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Crack problem within the context of implicitly constituted quasi-linear viscoelasticity

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A quasi-linear viscoelastic relation that stems from an implicit viscoelastic constitutive body containing a crack is considered. The abstract form of the response function is given first in L^p, $p > 1$, due to power-law hardening; second in L^1 due to limiting small strain. In both the cases, sufficient conditions on admissible response functions are formulated, and corresponding existence theorems are proved rigorously based on the variational theory and using monotonicity methods. Due to the presence of a Volterra convolution operator, an auxiliary-independent variable of velocity type is employed. In the case of limiting small strain, the generalized solution of the problem is provided within the context of bounded measures and expressed by a variational inequality.

Keywords: Implicit constitutive response; quasi-linear viscoelasticity; power-law hardening; limiting small strain; crack problem; variational theory; Volterra convolution operator; monotonicity method; existence theorem; generalized solution; variational inequality.

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1. Introduction

In linear viscoelasticity, one can either express the stress in terms of the history of the linearized strain or the linearized strain in terms of the history of the stress.[a] Of the above two modes of expressing the response of a body subject to external stimuli, the latter is in keeping with the demands of causality, as force is the cause and kinematics the response, while the former is not. As Truesdell[31] remarks: "A constitutive equation is a relation between forces and motions. In popular terms, force is applied to a body to 'cause' it to undergo a motion, and the motion 'caused' differs according to the nature of the body. In continuum mechanics, the forces of interest are contact forces, which are specified by the stress tensor **T**." The reason one chooses to express the stress in terms of the history of the linearized strain is possibly due to the mathematical simplicity that is achieved with regard to the governing equations which need to be solved that are obtained by substituting the constitutive relation for the stress into the balance of linear momentum. Expressing the linearized strain in terms of the stress requires one to solve the balance equations as well as the constitutive relation simultaneously. We shall not get into a lengthy discussion of the relevant issues here but refer the reader to Ref. 28 for a detailed discussion of the same.

Early models to describe the nonlinear viscoelastic response of bodies due to Green and Rivlin[7] and Green *et al.*[8] are expressions for the stress in terms of kinematic variables. Lockett[23,24] and later Pipkin and Rogers[26] developed models for the viscoelastic response of bodies wherein one can find expression for either stress in terms of the history of the strain or the strain in terms of the history of the stress. Wineman[32–34] used such nonlinear viscoelastic models to study several technologically important problems. The nonlinear viscoelastic models discussed above were essentially developed using analogies to extend the class of linear viscoelastic models, and moreover, the authors primarily developed models for the stress in terms of the history of the strain, and models for the strain in terms of the history of the stress were essentially obtained by inverting the specification for the stress, when such expressions were invertible. On the other hand, Prusa and Rajagopal[27] developed a more general theory which embeds both classes of models for viscoelastic response discussed above by considering an implicit functional relationship between the histories of the stress and strain.

Fung[6] used an approximation of a proper nonlinear viscoelastic model to describe the viscoelastic behavior of biological tissues which he referred to as a "Quasi-linear viscoelastic model". Fung developed a one-dimensional model wherein an explicit expression is provided for the stress in terms of the stretch. It is not

[a]Since it is the linearized strain that appears in the constitutive relation, the model refers to an approximation to a properly invariant nonlinear viscoelastic model. Thus, strictly speaking, we should refer to the model as the linearized viscoelasticity constitutive relation. However, in view of the current common usage, we shall refer to the model as the linear viscoelastic constitutive relation.

guaranteed that one can invert all such models and express the stretch in terms of the history of the stress. Recently, Muliana *et al.*[25] (see also Rajagopal and Wineman[29]) have developed three-dimensional models wherein expressions are provided for the linearized strain in terms of the history of the stress, the relationship being nonlinear. Since the linearized strain is not frame-indifferent, the quasi-linear constitutive relations,[6,25] in common with the classical linearized elastic model and the linear viscoelastic model, are not frame-indifferent. In this paper, we investigate the state of the stress and strain in the quasi-linear viscoelastic body developed in Refs. 25 and 29, which takes the following form for time $t > 0$. The solid response can be written with respect to the linearized strain $\boldsymbol{\varepsilon}$ and stress $\boldsymbol{\sigma}$ in the integral form

$$\boldsymbol{\varepsilon}(t) = J(t)\mathcal{F}(\boldsymbol{\sigma}(0)) + \int_0^t J(t-s)\frac{d}{ds}\mathcal{F}(\boldsymbol{\sigma}(s))\,ds, \tag{1.1a}$$

or, after integration by parts, equivalently as

$$\boldsymbol{\varepsilon}(t) = J(0)\mathcal{F}(\boldsymbol{\sigma}(t)) + \int_0^t J'(t-s)\mathcal{F}(\boldsymbol{\sigma}(s))\,ds. \tag{1.1b}$$

If the function $\mathcal{F}(\boldsymbol{\sigma})$ were to be linear, then (1.1a) and (1.1b) would lead to the classical linear viscoelastic model. In general, the representations (1.1a) and (1.1b) cannot be inverted. In fact even within the context of elasticity, a nonlinear expression for the linearized strain in terms of the stress cannot in general be inverted (see Ref. 16). The positive kernel J that appears in the Volterra convolution operator[3] in (1.1a) and (1.1b) is usually assumed to be given by the exponential sum

$$J(t) = J(0) + \sum_{n=1}^{N} J_n(1 - e^{-t/\tau_n}), \quad J'(t) = \sum_{n=1}^{N} \frac{J_n}{\tau_n} e^{-t/\tau_n} \tag{1.1c}$$

with parameters $J(0), J_1, \ldots, J_N, \tau_1, \ldots, \tau_N \geq 0$, that characterize the generalized creep. If $J(t) \equiv 1$, then the implicit response turns into the following nonlinear relation between the linearized strain and stress

$$\boldsymbol{\varepsilon} = \mathcal{F}(\boldsymbol{\sigma}). \tag{1.2}$$

Within the context of limiting small strain $\boldsymbol{\varepsilon}$, that is provided by the uniform bound

$$\|\mathcal{F}(\boldsymbol{\sigma})\| \leq M_1, \quad \text{where } M_1 > 0, \tag{1.3}$$

mathematical issues concerning the implicit model (1.2), (1.3) were studied by Beck *et al.*[1] On the other hand, if $N = 1$ and $J(0) = 0$ in (1.1c), on differentiating (1.1b) with respect to t, since $J''(t) = -\frac{1}{\tau_1}J'(t)$, we obtain a generalized Kelvin–Voigt model[5,13]

$$\boldsymbol{\varepsilon} + \frac{1}{\tau_1}\dot{\boldsymbol{\varepsilon}} = \mathcal{F}(\boldsymbol{\sigma}), \tag{1.4}$$

where dot stands for the time derivative, and $\frac{1}{\tau_1}$ characterizes the viscosity. See other relevant generalizations of the Kelvin–Voigt model in Ref. 2.

The fact that one has limited strains has interesting implications with regard to a very important class of practical problems, namely that of cracks. Classical theory of elasticity predicts singular strains contradicting the very assumptions which are used to derive the theory, namely the displacement gradient and hence the strain is small. The model with limiting strain allows one to study problems such as cracks in brittle materials in a rational manner. The variational theory of nonlinear cracks subject to non-penetration was developed in Refs. 9, 14 and 17–19 and other works by the authors. We refer to Ref. 15 for cracks in linear viscoelastic solids, and to Ref. 30 for a treatment of Signorini and friction conditions within linear viscoelasticity. It was proved that the linear creep model (in Ref. 17, Secs. 2.2 and 3.1, and in Ref. 19, Chap. 3, Sec. 10) is compatible with the non-penetration condition between crack faces. In the context of limiting small strain (1.3), cracks subject to non-penetration were treated[10–12] for the implicit response (1.2). For the Kelvin–Voigt model (1.4), stress-free cracks were dealt with in Ref. 13, while the presence of the time derivative $\dot{\boldsymbol{\varepsilon}}$ does not allow one to include the non-penetration condition. In the current paper, we extend the variational modeling of nonlinear cracks to the quasi-linear viscoelastic model (1.1).

One possible structure for the form of the response function $\mathcal{F}$ in (1.2) is the power-law hardening[21] defined over L^p-functions

$$\mathcal{F}_p(\boldsymbol{\sigma}) = \frac{1}{2\mu}\frac{\boldsymbol{\sigma}}{(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{2-p}{r}}}, \quad \kappa,\mu,r>0, \quad p\in(1,\infty), \tag{1.5}$$

where μ is the shear modulus, and $\|\boldsymbol{\sigma}\|^2 = \mathrm{tr}(\boldsymbol{\sigma}^2)$ the Frobenius norm. See also a related power-law equation adjacent to cracks.[20] We remark that $\kappa \searrow 0^+$ reduces (1.5) to the linear function. The other important limit $p \searrow 1^+$ turns (1.5) into the uniformly bounded function[1]

$$\mathcal{F}_1(\boldsymbol{\sigma}) = \frac{1}{2\mu}\frac{\boldsymbol{\sigma}}{(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{1}{r}}}, \quad \kappa,\mu,r>0, \tag{1.6a}$$

which fulfills (1.3) with the bound $M_1 = \frac{1}{2\mu\kappa^{\frac{1}{r}}}$. Indeed, with the help of (1.3) and (1.1c), we estimate (1.1b) as follows:

$$\|\boldsymbol{\varepsilon}(t)\| \le M_1 J(t) \le M_1 J_0, \quad J_0 := J(0) + \sum_{n=1}^{N} J_n > 0. \tag{1.6b}$$

Several other constitutive relations that describe limiting small strain can be found in Ref. 12. Based on the examples (1.5) and (1.6a), we explain below the main technical difficulties dealing with the quasi-linear viscoelastic model (1.1).

First, it is worth remarking that both the nonlinear functions $\mathcal{F}$ in (1.5) and in (1.6a) are monotone and coercive as will be proved in Appendix A. However, after entering the Volterra convolution operator in (1.1b), the monotonicity and coercivity properties are lost (except for the linear case as $\kappa = 0$). Thus, the Browder–Minty theorem is inapplicable to (1.1) as opposed to the situation with regard to the models (1.2) and (1.4). To remedy this difficulty, in Sec. 2, we introduce an auxiliary-independent variable $\boldsymbol{\epsilon}$ that splits (1.1b) into two equations

$$\boldsymbol{\varepsilon}(t) = J(0)\boldsymbol{\epsilon}(t) + \int_0^t J'(t-s)\boldsymbol{\epsilon}(s)\, ds, \tag{1.7a}$$

$$\boldsymbol{\epsilon} = \mathcal{F}(\boldsymbol{\sigma}), \tag{1.7b}$$

where (1.7a) is linear and (1.7b) keeps the properties of the function $\mathcal{F}$. The variable $\boldsymbol{\epsilon}$ is related to the time derivative of strain. In fact, if $N = 1$ and $J(0) = 0$ in (1.1c), then the integral Volterra equation (1.7a) can be solved analytically with respect to $\boldsymbol{\epsilon} = \boldsymbol{\varepsilon} + \frac{1}{\tau_1}\dot{\boldsymbol{\varepsilon}}$ (compare the result to (1.4)).

The second difficulty concerns the limiting small strain case. In fact, the domain of the operator $\mathcal{F}_1$ in (1.6a) is the L^1-space which is non-reflexive. For this reason, the generalized solution of the problem implying an accumulation point of the corresponding Galerkin approximations is provided within bounded measures and expressed by a variational inequality. This case is treated separately in Sec. 3.

2. The Quasi-Linear Viscoelastic Case Study

Let Ω be a bounded domain in the Euclidean space $\mathbb{R}^d$ of integer dimension $d \in \mathbb{N}$; the boundary $\partial\Omega$ be Lipschitz continuous with the normal vector $\mathbf{n} = (n_1, \ldots, n_d)$ outward to Ω. We assume that $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_D}$ consists of the Neumann part Γ_{N} and the Dirichlet part $\Gamma_{\mathrm{D}} \neq \emptyset$ which are disjoint. Let the crack Γ_c be a part of a $(d-1)$-dimensional oriented open manifold having the normal $\mathbf{n} = (n_1, \ldots, n_d)$, and Γ_c can be extended up to the external boundary $\partial\Omega$ such that Ω splits into two domains $\Omega^{\pm}$ with Lipschitz continuous boundaries $\partial\Omega^{\pm}$. The two crack faces $\Gamma_c^{\pm}$ can be distinguished as the corresponding parts of $\partial\Omega^{\pm}$. We call by "domain with crack" the geometric set $\Omega_c = \Omega \setminus \overline{\Gamma_c}$ obeying the boundary $\partial\Omega \cup \overline{\Gamma_c^+} \cup \overline{\Gamma_c^-}$. For a fixed final time $T > 0$, the time–space cylinder is denoted by $Q_c^T = (0,T) \times \Omega_c$.

Next we give an exact sense to relations (1.1). According to the example (1.1c), we assume a creep function $J \in C^1(0,T;\mathbb{R}) \cap C([0,T];\mathbb{R})$ such that

$$0 \le J(t) \le J_0, \quad J'(t) \ge 0 \quad \text{for } t \in (0,T), \quad J(0) > 0. \tag{2.1a}$$

With its help, we denote for short the integral operator in (1.1b) by

$$\mathcal{I} : C([0,T]) \times [0,T] \mapsto \mathbb{R}, \quad \mathcal{I}(\boldsymbol{\xi}, t) = J(0)\boldsymbol{\xi}(t) + \int_0^t J'(t-s)\boldsymbol{\xi}(s) ds \tag{2.1b}$$

for a time-dependent variable $\boldsymbol{\xi}$ with the image in a normed vector space to be defined later on. The response function in (1.1b) is determined as a mapping over the space of second-order symmetric d-by-d tensors

$$\mathcal{F}_p : \mathbb{R}^{d\times d}_{\mathrm{sym}} \mapsto \mathbb{R}^{d\times d}_{\mathrm{sym}}, \quad \mathcal{F}_p(\mathbf{0}) = \mathbf{0}, \tag{2.1c}$$

which is equipped with the inner product $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \sum_{i,j=1}^d \sigma_{ij}\varepsilon_{ij}$ and the associated matrix-norm $\|\boldsymbol{\sigma}\| = \sqrt{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}}$. The stress tensor $\boldsymbol{\sigma}(t, \mathbf{x}) = \{\sigma_{ij}\}_{i,j=1}^d$ and the linearized strain tensor $\boldsymbol{\varepsilon}(t, \mathbf{x}) = \{\varepsilon_{ij}\}_{i,j=1}^d$ are tensor functions of time t and spatial coordinates $\mathbf{x} = (x_1, \ldots, x_d)$. The linearized strain tensor is the symmetric part of the displacement gradient defined as follows:

$$\boldsymbol{\varepsilon} : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}_{\mathrm{sym}}, \quad \varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,\ldots,d, \tag{2.1d}$$

over the vector space $\mathbb{R}^d$ equipped with the inner product $\mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^d u_i v_i$ and the associated vector norm $\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$ for time-and-space-dependent vectors $\mathbf{u}(t,\mathbf{x}) = (u_1,\ldots,u_d)$ and $\mathbf{v}(t,\mathbf{x}) = (v_1,\ldots,v_d)$.

Let the body force $\mathbf{f}(t,\mathbf{x}) = (f_1,\ldots,f_d)$ and the boundary traction $\mathbf{g}(t,\mathbf{x}) = (g_1,\ldots,g_d)$ be given. Using notation (2.1), we formulate the implicitly constituted quasi-linear viscoelastic problem for solids with crack in the form of a nonlinear quasi-static boundary value problem

$$-\sum_{j=1}^d \frac{\partial \sigma_{ij}}{\partial x_j} = f_i, \quad i = 1,\ldots,d, \quad \text{in } Q_c^T, \tag{2.2a}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{I}(\mathcal{F}_p(\boldsymbol{\sigma})) \quad \text{in } Q_c^T, \tag{2.2b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } (0,T)\times\Gamma_{\mathrm{D}}, \tag{2.2c}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{on } (0,T)\times\Gamma_{\mathrm{N}}, \tag{2.2d}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \quad \text{on } (0,T)\times\Gamma_c^{\pm}. \tag{2.2e}$$

The system consists in the equilibrium equation (2.2a), the constitutive relation (2.2b), under the Dirichlet (2.2c) and the Neumann type (2.2d), and the stress-free crack (2.2e) boundary conditions.

We set the parameters characterizing the regularity of the unknown functions:

$$p \in [1,\infty), \quad p' \in (1,\infty], \quad \frac{1}{p} + \frac{1}{p'} = 1 \tag{2.3a}$$

in Sobolev spaces introduced later on. Let another parameter

$$q \geq p, \quad q \in (1,\infty), \quad q' \in (1,\infty), \quad \frac{1}{q} + \frac{1}{q'} = 1 \tag{2.3b}$$

characterizes regularity of the problem data

$$\mathbf{f} \in C([0,T]; L^q(\Omega_c;\mathbb{R}^d)) \subseteq C([0,T]; L^p(\Omega_c;\mathbb{R}^d)), \tag{2.3c}$$

$$\mathbf{g} \in C([0,T]; L^q(\Gamma_{\mathrm{N}};\mathbb{R}^d)) \subseteq C([0,T]; L^p(\Gamma_{\mathrm{N}};\mathbb{R}^d)). \tag{2.3d}$$

For convenience of dealing with $\mathbf{f}$ and $\mathbf{g}$, we set an auxiliary elastic stress tensor

$$\boldsymbol{\sigma}^E \in C([0,T]; L^q(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})) \subseteq C([0,T]; L^p(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{2.4a}$$

defined by the explicit power-law response

$$\boldsymbol{\sigma}^E = \|\boldsymbol{\varepsilon}(\mathbf{u}^E)\|^{q'-2}\boldsymbol{\varepsilon}(\mathbf{u}^E). \tag{2.4b}$$

Lemma 2.1. *There exists a displacement* $\mathbf{u}^E(t,\mathbf{x}) \in C([0,T]; W^{1,q'}(\Omega_c;\mathbb{R}^d))$ *such that*

$$\mathbf{u}^E = \mathbf{0} \quad \text{on } (0,T) \times \Gamma_{\mathrm{D}}, \tag{2.4c}$$

and Cauchy stress $\boldsymbol{\sigma}^E(t,\mathbf{x}) \in C([0,T]; L^q(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$, *which satisfy the auxiliary elasticity problem given by* (2.4b) *and the variational equation*

$$\int_{\Omega_c} \boldsymbol{\sigma}^E \cdot \boldsymbol{\varepsilon}(\overline{\mathbf{u}})\, d\mathbf{x} = \int_{\Omega_c} \mathbf{f} \cdot \overline{\mathbf{u}}\, d\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \overline{\mathbf{u}}\, dS_{\mathbf{x}} \quad \text{in } (0,T) \tag{2.4d}$$

for all test functions $\overline{\mathbf{u}} \in W^{1,q'}(\Omega_c;\mathbb{R}^d)$ *such that* $\overline{\mathbf{u}} = 0$ *on* Γ_{D}.

Proof. The existence follows after insertion of (2.4b) in (2.4d) resulting in

$$\int_{\Omega_c} \|\boldsymbol{\varepsilon}(\mathbf{u}^E)\|^{q'-2} \boldsymbol{\varepsilon}(\mathbf{u}^E) \cdot \boldsymbol{\varepsilon}(\overline{\mathbf{u}})\, d\mathbf{x} = \int_{\Omega_c} \mathbf{f} \cdot \overline{\mathbf{u}}\, d\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \overline{\mathbf{u}}\, dS_{\mathbf{x}} \tag{2.4e}$$

that implies the first-order optimality condition for minimization of the functional

$$\mathcal{J}(\mathbf{u}^E) = \frac{1}{q'} \int_{\Omega_c} \|\boldsymbol{\varepsilon}(\mathbf{u}^E)\|^{q'}\, d\mathbf{x} - \int_{\Omega_c} \mathbf{f} \cdot \mathbf{u}^E\, d\mathbf{x} - \int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{u}^E\, dS_{\mathbf{x}}. \tag{2.4f}$$

For $q' > 1$, the functional $\mathcal{J} : C([0,T]; W^{1,q'}(\Omega_c;\mathbb{R}^d)) \mapsto \mathbb{R}$ is evidently differentiable, convex (hence, weakly lower semi-continuous), and strictly coercive due to the Korn–Poincaré inequality as a consequence of (2.4c)

$$\|\mathbf{u}\|^{q'}_{L^{q'}(\Omega_c;\mathbb{R}^d)} \le K_{\mathrm{KP}}(q') \|\boldsymbol{\varepsilon}(\mathbf{u})\|^{q'}_{L^{q'}(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \quad \text{if } \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}, \tag{2.4g}$$

and using equivalence of the Frobenius norm to all other p-norms (see (A.4b) and (A.7a)). Therefore, the unique minimum of $\mathcal{J}$ exists (see Ref. 17, Sec. 1.2). □

Based on the examples (1.5) and (1.6a), we formulate the following conditions for the admissible response function $\mathcal{F}_p : L^p(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \mapsto L^{p'}(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})$: there exist constants $M_1(p), M_4(p) > 0$ and $M_0(p), M_3(p) \ge 0$ such that

$$\begin{aligned} &\text{boundedness: } \|\mathcal{F}_p(\boldsymbol{\sigma})\|^{p'}_{L^{p'}(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \le M_0(p) + M_1(p)\|\boldsymbol{\sigma}\|^p_{L^p(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})}, \\ &\qquad \|\mathcal{F}_1(\boldsymbol{\sigma})\|_{L^\infty(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \le M_1(1) \quad \text{for } p = 1, \end{aligned} \tag{2.5a}$$

$$\text{monotonicity: } \int_{\Omega_c} (\mathcal{F}_p(\boldsymbol{\sigma}) - \mathcal{F}_p(\overline{\boldsymbol{\sigma}})) \cdot (\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}})\, d\mathbf{x} \ge 0, \tag{2.5b}$$

$$\text{semi-continuity: } \lambda \mapsto \int_{\Omega_c} \mathcal{F}_p\big(\boldsymbol{\sigma} + \lambda\overline{\boldsymbol{\sigma}}\big) \cdot \overline{\boldsymbol{\sigma}}\, d\mathbf{x} \text{ is continuous at } \lambda = 0, \tag{2.5c}$$

$$\text{coercivity: } \int_{\Omega_c} \mathcal{F}_p(\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma}\, d\mathbf{x} \ge M_4(p)\|\boldsymbol{\sigma}\|^p_{L^p(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})} - M_3(p) \tag{2.5d}$$

hold for $\boldsymbol{\sigma}, \overline{\boldsymbol{\sigma}} \in L^p(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, where the standard norms are defined as

$$\|\boldsymbol{\varepsilon}\|^{p'}_{L^{p'}(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})} = \int_{\Omega_c} \sum_{i,j=1}^{d} |\varepsilon_{ij}|^{p'}\, d\mathbf{x},$$
$$\|\boldsymbol{\varepsilon}\|_{L^{\infty}(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})} = \sup_{i,j=1,\dots,d,\, \mathbf{x}\in\Omega_c} |\varepsilon_{ij}(\mathbf{x})|. \tag{2.5e}$$

The properties (2.5) are proven for the power-law hardening (1.5) in Appendix A.

We employ an auxiliary-independent variable $\mathbf{v}$ yielding $\boldsymbol{\epsilon} = \boldsymbol{\varepsilon}(\mathbf{v})$ in (1.7). Multiplying (2.2a) with a smooth test function, integrating it by parts over Ω_c with the help of Neumann boundary conditions (2.2d) and (2.2e), we get a variational formulation to problem (2.2): Find such functions

$$\mathbf{u}, \mathbf{v} \in C([0,T]; W^{1,p'}(\Omega_c;\mathbb{R}^d)), \quad \boldsymbol{\sigma} \in C([0,T]; L^p(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})) \tag{2.6a}$$

that satisfy the Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{v} = \mathbf{0} \quad \text{on } (0,T)\times\Gamma_{\mathrm{D}}, \tag{2.6b}$$

the variational equation (where we have used (2.4d))

$$\int_{\Omega_c} \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\overline{\mathbf{u}})\, d\mathbf{x} = \int_{\Omega_c} \mathbf{f}\cdot\overline{\mathbf{u}}\, d\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} \mathbf{g}\cdot\overline{\mathbf{u}}\, dS_{\mathbf{x}} = \int_{\Omega_c} \boldsymbol{\sigma}^E\cdot\boldsymbol{\varepsilon}(\overline{\mathbf{u}})\, d\mathbf{x} \quad \text{in } (0,T) \tag{2.6c}$$

for all test functions $\overline{\mathbf{u}} \in W^{1,p'}(\Omega_c;\mathbb{R}^d)$ such that $\overline{\mathbf{u}} = \mathbf{0}$ at Γ_{D}, and

$$\mathbf{u} = \mathcal{I}(\mathbf{v}) \quad \text{in } Q_c^T, \tag{2.6d}$$

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \mathcal{F}_p(\boldsymbol{\sigma}) \quad \text{in } Q_c^T. \tag{2.6e}$$

Indeed, if Eq. (2.6d) holds, then applying $\boldsymbol{\varepsilon}$ to its both sides, we can interchange $\boldsymbol{\varepsilon}$ and $\mathcal{I}$ and derive $\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{I}(\boldsymbol{\varepsilon}(\mathbf{v}))$. The converse holds true up to a rigid motion, which vanishes due to the homogeneous Dirichlet condition (2.6b). We also note that the variable $\mathbf{v}$ is redundant and can be excluded after solving problem (2.6) by substitution of (2.6e) in (2.6d) resulting in $\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{I}(\boldsymbol{\varepsilon}(\mathbf{v})) = \mathcal{I}(\mathcal{F}_p(\boldsymbol{\sigma}))$.

Theorem 2.1. *Let $p > 1$ in* (2.3a). *Under conditions* (2.5), *there exists a solution triple* $(\mathbf{u}, \mathbf{v}, \boldsymbol{\sigma})$ *to the implicitly constituted quasi-linear viscoelastic crack problem* (2.6). *If the monotone property* (2.5b) *is strict, then the stress $\boldsymbol{\sigma}$ is unique.*

Proof. We construct a Galerkin approximation to the problem (2.6). There exists a dense sequence forming the basis in the reflexive separable Banach spaces:

$$\mathbf{u}^k \in C([0,T]; W^{1,p'}(\Omega_c;\mathbb{R}^d)), \quad \boldsymbol{\sigma}^k \in C([0,T]; L^p(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \quad k\in\mathbb{N}, \tag{2.7a}$$

which also satisfies the Dirichlet boundary condition

$$\mathbf{u}^k = \mathbf{0} \quad \text{on } (0,T)\times\Gamma_{\mathrm{D}}. \tag{2.7b}$$

We look for an approximation of $(\mathbf{u}, \mathbf{v}, \boldsymbol{\sigma} - \boldsymbol{\sigma}^E)$ by the finite series

$$\mathbf{U}^m = \sum_{k=1}^m \alpha_k \mathbf{u}^k, \quad \mathbf{V}^m = \sum_{k=1}^m \beta_k \mathbf{u}^k, \quad \boldsymbol{\Sigma}^m = \sum_{k=1}^m \gamma_k \boldsymbol{\sigma}^k, \quad m \in \mathbb{N} \tag{2.7c}$$

with unknown coefficients $\boldsymbol{\alpha}^m = (\alpha_1, \ldots, \alpha_m)$, $\boldsymbol{\beta}^m = (\beta_1, \ldots, \beta_m)$, and $\boldsymbol{\gamma}^m = (\gamma_1, \ldots, \gamma_m)$ in $\mathbb{R}^m$. Using the ansatz (2.7c), test functions $\overline{\mathbf{u}} = \mathbf{u}^l$ in Eq. (2.6c) regularized with an elliptic term akin to (2.4e), multiplying Eq. (2.6e) with test functions $\boldsymbol{\sigma}^k$ and integrating it over Ω_c, we approximate Eqs. (2.6c)–(2.6e) by the finite-dimensional system

$$\int_{\Omega_c} \left(\frac{1}{m} \|\boldsymbol{\varepsilon}(\mathbf{V}^m)\|^{p'-2} \boldsymbol{\varepsilon}(\mathbf{V}^m) + \boldsymbol{\Sigma}^m \right) \cdot \boldsymbol{\varepsilon}(\mathbf{u}^l)\, d\mathbf{x} = 0, \quad l = 1, \ldots, m, \tag{2.7d}$$

$$\alpha_k \mathbf{u}^k = \beta_k \mathcal{I}(\mathbf{u}^k), \quad k = 1, \ldots, m, \tag{2.7e}$$

$$\int_{\Omega_c} (\mathcal{F}_p(\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E) - \boldsymbol{\varepsilon}(\mathbf{V}^m)) \cdot \boldsymbol{\sigma}^k \, d\mathbf{x} = 0, \quad k = 1, \ldots, m. \tag{2.7f}$$

First we prove the solvability of the Galerkin system (2.7d)–(2.7f).

The left-hand sides of Eqs. (2.7d) and (2.7f) form a nonlinear operator $\mathcal{L}(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) : \mathbb{R}^{2m} \mapsto \mathbb{R}^{2m}$ with respect to the coefficients $(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m)$. We compose the scalar product in $\mathbb{R}^{2m}$ such that the same terms with opposite signs are shortened

$$\mathcal{L}(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) \cdot (\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) = \int_{\Omega_c} \left(\frac{1}{m} \|\boldsymbol{\varepsilon}(\mathbf{V}^m)\|^{p'} + \mathcal{F}_p(\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E) \cdot \boldsymbol{\Sigma}^m \right) d\mathbf{x}. \tag{2.8a}$$

To estimate (2.8a) from below, we use the boundedness (2.5a), coercivity (2.5d), norm equivalence (A.4b) and (A.7a), Korn–Poincaré (2.4g) and weighted Young's inequalities

$$\left| \int_{\Omega_c} \mathcal{F}_p(\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E) \cdot \boldsymbol{\sigma}^E \, d\mathbf{x} \right| \\ \leq \frac{\delta^{p'}}{p'} \|\mathcal{F}_p(\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E)\|^{p'}_{L^{p'}(\Omega_c; \mathbb{R}^{d\times d}_{\text{sym}})} + \frac{1}{p\delta^p} \|\boldsymbol{\sigma}^E\|^p_{L^p(\Omega_c; \mathbb{R}^{d\times d}_{\text{sym}})} \tag{2.8b}$$

with sufficiently small weight $\delta > 0$ such that $M_4(p) - \frac{\delta^{p'} M_1(p)}{p'} > 0$, thus getting

$$\int_{\Omega_c} \left(\frac{1}{m} \|\boldsymbol{\varepsilon}(\mathbf{V}^m)\|^{p'} + \mathcal{F}_p(\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E) \cdot (\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E - \boldsymbol{\sigma}^E) \right) d\mathbf{x} \\ \geq \frac{d^{-2}}{m(1 + K_{\text{KP}}(p'))} \|\mathbf{V}^m\|^{p'}_{W^{1,p'}(\Omega_c; \mathbb{R}^d)} - M_3(p) - \frac{\delta^{p'} M_0(p)}{p'} \\ + \left(M_4(p) - \frac{\delta^{p'} M_1(p)}{p'} \right) \|\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E\|^p_{L^p(\Omega_c; \mathbb{R}^{d\times d}_{\text{sym}})} - \frac{1}{p\delta^p} \|\boldsymbol{\sigma}^E\|^p_{L^p(\Omega_c; \mathbb{R}^{d\times d}_{\text{sym}})}. \tag{2.8c}$$

For a sufficiently large value $R > 0$ of the norm of the coefficients, the expression in (2.8c) is positive

$$\mathcal{L}(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) \cdot (\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) > 0 \quad \text{when } \|(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m)\| = R. \tag{2.8d}$$

Henceforth, based on Brouwer's fixed point argument (also called hairy ball theorem in algebraic topology), from (2.8d), we conclude with the existence of such $(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) \in \overline{B_R(0)} \subset \mathbb{R}^{2m}$ in the closure of the ball $B_R(0)$ of radius R centered at origin that

$$\mathcal{L}(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) = 0 \tag{2.8e}$$

(see Ref. 22, Chap. 4, Sec. 8). In fact, if $\mathcal{L}(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) \neq 0$ would be true for all $(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) \in \overline{B_R(0)}$, then the following mapping is well defined:

$$-R\frac{\mathcal{L}(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m)}{\|\mathcal{L}(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m)\|} : \partial B_R(0) \mapsto \partial B_R(0)$$

and has a fixed point such that $\|(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m)\| = R$ satisfying

$$\mathcal{L}(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) \cdot (\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) = -\frac{\|\mathcal{L}(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m)\|}{R}\|(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m)\|^2 \leq 0$$

that contradicts (2.8d). The equality (2.8e) implies that (2.7d) and (2.7f) are solvable, then coefficients $\boldsymbol{\alpha}^m$ can be found in a straightforward manner from (2.7e).

Next we pass (2.7) to the limit when $m \nearrow \infty$. Since $\mathcal{L}(\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) \cdot (\boldsymbol{\beta}^m, \boldsymbol{\gamma}^m) = 0$ by virtue of (2.8e), from (2.8a) and (2.8c), we infer the uniform estimate

$$\begin{aligned} &\frac{d^{-2}}{m(1 + K_{\mathrm{KP}}(p'))}\|\mathbf{V}^m\|^{p'}_{W^{1,p'}(\Omega_c;\mathbb{R}^d)} + \left(M_4(p) - \frac{\delta^{p'} M_1(p)}{p'}\right)\|\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E\|^p_{L^p(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \\ &\leq M_3(p) + \frac{\delta^{p'} M_0(p)}{p'} + \frac{1}{p\delta^p}\|\boldsymbol{\sigma}^E\|^p_{L^p(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})}. \end{aligned} \tag{2.9a}$$

Due to the boundedness (2.5a), from (2.7f), we get the estimate for $\mathbf{V}^m$

$$\begin{aligned} \frac{1}{1 + K_{\mathrm{KP}}(p')}\|\mathbf{V}^m\|^{p'}_{W^{1,p'}(\Omega_c;\mathbb{R}^d)} &\leq \|\mathcal{F}_p(\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E)\|^{p'}_{L^{p'}(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \\ &\leq M_0(p) + M_1(p)\|\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E\|^p_{L^p(\Omega_c;\mathbb{R}^{d\times d}_{\mathrm{sym}})}. \end{aligned} \tag{2.9b}$$

Respectively, from (2.7e), using $J(0) + |\int_0^t J'(t-s)\,ds| = J(t) \leq J_0$ according to (2.1a) and (2.1b), after taking the maximum over time, the estimate for $\mathbf{U}^m$ is

$$\begin{aligned} \|\mathbf{U}^m\|^{p'}_{C([0,T];W^{1,p'}(\Omega_c;\mathbb{R}^d))} &= \|\mathcal{I}(\mathbf{V}^m)\|^{p'}_{C([0,T];W^{1,p'}(\Omega_c;\mathbb{R}^d))} \\ &\leq J_0\|\mathbf{V}^m\|^{p'}_{C([0,T];W^{1,p'}(\Omega_c;\mathbb{R}^d))}. \end{aligned} \tag{2.9c}$$

Because the basis functions are continuous in time, we can take the maximum over $t \in [0,T]$ in (2.9a) and (2.9b) (see Ref. 4, Remark 1, p. 555). Consequently, by the compactness principle, there exists an accumulation point $(\mathbf{u}, \mathbf{v}, \boldsymbol{\sigma})$ and a subsequence, still denoted by m for short, that converges as $m \nearrow \infty$

$$\mathbf{U}^m \rightharpoonup \mathbf{u}, \quad \mathbf{V}^m \rightharpoonup \mathbf{v} \quad \text{weakly in } C([0,T]; W^{1,p'}(\Omega_c; \mathbb{R}^d)), \tag{2.9d}$$

$$\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E \rightharpoonup \boldsymbol{\sigma} \quad \text{weakly in } C([0,T]; L^p(\Omega_c; \mathbb{R}^{d\times d}_{\text{sym}})). \tag{2.9e}$$

With the help of convergences (2.9d) and (2.9e), on taking the limit of Eqs. (2.7d) and (2.7e), where all terms are linear except for the nonlinear term in (2.7d) which disappears because of the factor $\frac{1}{m}$, thus yielding (2.6c) and (2.6d). To pass the nonlinear term in Eq. (2.7f) to the limit, the following Minty's argument is applied. For a test function $\widetilde{\boldsymbol{\sigma}} \in L^p(\Omega_c; \mathbb{R}^{d\times d}_{\text{sym}})$, using the monotone property (2.5b), from (2.7d) and (2.7f), we get the series of relations

$$\begin{aligned} 0 &= \int_{\Omega_c} (\mathcal{F}_p(\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E) - \boldsymbol{\varepsilon}(\mathbf{V}^m)) \cdot (\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E - \widetilde{\boldsymbol{\sigma}})\, d\mathbf{x} \\ &\geq \int_{\Omega_c} (\mathcal{F}_p(\widetilde{\boldsymbol{\sigma}}) \cdot (\boldsymbol{\Sigma}^m + \boldsymbol{\sigma}^E - \widetilde{\boldsymbol{\sigma}}) + \frac{1}{m} \|\boldsymbol{\varepsilon}(\mathbf{V}^m)\|^{p'} - \boldsymbol{\varepsilon}(\mathbf{V}^m) \cdot (\boldsymbol{\sigma}^E - \widetilde{\boldsymbol{\sigma}})) d\mathbf{x}. \end{aligned} \tag{2.10a}$$

Taking the limit of Eq. (2.10a), due to the convergences (2.9d) and (2.9e), yields

$$0 \geq \int_{\Omega_c} (\mathcal{F}_p(\widetilde{\boldsymbol{\sigma}}) - \boldsymbol{\varepsilon}(\mathbf{v})) \cdot (\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}) d\mathbf{x}, \tag{2.10b}$$

where we have used (2.6c) with $\overline{\mathbf{u}} = \mathbf{v}$. Inserting in (2.10b) $\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \pm \lambda \overline{\boldsymbol{\sigma}}$ with arbitrary $\overline{\boldsymbol{\sigma}} \in L^p(\Omega_c; \mathbb{R}^{d\times d}_{\text{sym}})$ and dividing the result by λ provides the inequality

$$0 \geq \mp \int_{\Omega_c} \big(\mathcal{F}_p(\boldsymbol{\sigma} \pm \lambda \overline{\boldsymbol{\sigma}}) - \boldsymbol{\varepsilon}(\mathbf{v})\big) \cdot \overline{\boldsymbol{\sigma}}\, d\mathbf{x}.$$

Taking the limit $\lambda \to 0$ due to the semi-continuity property (2.5c) results in Eq. (2.6e). The Dirichlet boundary conditions (2.6b) hold in virtue of (2.7b).

Finally, we assume two different solutions $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$ of (2.6c)

$$\int_{\Omega_c} \boldsymbol{\sigma}^k \cdot \boldsymbol{\varepsilon}(\overline{\mathbf{u}})\, d\mathbf{x} = \int_{\Omega_c} \boldsymbol{\sigma}^E \cdot \boldsymbol{\varepsilon}(\overline{\mathbf{u}}) d\mathbf{x}, \quad k = 1, 2, \tag{2.11a}$$

and the corresponding two different solutions $\mathbf{v}^1$ and $\mathbf{v}^2$ of (2.6e)

$$\boldsymbol{\varepsilon}(\mathbf{v}^k) = \mathcal{F}_p(\boldsymbol{\sigma}^k), \quad k = 1, 2. \tag{2.11b}$$

Inserting the difference $\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) = \boldsymbol{\varepsilon}(\mathbf{v}^1 - \mathbf{v}^2) = \mathcal{F}_p(\boldsymbol{\sigma}^1) - \mathcal{F}_p(\boldsymbol{\sigma}^2)$ from (2.11b) into (2.11a) for $k = 1$ and $k = 2$, after their subtraction, we get

$$\int_{\Omega_c} \big(\mathcal{F}_p(\boldsymbol{\sigma}^1) - \mathcal{F}_p(\boldsymbol{\sigma}^2)\big) \cdot (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2)\, d\mathbf{x} = 0. \tag{2.11c}$$

A strictly monotone function $\mathcal{F}_p$ implies $\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}^2$. The proof is completed. □

3. The Limiting Small Strain Case Study

In this section, we consider the limiting case $p \searrow 1^+$.

To deal with the non-reflexivity of the problem (2.6) as $p = 1$, for $q > 1$ from (2.3b) and for a small regularization parameter $\delta > 0$, we regularize the problem e.g. as follows: Find such functions

$$\mathbf{u}^\delta, \mathbf{v}^\delta \in C([0,T]; W^{1,q'}(\Omega_c; \mathbb{R}^d)), \quad \boldsymbol{\sigma}^\delta \in C([0,T]; L^q(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})) \tag{3.1a}$$

that satisfy the Dirichlet boundary conditions

$$\mathbf{u}^\delta = \mathbf{v}^\delta = \mathbf{0} \quad \text{on } (0,T) \times \Gamma_{\mathrm{D}}, \tag{3.1b}$$

the variational equation

$$\int_{\Omega_c} \boldsymbol{\sigma}^\delta \cdot \boldsymbol{\varepsilon}(\overline{\mathbf{u}})\, d\mathbf{x} = \int_{\Omega_c} \boldsymbol{\sigma}^E \cdot \boldsymbol{\varepsilon}(\overline{\mathbf{u}})\, d\mathbf{x} \quad \text{in } (0,T) \tag{3.1c}$$

for all test functions $\overline{\mathbf{u}} \in W^{1,q'}(\Omega_c; \mathbb{R}^d)$ such that $\overline{\mathbf{u}} = \mathbf{0}$ at Γ_{D}, and

$$\mathbf{u}^\delta = \mathcal{I}(\mathbf{v}^\delta) \quad \text{in } Q_c^T, \tag{3.1d}$$

$$\boldsymbol{\varepsilon}(\mathbf{v}^\delta) = \mathcal{F}_1(\boldsymbol{\sigma}^\delta) + \delta \mathcal{F}_q(\boldsymbol{\sigma}^\delta) \quad \text{in } Q_c^T, \tag{3.1e}$$

where the regularization operator $\mathcal{F}_q$ in (3.1e) is any of one satisfying properties (2.5). Since the sum $\mathcal{F}_1 + \delta\mathcal{F}_q$ satisfies (2.5) as well, we can apply Theorem 2.1 providing existence of a solution to the regularized problem (3.1). Further, we employ the space of bounded measures for tensor-valued functions $\mathcal{M}^1(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, which is dual to the space of continuous tensor-valued functions with compact support $C_c(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, with the duality pairing $\langle \cdot, \cdot \rangle_{\Omega_c}$ between them.

Theorem 3.1. *Let* $p = 1$ *in* (2.3a), *respectively,* $p' = \infty$.

(i) *Under conditions* (2.3b)–(2.3d) *and* (2.5), *there exists an accumulation point*

$$\mathbf{u}, \mathbf{v} \in C([0,T]; W^{1,q'}(\Omega_c; \mathbb{R}^d)), \quad \boldsymbol{\sigma} \in C([0,T]; \mathcal{M}^1(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})) \tag{3.2a}$$

of the sequence $(\mathbf{u}^\delta, \mathbf{v}^\delta, \boldsymbol{\sigma}^\delta)$ *as* $\delta \searrow 0^+$ *in* (3.1), *implying a generalized solution of the problem* (2.6), *which satisfies the Dirichlet boundary conditions*

$$\mathbf{u} = \mathbf{v} = \mathbf{0} \quad \text{on } (0,T) \times \Gamma_{\mathrm{D}}, \tag{3.2b}$$

the variational equation

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\overline{\mathbf{u}}) \rangle_{\Omega_c} = \int_{\Omega_c} \boldsymbol{\sigma}^E \cdot \boldsymbol{\varepsilon}(\overline{\mathbf{u}})\, d\mathbf{x} \quad \text{in } (0,T), \tag{3.2c}$$

the integral equation

$$\mathbf{u} = \mathcal{I}(\mathbf{v}) \quad \text{in } Q_c^T, \tag{3.2d}$$

and the variational inequality

$$\int_{\Omega_c} (\overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^E) \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, d\mathbf{x} \le \int_{\Omega_c} \overline{\boldsymbol{\sigma}} \cdot \mathcal{F}_1(\overline{\boldsymbol{\sigma}})\, d\mathbf{x} - \langle \boldsymbol{\sigma}, \mathcal{F}_1(\overline{\boldsymbol{\sigma}}) \rangle_{\Omega_c} \quad \text{in } (0,T) \tag{3.2e}$$

for all test functions such that $\boldsymbol{\varepsilon}(\overline{\mathbf{u}}), \overline{\boldsymbol{\sigma}} \in C_c(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ *and* $\overline{\mathbf{u}} = \mathbf{0}$ *at* Γ_{D}.

(ii) *If the stress in* (3.2a) *is regular such that*

$$\boldsymbol{\sigma} \in C([0,T]; L^{\rho}(\Omega_c; \mathbb{R}^{d\times d}_{\rm sym})), \quad \rho \in (1,q], \tag{3.3}$$

then the generalized solution $(\mathbf{u}, \mathbf{v}, \boldsymbol{\sigma})$ *satisfies all relations* (2.6) *with* $p = 1$.

Proof. Multiplying (3.1e) with $\boldsymbol{\sigma}^\delta - \boldsymbol{\sigma}^E$, integrating it over Ω_c, and testing (3.1c) with the function $\overline{\mathbf{u}} = \mathbf{v}^\delta$, their summation yields

$$\int_{\Omega_c} (\delta \mathcal{F}_q(\boldsymbol{\sigma}^\delta) + \mathcal{F}_1(\boldsymbol{\sigma}^\delta)) \cdot (\boldsymbol{\sigma}^\delta - \boldsymbol{\sigma}^E) d\mathbf{x} = 0. \tag{3.4a}$$

Using the boundedness (2.5a), the coercivity (2.5d), and Young's inequality

$$\left| \int_{\Omega_c} \delta \mathcal{F}_q(\boldsymbol{\sigma}^\delta) \cdot \boldsymbol{\sigma}^E \, d\mathbf{x} \right| \leq \frac{\delta^{q'}}{q'} \|\mathcal{F}_q(\boldsymbol{\sigma}^\delta)\|^{q'}_{L^{q'}(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} + \frac{1}{q} \|\boldsymbol{\sigma}^E\|^q_{L^q(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})}, \tag{3.4b}$$

for sufficiently small $0 < \delta < \delta_0$ such that $M_4(q) - \frac{\delta_0^{q'-1} M_1(q)}{q'} > 0$, from (3.4a), we derive the uniform estimate

$$\begin{aligned} &\delta \left(M_4(q) - \frac{\delta_0^{q'-1} M_1(q)}{q'} \right) \|\boldsymbol{\sigma}^\delta\|^q_{L^q(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} + M_4(1) \|\boldsymbol{\sigma}^\delta\|_{L^1(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} \\ &\quad \leq \delta M_3(q) + \frac{\delta^{q'} M_0(q)}{q'} + M_3(1) + \frac{1}{q} \|\boldsymbol{\sigma}^E\|^q_{L^q(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} + M_1(1) \|\boldsymbol{\sigma}^E\|_{L^1(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} \end{aligned} \tag{3.4c}$$

implying that $\delta \|\boldsymbol{\sigma}^\delta\|^q_{L^q(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})}$ and $\|\boldsymbol{\sigma}^\delta\|_{L^1(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})}$ are uniformly bounded. Due to the boundedness (2.5a), the consequence of Jensen's inequality applied to (3.1e)

$$\|\boldsymbol{\varepsilon}(\mathbf{v}^\delta)\|^{q'}_{L^{q'}(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} \leq 2^{q'-1} \Big(\|\mathcal{F}_1(\boldsymbol{\sigma}^\delta)\|^{q'}_{L^{q'}(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} + \|\delta \mathcal{F}_q(\boldsymbol{\sigma}^\delta)\|^{q'}_{L^{q'}(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} \Big),$$

and the Korn–Poincaré inequality (2.4g), from (3.1e), we get the estimate for $\mathbf{v}^\delta$

$$\begin{aligned} &\frac{1}{1 + K_{\rm KP}(q')} \|\mathbf{v}^\delta\|^{q'}_{W^{1,q'}(\Omega_c;\mathbb{R}^d)} \\ &\quad \leq 2^{q'-1} \Big(d^2 |\Omega_c| M_1(1)^{q'} + \delta^{q'} \big(M_0(q) + M_1(q) \|\boldsymbol{\sigma}^\delta\|^q_{L^q(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} \big) \Big), \end{aligned} \tag{3.4d}$$

and from (3.1d), taking the maximum over time, the estimate for $\mathbf{u}^\delta$

$$\|\mathbf{u}^\delta\|^{q'}_{C([0,T];W^{1,q'}(\Omega_c;\mathbb{R}^d))} \leq J_0 \|\mathbf{v}^\delta\|^{q'}_{C([0,T];W^{1,q'}(\Omega_c;\mathbb{R}^d))}. \tag{3.4e}$$

Utilizing the embedding of L^1-space in the $\mathcal{M}^1$-space of bounded measures, from (3.4), we conclude with existence of an accumulation point presented in (3.2a) and a subsequence, still denoted by δ for short, that converges as $\delta \searrow 0^+$

$$\|\delta \mathcal{F}_q(\boldsymbol{\sigma}^\delta)\|^{q'}_{L^{q'}(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} \leq \delta^{q'} \Big(M_0(q) + M_1(q) \|\boldsymbol{\sigma}^\delta\|^q_{L^q(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} \Big) \to 0, \tag{3.5a}$$

$$\mathbf{u}^\delta \rightharpoonup \mathbf{u}, \quad \mathbf{u}^\delta \rightharpoonup \mathbf{v} \quad \text{weakly in } C([0,T]; W^{1,q'}(\Omega_c; \mathbb{R}^d)), \tag{3.5b}$$

$$\boldsymbol{\sigma}^\delta \rightharpoonup \boldsymbol{\sigma} \quad \text{*-weakly in } C([0,T]; \mathcal{M}^1(\Omega_c; \mathbb{R}^{d\times d}_{\rm sym})). \tag{3.5c}$$

With the help of (3.5b), we pass to the limit in the boundary conditions (3.1b) and in the linear integral equation (3.1d) to get (3.2b) and (3.2d), respectively. For the test functions that satisfy $\boldsymbol{\varepsilon}(\overline{\mathbf{u}}), \overline{\boldsymbol{\sigma}} \in C_c(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, on taking the limit of (3.1c), the convergence (3.5c) gives (3.2c). Using the monotonicity (2.5b) and the equilibrium equation (3.1c), from (3.1e), we derive the relations

$$0 = \int_{\Omega_c} \big(\boldsymbol{\varepsilon}(\mathbf{v}^\delta) - \mathcal{F}_1(\boldsymbol{\sigma}^\delta) - \delta\mathcal{F}_q(\boldsymbol{\sigma}^\delta)\big)\cdot(\boldsymbol{\sigma}^\delta - \overline{\boldsymbol{\sigma}})\,d\mathbf{x}$$
$$\leq \int_{\Omega_c} (\boldsymbol{\varepsilon}(\mathbf{v}^\delta)\cdot(\boldsymbol{\sigma}^E - \overline{\boldsymbol{\sigma}}) - (\mathcal{F}_1(\overline{\boldsymbol{\sigma}}) + \delta\mathcal{F}_q(\overline{\boldsymbol{\sigma}}))\cdot(\boldsymbol{\sigma}^\delta - \overline{\boldsymbol{\sigma}}))d\mathbf{x}, \tag{3.6a}$$

where we pass to the lower limit using the convergences (3.5), thus arriving at the inequality (3.2e).

If the stress in (3.2a) is regular such that (3.3) holds true for some ρ, then the equilibrium equation (3.2c) reduces to (2.6c), since $C_c(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ is dense in both spaces of test functions $\overline{\boldsymbol{\sigma}} \in L^\rho(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) \in L^{\rho'}(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ for $\frac{1}{\rho} + \frac{1}{\rho'} = 1$ when $\rho, \rho' \in (1, \infty)$. Then we can insert $\overline{\mathbf{u}} = \mathbf{v} \in W^{1,q'}(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \subseteq W^{1,\rho'}(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ in the variational inequality (3.2e) which in virtue of (2.6c) turns into

$$\int_{\Omega_c} (\overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma})\cdot\boldsymbol{\varepsilon}(\mathbf{v})\,d\mathbf{x} \leq \int_{\Omega_c} (\overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma})\cdot\mathcal{F}_1(\overline{\boldsymbol{\sigma}})d\mathbf{x}. \tag{3.6b}$$

Repeating Minty's argument, we substitute in (3.6b) the test function $\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \pm \lambda\widetilde{\boldsymbol{\sigma}}$ with arbitrary $\widetilde{\boldsymbol{\sigma}} \in L^\rho(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\lambda > 0$, divide the result by λ such that

$$\pm\int_{\Omega_c} (\boldsymbol{\varepsilon}(\mathbf{v}) - \mathcal{F}_1(\boldsymbol{\sigma} \pm \lambda\widetilde{\boldsymbol{\sigma}}))\cdot\widetilde{\boldsymbol{\sigma}}\,d\mathbf{x} \leq 0. \tag{3.6c}$$

Taking the limit as $\lambda \to 0$ due to the semi-continuity property (2.5c), the equality (2.6e) follows by virtue of the fundamental lemma of calculus of variation. Moreover, if $p = 1$, then $\mathcal{F}_1(\boldsymbol{\sigma}(t))$ in (2.5a) is uniformly bounded, hence $\boldsymbol{\varepsilon}(\mathbf{u}(t))$ too, implying that $\mathbf{u}(t)$ is from $W^{1,\infty}$ for all t. The proof is completed. □

Appendix A.

Lemma A.1. *The following estimate*[1]

$$(1 + \kappa\|\boldsymbol{\sigma}\|^r)^{\frac{p}{r}} \leq C_{\frac{r}{p}}(1 + \kappa^{\frac{p}{r}}\|\boldsymbol{\sigma}\|^p), \quad p, r > 0, \tag{A.1}$$

holds, where $C_{\frac{r}{p}} = 2^{\frac{p}{r}-1}$ *for* $\frac{r}{p} < 1$, *and* $C_{\frac{r}{p}} = 1$ *for* $\frac{r}{p} \geq 1$.

Lemma A.2. *For* $p \in [1, \infty)$ *and* $p' \in (1, \infty]$ *such that* $\frac{1}{p} + \frac{1}{p'} = 1$, *the response function* $\mathcal{F}_p : L^p(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \mapsto L^{p'}(\Omega_c; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ *in* (1.5) *and* (1.6a) *given by*

$$\mathcal{F}_p(\boldsymbol{\sigma}) = \frac{1}{2\mu}\frac{\boldsymbol{\sigma}}{(1 + \kappa\|\boldsymbol{\sigma}\|^r)^{\frac{2-p}{r}}}, \quad \kappa, \mu, r > 0 \tag{A.2}$$

is (i) *bounded such that for* $p > 1$

$$\|\mathcal{F}_p(\boldsymbol{\sigma})\|^{p'}_{L^{p'}(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} \le M_0(p) + M_1(p)\|\boldsymbol{\sigma}\|^p_{L^p(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} \tag{A.3a}$$

with $M_0(p) = \frac{(p-p')d^2}{p(2\mu)^{p'}} C_{\frac{r}{p-p'}} |\Omega_c|$, $M_1(p) = \frac{1}{(2\mu)^{p'}} C_{\frac{r}{p-p'}} (\frac{p'}{p} + \kappa^{\frac{p-p'}{r}} d^{2p})$ *for* $p > 2$, $M_0(p) = 0$, $M_1(p) = \frac{1}{2\mu\kappa^{\frac{2-p}{r}}} d^{2p}$ *for* $1 < p \le 2$, *and for* $p = 1$

$$\|\mathcal{F}_1(\boldsymbol{\sigma})\|_{L^\infty(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} \le M_1(1) = \frac{1}{2\mu\kappa^{\frac{1}{r}}}; \tag{A.3b}$$

(ii) *strictly monotone, that is for* $\boldsymbol{\sigma}, \overline{\boldsymbol{\sigma}} \in L^p(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})$,

$$\int_{\Omega_c} \big(\mathcal{F}_p(\boldsymbol{\sigma}) - \mathcal{F}_p(\overline{\boldsymbol{\sigma}})\big) \cdot (\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}})\, d\mathbf{x} > 0, \quad \boldsymbol{\sigma} \ne \overline{\boldsymbol{\sigma}}; \tag{A.3c}$$

(iii) *semi-continuous that is*

$$\lambda \mapsto \int_{\Omega_c} \mathcal{F}_p(\boldsymbol{\sigma} + \lambda\overline{\boldsymbol{\sigma}}) \cdot \overline{\boldsymbol{\sigma}}\, d\mathbf{x} \quad \text{is continuous at } \lambda = 0; \tag{A.3d}$$

(iv) *coercive such that*

$$\int_{\Omega_c} \mathcal{F}_p(\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma}\, d\mathbf{x} \ge M_4(p)\|\boldsymbol{\sigma}\|^p_{L^p(\Omega_c;\mathbb{R}^{d\times d}_{\rm sym})} - M_3(p) \tag{A.3e}$$

with $M_3(p) = \frac{1}{2\mu p((2-p)C_{\frac{r}{p}})^{\frac{2-p}{p}} \kappa^{\frac{2}{r}}} |\Omega_c|$, $M_4(p) = \frac{1}{2\mu d^2 p((2-p)C_{\frac{r}{p}})^{\frac{2-p}{p}} \kappa^{\frac{2-p}{r}}}$ *for* $p < 2$, *and* $M_3(p) = 0$, $M_4(p) = \frac{1}{2\mu d^2}\kappa^{\frac{p-2}{r}}$ *for* $p \ge 2$.

Proof. First, we prove the assertion (i). From (A.2), we get the upper bound for $2 - p \ge 0$ estimating $|\mathcal{F}_p(\boldsymbol{\sigma})_{ij}| \le \|\mathcal{F}_p(\boldsymbol{\sigma})\| \le \frac{1}{2\mu\kappa^{\frac{2-p}{r}}} \|\boldsymbol{\sigma}\|^{p-1}$ componentwise for $i, j = 1, \ldots, d$, hence following (A.3b) as $p = 1$. For $1 < p \le 2$, using $p - 1 = \frac{p}{p'}$, after summation over $i, j = 1, \ldots, d$ we proceed with

$$\sum_{i,j=1}^d |\mathcal{F}_p(\boldsymbol{\sigma})_{ij}|^{p'} \le \frac{1}{(2\mu\kappa^{\frac{2-p}{r}})^{p'}} d^2 \|\boldsymbol{\sigma}\|^p, \tag{A.4a}$$

$$\|\boldsymbol{\sigma}\|^p \le d^{2(p-1)} \sum_{i,j=1}^d |\sigma_{ij}|^p; \tag{A.4b}$$

while for $2 - p < 0$ by using (A.1) and $(p-2)p' = p - p'$, we estimate

$$|\mathcal{F}_p(\boldsymbol{\sigma})_{ij}|^{p'} \le \frac{1}{(2\mu)^{p'}} C_{\frac{r}{p-p'}} \big(|\boldsymbol{\sigma}_{ij}|^{p'} + \kappa^{\frac{p-p'}{r}} \|\boldsymbol{\sigma}\|^p\big), \tag{A.4c}$$

and continue further by Young's inequality $|\boldsymbol{\sigma}_{ij}|^{p'} \le \frac{p'}{p}(|\boldsymbol{\sigma}_{ij}|^{p'})^{\frac{p}{p'}} + \frac{p-p'}{p}$ and (A.4b). On gathering (A.4) together, (A.3a) follows.

Next, we employ the integral representation:

$$\mathcal{F}_p(\boldsymbol{\sigma}) - \mathcal{F}_p(\overline{\boldsymbol{\sigma}}) = \int_0^1 \frac{d}{d\lambda} \mathcal{F}_p(\lambda\boldsymbol{\sigma} + (1-\lambda)\overline{\boldsymbol{\sigma}})\, d\lambda, \tag{A.5a}$$

and due to $\frac{d}{d\lambda}\|\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}}\| = \frac{\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}}}{\|\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}}\|}\cdot(\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}})$, we calculate the directional derivative in (A.5a) in the following way:

$$\begin{aligned}\frac{d}{d\lambda}\mathcal{F}_p(\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}}) = {} & \frac{1}{2\mu(1+\kappa\|\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}}\|^r)^{\frac{2-p}{r}+1}} \\ & \times((1+\kappa\|\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}}\|^r) \\ & \times(\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}}) - \kappa(2-p)\|\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}}\|^{r-2} \\ & \times[(\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}})\cdot(\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}})](\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}})). \end{aligned} \tag{A.5b}$$

Inserting the representation (A.5b) in (A.5a) and using the Cauchy–Schwarz inequality, the scalar product of $\mathcal{F}_p(\boldsymbol{\sigma}) - \mathcal{F}_p(\overline{\boldsymbol{\sigma}})$ with $\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}}$ is estimated from below as

$$\begin{aligned}(\mathcal{F}_p(\boldsymbol{\sigma}) - \mathcal{F}_p(\overline{\boldsymbol{\sigma}}))\cdot(\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}}) \geq {} & \int_0^1 \frac{1}{2\mu(1+\kappa\|\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}}\|^r)^{\frac{2-p}{r}+1}} \\ & \times(1+\kappa\min(1,p-1)\|\lambda\boldsymbol{\sigma}+(1-\lambda)\overline{\boldsymbol{\sigma}}\|^r)\|\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}}\|^2\,d\lambda, \end{aligned} \tag{A.5c}$$

thus justifying the strict monotone property (A.3b) and the assertion (ii).

The third assertion (iii) is the direct consequence of continuity as $\lambda \to 0$ of elementary functions entering the following expression:

$$\begin{aligned}&\int_{\Omega_c}\big(\mathcal{F}_p(\boldsymbol{\sigma}+\lambda\overline{\boldsymbol{\sigma}}) - \mathcal{F}_p(\boldsymbol{\sigma})\big)\cdot\overline{\boldsymbol{\sigma}}\,d\mathbf{x} \\ &\quad = \frac{1}{2\mu}\int_{\Omega_c}\Bigg\{\lambda\frac{\overline{\boldsymbol{\sigma}}\cdot\overline{\boldsymbol{\sigma}}}{(1+\kappa\|\boldsymbol{\sigma}+\lambda\overline{\boldsymbol{\sigma}}\|^r)^{\frac{2-p}{r}}} \\ &\qquad + \left(\frac{1}{(1+\kappa\|\boldsymbol{\sigma}+\lambda\overline{\boldsymbol{\sigma}}\|^r)^{\frac{2-p}{r}}} - \frac{1}{(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{2-p}{r}}}\right)(\boldsymbol{\sigma}\cdot\overline{\boldsymbol{\sigma}})\Bigg\}d\mathbf{x} \to 0. \end{aligned} \tag{A.6}$$

Finally, we obtain the assertion (iv) from the lower bound (compared to (A.4b))

$$\sum_{i,j=1}^{d}|\sigma_{ij}|^p \leq d^2\|\boldsymbol{\sigma}\|^p \tag{A.7a}$$

and $\mathcal{F}_p(\boldsymbol{\sigma})\cdot\boldsymbol{\sigma} = \frac{1}{2\mu}\frac{\|\boldsymbol{\sigma}\|^2}{(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{2-p}{r}}}$ by estimating the matrix norm for $p \geq 2$ as follows:

$$\|\boldsymbol{\sigma}\|^2 = \frac{\|\boldsymbol{\sigma}\|^2(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{p-2}{r}}}{(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{p-2}{r}}} \leq 2\mu(\mathcal{F}_p(\boldsymbol{\sigma})\cdot\boldsymbol{\sigma})\frac{1}{\kappa^{\frac{p-2}{r}}\|\boldsymbol{\sigma}\|^{p-2}}. \tag{A.7b}$$

For $p < 2$, we use (A.7a) and weighted Young's inequality with the powers $\rho = \frac{2}{p}$, $\rho' = \frac{2}{2-p}$, the weight $\alpha = ((2-p)C_{\frac{r}{p}}\kappa^{\frac{p}{r}})^{\frac{2-p}{2}}$, and apply (A.1) to derive

$$\|\boldsymbol{\sigma}\|^p = \left(\frac{\|\boldsymbol{\sigma}\|^2}{(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{2-p}{r}}}\right)^{\frac{p}{2}}(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{(2-p)p}{2r}}$$

$$\leq \frac{p}{2}\alpha^{\frac{2}{p}}\frac{\|\boldsymbol{\sigma}\|^2}{(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{2-p}{r}}} + \frac{2-p}{2\alpha^{\frac{2}{2-p}}}(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{p}{r}}$$

$$= \frac{p}{2}\big((2-p)C_{\frac{r}{p}}\kappa^{\frac{p}{r}}\big)^{\frac{2-p}{p}}2\mu(\mathcal{F}_p(\boldsymbol{\sigma})\cdot\boldsymbol{\sigma}) + \frac{1}{2C_{\frac{r}{p}}\kappa^{\frac{p}{r}}}(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{p}{r}}$$

$$\leq \mu p\big((2-p)C_{\frac{r}{p}}\big)^{\frac{2-p}{p}}\kappa^{\frac{2-p}{r}}(\mathcal{F}_p(\boldsymbol{\sigma})\cdot\boldsymbol{\sigma}) + \frac{1}{2\kappa^{\frac{p}{r}}} + \frac{1}{2}\|\boldsymbol{\sigma}\|^p. \tag{A.7c}$$

The estimates (A.7) result in (A.3e) which finishes the proof. □

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