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Contacting crack faces within the context of bodies exhibiting limiting strains

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Abstract

A nonlinear crack problem subject to a non-penetration inequality is considered within the framework of the limiting small strain approach, which does not suffer from the inconsistency of infinite strain at the crack tip. Based on the concept of a generalized solution, sufficient conditions proving the well-posedness of the problem are established and analyzed.

Keywords nonlinear elasticity, limiting small strain, nonlinear crack with non-penetration

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1. Introduction

Within the general theory of elastic bodies wherein the Cauchy–Green tensor can be a nonlinear function of the Cauchy stress (such as titanium alloys, which can respond nonlinearly even when the strains are "small"), a new subclass of models where the norm of the strain is limited apriori to a small value was previously introduced in [1, 2]. By applying the limiting small strain constitutive relations to fracture mechanics, the strains in the cracked body have been shown to be uniformly bounded, overcoming the drawback of strain blow-up at the crack tip inherent in the classical linearized theory of the fracture of brittle materials [3, 4].

In order to avoid the other drawback of the linearized theory, namely, the possibility of penetration between the crack faces, nonlinear crack problems subject to non-penetration conditions have been previously established within the framework of the variational theory [5–7]. The principal challenge here is finding singular solutions at the crack tip [8,9] and obtaining a formula for the energy release rate [10–12], which is relevant to brittle as well as quasi-brittle materials fracturing [13]. Numerical methods suitable for this class of nonlinear crack problems were developed in [14].

The principal difficulty in analyzing the limiting small strain model concerns the fact that the stresses live only in the non-reflexive L^1-space, which must be properly regularized. Various approaches to overcoming this mathematical difficulty have therefore been previously suggested [15, 16].

In a previous study [17], the nonlinear elasticity problem with limiting small strain was solved for cracks with contacting faces using penalization and elliptic regularization techniques. In the present paper, we extend this approach further in terms of a generalization of nonlinear functions, which are admissible for describing constitutive relations that exhibit limiting small strain behavior. Sufficient conditions are given in a general form and supported by concrete examples.

2. Problem formulation

In spatial dimensions $d = 2$ or 3, we consider a domain Ω in $\mathbb{R}^d$ with boundary $\partial\Omega$ and an outward-pointing normal $n = (n_1, \dots, n_d)$. Let $\partial\Omega$ be a $(d-1)$-dimensional Lipschitz manifold comprising two mutually disjoint parts Γ_N and Γ_D, such that Γ_D is non-empty.

We define the crack $\Gamma_c \subset \Omega$ as an oriented manifold such that its infinite extension splits Ω into two domains with Lipschitz boundaries. The positive face Γ_c^+ and the negative face Γ_c^- of the crack are distinguished by choosing an inward-pointing normal $n = (n_1, \dots, n_d)$ to Γ_c. We denote the jump across the crack by $[\![\cdot]\!] := \cdot|_{\Gamma_c^+} - \cdot|_{\Gamma_c^-}$, and the cracked domain by $\Omega_c := \Omega \setminus \Gamma_c$.

For a given body force $f(x) = (f_1, \dots, f_d)$, a boundary traction $g(x) = (g_1, \dots, g_d)$, and a symmetric d-by-d tensor $\mathcal{F} = \{\mathcal{F}_{ij}\}_{i,j=1}^d$ describing the constitutive response, we look for a displacement vector $u(x) = (u_1, \dots, u_d)$, symmetric d-by-d strain $\varepsilon(x) = \{\varepsilon_{ij}\}_{i,j=1}^d$ and stress $\sigma(x) = \{\sigma_{ij}\}_{i,j=1}^d$ tensors, which satisfy the following system for $i, j = 1, \dots, d$:

$$-\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\sigma_{ij} = f_i \quad \text{in } \Omega_c, \tag{1}$$

$$\varepsilon_{ij} = \mathcal{F}_{ij}(\sigma) \quad \text{in } \Omega_c, \tag{2}$$

$$\varepsilon_{ij} = \varepsilon_{ij}(u) := \frac{1}{2}\left(\frac{\partial}{\partial x_j}u_i + \frac{\partial}{\partial x_i}u_j\right) \quad \text{in } \Omega_c, \tag{3}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{4}$$

$$\sigma_n = g \quad \text{on } \Gamma_N, \tag{5}$$

$$[\![\sigma_n]\!] = 0 \quad \text{on } \Gamma_c, \tag{6}$$

$$\sigma_n - (\sigma_n \cdot n)n = 0 \quad \text{on } \Gamma_c^{\pm}, \tag{7}$$

$$[\![u_n]\!] \geq 0, \quad \sigma_n \cdot n \leq 0, \quad (\sigma_n \cdot n)[\![u_n]\!] = 0 \quad \text{on } \Gamma_c^{\pm}, \tag{8}$$

where the normal displacement $u_n := u \cdot n$ is given by the inner product $u \cdot n = \sum_{i=1}^d u_i n_i$, and the boundary traction $\sigma_n := \{\sum_{j=1}^d \sigma_{ij} n_j\}_{i=1}^d$.

In this system, (1) is the equilibrium equation, (2) is the implicit constitutive equation, and (3) represents the linearized strain. These equations are endowed with the homogeneous Dirichlet condition (4), the Neumann-type condition (5), and conditions (6)–(8) at the crack. The boundary traction σ_n is continuous across the crack according to (6) and has zero tangential component by (7). The non-penetration inequality $[\![u_n]\!] \geq 0$ enforces the complementarity conditions (8) involving the normal stress $\sigma_n \cdot n = \sum_{i,j=1}^d \sigma_{ij} n_j n_i$, see [5] for further details.

We consider a nonlinear tensor-valued function for (2)

$$\mathcal{F}: \ \mathrm{Sym}(\mathbb{R}^{d\times d}) \mapsto \mathrm{Sym}(\mathbb{R}^{d\times d}), \quad \mathcal{F}(0) = 0, \tag{9}$$

over symmetric d-by-d tensors $\mathrm{Sym}(\mathbb{R}^{d\times d})$, whose value implies the Cauchy–Green strain tensor corresponding to the Cauchy stress tensor given as the argument. The inner product on $\mathrm{Sym}(\mathbb{R}^{d\times d})$ assigns $\sigma : \varepsilon = \sum_{i,j=1}^d \sigma_{ij}\varepsilon_{ij}$ and is associated with the matrix norm $\|\sigma\| = \sqrt{\sigma : \sigma}$. Our aim is to specify a class of functions $\mathcal{F}$ in (9), which exhibit limiting small strain behavior and guarantee solvability of the problem (1)–(8).

In this study, we use the following properties of $\mathcal{F}$ to prove the existence theorem. Let constants $M_1, M_2, M_4 > 0$, and $M_3 \geq 0$ exist such that

$$\|\mathcal{F}(\sigma)\| \leq M_1, \tag{10}$$

$$0 \leq \big(\mathcal{F}(\sigma^1) - \mathcal{F}(\sigma^2)\big) : (\sigma^1 - \sigma^2) \leq M_2 \|\sigma^1 - \sigma^2\|^2, \tag{11}$$

$$-M_3 + M_4 \sum_{i,j=1}^d |\sigma_{ij}| \leq \mathcal{F}(\sigma) : \sigma \tag{12}$$

for all $\sigma, \sigma^1, \sigma^2 \in \mathrm{Sym}(\mathbb{R}^{d\times d})$. The uniform bound in (10) together with (2) leads to $\|\varepsilon\| \leq M_1$ implying limiting small strain for small M_1. The bounds in (11) describe the monotone and Lipschitz continuous properties of $\mathcal{F}$. After integration over Ω_c, property (12) provides σ with the following lower bound in the L^1-norm:

$$M_4 \int_{\Omega_c} \sum_{i,j=1}^d |\sigma_{ij}|\, dx \leq M_3 |\Omega| + \int_{\Omega_c} \mathcal{F}(\sigma) : \sigma \, dx \tag{13}$$

when $\int_{\Omega_c} \mathcal{F}(\sigma) : \sigma \, dx$ is uniformly bounded.

Many of constitutive models discussed in [1] lead to the response function $\mathcal{F}$ from (9) described by

$$\begin{aligned} &\mathcal{F}(\sigma) = \Psi_1\big(\mathrm{tr}(\sigma), \|\sigma\|\big) I + \Psi_2\big(\mathrm{tr}(\sigma), \|\sigma\|\big)\sigma, \\ &\Psi_1(0, 0) = 0, \end{aligned} \tag{14}$$

where the trace $\mathrm{tr}(\sigma) := \sum_{i=1}^d \sigma_{ii}$, and I stands for the unit d-by-d tensor.

To endow $\mathcal{F}$ given in (14) with the mathematical structure of a monotone function (the lower bound in (11)), depending on the two variables $\mathrm{tr}(\sigma)$ and $\|\sigma\|$, we first consider the following decomposition (see [2]):

$$\mathcal{F}(\sigma) = \Psi_1(\mathrm{tr}(\sigma)) I + \Psi_2(\|\sigma\|)\sigma, \tag{15}$$

given the functions

$$\Psi_1 : \mathbb{R} \mapsto \mathbb{R}, \quad \Psi_1(0) = 0, \quad \Psi_2 : \mathbb{R}_+ \mapsto \mathbb{R}. \tag{16}$$

Theorem 1 *Let Ψ_1 and Ψ_2 in* (16) *be continuous almost everywhere differentiable functions, and let the constants $a_1, a_2, b_1, b_2, b_4 > 0$ and $a_3, b_3 \geq 0$ exist such that*

$$|\Psi_1(y)| \leq a_1, \tag{17}$$

$$0 \leq \Psi_1'(y) \leq a_2, \tag{18}$$

$$-a_3 \leq y\Psi_1(y) \tag{19}$$

hold a.e. $y \in \mathbb{R}$, and

$$y|\Psi_2(y)| \leq b_1, \tag{20}$$

$$y\big|\Psi_2'(y)\big| + \Psi_2(y) \leq b_2, \tag{21}$$

$$-b_3 + b_4 y \leq y^2 \Psi_2(y) \tag{22}$$

hold a.e. $y \in \mathbb{R}_+$. Moreover, suppose that

$$\textit{either} \quad \Psi_2'(y) \geq 0, \quad \Psi_2(y) \geq 0, \tag{23}$$

$$\textit{or} \quad \Psi_2'(y) < 0, \quad \big(y\Psi_2(y)\big)' \geq 0, \tag{23'}$$

hold in subintervals of $\mathbb{R}_+$. Then, for the function $\mathcal{F}$ defined in (15)*, the properties* (10)*–*(12) *are true with*

$$\begin{aligned} &M_1 = a_1\sqrt{d} + b_1, \quad M_2 = a_2 d + b_2, \\ &M_3 = a_3 + b_3, \quad M_4 = \frac{b_4}{d}. \end{aligned} \tag{24}$$

Proof For $\mathcal{F}$ defined by (15), the upper bound (10) follows from (17) and (20), and the lower bound (12) from (19) and (22), due to $\sum_{i,j=1}^d |\sigma_{ij}| \leq d\|\sigma\|$.

To prove (11), for $\sigma^1, \sigma^2 \in \mathrm{Sym}(\mathbb{R}^{d\times d})$, we use the representation

$$\begin{aligned} &\big(\mathcal{F}(\sigma^1) - \mathcal{F}(\sigma^2)\big) : (\sigma^1 - \sigma^2) \\ &= \int_0^1 \frac{d}{dt}\mathcal{F}\big(t\sigma^1 + (1-t)\sigma^2\big) : (\sigma^1 - \sigma^2)\, dt \\ &= \int_0^1 \big\{ \Psi_1'\big(\mathrm{tr}(t\sigma^1 + (1-t)\sigma^2)\big)\mathrm{tr}^2(\sigma^1 - \sigma^2) \\ &\quad + \Psi_2'\big(\|t\sigma^1 + (1-t)\sigma^2\|\big) \\ &\qquad \cdot \frac{\big((t\sigma^1 + (1-t)\sigma^2) : (\sigma^1 - \sigma^2)\big)^2}{\|t\sigma^1 + (1-t)\sigma^2\|} \\ &\quad + \Psi_2\big(\|t\sigma^1 + (1-t)\sigma^2\|\big)\|\sigma^1 - \sigma^2\|^2 \big\}\, dt. \end{aligned}$$

For each of the cases $\Psi_2' \geq 0$ in (23) and $\Psi_2' < 0$ in (23'), applying the Cauchy–Schwarz inequality to the inner product and using (18), (21), and $|\mathrm{tr}(\sigma)| \leq \sqrt{d}\|\sigma\|$, the estimates (11) follow with the bounds given in (24). The proof is complete.

(QED)

For example, we can take the following function as Ψ_1 (see [2])

$$\Psi_1(y) = \alpha\Big\{1 - \exp\Big(\frac{-\lambda y}{(1+\gamma|y|^r)^{1/r}}\Big)\Big\} \tag{25}$$

where $\alpha, \lambda, \gamma, r > 0$. As a matter of fact, (17)–(19) are satisfied by

$$a_1 = \alpha\left(\exp\left(\frac{\lambda}{\gamma^{1/r}}\right) - 1\right), \quad a_2 = \alpha\lambda\exp\left(\frac{\lambda}{\gamma^{1/r}}\right),$$

$$a_3 = 0.$$

On the other hand, a possible function for Ψ_2 is [16,17]

$$\Psi_2(y) = \beta\frac{1}{(1+\kappa y^s)^{1/s}}, \quad \beta, \kappa, s > 0. \tag{26}$$

The conditions (20)–(22) and (23') hold with

$$b_1 = \frac{\beta}{\kappa^{1/s}}, \quad b_2 = 2\beta, \quad b_3 = \frac{b_4}{\kappa^{1/s}}, \quad b_4 = \frac{\beta}{c_s\kappa^{1/s}},$$

where $c_s = 2^{1/s-1}$ for $s \in (0,1)$ and $c_s = 1$ for $s \geq 1$.

Now, following [16], we consider the decomposition

$$\widetilde{\mathcal{F}}(\sigma) = \widetilde{\Psi}_1(\mathrm{tr}(\sigma))I + \Psi_2(\|\sigma^*\|)\sigma^* \tag{27}$$

employing the deviatoric stress tensor $\sigma^* := \sigma - \frac{\mathrm{tr}(\sigma)}{d}I \in \mathrm{Sym}(\mathbb{R}^{d\times d})$ with $\mathrm{tr}(\sigma^*) = 0$.

Theorem 2 *Let $\widetilde{\Psi}_1$ and Ψ_2 satisfy the assumptions of Theorem 1, and let the constant $a_4 > 0$ exist such that*

$$-a_3 + a_4|y| \leq y\widetilde{\Psi}_1(y) \tag{19'}$$

holds a.e. $y \in \mathbb{R}$. Then for $\widetilde{\mathcal{F}}$ defined as (27), the properties (10)–(12) hold with M_1, M_2, M_3 given by (24) and

$$M_4 = \min\{a_4, \frac{b_4}{d}\}. \tag{24'}$$

Proof The proof follows in a manner similar to that of Theorem 1 with the aid of the estimates

$$\|\sigma^*\| \leq \|\sigma\|, \quad \sum_{i,j=1}^{d}|\sigma_{ij}| \leq |\mathrm{tr}(\sigma)| + d\|\sigma^*\|,$$

and the equality

$$\begin{aligned}&\big(\widetilde{\mathcal{F}}(\sigma^1) - \widetilde{\mathcal{F}}(\sigma^2)\big) : (\sigma^1 - \sigma^2)\\ &= \int_0^1 \big\{\widetilde{\Psi}_1'\big(\mathrm{tr}(t\sigma^1 + (1-t)\sigma^2)\big)\mathrm{tr}^2(\sigma^1 - \sigma^2)\\ &+ \Psi_2'\big(\|(t\sigma^1 + (1-t)\sigma^2)^*\|\big)\frac{\big((t\sigma^1+(1-t)\sigma^2)^* : (\sigma^1-\sigma^2)^*\big)^2}{\|(t\sigma^1+(1-t)\sigma^2)^*\|}\\ &+ \Psi_2\big(\|(t\sigma^1 + (1-t)\sigma^2)^*\|\big)\|(\sigma^1 - \sigma^2)^*\|^2\big\}\,dt\end{aligned}$$

which holds by the virtue of the fact that $(\sigma^1)^* : \sigma^2 = (\sigma^1)^* : (\sigma^2)^*$.

(QED)

If we modify the function Ψ_1 from (25) such that

$$\widetilde{\Psi}_1(y) = \alpha\frac{y}{(1+\gamma|y|^r)^{1/r}}, \quad \alpha, \gamma, r > 0, \tag{28}$$

it satisfies the conditions (17), (18), and (19') with

$$a_1 = \tfrac{\alpha}{\gamma^{1/r}}, \quad a_2 = \alpha, \quad a_3 = \tfrac{a_4}{\gamma^{1/r}}, \quad a_4 = \tfrac{\alpha}{c_r\gamma^{1/r}},$$

where $c_r = 2^{1/r-1}$ for $r \in (0,1)$ and $c_r = 1$ for $r \geq 1$.

In this particular case, on taking the limit $\kappa \searrow 0^+$ in (26) and $\gamma \searrow 0^+$ in (28), from (27) we arrive at the limit function

$$\widetilde{\mathcal{F}}(\sigma) = \alpha\mathrm{tr}(\sigma)I + \beta\sigma^* = \left(\alpha - \frac{\beta}{d}\right)\mathrm{tr}(\sigma)I + \beta\sigma,$$

which coincides with the constitutive equation for linearized elasticity when $\alpha = (1-2\nu)/(3E)$ and $\beta = (1+\nu)/E$ for dimension $d = 3$.

3. Well-posedness theorem

Let $f \in L^2(\Omega_c;\mathbb{R}^d)$ and $g \in L^2(\Gamma_N;\mathbb{R}^d)$. According to the Dirichlet boundary condition (4) and the non-penetration conditions (8), we can introduce the cone of admissible displacements in the domain Ω_c as follows:

$$\begin{aligned}\mathcal{K} := \big\{&u \in H^1(\Omega_c;\mathbb{R}^d) :\\ &u = 0 \quad \text{on } \Gamma_D, [\![u_n]\!] \geq 0 \quad \text{on } \Gamma_c\big\}.\end{aligned} \tag{29}$$

By multiplying the equilibrium equation (1) by $\overline{u} - u$ (for a test function $\overline{u}$) and integrating the result by parts over Ω_c and by the virtue of the boundary conditions (5)–(8), we get the following weak formulation of the problem: For $p, p' \in (1,\infty)$ such that $1/p + 1/p' = 1$, find the displacement $u \in \mathcal{K}$, the strain $\varepsilon(u) \in L^\infty(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$, and the stress $\sigma \in L^p(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$ that satisfy the variational inequality

$$\begin{aligned}&\int_{\Omega_c}\sigma : \varepsilon(\overline{u} - u)\,dx\\ &\quad\geq \int_{\Omega_c} f\cdot(\overline{u} - u)\,dx + \int_{\Gamma_N} g\cdot(\overline{u} - u)\,dS_x\end{aligned} \tag{30}$$

for all $\overline{u} \in \mathcal{K}$ such that $\varepsilon(\overline{u}) \in L^{p'}(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$, and fulfill the constitutive equation (2), namely

$$\varepsilon(u) = \mathcal{F}(\sigma) \quad \text{in } \Omega_c. \tag{31}$$

The tensor function $\mathcal{F}$ is defined by (9), and the linearized strains $\varepsilon(u)$, $\varepsilon(\overline{u})$ are determined by the formula (3).

The solvability of (30) and (31) cannot be argued from standard existence theorems because the apriori estimate of the stress following from (13) is provided in non-reflexive L^1-space. Therefore, we now introduce a concept of a generalized solution.

By multiplying (31) by $\overline{\sigma} - \sigma$ (for a test function $\overline{\sigma}$) and after integrating the result over Ω_c, we have

$$\int_{\Omega_c}\mathcal{F}(\sigma) : (\overline{\sigma} - \sigma)\,dx = \int_{\Omega_c}(\overline{\sigma} - \sigma) : \varepsilon(u)\,dx. \tag{32}$$

Due to the cone property of $\mathcal{K}$, the variational inequality (30) is equivalent to the following two inequalities:

$$\int_{\Omega_c}\sigma : \varepsilon(u)\,dx \leq \int_{\Omega_c} f\cdot u\,dx + \int_{\Gamma_N} g\cdot u\,dS_x, \tag{33}$$

$$\int_{\Omega_c}\sigma : \varepsilon(\overline{u})\,dx \geq \int_{\Omega_c} f\cdot \overline{u}\,dx + \int_{\Gamma_N} g\cdot \overline{u}\,dS_x. \tag{34}$$

Adding (32) to (33) and using (11), we deduce the following variational inequality

$$\int_{\Omega_c} \mathcal{F}(\overline{\sigma}) : (\overline{\sigma} - \sigma)\, dx - \int_{\Omega_c} \overline{\sigma} : \varepsilon(u)\, dx \geq -\int_{\Omega_c} f \cdot u\, dx - \int_{\Gamma_N} g \cdot u\, dS_x, \tag{35}$$

which is advantageous since the terms $\mathcal{F}(\sigma) : \sigma$ in (32) and $\sigma : \varepsilon(u)$ in (33) are problematic to justify within the weak formulation. Therefore, we can utilize the compact embedding $L^1(\Omega_c) \hookrightarrow \mathcal{M}^1(\Omega_c)$ in the space of bounded measures, which is the dual space of the space $C_c(\Omega_c)$ of continuous functions with compact support in Ω_c.

Based on (34) and (35), we introduce the following conept of the generalized solution: Find the displacement $u \in \mathcal{K}$, the strain $\varepsilon(u) \in L^2(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$, and the stress $\sigma \in \mathcal{M}^1(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$ satisfying the inequalities

$$\langle \sigma : \varepsilon(\overline{u}) \rangle_{\Omega_c} \geq \int_{\Omega_c} f \cdot \overline{u}\, dx + \int_{\Gamma_N} g \cdot \overline{u}\, dS_x, \tag{36}$$

$$\langle (\sigma - \overline{\sigma}) : \mathcal{F}(\overline{\sigma}) \rangle_{\Omega_c} + \int_{\Omega_c} \overline{\sigma} : \varepsilon(u)\, dx \leq \int_{\Omega_c} f \cdot u\, dx + \int_{\Gamma_N} g \cdot u\, dS_x \tag{37}$$

for all $\overline{u} \in \mathcal{K}$ such that $\varepsilon(\overline{u}) \in C_c(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$ and $\overline{\sigma} \in C_c(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$. The brackets $\langle \cdot : \cdot \rangle_{\Omega_c}$ in (36) and (37) imply a duality pairing between $\mathcal{M}^1(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$ and $C_c(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$.

By constructing an elliptic regularization of (30) and (31) and penalizing $\mathcal{K}$ as follows [17], for fixed $\delta > 0$, there exist a displacement $u^\delta \in H^1(\Omega_c; \mathbb{R}^d)$ such that $u^\delta = 0$ at Γ_D, a strain $\varepsilon(u^\delta) \in L^2(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$, and a stress $\sigma^\delta \in L^2(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$, which satisfy the equations

$$\int_{\Omega_c} (\delta\varepsilon(u^\delta) + \sigma^\delta) : \varepsilon(\overline{u})\, dx + \frac{1}{\delta}\int_{\Gamma_c} \min\{0, [\![u_n^\delta]\!]\}[\![\overline{u}_n]\!]\, dS_x = \int_{\Omega_c} f \cdot \overline{u}\, dx + \int_{\Gamma_N} g \cdot \overline{u}\, dS_x, \tag{38}$$

$$\delta\sigma^\delta + \mathcal{F}(\sigma^\delta) = \varepsilon(u^\delta) \quad \text{in } \Omega_c, \tag{39}$$

for all $\overline{u} \in H^1(\Omega_c; \mathbb{R}^d)$ such that $\overline{u} = 0$ at Γ_D. As δ tends to zero, we can derive the existence theorem below.

Theorem 3 (i) *Let the tensor function $\mathcal{F}$ in* (9) *satisfy the assumptions* (10)–(12). *The generalized solution $(u, \varepsilon(u), \sigma)$ of the variational problem* (36) *and* (37) *exists as an accumulation point of the solution $(u^\delta, \varepsilon(u^\delta), \sigma^\delta)$ of* (38) *and* (39) *as $\delta \searrow 0^+$.*

(ii) *If the stress component is regular such that $\sigma \in L^p(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$ with $p \in (1, \infty)$, then the triple $(u, \varepsilon(u), \sigma)$ satisfies the weak formulation given by* (30) *and* (31).

The proof of Theorem 3 follows the proof given in [17] based on the properties (10)–(12) for the specific case $\mathcal{F}(\sigma) = \Psi_2(\|\sigma\|)\sigma$.

For further development, we note that the assumption of monotonicity (11) might be relaxed to an assumption of pseudo-monotonicity, following [18].

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