PROBLEM OF CRACK UNDER QUASIBRITTLE FRACTURE

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Overview:

Introduction

For the evolutionary problem describing crack propagation in a solid with allowance for the irreversible work due to cohesion contact between the crack opposite surfaces, general variational principles are proposed and investigated. The resulting crack model implies a quasibrittle fracture phenomenon.

From an optimization point of view, the model is described by the minimization of a nonconvex and nondifferentiable functional of the total potential energy subject to

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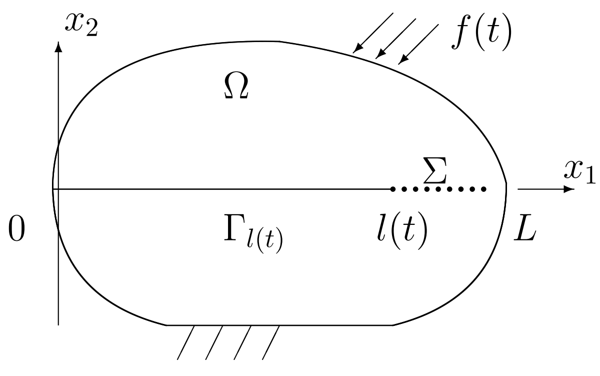
contact conditions. In this context, necessary and sufficient optimality conditions for the minimization problem do not coincide. The necessary optimality condition is represented as a hemivariational inequality. The sufficient optimality condition is expressed as a saddle point problem, which treats the displacement, contact and cohesion forces at the crack as independent state variables.

For the optimal crack, data on the H^2-smoothness of the displacement field in the solid and, hence, on the finiteness of the stress at the crack tip, are obtained. The solvability of the variational problem (i.e., the existence of an optimal crack) is proved for a curvilinear crack propagation path specified a-priori. For the particular case of a straight path, a generalized criterion of crack growth is proposed.

1. Motivation

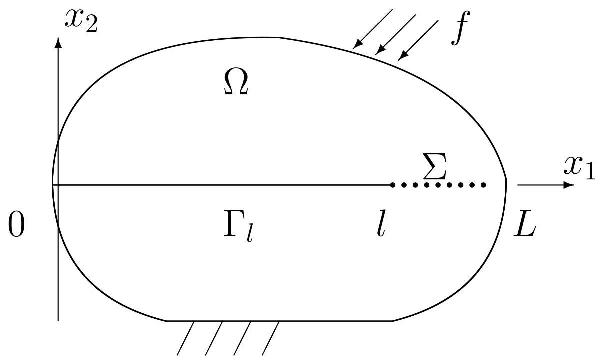
1.1. Evolutionary problem of crack propagation:

For solid occupying domain $\Omega \subseteq \mathbb{R}^2$ with fixed interface Σ, for load $f(t)$ with respect to time parameter $t \geq 0$ find a crack $\Gamma_{l(t)}$ of unknown length $l(t)$ propagating along Σ.

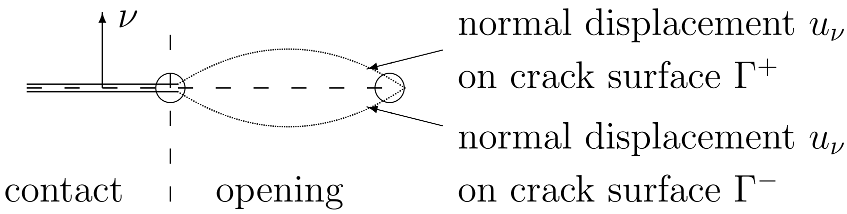


1.2. Stationary problem of crack equilibrium:

For solid occupying domain $\Omega \subseteq \mathbb{R}^2$ with fixed crack Γ_l of length l along Σ, for load f find a displacement u^l in $\Omega \setminus \Gamma_l$.



1.3. Interaction (contact+cohesion) between crack surfaces



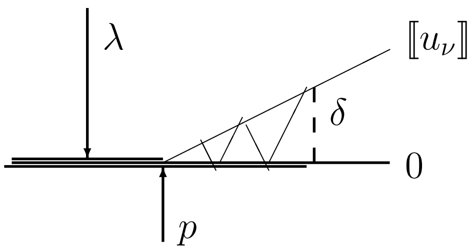
Possible contact between crack surfaces results in the constraint:

$$[\![u_\nu]\!] := (u_\nu)|_{\Gamma^+} - (u_\nu)|_{\Gamma^-} \geq 0.$$

Cohesion is expressed by forces: the interaction force

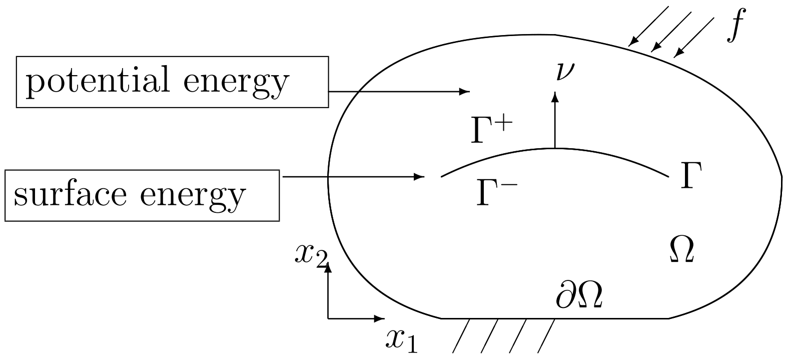
$$q = \lambda + p$$

is the sum of a contact force $\lambda \leq 0$ for $[\![u_\nu]\!] = 0$ and a cohesion force $p \geq 0$ for $[\![u_\nu]\!] < \delta$.



2. Stationary cohesion problem for crack

Consider linear elastic solid in bounded domain $\Omega \subset \mathbb{R}^2$ under load f in Ω and clamped at $\partial\Omega$, which contains crack Γ with normal ν and the crack surfaces $\Gamma^{\pm}$.



For displacement vector u and standard linear elasticity tensors $(i, j = 1, 2)$

$$\sigma_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u), \quad \varepsilon_{ij}(u) = 0,5(u_{i,j} + u_{j,i}),$$

energy (cost) functional of the total potential energy

$$T(u, \Gamma) = \Pi(u, \Gamma) + S([\![u_\nu]\!], \Gamma)$$

is the sum of the potential energy

$$\Pi(u, \Gamma) = \frac{1}{2} \int\limits_{\Omega\setminus\Gamma} \sigma_{ij}(u)\varepsilon_{ij}(u)\, dx - \int\limits_{\Omega\setminus\Gamma} f_i u_i\, dx,$$

and the surface energy

$$S([\![u_\nu]\!], \Gamma) = \int\limits_{\Gamma} g([\![u_\nu]\!])\, ds.$$

Function of distribution of the surface energy is given by

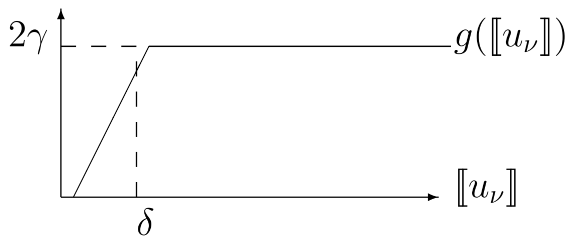
$$g([\![u_\nu]\!]) = \frac{2\gamma}{\delta} \min(\delta, [\![u_\nu]\!]) \quad \text{on}\, \Gamma$$

due to Barenblatt, Dugdale, Leonov–Panasyuk hypothesis, in contrast to the brittle fracture with Griffith hypothesis

$$g([\![u_\nu]\!]) \equiv 2\gamma$$

(γ and δ are material parameters).

Function g is non-differentiable and non-convex.



Nice properties of the function $0 \leq \xi \mapsto g$:

- uniformly bounded $\quad g(\xi) \leq 2\gamma$,
- nonnegative $\quad g(\xi) \geq 0$,
- Lipschitz continuous $\big|g(\zeta) - g(\xi)\big| \leq \frac{2\gamma}{\delta}|\zeta - \xi|$,
- concavity property $\quad g(\zeta) - g(\xi) \leq \frac{2\gamma}{\delta}\mathcal{H}(\delta - \xi)\,(\zeta - \xi)$,

where the Heaviside function

$$\mathcal{H}(\zeta) := \begin{cases} 1 & \text{for } \zeta \geq 0, \\ 0 & \text{for } \zeta < 0. \end{cases}$$

2.1. Variational formulation of cohesion problem

Taking into account possible contact between crack surfaces, let us introduce the set of admissible displacements

$$K(\Omega \setminus \Gamma) = \{v \in H(\Omega \setminus \Gamma), \quad [\![v_\nu]\!] \geq 0 \quad \text{on}\, \Gamma\},$$
$$H(\Omega \setminus \Gamma) = \{v \in H^1(\Omega \setminus \Gamma)^2, \quad v = 0 \quad \text{on } \partial\Omega\}.$$

For given $f \in L^2(\Omega)^2$, consider variational problem: Find $u \in K(\Omega \setminus \Gamma)$ such that

$$(VP) \qquad T(u,\Gamma) \leq T(v,\Gamma) \quad \text{for all } v \in K(\Omega \setminus \Gamma)$$

with nondifferentiable and nonconvex cost functional $T = \Pi + S$. Based on the following properties:

- set $K(\Omega \setminus \Gamma)$ is convex cone,
- $v \mapsto \Pi$ is quadratic and strictly convex,
- $v \mapsto S$ is nonnegative and Lipschitz continuous,

the following theorem holds true.

Theorem. *There exist a solution to problem* (VP).

However,
- solution may be not unique,
- necessary and sufficient optimality conditions do not coincide.

Hierarchy of the problems:

regularized (ε) variational problem

$\downarrow$

regularized (ε) saddle-point problem

$\downarrow \varepsilon \to 0$

saddle- point problem (SPP) $\searrow$

$\downarrow$

variational problem (VP) | primal-dual problem (PDP)

$\downarrow$ $\swarrow$ $\downarrow$

hemivariational inequality (HVI) $\leftrightarrow$ boundary-value formulation

2.2. Necessary optimality condition as a hemivariational inequality

- Due to the concavity of g we get the upper limit

$$\limsup_{\epsilon\to 0} \frac{g([\![u_\nu+\epsilon(v-u)_\nu]\!])-g([\![u_\nu]\!])}{\epsilon} \le \frac{2\gamma}{\delta}\mathcal{H}(\delta-[\![u_\nu]\!])[\![v_\nu-u_\nu]\!],$$

- using the Gâteaux differentiability of $v\mapsto\Pi$,

from (VP) we arrive at the necessary optimality condition as a hemivariational inequality:

(HVI)

$$\int\limits_{\Omega\setminus\Gamma} \big(\sigma_{ij}(u)\varepsilon_{ij}(v-u)-f_i(v-u)_i\big)\,dx$$
$$+\frac{2\gamma}{\delta}\int\limits_{\Gamma}\mathcal{H}\big(\delta-[\![u_\nu]\!]\big)[\![v_\nu-u_\nu]\!]\,ds\ge 0 \quad \text{for all } v\in K(\Omega\setminus\Gamma).$$

2.3. Sufficient optimality condition as a saddle-point problem

The sufficient optimality condition for (VP) is characterized by the following saddle-point problem: Find $(u,\lambda)\in H(\Omega\setminus\Gamma)\times M(\Gamma)$ such that

$$(SPP)\qquad \begin{array}{l}\mathcal{L}(u,\mu,\Gamma)\le\mathcal{L}(u,\lambda,\Gamma)\le\mathcal{L}(v,\lambda,\Gamma)\\ \text{for all } (v,\mu)\in H(\Omega\setminus\Gamma)\times M(\Gamma),\end{array}$$

where Lagrangian

$$\mathcal{L}(v,\mu,\Gamma):=T(v,\Gamma)+\langle\mu,[\![u_\nu]\!]\rangle_\Gamma,$$

and the dual cone

$$M(\Gamma)=\{\mu\in H^{1/2}_{00}(\Gamma_l)^\star:\ \langle\mu,\xi\rangle_\Gamma\le 0 \text{ for all } 0\le\xi\in H^{1/2}_{00}(\Gamma)\}.$$

To prove existence of a solution to (SPP) requires proper regularization of the nondifferentiable function g by a continuously differentiable function g_ε such that

$$0\le g_\varepsilon(\xi)\le g(\xi),\quad 0\le g'_\varepsilon(\xi)\le\frac{2\gamma}{\delta},\quad g_\varepsilon(\xi)=g(\xi)+O(\varepsilon).$$

2.4. Primal-dual formulation

• Due to the concavity property of g we get the upper limit

$$\limsup_{\epsilon\to 0} \frac{g(\llbracket u_\nu + \epsilon(v-u)_\nu\rrbracket) - g(\llbracket u_\nu\rrbracket)}{\epsilon} \le \frac{2\gamma}{\delta}\mathcal{H}(\delta - \llbracket u_\nu\rrbracket)\llbracket v_\nu - u_\nu\rrbracket,$$

• using the Gâteaux differentiability of $v \mapsto \Pi$,

from (SPP) we arrive at the primal-dual formulation of the problem:

(PDP)

$$\int\limits_{\Omega\setminus\Gamma} \big(\sigma_{ij}(u)\varepsilon_{ij}(v) - f_i v_i\big)\, dx + \Big\langle \lambda + \frac{2\gamma}{\delta}\mathcal{H}\big(\delta - \llbracket u_\nu\rrbracket\big), \llbracket v_\nu\rrbracket \Big\rangle_\Gamma = 0$$

$$\langle \mu - \lambda, \llbracket u_\nu\rrbracket\rangle_\Gamma \le 0 \quad \text{for all } (v,\mu) \in H(\Omega\setminus\Gamma)\times M(\Gamma).$$

Formulation (PDP) implies existence of the second Lagrange multiplier, associated to the cohesion force at Γ:

$$p := \frac{2\gamma}{\delta}\mathcal{H}\big(\delta - \llbracket u_\nu\rrbracket\big) \in L^2(\Gamma) \ge 0.$$

2.5. Boundary-value formulation

From (PDP) we find the normal stress at Γ:

$$\sigma_\nu(u) = \lambda + \frac{2\gamma}{\delta}\mathcal{H}\big(\delta - \llbracket u_\nu\rrbracket\big) = \lambda + p = q$$

and derive formally boundary-value formulation of the problem:

$$-\sigma_{ij,j}(u) = f_i \quad (i = 1,2) \quad \text{in } \Omega;$$

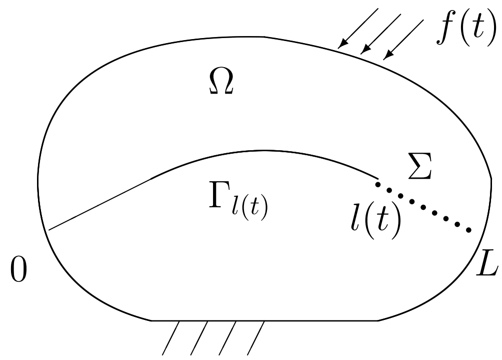
$$u = 0 \quad \text{on } \partial\Omega;$$

and on the crack surfaces $\Gamma^{\pm}$:

$$\sigma_\tau(u) = 0, \quad \sigma_\nu(u) \le \frac{2\gamma}{\delta} \quad \text{if } [\![u_\nu]\!] = 0,$$
$$\sigma_\nu(u) = \frac{2\gamma}{\delta} \quad \text{if } 0 < [\![u_\nu]\!] < \delta,$$
$$\sigma_\nu(u) = 0 \quad \text{if } [\![u_\nu]\!] > \delta.$$

3. Evolution of crack via optimization problem

Under load $f(t)$ in domain $\Omega \subset \mathbb{R}^2$, for $t \ge 0$ find crack $\Gamma_{l(t)}$ of length $l(t) \in [0, L]$ propagating along fixed path Σ of length L (one end-point of crack is fixed on the boundary).



Evolution of a crack along the fixed path is described by the optimization problem: For every $t \ge 0$ find $l(t) \in [0, L]$ such that

$$(OP) \qquad \begin{array}{l} T\big(f(t), u^{l(t)}, \Gamma_{l(t)}\big) \le T(f(t), u^l, \Gamma_l) \\ \text{for all } l \in [0, L], \end{array}$$

where $u^l \in K(\Omega \backslash \Gamma_l)$ is a solution of the variational problem:

$$(VP) \qquad \begin{array}{l} T\big(f(t), u^l, \Gamma_l\big) \le T(f(t), v, \Gamma_l) \\ \text{for all } v \in K(\Omega \setminus \Gamma_l). \end{array}$$

3.1. Properties of the energy function with respect to parameter of crack length

For fixed t, obtained from (VP) energy function $l \mapsto T(u^l, \Gamma_l) : [0, L] \mapsto \mathbb{R}$ is:

- monotone decreasing,
- uniformly bounded,
- continuous in $[0, L)$,
- lower semicontinuous in $[0, L]$,
- for the rectilinear crack path Σ, continuously differentiable in $(0, L)$, where derivative is defined by the shape derivative

$$\frac{\partial T}{\partial l}(u^l, \Gamma_l) := \lim_{\epsilon \to 0} \frac{T(u^{l+\epsilon}, \Gamma_{l+\epsilon}) - T(u^l, \Gamma_l)}{\epsilon}.$$

The derivative can be expressed with a velocity V tangential to Σ and $V|_{\partial\Omega} = 0$:

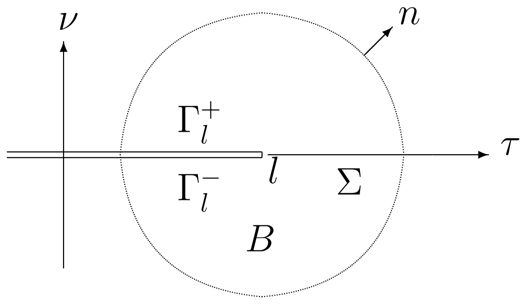
$$\frac{\partial T}{\partial l}(u^l, \Gamma_l) = \int\limits_{\Gamma_l} \mathrm{div}\,(V) \frac{2\gamma}{\delta} \min(\delta, [u^l_\nu])\, ds - \int\limits_{\Omega\setminus\Gamma_l} \mathrm{div}\,(V f_i) u^l_i\, dx$$

$$+ \frac{1}{2} \int\limits_{\Omega\setminus\Gamma_l} \Big(\mathrm{div}\,(V c_{ijkl}) \varepsilon_{kl}(u^l) \varepsilon_{ij}(u^l) - \sigma_{ij}(u^l) \big(u^l_{i,k} V_{k,j} + u^l_{j,k} V_{k,i} \big) \Big)\, dx$$

or, equivalently, with contour integral over ∂B around the crack tip:

$$\frac{\partial T}{\partial l}(u^l, \Gamma_l) = - \int\limits_{\partial B} \sigma_{ij}(u^l) \big(\frac{1}{2} \varepsilon_{ij}(u^l)(n_k V_k) - D_\tau(u^l_i) n_j \big)\, ds$$

$$+ \frac{2\gamma}{\delta} \min(\delta, [\![u^l_\nu]\!])|_{\partial B \cap \Gamma_l}.$$



3.2. Existence of a non-singular solution

Using boundedness and lower semicontinuity of $l \mapsto T(u^l, \Gamma_l)$ we have

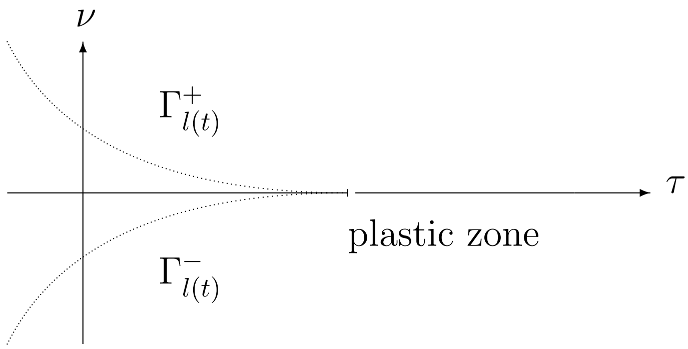
Theorem. *For every $t \geq 0$, there exist a solution to problem (OP).*

Existence of the minimum and monotone decreasing of $l \mapsto T(u^l, \Gamma_l)$ result in the following principle identity:

$$T\big(f(t), u^{l(t)}, \Gamma_{l(t)}\big) = T(f(t), u^l, \Gamma_l) \quad \text{for all } l \in [l(t), L].$$

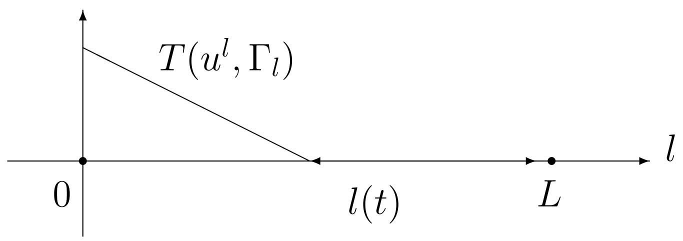
We derive the following consequences:

• **Theorem**. *The solution $u^{l(t)}$ of (VP) with $l(t)$ being the solution of (OP) obeys $u^{l(t)} \in H^2(\Omega \setminus \Gamma_{l(t)})^2$, i.e., has no singularity at the crack tip.*



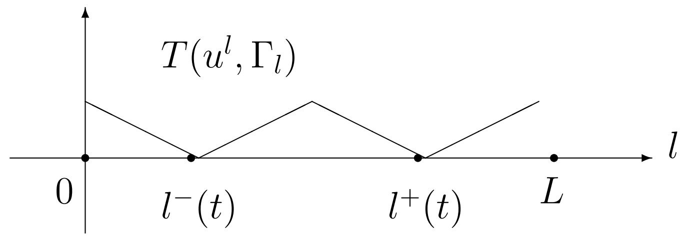
• The crack can become smaller during the quasibrittle evolution.

• Solutions $l(t)$ of (OP) are non-unique. A sketch for fixed t:



This may require an additional criterion to choose between solutions $l(t)$.

In contrast, under brittle fracture propagating crack can NOT become smaller due to the Griffith hypothesis. This hypothesis provides reasonable choice of the solution with possible jumps of $t \mapsto l(t)$, as sketch for fixed t:



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