{u}\cdot\mathbf{n} + p + \psi)\big)(\mathbf{x}) < 0\}; \tag{2.5a}$$

$$\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma\backslash\mathcal{A} = \{\mathbf{x}\in\Sigma \,|\, \big(\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} - c(\mathbf{u}\cdot\mathbf{n} + p + \psi)\big)(\mathbf{x}) \ge 0\} \tag{2.5b}$$

over the active set $\mathcal{A}(t)$, where the constraint is active, and its complementary inactive set $\Sigma\backslash\mathcal{A}(t)$. Then the integral in (2.3c) can be rewritten over $\mathcal{A}$. The mixed formulation (2.5) is advantageous for numerical implementation, see the semi-smooth Newton method in Ref. 15.

For illustration, the axisymmetric cone with the side shape $\psi(r) = -r/\tan(\alpha)$, $\alpha \in (0,\pi/2)$, pressed into the half-space in the normal direction is portrayed versus radius $r = \sqrt{x_1^2 + x_2^2}$ in Fig. 1, where the coincidence set is determined by $r < h$.

Now we give a weak formulation of the boundary-value problem (2.2) for fixed t. According to the Dirichlet boundary condition (2.2d) we set the Sobolev space

$$H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^d) = \{\mathbf{v} = (v_1,\dots,v_d)(\mathbf{x}) \in H^1(\Omega;\mathbb{R}^d) \,|\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{\rm D}\}. \tag{2.6}$$

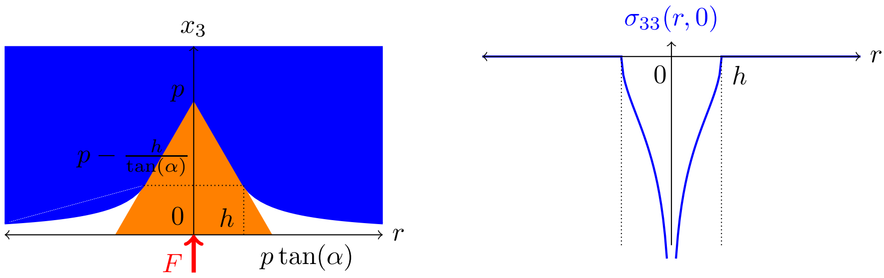

Fig. 1. The cross-section (r, x_3) of indentation by conical punch.

Due to the non-penetration condition in (2.2g) admissible displacements lie in the function set which depends on the indentation depth p:

$$\mathcal{K}(p) = \{\mathbf{v} \in H^1_{\Gamma_\mathrm{D}}(\Omega; \mathbb{R}^d) \,|\, \mathbf{v}\cdot\mathbf{n} + p + \psi \le 0 \text{ on } \Sigma\}. \tag{2.7}$$

Multiplying the equilibrium equation (2.2a) by $v_i - u_i$, summing over $i = 1, \ldots, d$, integrating the sum by parts over Ω and denoting the inner product of linear tranformation by dot, with the help of (2.2b) and boundary conditions (2.2d)–(2.2f) we get

$$\begin{aligned} 0 &= -\int_\Omega \sum_{i,j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}(v_i - u_i)\,d\mathbf{x} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\,d\mathbf{x} - \int_{\partial\Omega} (\boldsymbol{\sigma}\mathbf{n})\cdot(\mathbf{v}-\mathbf{u})dS_\mathbf{x} \\ &= \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\,d\mathbf{x} - \int_\Sigma (\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n} + p + \psi - \mathbf{u}\cdot\mathbf{n} - p - \psi)dS_\mathbf{x} \end{aligned} \tag{2.8}$$

for arbitrary $\mathbf{v} \in H^1_{\Gamma_\mathrm{D}}(\Omega; \mathbb{R}^d)$. Using the complementarity relations (2.2g) we derive from (2.8) the quasi-variational inequality, which together with (2.2c) and (2.2h) is given by the mixed formulation: find a triple $\mathbf{u} \in \mathcal{K}(p)$, $\boldsymbol{\sigma} \in L^2(\Omega; \mathbb{R}^{d\times d}_\mathrm{sym})$ and $p \in \mathbb{R}$ such that

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\,d\mathbf{x} \ge 0 \quad \text{for all } \mathbf{v} \in \mathcal{K}(p); \tag{2.9a}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{I}[\mathcal{F}\boldsymbol{\sigma}] \quad \text{in } \Omega; \tag{2.9b}$$

$$F + \langle \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n}, \eta\rangle_{\Sigma\cup\Gamma_\mathrm{N}} = 0; \tag{2.9c}$$

$$p = \max\{q \in \mathbb{R} \,|\, \mathbf{u} \in \mathcal{K}(q)\}. \tag{2.9d}$$

In (2.9c), which generalizes the integral in (2.3c), the brackets $\langle\,\cdot\,,\,\cdot\,\rangle_{\Sigma\cup\Gamma_\mathrm{N}}$ stand for the duality between the Lions–Magenes space of traces $H^{1/2}_{00}(\Sigma\cup\Gamma_\mathrm{N})$ continued by zero on Γ_D, and its dual $H^{1/2}_{00}(\Sigma\cup\Gamma_\mathrm{N})^\star$ (see, e.g. Ref. 28, Sec. 1.4). A cut-off function $\eta \in H^{1/2}_{00}(\Sigma\cup\Gamma_\mathrm{N})$ is such that $\eta(\mathbf{x}) = 1$ on Σ, and arbitrary on Γ_N on recalling that $\boldsymbol{\sigma}\mathbf{n} = \mathbf{0}$.

The condition of maximal indentation depth (2.9d) comes from the following consideration. If the non-penetration $\mathbf{u}\cdot\mathbf{n} + p + \psi \le 0$ holds on Σ according to

(2.7), then it follows $\mathbf{u}\cdot\mathbf{n}+q+\psi<0$, hence $\mathbf{u}\in\mathcal{K}(q)$ for all $q<p$. But in the case of strict inequality, the punch loses contact with the body, that contradicts the prescribed contact force. Therefore, on taking the maximum over all possible q in (2.9d) we ensure that the punch contacts the body.

It is worth noting that condition (2.9d) follows the lower bound $p\geq -p_0$. Indeed, if $F=0$, then $\mathbf{u}=\mathbf{0}$, $\boldsymbol{\sigma}=\mathbf{0}$, $p=-p_0$ solves (2.9), and this trivial solution is feasible because $\mathbf{u}\cdot\mathbf{n}+p+\psi=-p_0+\psi\leq 0$ according to (2.1), i.e. the punch touching the unstressed body. If $F>0$, then contact is geometrically possible for $p>-p_0$, otherwise the coincidence set $\mathcal{C}=\emptyset$ contradicts (2.3c) and (2.9c).

In the next section we establish the existence of a weak solution to (2.9).

3. Well-Posedness of the Unilateral Indentation Problem

For a small penalty parameter $\delta>0$, we introduce an augmented Lagrangian $\mathscr{L}_\delta : V(\Omega)\mapsto\mathbb{R}$ in the space $V(\Omega)=H^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})\times\mathbb{R}$ by

$$\begin{aligned}\mathscr{L}_\delta(\mathbf{v},\mathbf{s},q) := \int_\Omega \mathbf{s}\cdot\left(\boldsymbol{\varepsilon}(\mathbf{v})-\frac{1}{2}\mathcal{I}[\mathcal{F}\mathbf{s}]\right)d\mathbf{x}\\ +\frac{1}{2\delta}\int_\Sigma([\mathbf{v}\cdot\mathbf{n}+q+\psi]^+)^2\,dS_{\mathbf{x}}-Fq,\end{aligned}\tag{3.1}$$

where the standard decomposition into positive and negative parts means

$$\mathbf{v}\cdot\mathbf{n}+q+\psi=[\mathbf{v}\cdot\mathbf{n}+q+\psi]^+-[\mathbf{v}\cdot\mathbf{n}+q+\psi]^-.\tag{3.2}$$

We prove two auxiliary lemmas.

Lemma 3.1. (Existence of argminimum) *Let the linear mapping* $\mathbf{s}\mapsto\mathcal{I}[\mathcal{F}\mathbf{s}]:L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})\mapsto L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ *be bounded with constant* $M_0,M_1\geq 0$, $M_0^2+M_1^2\neq 0$:

$$\|\mathcal{I}[\mathcal{F}\mathbf{s}]\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}\leq M_0+M_1\|\mathbf{s}\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})};\tag{3.3a}$$

coercive with constant $M_3\geq 0$, $M_4>0$ *such that*

$$\frac{1}{2}\int_\Omega \mathbf{s}\cdot\mathcal{I}[\mathcal{F}\mathbf{s}]\,d\mathbf{x}\geq M_4\|\mathbf{s}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}-M_3;\tag{3.3b}$$

and weakly lower semi-continuous (w.l.s.c.) such that convergence $\boldsymbol{\sigma}^n\rightharpoonup\boldsymbol{\sigma}$ *weakly in* $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ *as* $n\to\infty$ *implies that*

$$\liminf_{n\to\infty}\int_\Omega\boldsymbol{\sigma}^n\cdot\mathcal{I}[\mathcal{F}\boldsymbol{\sigma}^n]\,d\mathbf{x}\geq\int_\Omega\boldsymbol{\sigma}\cdot\mathcal{I}[\mathcal{F}\boldsymbol{\sigma}]d\mathbf{x}.\tag{3.3c}$$

Then there exists $(\mathbf{u}^\delta,\boldsymbol{\sigma}^\delta,p^\delta)\in\mathcal{M}(\Omega)$ *of the minimum*:

$$\mathscr{L}_\delta(\mathbf{u}^\delta,\boldsymbol{\sigma}^\delta,p^\delta)=\min_{(\mathbf{v},\mathbf{s},q)\in\mathcal{M}(\Omega)}\mathscr{L}_\delta(\mathbf{v},\mathbf{s},q)\tag{3.4}$$

in the admissible set expressed by the convex closed cone

$$\mathcal{M}(\Omega)=\big\{(\mathbf{v},\mathbf{s},q)\in V(\Omega)\,|\,\boldsymbol{\varepsilon}(\mathbf{v})=\mathcal{I}[\mathcal{F}\mathbf{s}]\ \text{in}\ \Omega,\ q\geq -p_0\big\},\tag{3.5}$$

satisfying the a-priori estimate with constant $C > 0$:

$$\|\mathbf{u}^\delta\|_{H^1(\Omega;\mathbb{R}^d)} + \|\boldsymbol{\sigma}^\delta\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} + |p^\delta| + \frac{1}{\sqrt{\delta}}\|[\mathbf{u}^\delta\cdot\mathbf{n} + p^\delta + \psi]^+\|_{L^2(\Sigma)} \le C. \tag{3.6}$$

Proof. Let $(\mathbf{u}^n, \boldsymbol{\sigma}^n, p^n) \in \mathcal{M}(\Omega)$ be an infimal sequence such that

$$\mathscr{L}_\delta(\mathbf{u}^n, \boldsymbol{\sigma}^n, p^n) \to \inf_{(\mathbf{v},\mathbf{s},q)\in\mathcal{M}(\Omega)} \mathscr{L}_\delta(\mathbf{v}, \mathbf{s}, q) =: l_\delta \quad \text{as } n \to \infty. \tag{3.7}$$

Using the coercivity (3.3b), continuous embedding and trace theorem with estimate

$$\|\mathbf{v}\cdot\mathbf{n}\|_{L^1(\Sigma)} \le K_{\rm emb}\|\mathbf{v}\cdot\mathbf{n}\|_{L^2(\Sigma)} \le K_{\rm emb}K_{\rm tr}\|\mathbf{v}\|_{H^1(\Omega;\mathbb{R}^d)}, \quad K_{\rm emb}, K_{\rm tr} > 0, \tag{3.8}$$

and substituting $\boldsymbol{\varepsilon}(\mathbf{u}^n) = \mathcal{I}[\mathcal{F}\boldsymbol{\sigma}^n]$ we get the lower bound for the Lagrangian in (3.1):

$$\begin{aligned} \mathscr{L}_\delta(\mathbf{u}^n, \boldsymbol{\sigma}^n, p^n) \ge{} & M_4\|\boldsymbol{\sigma}^n\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} - M_3 \\ & + \frac{1}{2\delta}\|[\mathbf{u}^n\cdot\mathbf{n} + p^n + \psi]^+\|^2_{L^2(\Sigma)}\, dS_{\mathbf{x}} - Fp^n. \end{aligned} \tag{3.9a}$$

From the Korn–Poincare inequality with constant $K_{\rm KP} > 0$, the boundedness (3.3a) and the constitutive equation in (3.5) it follows that

$$\begin{aligned} K_{\rm KP}\|\mathbf{u}^n\|_{H^1(\Omega;\mathbb{R}^d)} &\le \|\boldsymbol{\varepsilon}(\mathbf{u}^n)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} = \|\mathcal{I}[\mathcal{F}\boldsymbol{\sigma}^n]\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} \\ &\le M_0 + M_1\|\boldsymbol{\sigma}^n\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}. \end{aligned} \tag{3.9b}$$

For $p^n \in \mathcal{M}(\Omega)$, by the virtue of decomposition (3.2) we obtain that

$$-p_0 \le p^n \le -\mathbf{u}^n\cdot\mathbf{n} - \psi + [\mathbf{u}^n\cdot\mathbf{n} + p^n + \psi]^+,$$

and integration of these inequalities over Σ using (3.8) provides the estimate

$$\begin{aligned} |\Sigma||p^n| = \|p^n\|_{L^1(\Sigma)} \le{} & |\Sigma||p_0| + K_{\rm emb}K_{\rm tr}\|\mathbf{u}^n\|_{H^1(\Omega;\mathbb{R}^d)} + \|\psi\|_{L^1(\Sigma)} \\ & + K_{\rm emb}\|[\mathbf{u}^n\cdot\mathbf{n} + p^n + \psi]^+\|_{L^2(\Sigma)}. \end{aligned} \tag{3.9c}$$

From inequalities (3.9) we conclude with uniform $C > 0$ in the upper bound

$$\|\mathbf{u}^n\|_{H^1(\Omega;\mathbb{R}^d)} + \|\boldsymbol{\sigma}^n\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} + |p^n| + \frac{1}{\sqrt{\delta}}\|[\mathbf{u}^n\cdot\mathbf{n} + p^n + \psi]^+\|_{L^2(\Sigma)} \le C. \tag{3.10}$$

Otherwise, if any of the norm in (3.10) tends to infinity, then $\mathscr{L}_\delta(\mathbf{u}^n, \boldsymbol{\sigma}^n, p^n) \to \infty$ in (3.9a) that contradicts (3.7).

By the compactness principle, from (3.10) we assert the existence of an accumulation point $(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, p^\delta) \in V(\Omega)$ and a subsequence, still denoted by n, such that

$$\begin{gathered} (\mathbf{u}^n, \boldsymbol{\sigma}^n) \rightharpoonup (\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta) \text{ weakly in } H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}), \\ \mathbf{u}^n \to \mathbf{u}^\delta \text{ strongly in } L^2(\Sigma;\mathbb{R}^d), \quad p^n \to p^\delta \quad \text{as } n \to \infty, \end{gathered} \tag{3.11}$$

and $(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, p^\delta) \in \mathcal{M}(\Omega)$ since $\mathcal{M}(\Omega)$ is weakly closed. On taking the limit inferior of (3.7), after substitution of $\boldsymbol{\varepsilon}(\mathbf{u}^\delta) = \mathcal{I}[\mathcal{F}\boldsymbol{\sigma}^\delta]$, the w.l.s.c. assumption (3.3a) and the fact that the quadratic penalty term is w.l.s.c. it follows that

$$l_\delta = \liminf_{n\to\infty} \mathscr{L}_\delta(\mathbf{u}^n, \boldsymbol{\sigma}^n, p^n) \geq \mathscr{L}_\delta(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, p^\delta) \geq l_\delta. \tag{3.12}$$

Thus, the equality in (3.12) holds, which guarantees the argminimum in (3.4). The limit of (3.10) leads to the *a-priori* estimate (3.6) and completes the proof. □

We note that the trivial solution $(\mathbf{0}, \mathbf{0}, -p_0)$ to (3.4) is provided by the solvability condition (2.1) when $F = 0$.

Lemma 3.2. (Optimality conditions) *Under the assumptions* (3.3), *let the linear operator* $\mathcal{I}[\mathcal{F}(\,\cdot\,)]$ *be self-adjoint such that for* $\mathbf{s}, \boldsymbol{\chi} \in L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$

$$\int_\Omega \mathbf{s} \cdot \mathcal{I}[\mathcal{F}\boldsymbol{\chi}]\, d\mathbf{x} = \int_\Omega \boldsymbol{\chi} \cdot \mathcal{I}[\mathcal{F}\mathbf{s}]\, d\mathbf{x} \tag{3.13a}$$

holds, and let the operator be surjective: *for arbitrary* $\mathbf{v} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d)$

$$\text{there exists } \mathbf{s} \in L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \quad \text{such that } \boldsymbol{\varepsilon}(\mathbf{v}) = \mathcal{I}[\mathcal{F}\mathbf{s}] \quad \text{in } \Omega. \tag{3.13b}$$

Then necessary and sufficient optimality conditions of the minimum in (3.4) *reads*

$$\int_\Omega \boldsymbol{\sigma}^\delta \cdot \boldsymbol{\varepsilon}(\mathbf{v}) d\mathbf{x} + \frac{1}{\delta} \int_\Sigma [\mathbf{u}^\delta \cdot \mathbf{n} + p^\delta + \psi]^+ (\mathbf{v} \cdot \mathbf{n})\, dS_{\mathbf{x}} = 0 \quad \text{for all } \mathbf{v} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d); \tag{3.14a}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}^\delta) = \mathcal{I}[\mathcal{F}\boldsymbol{\sigma}^\delta] \quad \text{in } \Omega; \tag{3.14b}$$

$$F = \frac{1}{\delta} \int_\Sigma [\mathbf{u}^\delta \cdot \mathbf{n} + p^\delta + \psi]^+\, dS_{\mathbf{x}}. \tag{3.14c}$$

If the coercivity estimate (3.3b) *is strict, i.e.* $M_3 = 0$, *then the solution* $(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, p^\delta) \in V(\Omega)$ *to* (3.14) *is unique. The variational solution* $(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta)$ *to* (3.14) *satisfies the following boundary-value relations*:

$$-\sum_{j=1}^d \frac{\partial \sigma^\delta_{ij}}{\partial x_j} = 0, \quad i = 1\ldots, d, \quad \text{in } \Omega; \tag{3.15a}$$

$$\mathbf{u}^\delta = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}; \quad \boldsymbol{\sigma}^\delta\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{N}}; \tag{3.15b}$$

$$\boldsymbol{\sigma}^\delta\mathbf{n} - (\boldsymbol{\sigma}^\delta\mathbf{n} \cdot \mathbf{n})\mathbf{n} = \mathbf{0}, \quad \boldsymbol{\sigma}^\delta\mathbf{n} \cdot \mathbf{n} = -\frac{1}{\delta}[\mathbf{u}^\delta \cdot \mathbf{n} + p^\delta + \psi]^+ \quad \text{on } \Sigma. \tag{3.15c}$$

Proof. The constitutive equation (3.14b) follows from the definition of set $\mathcal{M}(\Omega)$.

For arbitrary $\mathbf{v} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d)$ and tensor $\mathbf{s}$ satisfying (3.13b), by the linearity of $\mathcal{I}[\mathcal{F}(\,\cdot\,)]$ we have $(\mathbf{u}^\delta + \tau\mathbf{v}, \boldsymbol{\sigma}^\delta + \tau\mathbf{s}, p^\delta) \in \mathcal{M}(\Omega)$ according to (3.5), where $\tau \in \mathbb{R}$. Substituting it as a test function into (3.4), using the constitutive equation (3.14b) and $\boldsymbol{\varepsilon}(\mathbf{v}) = \mathcal{I}[\mathcal{F}\mathbf{s}]$, the self-adjointness property (3.13a) provides

$$\begin{aligned} 0 &\le \mathscr{L}_\delta(\mathbf{u}^\delta + \tau\mathbf{v}, \boldsymbol{\sigma}^\delta + \tau\mathbf{s}, p^\delta) - \mathscr{L}_\delta(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, p^\delta) \\ &= \int_\Omega \left(\tau\boldsymbol{\sigma}^\delta + \frac{\tau^2}{2}\mathbf{s}\right) \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, d\mathbf{x} \\ &\quad + \frac{1}{2\delta}\int_\Sigma \big\{([(\mathbf{u}^\delta + \tau\mathbf{v}) \cdot \mathbf{n} + p^\delta + \psi]^+)^2 - ([\mathbf{u}^\delta \cdot \mathbf{n} + p^\delta + \psi]^+)^2\big\} dS_\mathbf{x}. \end{aligned} \tag{3.16}$$

Since τ can be positive as well as negative, dividing (3.16) by $\tau \neq 0$ and passing $\tau \to 0$ we get the variational equation (3.14a).

On the other hand, for $p^\delta > -p_0$ and $p^\delta + \tau q \ge -p_0$, the variation

$$\begin{aligned} 0 &\le \mathscr{L}_\delta(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, p^\delta + \tau q) - \mathscr{L}_\delta(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, p^\delta) \\ &= \frac{1}{2\delta}\int_\Sigma \big\{([\mathbf{u}^\delta \cdot \mathbf{n} + p^\delta + \tau q + \psi]^+)^2 - ([\mathbf{u}^\delta \cdot \mathbf{n} + p^\delta + \psi]^+)^2\big\}\, dS_\mathbf{x} - \tau F q \end{aligned}$$

after division by $\tau \neq 0$ and taking the limit $\tau \to 0$ leads to the integral equality (3.14c). If $p^\delta = -p_0$, then $\mathbf{u}^\delta = \mathbf{0}$, $\boldsymbol{\sigma}^\delta = \mathbf{0}$ solves (3.4) provided by (2.1) and $F = 0$, thus (3.14c) holds trivially.

Conversely, let relations (3.14) hold. We insert here $\mathbf{v} = \mathbf{u}^\delta - \mathbf{w}$ and $\mathbf{s} = \boldsymbol{\sigma}^\delta - \boldsymbol{\chi}$ with arbitrary $\mathbf{w} \in H^1_{\Gamma_\mathrm{D}}(\Omega; \mathbb{R}^d)$ and tensor $\boldsymbol{\chi}$ satisfying $\boldsymbol{\varepsilon}(\mathbf{w}) = \mathcal{I}[\mathcal{F}\boldsymbol{\chi}]$. By the virtue of convexity of non-negative quadratic functions and using (3.14c) it follows that

$$\begin{aligned} 0 &= \int_\Omega \boldsymbol{\sigma}^\delta \cdot \mathcal{I}[\mathcal{F}(\boldsymbol{\sigma}^\delta - \boldsymbol{\chi})]\, d\mathbf{x} \\ &\quad + \frac{1}{\delta}\int_\Sigma [\mathbf{u}^\delta \cdot \mathbf{n} + p^\delta + \psi]^+ \big((\mathbf{u}^\delta - \mathbf{w}) \cdot \mathbf{n} + p^\delta - q - (p^\delta - q)\big)\, dS_\mathbf{x} \\ &\ge \frac{1}{2}\int_\Omega \big(\boldsymbol{\sigma}^\delta \cdot \mathcal{I}[\mathcal{F}\boldsymbol{\sigma}^\delta] - \boldsymbol{\chi} \cdot \mathcal{I}[\mathcal{F}\boldsymbol{\chi}]\big)\, d\mathbf{x} \\ &\quad + \frac{1}{2\delta}\int_\Sigma \big(([\mathbf{u}^\delta \cdot \mathbf{n} + p^\delta + \psi]^+)^2 - ([\mathbf{w} \cdot \mathbf{n} + q + \psi]^+)^2\big)\, dS_\mathbf{x} - F(p^\delta - q) \\ &= \mathscr{L}_\delta(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, p^\delta) - \mathscr{L}_\delta(\mathbf{w}, \boldsymbol{\chi}, q), \end{aligned} \tag{3.17}$$

which proves the minimum in (3.4).

Assume that there are two solutions to (3.14), which we mark by $k = 1, 2$:

$$\int_\Omega \boldsymbol{\sigma}^k \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, d\mathbf{x} + \frac{1}{\delta}\int_\Sigma [\mathbf{u}^k \cdot \mathbf{n} + p^k + \psi]^+ (\mathbf{v} \cdot \mathbf{n}) dS_\mathbf{x} = 0. \tag{3.18}$$

Testing (3.18) with $\mathbf{v} = \mathbf{u}^1 - \mathbf{u}^2$, using $\boldsymbol{\varepsilon}(\mathbf{u}^k) = \mathcal{I}[\mathcal{F}\boldsymbol{\sigma}^k]$, monotony of the maximum-based penalty operator and (3.14c), we estimate the difference

$$\begin{aligned} &2M_4\|\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \\ &\quad\le \int_\Omega (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \cdot \boldsymbol{\varepsilon}(\mathbf{u}^1 - \mathbf{u}^2) d\mathbf{x} \\ &\quad= -\frac{1}{\delta}\int_\Sigma \big([\mathbf{u}^1 \cdot \mathbf{n} + p^1 + \psi]^+ - [\mathbf{u}^2 \cdot \mathbf{n} + p^2 + \psi]^+\big)\big((\mathbf{u}^1 - \mathbf{u}^2) \cdot \mathbf{n} + p^1 - p^2\big) dS_\mathbf{x} \\ &\quad\le 0 \end{aligned}$$

when the strict coercivity in (3.3b) holds with $M_3 = 0$. It then follows that $\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}^2 = \boldsymbol{\sigma}^\delta$, and $\mathbf{u}^1 = \mathbf{u}^2 = \mathbf{u}^\delta$ according to (3.14b). From (3.14c) we conclude that

$$[\mathbf{u}^\delta \cdot \mathbf{n} + p^1 + \psi]^+ = [\mathbf{u}^\delta \cdot \mathbf{n} + p^2 + \psi]^+,$$

and $p^1 = p^2$ by the continuity: if $p^1 - p^2 > 0$ and $\max_{\mathbf{x}\in\Sigma}[\mathbf{u}^\delta \cdot \mathbf{n} + p^1 + \psi]^+(\mathbf{x}) > 0$, then $\mathbf{x} \in \Sigma$ exists such that $0 < (\mathbf{u}^\delta \cdot \mathbf{n} + p^1 + \psi)(\mathbf{x}) \le p^1 - p^2$, then $(\mathbf{u}^\delta \cdot \mathbf{n} + p^2 + \psi)(\mathbf{x}) \le 0$ leads to the contradiction $[\mathbf{u}^\delta \cdot \mathbf{n} + p^2 + \psi]^+(\mathbf{x}) = 0$. This proves uniqueness to (3.14). The derivation of boundary value problem (3.15) is standard. □

It is worth noting that substitution of the boundary condition from (3.15c) into (3.14c) follows its representation in the form akin to (2.2h):

$$F + \int_\Sigma \boldsymbol{\sigma}^\delta \mathbf{n} \cdot \mathbf{n}\, dS_{\mathbf{x}} = 0. \tag{3.19}$$

In other words, the total contact force F is achieved exactly by the penalty approximation when $\delta > 0$, rather than in the limit as $\delta \to 0^+$.

From Lemmas 3.1 and 3.2 we derive the main result on well-posedness.

Theorem 3.1. (Well-posedness of the indentation problem) *Under the assumptions* (3.3) *and* (3.13) *on the operator* $\mathcal{I}[\mathcal{F}(\,\cdot\,)]$, *at every* $t \in [0, T]$ *there exists a weak solution* $\mathbf{u}(t) \in \mathcal{K}(p(t))$, $\boldsymbol{\sigma}(t) \in L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})$, $p(t) \ge -p_0$ *to the quasi-static unilateral indentation problem* (2.9), *which is uniformly bounded according to* (3.6):

$$\|\mathbf{u}(t)\|_{H^1(\Omega;\mathbb{R}^d)} + \|\boldsymbol{\sigma}(t)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})} + |p(t)| \le C, \quad C > 0. \tag{3.20}$$

When $M_3 = 0$ *in the coercivity estimate* (3.3b), *the solution is unique. Moreover, for* $F \in C([0, T])$ *it is continuous in time*:

$$\mathbf{u} \in C([0, T]; H^1_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d)), \quad \boldsymbol{\sigma} \in C([0, T]; L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})), \quad p \in C([0, T]). \tag{3.21}$$

If the variational solution $(\mathbf{u}, \boldsymbol{\sigma})$ *is* $H^2 \times H^1$*-smooth, then it satisfies the boundary value problem* (2.2), *complementarity conditions* (2.3), *and NLCP conditions* (2.5).

Proof. For fixed t, from the uniform estimate (3.6) we assert existence of an accumulation point $(\mathbf{u}, \boldsymbol{\sigma}, p) \in V(\Omega)$ and a subsequence denoted by δ_n such that

$$\begin{gathered}(\mathbf{u}^{\delta_n}, \boldsymbol{\sigma}^{\delta_n}) \rightharpoonup (\mathbf{u}, \boldsymbol{\sigma}) \quad \text{weakly in } H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}}), \\ \mathbf{u}^{\delta_n} \to \mathbf{u} \quad \text{strongly in } L^2(\Sigma; \mathbb{R}^d), \quad p^{\delta_n} \to p \quad \text{as } \delta_n \to 0^+,\end{gathered} \tag{3.22}$$

and $(\mathbf{u}, \boldsymbol{\sigma}, p) \in \mathcal{M}(\Omega)$ since $\mathcal{M}(\Omega)$ in (3.5) is weakly closed. Thus, (2.9b) holds. Moreover, in the limit $[\mathbf{u} \cdot \mathbf{n} + p + \psi]^+ = 0$ on Σ, and we obtain $\mathbf{u} \in \mathcal{K}(p)$.

Testing (3.14a) with $\mathbf{v}-\mathbf{u}^{\delta_n}$, due to the monotony of the $[\,\cdot\,]^+$-function and the total contact force equality (3.14c) we derive

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma}^{\delta_n}\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}^{\delta_n})\,d\mathbf{x} &= -\frac{1}{\delta_n}\int_\Sigma [\mathbf{u}^{\delta_n}\cdot\mathbf{n}+p^{\delta_n}+\psi]^+\big((\mathbf{v}-\mathbf{u}^{\delta_n})\cdot\mathbf{n}\big)\,dS_{\mathbf{x}}\\ &\geq F(p-p^{\delta_n})-\frac{1}{\delta_n}\int_\Sigma[\mathbf{v}\cdot\mathbf{n}+p+\psi]^+\\ &\quad\times\big((\mathbf{v}-\mathbf{u}^{\delta_n})\cdot\mathbf{n}+p-p^{\delta_n}\big)dS_{\mathbf{x}}\geq F(p-p^{\delta_n})\end{aligned}\tag{3.23}$$

for $\mathbf{v}\in\mathcal{K}(p)$, where $[\mathbf{v}\cdot\mathbf{n}+p+\psi]^+=0$ on Σ. On taking the limit superior of (3.23) this implies the variational inequality (2.9a). By the trace theorem, the left-hand side of Eq. (3.14a) leads to a linear continuous functional over $H^{1/2}_{00}(\Sigma\cup\Gamma_{\mathrm{N}})$. Therefore, there exists a limit in its dual space such that as $\delta_n\to 0^+$:

$$-\frac{1}{\delta_n}[\mathbf{u}^{\delta_n}\cdot\mathbf{n}+p^{\delta_n}+\psi]^+=\boldsymbol{\sigma}^{\delta_n}\mathbf{n}\cdot\mathbf{n}\rightharpoonup\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n}\quad\star\text{-weakly in } H^{1/2}_{00}(\Sigma\cup\Gamma_{\mathrm{N}})^\star,\tag{3.24}$$

which results in the expression (2.9c) by duality pairing. If the solution is smooth, then the boundary stress $\boldsymbol{\sigma}\mathbf{n}$ implies a function, and all relations (2.2), (2.3), (2.5) are fulfilled in the point-wise sense.

We prove the maximal indentation depth condition (2.9d) by contradiction. Assume that $\max_{\mathbf{x}\in\Sigma}(\mathbf{u}\cdot\mathbf{n}+p+\psi)(\mathbf{x})<0$, which is possible in the limit (3.22) only when $\max_{\mathbf{x}\in\Sigma}(\mathbf{u}^{\delta_n}\cdot\mathbf{n}+p^{\delta_n}+\psi)(\mathbf{x})<0$ for small δ_n. This follows from $[\mathbf{u}^{\delta_n}\cdot\mathbf{n}+p^{\delta_n}+\psi]^+=0$ and $F=0$ in (3.14c), then $\mathbf{u}^{\delta_n}=\mathbf{0}$, $\boldsymbol{\sigma}^{\delta_n}=\mathbf{0}$, $p^{\delta_n}=-p_0$ contradicts $\mathbf{u}^{\delta_n}\cdot\mathbf{n}+p^{\delta_n}+\psi=-p_0+\psi$, which is equal to zero at the point of maximum of ψ according to the definition (2.1).

Let $M_3=0$ in the coercivity estimate (3.3b) and $F\in C([0,T])$. To show the continuity of the solution in time (3.21), consider $t_1\neq t_2\in[0,T]$. We construct $\mathbf{w}\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)$ such that $\mathbf{w}\cdot\mathbf{n}=p(t_1)-p(t_2)$ on Σ solving the well-posed problem:

$$\int_\Omega\boldsymbol{\varepsilon}(\mathbf{w})\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{w})\,d\mathbf{x}\geq 0\quad\text{for all }\mathbf{v}\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)\text{: }\mathbf{v}\cdot\mathbf{n}=p(t_1)-p(t_2)\text{ on }\Sigma.$$

At time t_1 we have $(\mathbf{u}(t_2)-\mathbf{w})\cdot\mathbf{n}+p(t_1)+\psi=\mathbf{u}(t_2)\cdot\mathbf{n}+p(t_2)+\psi\leq 0$ on Σ, hence can test (2.9a) with $\mathbf{v}=\mathbf{u}(t_2)-\mathbf{w}$ such that

$$\int_\Omega\boldsymbol{\sigma}(t_1)\cdot\boldsymbol{\varepsilon}(\mathbf{u}(t_2)-\mathbf{w}-\mathbf{u}(t_1))\,d\mathbf{x}\geq 0.\tag{3.25a}$$

Similarly, $(\mathbf{u}(t_1)+\mathbf{w})\cdot\mathbf{n}+p(t_2)+\psi=\mathbf{u}(t_1)\cdot\mathbf{n}+p(t_1)+\psi\leq 0$ on Σ, and

$$\int_\Omega\boldsymbol{\sigma}(t_2)\cdot\boldsymbol{\varepsilon}(\mathbf{u}(t_1)+\mathbf{w}-\mathbf{u}(t_2))\,d\mathbf{x}\geq 0.\tag{3.25b}$$

After summation of (3.25) and integration by parts the estimate

$$2M_4\|\boldsymbol{\sigma}(t_1)-\boldsymbol{\sigma}(t_2)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})}$$

$$\begin{aligned}
&\le \int_\Omega (\boldsymbol{\sigma}(t_1) - \boldsymbol{\sigma}(t_2)) \cdot \boldsymbol{\varepsilon}\big(\mathbf{u}(t_1) - \mathbf{u}(t_2)\big)\, d\mathbf{x} \\
&\le - \int_\Omega (\boldsymbol{\sigma}(t_1) - \boldsymbol{\sigma}(t_2)) \cdot \boldsymbol{\varepsilon}(\mathbf{w})\, d\mathbf{x} = -\langle (\boldsymbol{\sigma}(t_1) - \boldsymbol{\sigma}(t_2))\mathbf{n}\cdot\mathbf{n}, \mathbf{w}\cdot\mathbf{n}\rangle_{\Sigma\cup\Gamma_{\mathrm{N}}} \\
&= (F(t_1) - F(t_2))(p(t_1) - p(t_2)) \qquad (3.26)
\end{aligned}$$

follows, where we have used the equality (2.9c). Since $p(t_1) - p(t_2)$ is bounded, the continuity $F(t_1) - F(t_2) \to 0$ as $t_1 - t_2 \to 0$ in (3.26) implies that $\boldsymbol{\sigma}(t_1) - \boldsymbol{\sigma}(t_2) \to 0$. From (2.9b) we conclude that $\boldsymbol{\varepsilon}\big(\mathbf{u}(t_1) - \mathbf{u}(t_2)\big) = \mathcal{I}[\mathcal{F}(\boldsymbol{\sigma}(t_1) - \boldsymbol{\sigma}(t_2))] \to 0$, and $p(t_1) - p(t_2) \to 0$ by the virtue of the maximal indentation depth condition (2.9d).

In particular, when $F(t_1) = F(t_2)$ in (3.26) the solution is unique. This finishes the proof. ∎

We now apply Theorem 3.1 to linear elastic models to illustrate its implications.

Corollary 3.1. (Elastic indentation problem) *Let the compliance tensor* $\mathcal{F}(\mathbf{x})$ *mapping* $\mathbf{s} \mapsto \mathcal{F}\mathbf{s} : L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \mapsto L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ *be bounded such that*

$$\|\mathcal{F}\mathbf{s}\|_{L^2(\mathbb{R}^{d\times d}_{\mathrm{sym}})} \le M_1 \|\mathbf{s}\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})}, \quad M_1 > 0, \tag{3.27a}$$

for $\mathbf{s} \in L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, *and positive definite such that*

$$\frac{1}{2}\int_\Omega \mathbf{s}\cdot\mathcal{F}\mathbf{s}\, d\mathbf{x} \ge M_4 \|\mathbf{s}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})}, \quad M_4 > 0. \tag{3.27b}$$

By Theorem 3.1, *for* $\mathcal{I} = \mathcal{I}_{\mathrm{E}}$ *in* (1.2) *there exists the unique solution*

$$\begin{aligned}
\mathbf{u}^{\mathrm{E}}(t,\mathbf{x}) &\in L^\infty(0,T; H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)), \\
\boldsymbol{\sigma}^{\mathrm{E}}(t,\mathbf{x}) &\in L^\infty(0,T; L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \qquad (3.28) \\
p^{\mathrm{E}}(t) &\in L^\infty(0,T), \quad p^{\mathrm{E}} \ge -p_0
\end{aligned}$$

to the elastic indentation problem (2.9) *which holds at every* $t \in [0,T]$ *meeting*

$$\mathbf{u}^{\mathrm{E}}\cdot\mathbf{n} + p^{\mathrm{E}} + \psi \le 0 \text{ on } \Sigma, \quad \int_\Omega \boldsymbol{\sigma}^{\mathrm{E}}\cdot\boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}^{\mathrm{E}})\, d\mathbf{x} \ge 0$$

$$\text{for all } \mathbf{v} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d) \text{ such that } \mathbf{v}\cdot\mathbf{n} + p^{\mathrm{E}} + \psi \le 0 \text{ on } \Sigma; \tag{3.29a}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{E}}) = \mathcal{F}\boldsymbol{\sigma}^{\mathrm{E}} \quad \text{in } \Omega; \tag{3.29b}$$

$$F + \langle \boldsymbol{\sigma}^{\mathrm{E}}\mathbf{n}\cdot\mathbf{n}, \eta\rangle_{\Sigma\cup\Gamma_{\mathrm{N}}} = 0; \tag{3.29c}$$

$$p^{\mathrm{E}} = \max\{q \in \mathbb{R} \,|\, \mathbf{u}^{\mathrm{E}}\cdot\mathbf{n} + q + \psi \le 0 \text{ on } \Sigma\}. \tag{3.29d}$$

For $H^2 \times H^1$*-smooth* $(\mathbf{u}^{\mathrm{E}}, \boldsymbol{\sigma}^{\mathrm{E}})$, *the variational inequality* (3.29a) *is expressed as*

$$-\sum_{j=1}^{d} \frac{\partial \sigma^{\mathrm{E}}_{ij}}{\partial x_j} = 0, \quad i = 1\ldots, d, \quad \text{in } \Omega; \tag{3.30a}$$

$$\mathbf{u}^{\mathrm{E}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}; \quad \boldsymbol{\sigma}^{\mathrm{E}}\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{N}}; \quad \boldsymbol{\sigma}^{\mathrm{E}}\mathbf{n} - (\boldsymbol{\sigma}^{\mathrm{E}}\mathbf{n}\cdot\mathbf{n})\mathbf{n} = \mathbf{0} \quad \text{on } \Sigma; \tag{3.30b}$$

$$\mathbf{u}^{\mathrm{E}}\cdot\mathbf{n} + p^{\mathrm{E}} + \psi = 0, \quad \boldsymbol{\sigma}^{\mathrm{E}}\mathbf{n}\cdot\mathbf{n} < 0 \quad \text{on } \mathcal{C}^{\mathrm{E}}; \tag{3.30c}$$

$$\mathbf{u}^{\mathrm{E}}\cdot\mathbf{n} + p^{\mathrm{E}} + \psi \le 0, \quad \boldsymbol{\sigma}^{\mathrm{E}}\mathbf{n}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma\backslash\mathcal{C}^{\mathrm{E}} \tag{3.30d}$$

over the coincidence set $\mathcal{C}^{\mathrm{E}}(t)$ *and its complement* $\Sigma\backslash\mathcal{C}^{\mathrm{E}}(t)$ *following* (2.3).

Indeed, based on properties (3.27) it is easy to check that all the conditions (3.3) and (3.13) are satisfied for the identity mapping $\mathcal{I}_{\mathrm{E}}$ in (1.2). The additional smoothness for the contact problem is provided by smooth problem data.[28]

On the contrary, the mapping $\mathcal{I}_{\mathrm{VE}}$ in (1.3) corresponding to the linear viscoelastic model fails the conditions (3.3b), (3.3c) and (3.13) because of the presence of the Volterra convolution operator. For this reason, in the next section we extend Theorem 3.1 to a specific case when the contact area does not increase.

4. Solution for a Viscoelastic Body for Non-Increasing Contact Area

The linear viscoelastic model defined by $\mathcal{I} = \mathcal{I}_{\mathrm{VE}}$ in (1.3) satisfies the following properties. The operator is sign-preserving for $\boldsymbol{\chi}(t,\mathbf{x}) \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$:

$$\text{if } \boldsymbol{\chi}(s) \ge 0 \text{ for all } s\in[0,t], \quad \text{then } \mathcal{I}[\boldsymbol{\chi}](t) \ge 0; \tag{4.1a}$$

and commutes with the strain tensor for $\mathbf{v} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)$:

$$\mathcal{I}[\boldsymbol{\varepsilon}(\mathbf{v})] = \boldsymbol{\varepsilon}(\mathcal{I}[\mathbf{v}]). \tag{4.1b}$$

It is worth noting that by (4.1a), $\boldsymbol{\chi} = 0$ implies that $\mathcal{I}[\boldsymbol{\chi}] = 0$. Based on (4.1), from (3.29) and (3.30) we construct the corresponding solution to (2.9).

Theorem 4.1. (Well-posedness of the indentation problem for non-increasing contact area) *Let the coincidence sets in* (3.30) *for the elastic indentation problem* (3.29) *be non-increasing in time such that*

$$\Sigma = \mathcal{C}^{\mathrm{E}}(0) \supseteq \mathcal{C}^{\mathrm{E}}(s) \supseteq \mathcal{C}^{\mathrm{E}}(t) \quad \text{for all } 0 \le s \le t \le T, \tag{4.2}$$

where when $t = 0$ *the elastic solution* $(\mathbf{u}^{\mathrm{E}}(0,\mathbf{x}), \boldsymbol{\sigma}^{\mathrm{E}}(0,\mathbf{x}), p^{\mathrm{E}}(0))$ *satisfies*

$$\mathbf{u}^{\mathrm{E}}(0)\cdot\mathbf{n} + p^{\mathrm{E}}(0) + \psi = 0 \text{ on } \Sigma, \quad \int_\Omega \boldsymbol{\sigma}^{\mathrm{E}}(0)\cdot\boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}^{\mathrm{E}}(0))\,d\mathbf{x} \ge 0$$

$$\text{for all } \mathbf{v} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d) \text{ such that } \mathbf{v}\cdot\mathbf{n} + p^{\mathrm{E}}(0) + \psi \le 0 \text{ on } \Sigma; \tag{4.3a}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{E}}(0)) = \mathcal{F}\boldsymbol{\sigma}^{\mathrm{E}}(0) \quad \text{in } \Omega; \tag{4.3b}$$

$$F(0) + \langle \boldsymbol{\sigma}^{\mathrm{E}}(0)\mathbf{n}\cdot\mathbf{n}, \eta\rangle_{\Sigma\cup\Gamma_{\mathrm{N}}} = 0; \tag{4.3c}$$

$$p^{\mathrm{E}}(0) = \max\{q\in\mathbb{R} \,|\, \mathbf{u}^{\mathrm{E}}(0)\cdot\mathbf{n} + q + \psi \le 0 \text{ on } \Sigma\}, \tag{4.3d}$$

and the variational inequality (4.3a) *implies that*

$$-\sum_{j=1}^{d} \frac{\partial \sigma_{ij}^{\mathrm{E}}}{\partial x_j}(0) = 0, \quad i = 1\ldots, d, \quad \text{in } \Omega; \tag{4.4a}$$

$$\mathbf{u}^{\mathrm{E}}(0) = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}; \quad \boldsymbol{\sigma}^{\mathrm{E}}(0)\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{N}}; \tag{4.4b}$$

$$\boldsymbol{\sigma}^{\mathrm{E}}(0)\mathbf{n} - (\boldsymbol{\sigma}^{\mathrm{E}}(0)\mathbf{n}\cdot\mathbf{n})\mathbf{n} = \mathbf{0}, \quad \boldsymbol{\sigma}^{\mathrm{E}}(0)\mathbf{n}\cdot\mathbf{n} \le 0 \quad \text{on } \Sigma. \tag{4.4c}$$

Under the assumptions (3.27) *on* $\mathcal{F}$ *and* (4.1) *on* $\mathcal{I}$, *there exists a solution to the unilateral indentation problem* (2.9) *given semi-explicitly by the formulas*

$$\mathbf{u} = \mathbf{u}^{\mathrm{E}}(0) + \mathcal{I}[\mathbf{u}^{\mathrm{E}} - \mathbf{u}^{\mathrm{E}}(0)], \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{E}}, \quad p = p^{\mathrm{E}}(0) + \mathcal{I}[p^{\mathrm{E}} - p^{\mathrm{E}}(0)], \quad \mathcal{C} = \mathcal{C}^{\mathrm{E}}. \tag{4.5}$$

Proof. Using equations in (4.3) and (4.4), from (3.29) and (3.30) we derive for

$$\overline{\mathbf{u}}^{\mathrm{E}} := \mathbf{u}^{\mathrm{E}} - \mathbf{u}^{\mathrm{E}}(0), \quad \overline{\boldsymbol{\sigma}}^{\mathrm{E}} := \boldsymbol{\sigma}^{\mathrm{E}} - \boldsymbol{\sigma}^{\mathrm{E}}(0), \quad \overline{p}^{\mathrm{E}} := p^{\mathrm{E}} - p^{\mathrm{E}}(0) \tag{4.6}$$

the following boundary-value relations written in the incremental form:

$$-\sum_{j=1}^{d} \frac{\partial \overline{\sigma}_{ij}^{\mathrm{E}}}{\partial x_j} = 0, \quad i = 1\ldots, d, \quad \text{in } \Omega; \tag{4.7a}$$

$$\overline{\mathbf{u}}^{\mathrm{E}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}; \quad \overline{\boldsymbol{\sigma}}^{\mathrm{E}}\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{N}}; \quad \overline{\boldsymbol{\sigma}}^{\mathrm{E}}\mathbf{n} - (\overline{\boldsymbol{\sigma}}^{\mathrm{E}}\mathbf{n}\cdot\mathbf{n})\mathbf{n} = \mathbf{0} \quad \text{on } \Sigma; \tag{4.7b}$$

$$\overline{\mathbf{u}}^{\mathrm{E}}\cdot\mathbf{n} + \overline{p}^{\mathrm{E}} = 0 \text{ on } \mathcal{C}^{\mathrm{E}}; \quad \overline{\mathbf{u}}^{\mathrm{E}}\cdot\mathbf{n} + \overline{p}^{\mathrm{E}} \le 0 \quad \text{on } \Sigma\backslash\mathcal{C}^{\mathrm{E}}; \tag{4.7c}$$

$$\boldsymbol{\varepsilon}(\overline{\mathbf{u}}^{\mathrm{E}}) = \mathcal{F}\overline{\boldsymbol{\sigma}}^{\mathrm{E}} \quad \text{in } \Omega; \tag{4.7d}$$

$$\overline{p}^{\mathrm{E}} = \max\{q \in \mathbb{R} \,|\, \overline{\mathbf{u}}^{\mathrm{E}}\cdot\mathbf{n} + q \le 0 \text{ on } \Sigma\}. \tag{4.7e}$$

For $t \in (0, T]$, according to (4.7c) and (4.2) we have the contact $(\overline{\mathbf{u}}^{\mathrm{E}}\cdot\mathbf{n} + \overline{p}^{\mathrm{E}})(s) = 0$ on $\mathcal{C}^{\mathrm{E}}(s)$ for all $s \in (0, t]$. By the virtue of sign-preserving property (4.1a), applying the linear operator $\mathcal{I}$, from (3.30c) and (4.7c) it follows that

$$\mathcal{I}[\overline{\mathbf{u}}^{\mathrm{E}}]\cdot\mathbf{n} + \mathcal{I}[\overline{p}^{\mathrm{E}}] = 0, \quad \boldsymbol{\sigma}^{\mathrm{E}}\mathbf{n}\cdot\mathbf{n} < 0 \quad \text{on } \mathcal{C}^{\mathrm{E}}(t), \tag{4.8a}$$

and, similarly, on the complementary set according to (3.30d)

$$\mathcal{I}[\overline{\mathbf{u}}^{\mathrm{E}}]\cdot\mathbf{n} + \mathcal{I}[\overline{p}^{\mathrm{E}}] \le 0, \quad \boldsymbol{\sigma}^{\mathrm{E}}\mathbf{n}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma\backslash\mathcal{C}^{\mathrm{E}}(t) \tag{4.8b}$$

holds. Due to the commutative property (4.1b), from (4.7d) we get

$$\boldsymbol{\varepsilon}(\mathcal{I}[\overline{\mathbf{u}}^{\mathrm{E}}]) = \mathcal{I}[\mathcal{F}\overline{\boldsymbol{\sigma}}^{\mathrm{E}}] \quad \text{in } \Omega. \tag{4.9}$$

Applying $\mathcal{I}$ to the maximum conditions (4.7e) leads to

$$\mathcal{I}[\overline{p}^{\mathrm{E}}] = \max\{q \in \mathbb{R} \,|\, \mathcal{I}[\overline{\mathbf{u}}^{\mathrm{E}}]\cdot\mathbf{n} + q \le 0 \text{ on } \Sigma\}. \tag{4.10}$$

Using the definition of $(\mathbf{u}, \boldsymbol{\sigma}, p)$ in (4.5), from (3.30a), (3.30b), (4.8) we arrive at the relations (2.2a), (2.2d)–(2.2g), which are satisfied weakly within the quasi-variational inequality (2.9a). The linear equations (4.3b), (4.9) lead to the response

(2.9b), and (3.29c) implies the total contact force given by (2.9c). From the maximum conditions (4.3d) and (4.10) we obtain (2.9d). The proof is complete. □

We note that, for the initialization in (4.2), it suffices to solve the elastic problem (3.29) for some $F(0) > 0$, to determine its correspondence set $\mathcal{C}^{\mathrm{E}}(0)$ where the contact occurs, and then to reset the contact boundary as $\Sigma := \mathcal{C}^{\mathrm{E}}(0)$. Whereas the subsequent non-increase of $\mathcal{C}^{\mathrm{E}}(t)$ can be achieved by a non-increasing total contact force $F(t)$ due to the continuity in time as stated in Theorem 3.1.

We apply Theorem 4.1 to the linear viscoelastic model to illustrate its use.

Corollary 4.1. (Viscoelastic indentation problem for non-increasing contact area) *Under the assumptions of Theorem* 4.1, *for* $\mathcal{I} = \mathcal{I}_{\mathrm{VE}}$ *in* (1.3) *there exists a solution to the viscoelastic indentation problem* (2.9) *given by*

$$\begin{aligned} \mathbf{u}^{\mathrm{VE}} &= \mathbf{u}^{\mathrm{E}}(0) + \mathcal{I}[\mathbf{u}^{\mathrm{E}} - \mathbf{u}^{\mathrm{E}}(0)], \quad \boldsymbol{\sigma}^{\mathrm{VE}} = \boldsymbol{\sigma}^{\mathrm{E}}, \\ p^{\mathrm{VE}} &= p^{\mathrm{E}}(0) + \mathcal{I}[p^{\mathrm{E}} - p^{\mathrm{E}}(0)], \quad \mathcal{C}^{\mathrm{VE}} = \mathcal{C}^{\mathrm{E}}. \end{aligned} \tag{4.11}$$

Next, on the basis of Corollary 4.1 we construct explicitly an analytical solution.

4.1. *Axisymmetric indentation of viscoelastic half-space by a cone*

Let Ω be given in a cylindrical coordinate system $(r, \theta, z) \in \mathbb{R}_+ \times [-\pi, \pi] \times \mathbb{R}$, namely the half-space $z > 0$. The boundary $\Sigma \cup \Gamma_{\mathrm{N}}$ implies that at the plane $z = 0$ the normal vector is given by $\mathbf{n} = (0, 0, -1)$, and the Dirichlet boundary Γ_{D} applies as $r + z \to \infty$. We consider the axisymmetric cone of shape $\psi(r) = -r/\tan(\alpha)$ with $\alpha \in (0, \pi/2)$, which is impressed with the prescribed total contact force F at the origin $r = z = 0$ in a direction opposite to $\mathbf{n}$ as shown in the left picture in Fig. 1. The indentation exhibits symmetry about the z axis and does not depend on θ.

Ignoring the components u_θ^{VE}, $\sigma_{r\theta}^{\mathrm{VE}}$, $\sigma_{z\theta}^{\mathrm{VE}}$ that are zero, the indentation problem (2.2) for an isotropic viscoelastic body in the axisymmetric configuration reads: find displacements $u_r^{\mathrm{VE}}(t, r, z)$, $u_z^{\mathrm{VE}}(t, r, z)$, stresses $\sigma_{rr}^{\mathrm{VE}}(t, r, z)$, $\sigma_{\theta\theta}^{\mathrm{VE}}(t, r, z)$, $\sigma_{rz}^{\mathrm{VE}}(t, r, z)$, $\sigma_{zz}^{\mathrm{VE}}(t, r, z)$, the punch indentation depth $p^{\mathrm{VE}}(t) \geq 0$ and the radius of contact $h^{\mathrm{VE}}(t) \geq 0$, which satisfy for $z > 0$ the balance equations:

$$\sigma_{rr,r}^{\mathrm{VE}} + \sigma_{rz,z}^{\mathrm{VE}} + \frac{1}{r}(\sigma_{rr}^{\mathrm{VE}} - \sigma_{\theta\theta}^{\mathrm{VE}}) = 0, \quad \sigma_{rz,r}^{\mathrm{VE}} + \sigma_{zz,z}^{\mathrm{VE}} + \frac{1}{r}\sigma_{rz}^{\mathrm{VE}} = 0; \tag{4.12a}$$

the constitutive equations (1.3) and expression for the linearized strain:

$$\begin{aligned} &\varepsilon_{rr}(\mathbf{u}^{\mathrm{VE}}) = u_{r,r}^{\mathrm{VE}}, \quad \varepsilon_{\theta\theta}(\mathbf{u}^{\mathrm{VE}}) = \frac{1}{r}u_r^{\mathrm{VE}}, \quad \varepsilon_{zz}(\mathbf{u}^{\mathrm{VE}}) = u_{z,z}^{\mathrm{VE}}, \\ &\varepsilon_{rz}(\mathbf{u}^{\mathrm{VE}}) = \frac{1}{2}(u_{r,z}^{\mathrm{VE}} + u_{z,r}^{\mathrm{VE}}). \end{aligned} \tag{4.12b}$$

The asymptotic condition at infinity is given by

$$u_r^{\mathrm{VE}}, u_z^{\mathrm{VE}} \to 0 \quad \text{as } r + z \to \infty. \tag{4.12c}$$

At $z = 0$, the stress-free boundary conditions hold:

$$\sigma_{rz}^{\mathrm{VE}} = 0 \text{ for } r \geq 0; \quad \sigma_{zz}^{\mathrm{VE}} = 0 \text{ for } r \geq h_\Sigma; \tag{4.12d}$$

where at large radius $h_\Sigma > 0$ separates Σ from Γ_{N}, the contact conditions are:

$$u_z^{\mathrm{VE}} = p^{\mathrm{VE}} - \frac{r}{\tan(\alpha)} = 0, \quad \sigma_{zz}^{\mathrm{VE}} < 0 \quad \text{for } 0 \leq r < h^{\mathrm{VE}}; \tag{4.12e}$$

$$u_z^{\mathrm{VE}} \geq p^{\mathrm{VE}} - \frac{r}{\tan(\alpha)}, \quad \sigma_{zz}^{\mathrm{VE}} = 0 \quad \text{for } h^{\mathrm{VE}} \leq r < h_\Sigma; \tag{4.12f}$$

the total contact force prescribed at $z = 0$ is

$$F + 2\pi \int_0^{h^{\mathrm{VE}}} \sigma_{zz}^{\mathrm{VE}} r \, dr = 0; \tag{4.12g}$$

and the maximal indentation depth is

$$p^{\mathrm{VE}} = \max \left\{ q \in \mathbb{R} \,|\, u_z^{\mathrm{VE}} \geq q - \frac{r}{\tan(\alpha)} \right\}. \tag{4.12h}$$

The analytical solution to (4.12) is constructed based on Corollary 4.1.

Theorem 4.2. (Solution to the viscoelastic half-space indented by a cone) *Let the total contact force be non-increasing such that*

$$F(0) > 0, \quad F(s) \geq F(t) \geq 0 \quad \text{for all } 0 \leq s \leq t \leq T. \tag{4.13}$$

Then the viscoelastic indentation problem (4.12) *has a solution for* $h_\Sigma = h^{\mathrm{VE}}(0)$. *At every* $t \in [0, T]$, *the indentation depth and the radius of contact are*

$$\begin{aligned} p^{\mathrm{VE}} &= \sqrt{\frac{\pi}{2M \tan(\alpha)}} \big(\sqrt{F(0)} + \mathcal{I}_{\mathrm{VE}} \big[\sqrt{F} - \sqrt{F(0)} \big] \big); \\ h^{\mathrm{VE}} &= \sqrt{\frac{2\tan(\alpha)}{\pi M} F}; \end{aligned} \tag{4.14a}$$

where the indentation modulus for the elastic problem is given by $M := E/(1 - \nu^2)$ *for Young's modulus and Poisson's ratio* E *and* ν, *and the displacement is*

$$u_r^{\mathrm{VE}} = u_r^{\mathrm{E}}(0) + \mathcal{I}_{\mathrm{VE}}[u_r^{\mathrm{E}} - u_r^{\mathrm{E}}(0)], \quad u_z^{\mathrm{VE}} = u_z^{\mathrm{E}}(0) + \mathcal{I}_{\mathrm{VE}}[u_z^{\mathrm{E}} - u_z^{\mathrm{E}}(0)]; \tag{4.14b}$$

where u_r^{E}, u_z^{E} *are given by* (4.31a), *and the stress yields expressions*

$$\begin{aligned} \sigma_{rr}^{\mathrm{VE}} &= \frac{M}{2\tan(\alpha)} \int_0^{h^{\mathrm{VE}}} \left(\frac{1 - 2\nu}{r} J_0^1(a) - J_1^0(a) - \frac{z}{r} J_1^1(a) + z J_2^0(a) \right) da, \\ \sigma_{rz}^{\mathrm{VE}} &= -\frac{M}{2\tan(\alpha)} z \int_0^{h^{\mathrm{VE}}} J_2^1(a) da, \\ \sigma_{zz}^{\mathrm{VE}} &= -\frac{M}{2\tan(\alpha)} \int_0^{h^{\mathrm{VE}}} \big(J_1^0(a) + z J_2^0(a) \big) da, \\ \sigma_{\theta\theta}^{\mathrm{VE}} &= \frac{M}{2\tan(\alpha)} \int_0^{h^{\mathrm{VE}}} \left(-\frac{1 - 2\nu}{r} J_0^1(a) - 2\nu J_1^0(a) + \frac{z}{r} J_1^1(a) \right) da. \end{aligned} \tag{4.14c}$$

Sneddon's integrals $J_n^m(a)$ *for integer* $m, n \geq 0$ *are defined by the imaginary part of the mth-order Hanckel transform*

$$J_n^m(a, r, z) := \operatorname{Im} \int_0^\infty e^{-(z-\imath a)\xi} \xi^{n-1} J_m(r\xi)\, d\xi, \quad \imath^2 = -1, \tag{4.15}$$

which can be calculated analytically in (4.14) *as*

$$J_0^0(a) = \arcsin \frac{2a}{\sqrt{(r+a)^2 + z^2} + \sqrt{(r-a)^2 + z^2}}, \tag{4.16a}$$

$$J_0^1(a) = \operatorname{Im} \frac{r}{z - \imath a + \sqrt{r^2 + (z-\imath a)^2}}, \quad J_1^0(a) = \operatorname{Im} \frac{1}{\sqrt{r^2 + (z-\imath a)^2}},$$

$$J_1^1(a) = \operatorname{Im} \frac{r}{\sqrt{r^2 + (z-\imath a)^2}\big(z - \imath a + \sqrt{r^2 + (z-\imath a)^2}\big)},$$

$$J_2^0(a) = \operatorname{Im} \frac{z - \imath a}{\big(r^2 + (z-\imath a)^2\big)^{3/2}}, \quad J_2^1(a) = \operatorname{Im} \frac{r}{\big(r^2 + (z-\imath a)^2\big)^{3/2}}. \tag{4.16b}$$

In particular, on the plane $z = 0$ *the normal displacement for* $r \geq h^{\mathrm{VE}}$ *is*

$$\begin{aligned} u_z^{\mathrm{VE}} = {} & \sqrt{\frac{2}{\pi M \tan(\alpha)}} \bigg\{ \sqrt{F(0)} \arcsin\left(\frac{h^{\mathrm{VE}}(0)}{r}\right) \\ & + I_{\mathrm{VE}}\left[\sqrt{F} \arcsin\left(\frac{h^{\mathrm{VE}}}{r}\right) - \sqrt{F(0)} \arcsin\left(\frac{h^{\mathrm{VE}}(0)}{r}\right)\right]\bigg\} \\ & + \frac{\sqrt{r^2 - (h^{\mathrm{VE}}(0))^2} - r + I_{\mathrm{VE}}\big[\sqrt{r^2 - (h^{\mathrm{VE}})^2} - \sqrt{r^2 - (h^{\mathrm{VE}}(0))^2}\big]}{\tan(\alpha)}; \end{aligned} \tag{4.17a}$$

and the normal stress has the form

$$\sigma_{zz}^{\mathrm{VE}} = -\frac{M}{2\tan(\alpha)} \operatorname{arcosh}\left(\frac{h^{\mathrm{VE}}}{r}\right) \quad \textit{for } 0 \leq r < h^{\mathrm{VE}}. \tag{4.17b}$$

Proof. The elastic indentation problem (3.29) and (3.30) for the corresponding isotropic body implies: find displacements $u_r^{\mathrm{E}}(t, r, z)$, $u_z^{\mathrm{E}}(t, r, z)$, stresses $\sigma_{rr}^{\mathrm{E}}(t, r, z)$, $\sigma_{\theta\theta}^{\mathrm{E}}(t, r, z)$, $\sigma_{rz}^{\mathrm{E}}(t, r, z)$, $\sigma_{zz}^{\mathrm{E}}(t, r, z)$, the punch indentation depth $p^{\mathrm{E}}(t) \geq 0$ and the contact radius $h^{\mathrm{E}}(t) \geq 0$, which satisfy for $z > 0$ the balance equations:

$$\sigma_{rr,r}^{\mathrm{E}} + \sigma_{rz,z}^{\mathrm{E}} + \frac{1}{r}(\sigma_{rr}^{\mathrm{E}} - \sigma_{\theta\theta}^{\mathrm{E}}) = 0, \quad \sigma_{rz,r}^{\mathrm{E}} + \sigma_{zz,z}^{\mathrm{E}} + \frac{1}{r}\sigma_{rz}^{\mathrm{E}} = 0; \tag{4.18a}$$

the constitutive equations:

$$\begin{aligned} u_{r,r}^{\mathrm{E}} &= \frac{1}{E}\big(\sigma_{rr}^{\mathrm{E}} - \nu(\sigma_{\theta\theta}^{\mathrm{E}} + \sigma_{zz}^{\mathrm{E}})\big), \quad \frac{1}{r}u_r^{\mathrm{E}} = \frac{1}{E}\big(\sigma_{\theta\theta}^{\mathrm{E}} - \nu(\sigma_{rr}^{\mathrm{E}} + \sigma_{zz}^{\mathrm{E}})\big), \\ u_{z,z}^{\mathrm{E}} &= \frac{1}{E}\big(\sigma_{zz}^{\mathrm{E}} - \nu(\sigma_{rr}^{\mathrm{E}} + \sigma_{\theta\theta}^{\mathrm{E}})\big), \quad \frac{1}{2}(u_{r,z}^{\mathrm{E}} + u_{z,r}^{\mathrm{E}}) = \frac{1+\nu}{E}\sigma_{rz}^{\mathrm{E}}; \end{aligned} \tag{4.18b}$$

and the asymptotic condition at infinity

$$u_r^{\mathrm{E}}, u_z^{\mathrm{E}} \to 0 \quad \text{as } r+z \to \infty. \tag{4.18c}$$

At $z=0$, the tangential stress-free boundary condition satisfies

$$\sigma_{rz}^{\mathrm{E}} = 0; \tag{4.18d}$$

the complementary contact conditions:

$$u_z^{\mathrm{E}} = p^{\mathrm{E}} - \frac{r}{\tan(\alpha)} = 0, \quad \sigma_{zz}^{\mathrm{E}} < 0 \quad \text{for } 0 \le r < h^{\mathrm{E}}; \tag{4.18e}$$

$$u_z^{\mathrm{E}} \ge p^{\mathrm{E}} - \frac{r}{\tan(\alpha)}, \quad \sigma_{zz}^{\mathrm{E}} = 0 \quad \text{for } r \ge h^{\mathrm{E}}; \tag{4.18f}$$

the prescribed total contact force at $z=0$:

$$F + 2\pi \int_0^{h^{\mathrm{E}}} \sigma_{zz}^{\mathrm{E}} r\, dr = 0; \tag{4.18g}$$

and the maximal indentation depth condition:

$$p^{\mathrm{E}} = \max\left\{ q \in \mathbb{R} \,|\, u_z^{\mathrm{E}} \ge q - \frac{r}{\tan(\alpha)} \right\}. \tag{4.18h}$$

To solve (4.18) we sketch the Papkovich–Neuber representation technique (see Ref. 23). With the help of two unknown potentials $\phi(t,r,z)$ and $\psi_z(t,r,z)$ such that

$$\psi_{z,rr} + \frac{1}{r}\psi_{z,r} + \psi_{z,zz} = 0, \quad \phi_{,rr} + \frac{1}{r}\phi_{,r} + \phi_{,zz} = 0 \quad \text{for } z > 0, \tag{4.19}$$

the relations (4.18a) and (4.18b) are satisfied with the displacement

$$u_r^{\mathrm{E}} = -z\psi_{z,r} - \phi_{,r}, \quad u_z^{\mathrm{E}} = (3-4\nu)\psi_z - z\psi_{z,z} - \phi_{,z} \tag{4.20a}$$

and, using the Lamé parameter $\mu = E/(2(1+\nu))$, the stresses are

$$\begin{aligned} \sigma_{rz}^{\mathrm{E}} &= 2\mu\big((1-2\nu)\psi_{z,r} - z\psi_{z,rz} - \phi_{,rz}\big), \\ \sigma_{zz}^{\mathrm{E}} &= 2\mu\big(2(1-\nu)\psi_{z,z} - z\psi_{z,zz} - \phi_{,zz}\big), \\ \sigma_{rr}^{\mathrm{E}} &= 2\mu(-z\psi_{z,rr} + 2\nu\psi_{z,z} - \phi_{,rr}), \\ \sigma_{\theta\theta}^{\mathrm{E}} &= 2\mu\left(-\frac{z}{r}\psi_{z,r} + 2\nu\psi_{z,z} - \frac{1}{r}\phi_{,r}\right). \end{aligned} \tag{4.20b}$$

We apply to (4.19) the Fourier–Bessel transform with the Bessel functions of the first kind J_m, $m=0,1$, and get its general solution, decaying at the infinity according to (4.18c), in the form of the Laplace transform

$$\psi_z = \int_0^\infty e^{-z\xi} C(t,\xi) J_0(r\xi)\xi\, d\xi, \quad \phi = \int_0^\infty e^{-z\xi} A(t,\xi) J_0(r\xi)\xi\, d\xi, \tag{4.21}$$

with free coefficients C and A. From the tangential stress-free boundary condition (4.18d) at $z=0$, we find that the coefficient

$$A = -(1-2\nu)\xi^{-1}C. \tag{4.22}$$

The other coefficient is given in the terms of the cosine Fourier transform as

$$C = \frac{1}{2(1-\nu)\xi}\int_0^{h^{\mathrm{E}}} f(l)\cos(\xi l)\,dl, \tag{4.23}$$

with unknown function $f(l) = f(t,l)$. Using discontinuous Weber–Schafheitlin integrals this leads to the representation at $z = 0$ of the normal stress as

$$\sigma_{zz}^{\mathrm{E}} = \frac{E}{2(1-\nu^2)}\int_r^{h^{\mathrm{E}}} \frac{f'(l)}{\sqrt{l^2-r^2}}\,dl - \frac{f(h^{\mathrm{E}})}{\sqrt{(h^{\mathrm{E}})^2-r^2}} \quad \text{if } 0 \le r < h^{\mathrm{E}}, \tag{4.24a}$$

satisfying the condition $\sigma_{zz}^{\mathrm{E}} = 0$ if $r \ge h^{\mathrm{E}}$ in (4.18f), and the normal displacement:

$$\begin{aligned} u_z^{\mathrm{E}} &= \int_0^r \frac{f(l)}{\sqrt{r^2-l^2}}\,dl \quad \text{if } 0 \le r < h^{\mathrm{E}}, \\ u_z^{\mathrm{E}} &= \int_0^{h^{\mathrm{E}}} \frac{f(l)}{\sqrt{r^2-l^2}}\,dl \quad \text{if } r \ge h^{\mathrm{E}}. \end{aligned} \tag{4.24b}$$

From (4.18e) and (4.24b) we obtain the Abel integral equation

$$\int_0^r \frac{f(l)}{\sqrt{r^2-l^2}}\,dl = p^{\mathrm{E}} - \frac{r}{\tan(\alpha)}, \quad 0 \le r < h^{\mathrm{E}}, \tag{4.25}$$

which leads to the analytic solution

$$f(l) = \frac{2}{\pi}p^{\mathrm{E}} - \frac{l}{\tan(\alpha)}, \quad 0 \le l < h^{\mathrm{E}}. \tag{4.26}$$

On inserting (4.26) into (4.24a) we find that $\sigma_{zz}^{\mathrm{E}} = 0$ at $r = h^{\mathrm{E}}$ when

$$p^{\mathrm{E}} = \frac{\pi}{2\tan(\alpha)}h^{\mathrm{E}}. \tag{4.27}$$

We can then calculate the normal stress as

$$\sigma_{zz}^{\mathrm{E}} = -\frac{M}{2\tan(\alpha)}\operatorname{arcosh}\left(\frac{h^{\mathrm{E}}}{r}\right) \quad \text{for } 0 \le r < h^{\mathrm{E}}, \tag{4.28a}$$

and from (4.24b) the normal displacement is

$$u_z^{\mathrm{E}} = \frac{2p^{\mathrm{E}}}{\pi}\arcsin\left(\frac{h^{\mathrm{E}}}{r}\right) + \frac{\sqrt{r^2-(h^{\mathrm{E}})^2}-r}{\tan(\alpha)} \quad \text{for } r \ge h^{\mathrm{E}}. \tag{4.28b}$$

The substitution of (4.28a) into the integral (4.18g) results in

$$F = \frac{\pi M}{2\tan(\alpha)}(h^{\mathrm{E}})^2. \tag{4.29}$$

From (4.23) and (4.27) we conclude that the coefficient

$$C = \frac{1-\cos(h^{\mathrm{E}}\xi)}{\xi^3\tan(\alpha)} = \int_0^{h^{\mathrm{E}}} \frac{\sin(a\xi)}{\xi^2\tan(\alpha)}\,da = \operatorname{Im}\int_0^{h^{\mathrm{E}}} \frac{e^{\imath a\xi}}{\xi^2\tan(\alpha)}\,da. \tag{4.30}$$

On substituting (4.30) into equations (4.20)–(4.22) and using the notation $J_n^m(a)$ as defined in (4.15) yields the integral representations for the elastic displacement:

$$\begin{aligned} u_r^{\mathrm{E}} &= \frac{1}{2(1-\nu)\tan(\alpha)} \int_0^{h^{\mathrm{E}}} \big(-(1-2\nu) J_0^1(a) + z J_1^1(a)\big)\, da, \\ u_z^{\mathrm{E}} &= \frac{1}{2(1-\nu)\tan(\alpha)} \int_0^{h^{\mathrm{E}}} \big(2(1-\nu) J_0^0(a) + z J_1^0(a)\big)\, da; \end{aligned} \tag{4.31a}$$

and for the elastic stresses:

$$\begin{aligned} \sigma_{rr}^{\mathrm{E}} &= \frac{M}{2\tan(\alpha)} \int_0^{h^{\mathrm{E}}} \left(\frac{1-2\nu}{r} J_0^1(a) - J_1^0(a) - \frac{z}{r} J_1^1(a) + z J_2^0(a) \right) da, \\ \sigma_{rz}^{\mathrm{E}} &= -\frac{M}{2\tan(\alpha)} \int_0^{h^{\mathrm{E}}} z J_2^1(a)\, da, \quad \sigma_{zz}^{\mathrm{E}} = -\frac{M}{2\tan(\alpha)} \int_0^{h^{\mathrm{E}}} \big(J_1^0(a) + z J_2^0(a)\big)\, da, \\ \sigma_{\theta\theta}^{\mathrm{E}} &= \frac{M}{2\tan(\alpha)} \int_0^{h^{\mathrm{E}}} \left(-\frac{1-2\nu}{r} J_0^1(a) - 2\nu J_1^0(a) + \frac{z}{r} J_1^1(a) \right) da. \end{aligned} \tag{4.31b}$$

Sneddon's integrals in (4.15) for $n+m>0$ can be calculated as

$$J_n^m(a) = \mathrm{Im} \frac{(n-1-m)!}{\big(r^2 + (z-\imath a)^2\big)^{n/2}} P_{n-1}^{\beta m} \left(\frac{z-\imath a}{\sqrt{r^2+(z-\imath a)^2}} \right) \tag{4.32}$$

with the help of the associated Legendre function $P_{n-1}^{\beta m}$, where the binary index $\beta = 1$ if $m \le n-1$, otherwise $\beta = -1$. For the particular indices $m \in \{0,1\}$, $n \in \{0,1,2\}$ in (4.32) we obtain the formula (4.16b) (see Ref. 42 for more details). In the singular case $m = n = 0$ (such that the factorial in (4.32) does not exist), the special formula (4.16a) is due to Fabrikant Ref. 9 (Equation (0.13)).

We set $h_\Sigma := h^{\mathrm{E}}(0)$ and define the contact boundary as $\Sigma := \{r \in [0, h_\Sigma), z = 0\}$. The assumption (4.13) implies that the length of the coincidence sets decays:

$$h_\Sigma = h^{\mathrm{E}}(0) \ge h^{\mathrm{E}}(s) \ge h^{\mathrm{E}}(t) \quad \text{for all } 0 \le s \le t \le T, \tag{4.33}$$

thus providing the non-increasing property (4.2). Applying Corollary 4.1, according to the construction procedure provided in (4.11) we set $h^{\mathrm{VE}} = h^{\mathrm{E}}$ and $\boldsymbol{\sigma}^{\mathrm{VE}} = \boldsymbol{\sigma}^{\mathrm{E}}$, hence from (4.31b) and (4.29) we obtain (4.14c) and the second equality in (4.14a), respectively. On setting $\mathbf{u}^{\mathrm{VE}} = \mathbf{u}^{\mathrm{E}}(0) + \mathcal{I}[\mathbf{u}^{\mathrm{E}} - \mathbf{u}^{\mathrm{E}}(0)]$ and $p^{\mathrm{VE}} = p^{\mathrm{E}}(0) + \mathcal{I}[p^{\mathrm{E}} - p^{\mathrm{E}}(0)]$, the representations (4.31a) and (4.27) provide (4.14b) and the first equality in (4.14a).

In particular, the well-known formula of the elastic indentation by cone (4.28) and (4.11) follow the representation (4.17) for the normal displacement and the normal stress on the plane $z = 0$. This finishes the proof. □

4.2. *Numerical test of the viscoelastic indentation by cone*

Integrating by parts the function in (1.3) for the viscoelastic response allows us to rewrite it as

$$\mathcal{I}_{\mathrm{VE}}[f](t) = K(t)f(0) + \int_0^t K(t-s)f'(s)\,ds. \tag{4.34a}$$

If $f'(s) \equiv 1$ for $s \in (T_{\mathrm{H}}, t)$, where $0 \le T_{\mathrm{H}} < t \le T$, then the integral in (4.34a) can be calculated explicitly based on the form $K(t)$ from (1.4):

$$\int_{T_{\mathrm{H}}}^t K(t-s)\,ds = (t - T_{\mathrm{H}})\left(K(0) + \sum_{n=1}^N K_n\right) - \sum_{n=1}^N \frac{K_n}{\tau_n}\left(1 - e^{-(t-T_{\mathrm{H}})/\tau_n}\right). \tag{4.34b}$$

We suggest the piecewise-quadratic holding-unloading procedure

$$F = F(0) \text{ for } t \in [0, T_{\mathrm{H}}], \quad F = F(0)\left(\frac{T-t}{T-T_{\mathrm{H}}}\right)^2 \text{ for } t \in (T_{\mathrm{H}}, T] \tag{4.35}$$

as portrayed in the left plot in Fig. 2.

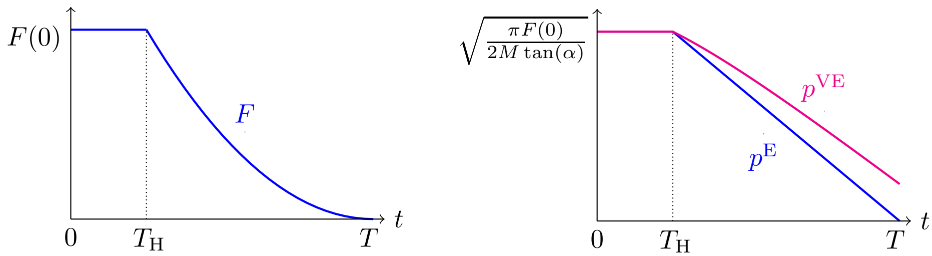

Fig. 2. Holding-unloading indentation testing by Berkovich's cone.

According to Eqs. (4.27) and (4.29) for the isotropic elastic body, the indentation depth subject to the total contact force F from (4.35) is found as the following piecewise-linear function:

$$\begin{aligned} p^{\mathrm{E}} &= \sqrt{\frac{\pi F}{2M\tan(\alpha)}} \\ &= \sqrt{\frac{\pi F(0)}{2M\tan(\alpha)}}\begin{cases} 1 & \text{for } t \in [0, T_{\mathrm{H}}], \\ \dfrac{T-t}{T-T_{\mathrm{H}}} & \text{for } t \in (T_{\mathrm{H}}, T]. \end{cases} \end{aligned} \tag{4.36a}$$

From Eqs. (4.14a) and (4.34b) we calculate

$$\begin{aligned} p^{\mathrm{VE}} &= \sqrt{\frac{\pi}{2M\tan(\alpha)}}\left((1 - K(t))\sqrt{F(0)} + I_{\mathrm{VE}}[\sqrt{F}]\right) \\ &= \sqrt{\frac{\pi F(0)}{2M\tan(\alpha)}} \quad \text{for } t \in [0, T_{\mathrm{H}}]; \end{aligned}$$

$$p^{\mathrm{VE}} = \sqrt{\frac{\pi F(0)}{2M\tan(\alpha)}}\left\{1 - \frac{1}{T - T_{\mathrm{H}}}\left[(t - T_{\mathrm{H}})\left(K(0) + \sum_{n=1}^{N} K_n\right)\right.\right.$$
$$\left.\left.- \sum_{n=1}^{N}\frac{K_n}{\tau_n}\left(1 - e^{-(t-T_{\mathrm{H}})/\tau_n}\right)\right]\right\} \text{ for } t \in (T_{\mathrm{H}}, T]. \tag{4.36b}$$

For the numerical test, the following values are assumed for the material parameters: $\nu = 0.5$ and rather small $E = 1\,\mathrm{MPa}$ that is typical for soft materials, such that the indentation modulus $M = 4/3$. We choose the three-parameter viscoelastic model according to Ref. 37 with $K(0) = 1$, $K_1 = 0.5$, $\tau_1 = 1$ as $N = 1$. The Berkovich's indenter with $\alpha = 65.03°$, $\tan(\alpha) \approx 2.147$ and $F(0) = 300\,\mathrm{N}$ are chosen. Since within the holding phase as $t \in [0, T_{\mathrm{H}}]$ both p^{E} and p^{VE} in (4.36) remain constant, we focus on the unloading phase as $t \in (T_{\mathrm{H}}, T]$ within $T - T_{\mathrm{H}} = 30$ sec. The corresponding indentation depth functions computed by (4.36) are depicted versus time t in the right plot in Fig. 2. In this picture we observe a delay in time by unloading the viscoelastic body by comparing p^{VE} to the elastic case p^{E}.

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