MIXED VARIATIONAL PROBLEM FOR A GENERALIZED DARCY–FORCHHEIMER MODEL DRIVEN BY HYDRAULIC FRACTURE

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Abstract. The model of a stationary flow in porous media stemming from hydraulic fracking and accounting for inertial phenomena is considered. The incompressible fluid is modeled by a nonlinear Darcy–Forchheimer (DF) equation under mixed boundary conditions, which are appropriate for a fluid-driven fracture. The classical DF equation is generalized with respect to a growth exponent m and inhomogeneous coefficients. Using mixed variational formulation of the problem for unknown fluid velocity and fluid pressure, the well-posedness theorem is proved for arbitrary $m > 1$. The developed Lagrange multiplier formalism is advantageous for optimal shape design of fractures.

1. Introduction

We present a mathematical modeling for the filtration of incompressible fluids through porous media (the earth) motivated by engineering technologies using hydraulic fracturing for mining natural gas and oil from reservoirs. A porous medium filled with an incompressible fluid is described, which contains inside a hydraulic fracture (HF) created by pumping a fracturing fluid into a well-bore. In practice, the fracture width is negligible, its length and height may differ by two orders of magnitude. Therefore, in simulations, the reservoir is often simplified to a plane 2d geometry with a straight fracture as sketched in Fig. 1. For modeling the flow in reservoir and

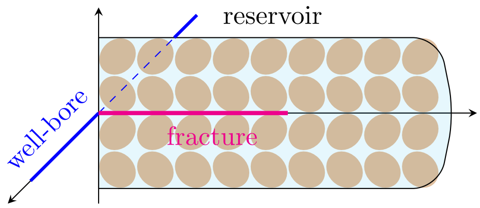

Figure 1. Sketch of a hydraulic fracture in a reservoir.

in the fracture, this typically requires different types of partial differential equations (PDEs) and their coupling along fracture walls by transmission boundary conditions.[3, 17] However, treatment of the coupled models on different scales is numerically not attainable. We assume a pressure p^0 prescribed on the fracture walls, which is obtained from experimental data or by proper modeling in the fracture,[19, 9] and we study separately the flow in the reservoir. The complementary modeling of solid phase by the coupled poroelastic equations driven by the hydraulic fracture and non-penetration of the fracture walls see in the recent research.[11]

For physical consistence of a flow model,[16] inertial phenomena can be quantified with the help of the Reynolds number

$$\mathrm{Re} = \frac{\rho U \delta}{\eta}, \tag{1}$$

where ρ is the density, U is a characteristic velocity, δ is a characteristic pore size, and η is the effective viscosity of the fluid. Whereas the Darcy law is suitable for $\mathrm{Re} < 1$ in (1), for $\mathrm{Re} > 1$ inertial phenomena should be

Key words and phrases. porous medium, filtration, incompressible fluid, hydro-fracking, variational principle, Lagrange multiplier.

taken into consideration, typically within the Darcy–Forchheimer (DF) law[4] given for the fluid pressure p and velocity vector $\mathbf{u}$ by

$$\nabla p + \frac{\eta}{\kappa}\mathbf{u} + \frac{\rho C}{\sqrt{\kappa}}|\mathbf{u}|\mathbf{u} = 0, \tag{2}$$

where κ is the permeability, C is the inertial resistance, and $|\cdot|$ denotes the Euclidean norm. Setting $C = 0$ in (2) turns it into the Darcy law. Equation (2) implies momentum conservation, and continuity equation leads to the incompressibility condition

$$\operatorname{div}(\mathbf{u}) = 0. \tag{3}$$

In general, the nonlinear relation (2) cannot be inverted with respect to $\mathbf{u}$ in order to substitute it into (3). Thus, we have mixed $(p, \mathbf{u})$ constitutive equations (2) and (3). Further we treat a generalization of the classical DF law (2) as

$$\nabla p + A\mathbf{u} + \beta|\mathbf{u}|^{m-2}\mathbf{u} = 0 \tag{4}$$

with respect to the growth exponent m, where inhomogeneous coefficient matrix A and scalar β characterize the linear and nonlinear resistance, respectively. In general, A and β may depend on the pressure p.[1] If $m = 3$ and $A = \alpha\mathbf{I}$ with the identity transformation $\mathbf{I}$, then (4) turns into (2) with the Darcy coefficient $\alpha = \eta/\kappa$ and the Forchheimer coefficient $\beta = \rho C/\sqrt{\kappa}$. Compared to the literature,[5, 8] where various ranges of m were considered, we provide well-posed to the flow model (3) and (4) in a unified way for arbitrary growth exponents $m > 1$.

From a mathematical viewpoint, we utilize the Lagrange multiplier approach[10] resulting in a mixed, primal-dual variational formulation of the underlying problem. Based on the coercivity and Ladyzhenskaya–Babuška–Brezzi inf-sup condition[15], the differentiability and convexity of the corresponding Lagrangian, from the minimax theorem existence of a unique solution follows. The primal-dual formalism is helpful for the ultimate reason of shape design and identification.[2, 12] We cite the Lagrangian-based shape optimal control in flow models according to Stokes,[13, 14] Brinkman,[6] and Forchheimer[7] laws. In this respect, shape optimization of hydraulic fractures (HFs) in the DF model would be of practical interest.

2. Mixed variational problem

In the Euclidean space $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, where either $d = 2$ or $d = 3$ depending on a specific model, let a domain Ω having the Lipschitz boundary $\partial\Omega$ and outward normal vector $\mathbf{n} = (n_1, \dots, n_d)$ associate the reservoir. We assume that Ω can be split by an interface Σ into two sub-domains Ω^+ and Ω^- with Lipschitz boundaries $\partial\Omega^\pm$ as illustration in 2d in Fig. 2. Further we associate a part Γ_{f} of the interface to the fracture

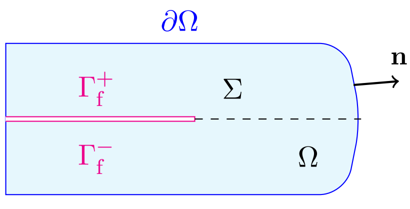

Figure 2. Geometry of fractured domain Ω_{f}.

with two fracture walls $\Gamma_{\mathrm{f}}^\pm$ such that

$$\partial\Omega^+ \cap \partial\Omega^- = \Sigma, \quad \Omega = \Omega^+ \cup \Omega^- \cup \Sigma, \quad \Gamma_{\mathrm{f}} \subset \Sigma, \quad \Gamma_{\mathrm{f}}^\pm \subset \Sigma^\pm \subset \partial\Omega^\pm, \quad \Omega_{\mathrm{f}} = \Omega \setminus \overline{\Gamma_{\mathrm{f}}},$$

where the fractured domain Ω_{f} has the boundary $\partial\Omega_{\mathrm{f}} = \partial\Omega \cup \Gamma_{\mathrm{f}}^+ \cup \Gamma_{\mathrm{f}}^-$. Let the boundary split into two disjoint parts $\partial\Omega_{\mathrm{f}} = \overline{\Gamma_{\mathrm{N}}} \cup \overline{\Gamma_{\mathrm{D}}}$. For modeling of the fluid-driven fracture, both fracture walls are assumed to be parts of the Dirichlet boundary $\Gamma_{\mathrm{f}}^\pm \subset \Gamma_{\mathrm{D}}$.

The growth exponents $m \in (1, \infty)$ in equation (4) determines its conjugate number $m' \in (1, \infty)$ with $1/m + 1/m' = 1$. For $m > 2$ it follows $m' \in (1, 2)$. If $m \in (1, 2)$, hence $m' > 2$, then the continuous embeddings

$L^2(\Omega_{\rm f};\mathbb{R}^d) \subset L^m(\Omega_{\rm f};\mathbb{R}^d)$ and $L^{m'}(\Omega_{\rm f};\mathbb{R}) \subset L^2(\Omega_{\rm f};\mathbb{R})$ hold with the uniform constants $K_{(m,2)} > 0$ and $K_{(2,m')} > 0$ providing

$$\|\mathbf{u}\|_m \le K_{(m,2)}\|\mathbf{u}\|_2, \quad \|\nabla p\|_2 \le K_{(2,m')}\|\nabla p\|_{m'} \quad \text{for } \mathbf{u} \in L^2(\Omega_{\rm f};\mathbb{R}^d),\ \nabla p \in L^{m'}(\Omega_{\rm f};\mathbb{R}^d), \quad m \in (1,2). \tag{5}$$

Based on estimates (5) we consider the conjugate numbers $\max(2,m)$ and $\min(2,m')$ with $1/\max(2,m) + 1/\min(2,m') = 1$, which correspond to $\mathbf{u}$ and ∇p entering the generalized DF law (4).

We assume existence of a function $p^0(\mathbf{x})$ in the reservoir, which jump across the fracture walls is denoted by $[\![p^0]\!]$, such that

$$p^0 \in W^{1,\min(2,m')}(\Omega_{\rm f};\mathbb{R}), \quad [\![p^0]\!] := p^0|_{\Gamma_{\rm f}^+} - p^0|_{\Gamma_{\rm f}^-}. \tag{6}$$

From (6) the trace $p^0 \in W^{1/\max(2,m),\min(2,m')}(\partial\Omega_{\rm f};\mathbb{R})$ is well defined and allows a non-zero jump across $\Gamma_{\rm f}$ except for the fracture tip/front. The assumption (6) is used to prescribe the inhomogeneous Dirichlet data within the mixed boundary conditions

$$p = p^0 \text{ on } \Gamma_{\rm D}, \quad \mathbf{n}^\top\mathbf{u} = 0 \text{ on } \Gamma_{\rm N}, \tag{7}$$

where $^\top$ swaps between columns and rows, and $\mathbf{n}^\top\mathbf{u}$ implies the scalar product. The function p^0 means an extension into the fractured domain $\Omega_{\rm f}$ of the boundary data from (7). Physically, the former condition in (7) describes continuity of the fluid pressure through $\Gamma_{\rm D}$, whereas the latter condition corresponds to impermeability at $\Gamma_{\rm N}$. It is worth noting that for the Darcy law $\nabla p + \alpha\mathbf{u} = 0$ it implies the homogeneous Neumann condition $\mathbf{n}^\top\alpha^{-1}\nabla p = 0$ at the free surface $\Gamma_{\rm N}$.

The linear $A(\mathbf{x})$ and nonlinear $\beta(\mathbf{x})$ resistance coefficients are assumed inhomogeneous functions in the reservoir, A is given by a symmetric $d\times d$ matrix, which is uniformly bounded and positive definite with constants $0 < \underline{\alpha} \le \overline{\alpha}$:

$$A \in L^\infty(\Omega;\mathbb{R}^{d\times d}_{\rm sym}), \quad \underline{\alpha}|\xi|^2 \le \xi^\top A\xi, \quad |A\xi| \le \overline{\alpha}|\xi| \quad \text{for } \xi \in \mathbb{R}^d, \quad \beta \in L^\infty(\Omega;\mathbb{R}), \quad 0 < \underline{\beta} \le \beta \le \overline{\beta}, \tag{8}$$

where $A\xi$ stands for the matrix-vector multiplication in (8) and in what follows.

After these assumptions we consider the nonlinear boundary value problem consisted of the governing equations (3), (4) fulfilled in the fractured domain $\Omega_{\rm f}$ and supported with boundary conditions (7) at $\partial\Omega_{\rm f}$, with respect to unknown the fluid pressure $p(\mathbf{x})$ and the fluid velocity $\mathbf{u}(\mathbf{x})$. Because of the Dirichlet condition we introduce the closed subspace

$$W^{1,\min(2,m')}_{\Gamma_{\rm D}}(\Omega_{\rm f};\mathbb{R}) = \{q \in W^{1,\min(2,m')}(\Omega_{\rm f};\mathbb{R}) | \quad q = 0 \text{ on } \Gamma_{\rm D}\}.$$

Multiplying the equations with smooth test functions and integrating $\operatorname{div}(\mathbf{u})$ by parts using boundary conditions (7), the mixed variational problem seeks for a solution in the weak form: find $\mathbf{u} \in L^{\max(2,m)}(\Omega_{\rm f};\mathbb{R}^d)$ and $p - p^0 \in W^{1,\min(2,m')}_{\Gamma_{\rm D}}(\Omega_{\rm f};\mathbb{R})$ satisfying

$$\int_{\Omega_{\rm f}} \mathbf{v}^\top\big(\nabla p + A\mathbf{u} + \beta|\mathbf{u}|^{m-2}\mathbf{u}\big)\,d\mathbf{x} = 0 \quad \text{for all } \mathbf{v} \in L^{\max(2,m)}(\Omega_{\rm f};\mathbb{R}^d), \tag{9}$$

$$\int_{\Omega_{\rm f}} \mathbf{u}^\top\nabla q\,d\mathbf{x} = 0 \quad \text{for all } q \in W^{1,\min(2,m')}_{\Gamma_{\rm D}}(\Omega_{\rm f};\mathbb{R}). \tag{10}$$

We note that the nonlinear term in (9) can be estimated using Hölder's inequality and the upper bound in (8) as

$$\Big|\int_{\Omega_{\rm f}} \mathbf{v}^\top\beta|\mathbf{u}|^{m-2}\mathbf{u}\,d\mathbf{x}\Big| \le \overline{\beta}\|\mathbf{u}\|_m^{m-1}\|\mathbf{v}\|_m. \tag{11}$$

Conversely, relations (3), (4) and (7) can be derived from (9) and (10) only for a smooth velocity whose $\operatorname{div}(\mathbf{u}) \in L^n(\Omega_{\rm f};\mathbb{R})$ with $1/n = 1/\max(2,m) + 1/d$, by virtue of the Sobolev embedding $W^{1,\min(2,m')}(\Omega_{\rm f};\mathbb{R}) \subset L^{n'}(\Omega_{\rm f};\mathbb{R})$ for the conjugate number $1/n' = 1/\min(2,m') - 1/d$.

3. Lagrangian formalism

Based on the variational equations (9) and (10) we introduce the Lagrange function

$$\mathcal{L}: L^{\max(2,m)}(\Omega_{\rm f};\mathbb{R}^d)\times W^{1,\min(2,m')}(\Omega_{\rm f};\mathbb{R})\mapsto\mathbb{R},\quad \mathcal{L}(\mathbf{u},p)=\int_{\Omega_{\rm f}}\Bigl(\mathbf{u}^\top\bigl(\nabla p+\frac{1}{2}A\mathbf{u}\bigr)+\frac{\beta}{m}|\mathbf{u}|^m\Bigr)\,d\mathbf{x}. \tag{12}$$

Further we use the notation of the closed subspace $U(\Omega_{\rm f}):=L^{\max(2,m)}(\Omega_{\rm f};\mathbb{R}^d)\times W^{1,\min(2,m')}_{\Gamma_{\rm D}}(\Omega_{\rm f};\mathbb{R})$ for brevity.

Lemma 1 (properties of Lagrangian)**.** *(i) The Lagrangian $\mathcal{L}(\mathbf{u},p)$ in* (12) *is* differentiable*:*

$$\langle\frac{\partial\mathcal{L}}{\partial\mathbf{u}}(\mathbf{u},p),\mathbf{v}\rangle=\int_{\Omega_{\rm f}}\mathbf{v}^\top\bigl(\nabla p+A\mathbf{u}+\beta|\mathbf{u}|^{m-2}\mathbf{u}\bigr)\,d\mathbf{x},\quad \langle\frac{\partial\mathcal{L}}{\partial p}(\mathbf{u},p),q\rangle=\int_{\Omega_{\rm f}}\mathbf{u}^\top\nabla q\,d\mathbf{x}. \tag{13}$$

(ii) If the following condition for the growth exponent m and bounds $\underline{\beta}$, $\overline{\beta}$ in (8) *holds*

$$\underline{\beta}+(m-2)\overline{\beta}\ge 0, \tag{14}$$

then the function $\mathbf{u}\mapsto\mathcal{L}(\mathbf{u},p)$ is strictly convex *with the second derivatives satisfying*

$$\frac{\partial^2\mathcal{L}}{\partial\mathbf{u}^2}(\mathbf{u},p)[\mathbf{v},\mathbf{v}]=\int_{\Omega_{\rm f}}\Bigl(\mathbf{v}^\top\bigl(A+\beta|\mathbf{u}|^{m-2}\bigr)\mathbf{v}+(m-2)\beta|\mathbf{u}|^{m-4}(\mathbf{v}^\top\mathbf{u})^2\Bigr)\,d\mathbf{x}>0\quad \textit{for } \mathbf{v}\neq 0. \tag{15}$$

(iii) The function $\mathbf{u}\mapsto\mathcal{L}(\mathbf{u},p): L^{\max(2,m)}(\Omega_{\rm f};\mathbb{R}^d)\mapsto\mathbb{R}$ is coercive*: for arbitrary fixed $p\in W^{1,\min(2,m')}(\Omega_{\rm f};\mathbb{R})$*

$$\mathcal{L}(\mathbf{u},p)\ge\frac{\underline{\alpha}}{2}\|\mathbf{u}\|_2^2+\frac{\underline{\beta}}{m}\|\mathbf{u}\|_m^m-\|\nabla p\|_{\min(2,m')}\|\mathbf{u}\|_{\max(2,m)}\to+\infty\quad \textit{as } \|\mathbf{u}\|_{\max(2,m)}\to\infty. \tag{16}$$

(iv) The following inf-sup condition *holds for $p\in W^{1,\min(2,m')}_{\Gamma_{\rm D}}(\Omega_{\rm f};\mathbb{R})$:*

$$\sup_{0\neq\mathbf{u}\in L^{\max(2,m)}(\Omega_{\rm f};\mathbb{R}^d)}\frac{1}{\|\mathbf{u}\|_{\max(2,m)}}\int_{\Omega_{\rm f}}\mathbf{u}^\top\nabla p\,d\mathbf{x}\ge K_{\rm LBB}\|p\|_{W^{1,\min(2,m')}(\Omega_{\rm f})},\quad K_{\rm LBB}>0. \tag{17}$$

Proof. The properties (i) and (ii) can be easily checked directly by the differentiation for

$$\frac{\partial\mathcal{L}}{\partial\mathbf{u}}(\mathbf{u},p)=\nabla p+A\mathbf{u}+\beta|\mathbf{u}|^{m-2}\mathbf{u},\quad \frac{\partial^2\mathcal{L}}{\partial\mathbf{u}^2}(\mathbf{u},p)=A+\beta|\mathbf{u}|^{m-2}\mathbf{I}+(m-2)\beta|\mathbf{u}|^{m-4}\mathbf{u}\mathbf{u}^\top,$$

where $\mathbf{u}\mathbf{u}^\top$ implies the dyadic product. The property (iii) follows straightforwardly from (12) when using the lower bounds in (8) and Hölder's inequality for $\mathbf{u}^\top\nabla p$. The estimate from below (17) is the consequence of definition of the dual norm in the space $L^{\min(2,m')}(\Omega_{\rm f};\mathbb{R})$, provided by the surjectivity of the mapping $q\mapsto\nabla q: W^{1,\min(2,m')}_{\Gamma_{\rm D}}(\Omega_{\rm f};\mathbb{R})\mapsto L^{\min(2,m')}(\Omega_{\rm f};\mathbb{R}^d)$, thus proving the property (iv). □

From Lemma 1 we derive the main theorem on existence.

Theorem 1 (solution existence)**.** *Under condition* (14)*, there exists a unique saddle-point to the following minimax problem: For given p^0 from* (6)*, find $(\mathbf{u},p-p^0)\in U(\Omega_{\rm f})$ satisfying*

$$\mathcal{L}(\mathbf{u},q)\le\mathcal{L}(\mathbf{u},p)\le\mathcal{L}(\mathbf{v},p)\quad \textit{for all } (\mathbf{v},q)\in U(\Omega_{\rm f}). \tag{18}$$

The pair $(\mathbf{u},p)$ fulfills optimality conditions in the form of variational equations (9), (10)*, which imply a weak solution to the generalized DF equations* (3), (4) *under boundary conditions* (7)*. The following a-priori estimates hold:*

$$\min\bigl(1,\frac{1}{K_{(m,2)}}\bigr)\underline{\alpha}\|\mathbf{u}\|_2+\min\bigl(1,\frac{1}{K_{(2,m)}}\bigr)\underline{\beta}\|\mathbf{u}\|_m^{m-1}\le\|\nabla p^0\|_{\min(2,m')},\quad K_{\rm LBB}\|p\|_{W^{1,\min(2,m')}(\Omega_{\rm f})}\le\|\nabla p^0\|_{\min(2,m')}, \tag{19}$$

where constants $K_{(m,2)}$, $K_{(2,m')}$, $\underline{\alpha}$, $\underline{\beta}$, and $K_{\rm LBB}$ are from (5), (8)*, and* (17)*.*

Proof. The mapping $\mathbf{u} \mapsto \mathcal{L}(\mathbf{u}, p)$ is coercive by (16), convex by (15) and differentiable by (13), hence weakly lower semi-continuous, see Theorem 1.7.[10] The linear mapping $p \mapsto \mathcal{L}(\mathbf{u}, p)$ satisfies the inf-sup condition (17). These properties guarantee a solution to the nonlinear saddle-point problem (18),[18] which is unique due to the strict convexity in (15). Applying the differential calculus (13) we arrive at the variational equations (9) and (10).

Testing (9) with $\mathbf{v} = \mathbf{u}$, and (10) with admissible $q = p^0 - p$, after their summation we have

$$\underline{\alpha}\|\mathbf{u}\|_2^2 + \underline{\beta}\|\mathbf{u}\|_m^m \leq \int_{\Omega_{\mathrm{f}}} \big(\mathbf{u}^\top A\mathbf{u} + \beta|\mathbf{u}|^m\big)\, d\mathbf{x} = -\int_{\Omega_{\mathrm{f}}} \mathbf{u}^\top \nabla p^0\, d\mathbf{x} \leq \|\nabla p^0\|_{\min(2,m')}\|\mathbf{u}\|_{\max(2,m)}, \tag{20}$$

where the lower bounds in (8) and Hölder's inequality were used for the estimates from below and above. With the help of continuous embeddings in (5), from (20) the two cases follow:

$$\underline{\alpha}\|\mathbf{u}\|_2 + \frac{\underline{\beta}}{K_{(2,m)}}\|\mathbf{u}\|_m^{m-1} \leq \|\nabla p^0\|_{\min(2,m')} \text{ for } m < 2, \quad \frac{\underline{\alpha}}{K_{(m,2)}}\|\mathbf{u}\|_2 + \underline{\beta}\|\mathbf{u}\|_m^{m-1} \leq \|\nabla p^0\|_{\min(2,m')} \text{ for } m \geq 2,$$

which combined together are the former inequality in (19). Inserting (10) with $q = p - p^0$ into (17) estimates the norms as

$$K_{\mathrm{LBB}}\|p\|_{W^{1,\min(2,m')}(\Omega_{\mathrm{f}})} \leq \sup_{0 \neq \mathbf{u} \in L^{\max(2,m)}(\Omega_{\mathrm{f}};\mathbb{R}^d)} \frac{1}{\|\mathbf{u}\|_{\max(2,m)}} \int_{\Omega_{\mathrm{f}}} \mathbf{u}^\top \nabla p^0\, d\mathbf{x} = \|\nabla p^0\|_{\min(2,m')}$$

implying the latter inequality in (19). The proof is complete. $\square$

The following corollary of Theorem 1 deals with the case of $\underline{\beta} = 0$ in the uniform estimate of coefficients (8), which needs $m - 2 \geq 0$ in order to guarantee the fulfillment of assumption (14).

Corollary 1 (case $\underline{\beta} = 0$)**.** *If $m \geq 2$, then the lower bound $\underline{\beta} = 0$ is admissible in* (8) *for the assertion of Theorem 1.*

We finish with few concluding remarks on further modeling perspectives.

4. Concluding remarks

The importance of Corollary 1 concerns the fact that, the inhomogeneous Forchheimer coefficient $0 \leq \beta(\mathbf{x}) \leq \overline{\beta}$ is capable to describe a mixed laminar-turbulent model as follows. The laminar flow is described by Darcy's law in those $\mathbf{x} \in \Omega_{\mathrm{f}}$ where $\beta(\mathbf{x}) = 0$ (e.g., in the reservoir far from the fracture). Whereas the turbulent flow is presented by Darcy–Forchheimer's law in the complement $\mathbf{x} \in \Omega_{\mathrm{f}}$ where $\beta(\mathbf{x}) > 0$ (e.g., near the fracture).

The continuation of the model to a non-stationary one is given by replacing the continuity equation (3) with

$$\frac{\partial p}{\partial t} + \gamma \mathrm{div}(\mathbf{u}) = 0, \tag{21}$$

where γ is the compressibility coefficient (in general, inhomogeneous). This implies the parabolic equations (4) and (21) under mixed boundary conditions(7) and an initial condition for p, which constitute the evolutionary Darcy–Forchheimer's problem. The model is reasonable for the task of hydraulic fracturing.

Acknowledgments

The work is supported by the European Union's Horizon 2020 Research and Innovation Programme of the European Research Council (ERC) advanced grant No. 668998 (OCLOC), partially supported by the Austrian Science Fund (FWF) project P26147-N26: PION and the Russian Foundation for Basic Research (RFBR) project 18-29-10007.

The author thanks P.V. Kovtunenko for discussion of hydro-fracking.

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