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Evolution of a crack with kink and non-penetration

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Abstract. The nonlinear evolution problem for a crack with a kink in elastic body is considered. This nonlinear formulation accounts the condition of mutual non-penetration between the crack faces. The kinking crack is presented with the help of two unknown shape parameters of the kink angle and of the crack length, which minimize an energy due to the Griffith hypothesis. Based on the obtained results of the shape sensitivity analysis, solvability of the evolutionary minimization problem is proved, and the necessary conditions for the optimal crack are derived.

1. Introduction.

The problem of kink is of special interest, because it represents a change of topology from a smooth crack to the non-smooth one. This specialty is inherently connected with the phenomenon of crack appearance in a homogeneous body. The topology change is the main difficulty for mathematical consideration of cracks with a kink.

The known approaches to kinking cracks in fracture mechanics for linear models deal with local asymptotic representations, see [**2**], [**5**], [**24**], [**25**]. The term local implies that the consideration is restricted to local crack changes close to a point of kink. For overview of asymptotic methods used in singular domains, see [**23**]. In contrast to local methods, we suggest a global approach, which is based on shape optimization, thus managing also global changes during the crack evolution.

As a mathematical tool we employ regular perturbations, see [**13**] for their foundation. In the context of shape optimization, the suitable description of regular perturbations via almost identical coordinate transformations (thus, homeomorphic maps) was developed in [**7**], [**16**]. For calculation of the so-called J-integrals in fracture mechanics, perturbation technique was specified in [**8**], [**26**]. Shape optimization methods for the close problem of crack identification in a solid are presented in [**3**], [**10**].

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The other specialty of our consideration concerns non-penetration conditions, which allow contact between the opposite crack surfaces, but not their mutual interpenetration. This results in constrained (nonlinear) variational problems describing equilibrium of a crack with non-penetration, see [**15**]. We stress on the point that standard results of the shape sensitivity analysis are not applicable to nonlinear problems with cracks.

For crack problems constrained by non-penetration, applying a coordinate transformation of tangential shift along planar cracks, formula of the shape derivative (the energy release rate) for the energy functional follows from the results of [**15**], [**17**], [**18**]. For curvilinear cracks described by parameterized curves, the shape derivative was deduced in [**27**], and in [**19**] for general (smooth) codimensional-one manifolds representing cracks. However, these results are not applicable to describe the kink of a crack, because the tangential shift is not smooth in this case. For this reason, in the present work we employ coordinate transformations of rotation and extension adopted to the bounded domain with kinking crack.

Revisiting brittle fracture as an energy minimization problem, the optimization approach to description of crack evolution was developed in [**6**], [**9**]. The principal difficulty of this approach consists in a suitable measuring of moving geometrical objects (cracks in our case) in the function sense. In [**21**], there is suggested the measuring of cracks by means of kinematic velocities within the level-set context. For the specific case of a pre-defined crack path (thus, the velocity is given) during the delamination process in a composite, one-parametric optimization of the crack length is investigated in [**20**], and in [**11**] with account of non-penetration between the crack surfaces. A quasi-brittle fracture within the optimization approach is studied in [**22**]. For optimal control problems with respect to shape parameters of a crack, see [**4**], [**14**], and [**12**] for time-dependent problems with cracks.

In the present work, we apply the shape optimization approach to a two-parametric problem for the kinking crack. By this, we fix a point of kink and look for unknown shape parameters of the kink angle and the crack length, which minimize an energy (the total potential energy of the solid with crack under non-penetration conditions) due to Griffith. The nonlinear minimization problem describes evolution of the crack with kink with respect to time-like (loading) parameter. To prove its solvability in Section 4, the continuity properties of the energy function are obtained in Section 3.1, and to provide necessary optimality conditions, the shape derivatives are derived in Section 3.2. As a tool, we construct homeomorphic maps for kinking cracks in Section 2.

In comparison with the previous related results in the field of crack problems, in this paper we propose the unified mathematical model for description of the process of crack initiation as well as its evolution which admits kink of a crack with contact between the crack faces. For simplification we relax on the assumption that the crack path draws a straight segment after kinking. Although avoiding curving we find the kinking angle from minimization of the energy over all possible rectilinear extensions of the crack inside the solid.

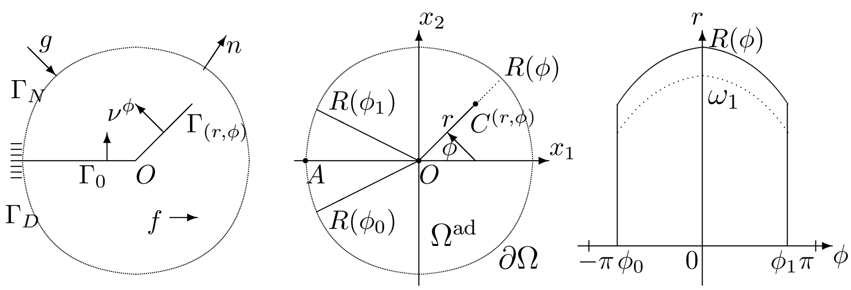
On the one hand, the result of global minimization is not directly comparable with known fracture criteria which are derived from the local asymptotic expansion due to the crack extension. Nevertheless, for the crack extension without kink in the numerical experiments of [**11**], [**20**] it was observed that the global minimization refined the Griffith fracture law in the situations where the latter was not applicable. On the other hand, known criteria of kinking operate with the stress intensity factors, see, e.g. [**1**]. These characteristics are not determined mathematically for general models of solids with cracks accounting structural inhomogeneity, phenomena of contact, etc. For such common situations our optimization approach still remains applicable.

2. Kinematic description of kinking cracks.

Let $\Omega \subset \{x = (x_1, x_2)^\top \in \boldsymbol{R}^2\}$ be a bounded domain, the origin $O = (0,0)^\top$ belong to $\overline{\Omega}$, and $n = (n_1, n_2)^\top$ be the outward unit normal vector at the boundary $\partial\Omega$. Let the initial crack Γ_0 be given as the segment AO at the x_1-axis, where point A is posed at $\partial\Omega$. We define the domain $\Omega^{\text{ad}} \subset \Omega$ of admissible crack evolution by the image of domain ω of two parameters

$$\omega = \big\{(r,\phi): \quad 0 < r < R(\phi) \quad \text{for } \phi \in (\phi_0, \phi_1)\big\}, \quad [\phi_0, \phi_1] \subset (-\pi, \pi),$$

where a periodic function $R \in W^{2,\infty}(-\pi,\pi)$ represents the boundary $\partial\Omega$, and $R(\phi) > 0$ for $\phi \in [\phi_0, \phi_1]$, see Figure 1 for example configuration. The admissible



a) domain $\Omega_{(r,\phi)}$ with crack b) admissible cracks in Ω c) domain ω of parameters

Figure 1. Example configuration.

crack tips

$$C^{(r,\phi)} = r(\cos\phi, \sin\phi)^\top \quad \text{for } (r,\phi) \in \overline{\omega}$$

determines a crack as the union

$$\Gamma_{(r,\phi)} = \Gamma_0 \cup OC^{(r,\phi)} \subset \Omega. \tag{1}$$

It obeys a kink with the angle ϕ to the x_1-axis at the origin O. Representation (1) defines a two-parametric family of kinking cracks with respect to $(r,\phi) \in \overline{\omega}$. At $r = 0$, we have $\Gamma_{(0,\phi)} = \Gamma_0$. At $r = R(\phi)$, the domain Ω is split into two separate parts by $\Gamma_{(R(\phi),\phi)}$. The particular case of $\phi = 0$ represents the crack $\Gamma_{(r,0)}$ without kink. The other specific case of $A = O$, with reasonable choice of ϕ_0 and ϕ_1, implies the rectilinear crack $\Gamma_{(r,\phi)} = OC^{(r,\phi)}$. For the following use, we denote the domain with crack by $\Omega_{(r,\phi)} = \Omega \setminus \Gamma_{(r,\phi)}$.

To describe evolution of the crack with kink, we employ global coordinate transformations of the crack rotation and extension, following the velocity approach of [**21**].

We introduce a velocity field, which is tangential to $\partial\Omega$:

$$W = (W_1, W_2)^\top(x) \in W^{1,\infty}(\boldsymbol{R}^2)^2, \tag{2a}$$

$$W_i n_i = 0 \text{ at } \partial\Omega, \tag{2b}$$

and, for a time-like kinematic parameter $t \in \boldsymbol{R}$, consider the Cauchy problem for a nonlinear ODE system

$$\frac{d}{dt}\Phi_W(t,\,\cdot\,) = W(\Phi_W(t,\,\cdot\,)) \text{ for } t \neq 0, \quad \Phi_W(0,x) = x. \tag{3}$$

By (2a) there exists a unique solution to (3),

$$\Phi_W = \big((\Phi_W)_1, (\Phi_W)_2\big)^\top(t,x) \in C^1\big([0,T]; W^{1,\infty}_{loc}(\boldsymbol{R}^2)\big)^2, \quad |T| > 0. \tag{4}$$

For fixed t, an inverse function to (4) is defined by means of the identities

$$y = \Phi_W\big(t, \Phi_W^{-1}(t,y)\big), \quad x = \Phi_W^{-1}(t, \Phi_W(t,x)), \quad x,y \in \boldsymbol{R}^2. \tag{5}$$

The inverse function can be determined similarly to Φ_W as the solution $\Phi_W^{-1} = \Phi_{-W}$ for the Cauchy problem

$$\frac{d}{dt}\Phi_W^{-1}(t,\,\cdot\,) = -W\big(\Phi_W^{-1}(t,\,\cdot\,)\big) \text{ for } t \neq 0, \quad \Phi_W^{-1}(0,y) = y, \tag{6}$$

with the same regularity

$$\Phi_W^{-1} = \big((\Phi_W^{-1})_1,(\Phi_W^{-1})_2\big)^\top(t,y) \in C^1\big([0,T];W_{loc}^{1,\infty}(\boldsymbol{R}^2)\big)^2. \tag{7}$$

To check the property (5), we observe that the following relation is satisfied

$$\Phi_W(t-s,x) = \Phi_{-W}(s,\Phi_W(t,x)) \quad \text{for } s \in [\min\{t,0\},\max\{0,t\}]. \tag{8}$$

In fact, the differentiation of $s \mapsto \psi(s) = \Phi_W(t-s,x)$ with respect to s yields the equality for the derivative

$$\frac{d}{ds}\psi(s) = -\frac{d}{d(t-s)}\Phi_W(t-s,x) = -W(\Phi_W(t-s,x)) = -W(\psi(s))$$

due to (3). Together with the initial condition $\psi(0) = \Phi_W(t,x)$ it implies (8). With the help of (6) and (8) at $s=t$ we derive relations

$$\Phi_W^{-1}(t,\Phi_W(t,x)) = \Phi_{-W}(t,\Phi_W(t,x)) = \Phi_W(t-t,x) = x,$$

hence

$$\Phi_W\big(t,\Phi_W^{-1}(t,y)\big) = \Phi_{-W}^{-1}(t,\Phi_{-W}(t,y)) = y,$$

thus (5) holds true.

To realize condition (2b) for the crack in (1), we rely on a description in polar coordinates $x_1 = \rho\cos\theta$, $x_2 = \rho\sin\theta$ for $\rho \geq 0$, $\theta \in [-\pi,\pi]$. With the aim of preserving of Γ_0, let us take a smooth cut-off function χ such that $0 \leq \chi(\xi) \leq 1$, $\chi(\xi) = 1$ for $\xi \in [\phi_0,\phi_1]$, and $\chi(\xi) = 0$ near the end points $\xi = \pm\pi$. Also, let $\mu(x), x \in \boldsymbol{R}^2$, be a smooth function with compact support D, $\Omega \subset D$, and $\mu = 1$ on Ω.

Lemma 1. *For every fixed $(r,\phi) \in \overline{\omega}$ and for all $t \in [0,T]$, $T > 0$ such that $(r_\phi(T),\phi+T) \in \overline{\omega}$, where*

$$r_\phi(t) = rR(\phi+t)/R(\phi), \tag{9}$$

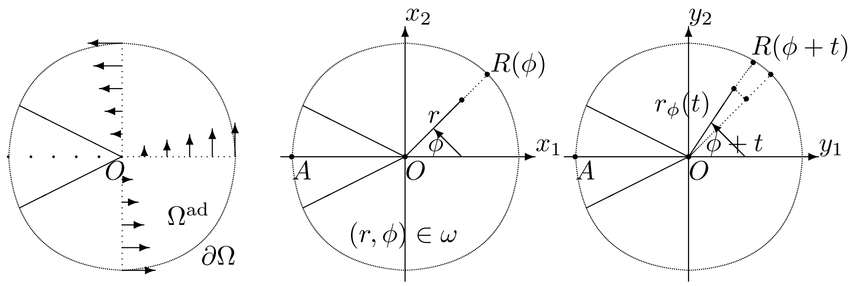
the solutions (4) and (7) of problems (3) and (6) with the velocity

$$W(x) = \mu(x)\chi(\theta)\Big(\frac{R'(\theta)}{R(\theta)}x_1 - x_2, \frac{R'(\theta)}{R(\theta)}x_2 + x_1\Big)^\top \tag{10}$$

determine a bijective mapping between the geometric domains:

$$\begin{aligned} y = \Phi_W(t,x) :\ \Omega_{(r,\phi)} \mapsto \Omega_{(r_\phi(t),\phi+t)}, \\ x = \Phi_W^{-1}(t,y) :\ \Omega_{(r_\phi(t),\phi+t)} \mapsto \Omega_{(r,\phi)}. \end{aligned} \tag{11}$$

Proof. The transformation of rotation (11) is illustrated in Figure 2. To check (2b) for W from (10), we represent $\partial\Omega$ by a nonnegative distance function,



a) velocity field W b) coordinate transformation $y = \Phi_W(t,x)$

Figure 2. Crack rotation.

$$\partial\Omega = \{x \in \boldsymbol{R}^2 :\ d(x) = 0\}, \quad d(x) = |\rho - R(\theta)|.$$

The normal direction of $\partial\Omega$ can be expressed by its gradient

$$\nabla d(x) = \Big(\cos\theta + \frac{\sin\theta}{\rho}R'(\theta),\ \sin\theta - \frac{\cos\theta}{\rho}R'(\theta)\Big)^\top \operatorname{sign}(\rho - R(\theta)).$$

At $\partial\Omega$, where $\rho = R(\theta)$, we obtain the unit normal vector

$$n = \frac{\nabla d}{|\nabla d|}, \quad \nabla d = \frac{1}{R}\Big(\frac{R'(\theta)}{R(\theta)}x_2 + x_1, -\frac{R'(\theta)}{R(\theta)}x_1 + x_2\Big)^\top,$$

that is orthogonal to W in (10). Thus, due to (2b), the maps in (11) preserve the external boundary $\partial\Omega$ for all $t \in \boldsymbol{R}$.

For $y \in \overline{\Omega}^{\mathrm{ad}}$, where $\chi = 1$, system (3) with W from (10) takes the particular form:

$$\frac{d}{dt}\begin{pmatrix} (\Phi_W)_1 \\ (\Phi_W)_2 \end{pmatrix} = \begin{pmatrix} \dfrac{R'(\theta_t)}{R(\theta_t)}(\Phi_W)_1 - (\Phi_W)_2 \\ \dfrac{R'(\theta_t)}{R(\theta_t)}(\Phi_W)_2 + (\Phi_W)_1 \end{pmatrix}, \quad \Phi_W(0,x) = x,$$

where $\tan\theta_t = (\Phi_W)_2/(\Phi_W)_1$. Its solution can be calculated as

$$y = \Phi_W(t,x) = \frac{R(\theta+t)}{R(\theta)}(x_1\cos t - x_2\sin t, x_1\sin t + x_2\cos t)^\top, \tag{12}$$

and $\theta_t = \theta + t$. Direct calculations show that inverse to (12) function

$$x = \Phi_W^{-1}(t,y) = \frac{R(\theta_t - t)}{R(\theta_t)}(y_1\cos t + y_2\sin t, -y_1\sin t + y_2\cos t)^\top \tag{13}$$

satisfies system (6) with $\chi = 1$. In (13), the polar angle θ_t is given with respect to Lagrange coordinates $y_1 = \rho_t\cos\theta_t$, $y_2 = \rho_t\sin\theta_t$. The maps in (12) and (13) transform the point (r,ϕ) to $(r_\phi(t), \phi+t)$, where $r_\phi(t)$ is defined by (9), and conversely. Therefore, y from (12) is located in $\overline{\Omega}^{\mathrm{ad}}$ for all $t \in [0,T]$ with the upper estimate of $|T|$ such that $(r_\phi(T), \phi+T) \in \overline{\omega}$. In this interval, (11) establishes the one-to-one correspondence between the segments $OC^{(r,\phi)}$ and $OC^{(r_\phi(t),\phi+t)}$.

In a neighborhood of Γ_0, where $\chi = 0$ and $W = 0$, the solutions of (3) and (6) describe the identity transformation $y = x$, which preserves Γ_0 for all $t \in \boldsymbol{R}$. □

For crack extension, which is not tangential to $\partial\Omega$, we restrict a support of velocity in (2) to Ω with the help of a smooth cut-off function $0 \le \eta(x) \le 1$ such that:

$$\begin{aligned} &\mathrm{supp}(\eta) = B_0 \subset \Omega, \\ &\eta = 1 \text{ in } B_1, \quad B_1 \subset B_0, \\ &\text{segments } OC^{(r,\phi)} \subset B_1 \cap \Omega^{\mathrm{ad}} \text{ for } (r,\phi) \in \omega_1, \quad \omega_1 \subset \omega, \end{aligned} \tag{14}$$

where B_1 and ω_1 are open sets.

Lemma 2. *For every fixed $(r,\phi) \in \overline{\omega}_1$ and for all $t \in [0,T]$, $T > 0$ such that*

$(re^T, \phi) \in \overline{\omega}_1$, *the solutions* Φ_V *and* Φ_V^{-1} *to the problems*:

$$\frac{d}{dt}\Phi_V(t, \cdot) = V(\Phi_V(t, \cdot)) \text{ for } t \neq 0, \quad \Phi_V(0, x) = x; \tag{15a}$$

$$\frac{d}{dt}\Phi_V^{-1}(t, \cdot) = -V(\Phi_V^{-1}(t, \cdot)) \text{ for } t \neq 0, \quad \Phi_V^{-1}(0, y) = y, \tag{15b}$$

with the velocity

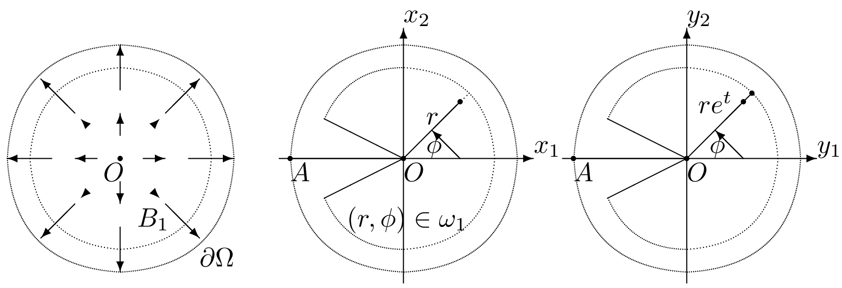
$$V(x) = x\eta(x), \tag{16}$$

determine a bijective mapping between the geometric domains:

$$\begin{aligned} y = \Phi_V(t, x) &: \ \Omega_{(r,\phi)} \mapsto \Omega_{(re^t,\phi)}, \\ x = \Phi_V^{-1}(t, y) &: \ \Omega_{(re^t,\phi)} \mapsto \Omega_{(r,\phi)}. \end{aligned} \tag{17}$$

Proof. We illustrate the transformation of extension (17) in Figure 3. For $y \in \text{cl}(B_1 \cap \Omega^{\text{ad}})$, where $\eta(y) = 1$ and $V(y) = y$, the solutions of (15a) and (15b) are:

$$y = \Phi_V(t, x) = xe^t, \quad x = \Phi_V^{-1}(t, y) = ye^{-t}.$$



a) velocity field V b) coordinate transformation $y = \Phi_V(t, x)$

Figure 3. Crack extension.

Therefore, these transformations map between the segments $OC^{(r,\phi)}$ and $OC^{(re^t,\phi)}$ for $t \in [0, T]$, when $y = C^{(re^T,\phi)} \in \text{cl}(B_1 \cap \Omega^{\text{ad}})$. Using (14) provides the upper bound of $|T|$ such that $(re^T, \phi) \in \overline{\omega}_1$.

For $y \in \Omega \setminus B_0$, where $\eta(y) = 0$ and $V(y) = 0$, the identity solutions of (15) preserve the external boundary $\partial\Omega$.

The velocity field V in (16) is tangential to all rectilinear line passing through the origin O, thus also to Γ_0. Due to $V = 0$ at points A and O, which end the segment, Γ_0 remains unchanged by the maps in (17) for all t. □

3. Static problem for a kinking crack.

In this section we describe equilibrium problem for an elastic solid with the kinking crack under non-penetration conditions and investigate properties of the reduced potential energy function with respect to parameters r and ϕ.

We start with geometric assumptions required to state properly boundary conditions at $\partial\Omega$. Let $\partial\Omega = \Gamma_D \cup \Gamma_N$ be such that:

$$\begin{aligned} \Gamma_D \neq \emptyset, \quad \Gamma_D \cap (\overline{\Omega} \setminus \Omega^{\mathrm{ad}}) \cap \{x_2 > 0\} \neq \emptyset, \\ \Gamma_D \cap (\overline{\Omega} \setminus \Omega^{\mathrm{ad}}) \cap \{x_2 < 0\} \neq \emptyset; \end{aligned} \tag{18a}$$

$$\Gamma_D \circ \Phi_W(t) = \Gamma_D, \quad \Gamma_N \circ \Phi_W(t) = \Gamma_N \quad \text{for all } t. \tag{18b}$$

Assumption (18a) is needed to fix the solid in the case when it is split by $\Gamma_{(R(\phi),\phi)}$ into two separate parts, and assumption (18b) preserves boundary conditions after the rotation of the domain.

Let $(r, \phi) \in \overline{\omega}$ be fixed. In the domain $\Omega_{(r,\phi)}$ with crack, we consider a linear elasticity model for the displacement vector $u = (u_1, u_2)^\top(x)$, the stress and strain tensors

$$\sigma_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u), \quad \varepsilon_{ij}(u) = 0.5(u_{i,j} + u_{j,i}), \quad i, j = 1, 2. \tag{19}$$

The elasticity coefficients c_{ijkl}, $i, j, k, l = 1, 2$, are assumed to be symmetric, positive definite, and constant for simplicity. The convention of summation over repeated indices is used. For given volume load $f = (f_1, f_2)^\top(x) \in C^1(\overline{\Omega})^2$ and traction force $g = (g_1, g_2)^\top(x) \in C^1(\overline{\Gamma}_N)^2$, see Figure 1 for illustration, we consider the equilibrium problem:

$$-\sigma_{ij,j}(u) = f_i, \quad i = 1, 2, \quad \text{in } \Omega_{(r,\phi)}; \tag{20a}$$

$$u = 0 \quad \text{on } \Gamma_D; \tag{20b}$$

$$\sigma_{ij}(u)n_j = g_i, \quad i = 1, 2, \quad \text{on } \Gamma_N; \tag{20c}$$

$$\sigma_{\tau^\phi}(u) = 0 \quad \text{on } \Gamma^{\pm}_{(r,\phi)}; \tag{20d}$$

$$\begin{aligned} &[\![\sigma_{\nu^\phi}(u)]\!] = 0, \quad \sigma_{\nu^\phi}(u) \le 0, \\ &[\![u_{\nu^\phi}]\!] \ge 0, \quad \sigma_{\nu^\phi}(u)[\![u_{\nu^\phi}]\!] = 0 \quad \text{on } \Gamma_{(r,\phi)}. \end{aligned} \tag{20e}$$

Here the normal and tangential components of the stress vector at the crack:

$$\sigma_{\nu^\phi}(u) = \sigma_{ij}(u)\nu_j^\phi \nu_i^\phi, \quad (\sigma_{\tau^\phi}(u))_i = \sigma_{ij}(u)\nu_j^\phi - \sigma_{\nu^\phi}(u)\nu_i^\phi,\ i = 1, 2,$$

are given with respect to the normal and tangential vectors for $\Gamma_{(r,\phi)}$ from (1):

$$\begin{aligned} \nu^\phi &= \begin{cases} (0,1)^\top & \text{on } \Gamma_0, \\ (-\sin\phi, \cos\phi)^\top & \text{on } OC^{(r,\phi)}, \end{cases} \\ \tau^\phi &= \begin{cases} (1,0)^\top & \text{on } \Gamma_0, \\ (\cos\phi, \sin\phi)^\top & \text{on } OC^{(r,\phi)}. \end{cases} \end{aligned} \tag{21}$$

Notation $[\![\,\cdot\,]\!]$ is used for the jump across $\Gamma_{(r,\phi)}$, for instance

$$[\![u_{\nu^\phi}]\!] = u_{\nu^\phi}|_{\Gamma^+_{(r,\phi)}} - u_{\nu^\phi}|_{\Gamma^-_{(r,\phi)}}, \quad u_{\nu^\phi} = u_i \nu_i^\phi. \tag{22}$$

The crack surfaces $\Gamma^\pm_{(r,\phi)}$ correspond to the directions $\mp\nu^\phi$ of the normal vector. Relations (20e) imply conditions of mutual non-penetration between the opposite crack surfaces $\Gamma^\pm_{(r,\phi)}$.

Introducing the Sobolev space

$$H(\Omega_{(r,\phi)}) = \big\{u \in H^1(\Omega_{(r,\phi)})^2 : \quad u = 0 \quad \text{on } \Gamma_D\big\}, \tag{23}$$

which includes the Dirichlet boundary condition (20b), we define the set (a convex cone) of admissible displacements with the non-penetration condition by

$$K_{\nu^\phi}(\Omega_{(r,\phi)}) = \big\{u \in H(\Omega_{(r,\phi)}) : \quad [\![u_{\nu^\phi}]\!] \ge 0 \quad \text{on } \Gamma_{(r,\phi)}\big\}. \tag{24}$$

The lower subscription in notation (24) marks also dependence of the set K of the vector ν^ϕ used for the product in (22). The weak formulation of the equilibrium problem (20) is presented by a constrained minimization problem: Find $u^{(r,\phi)} \in K_{\nu^\phi}(\Omega_{(r,\phi)})$ such that

$$\Pi\big(u^{(r,\phi)}; \Omega_{(r,\phi)}\big) \le \Pi(v; \Omega_{(r,\phi)}) \quad \text{for all } v \in K_{\nu^\phi}(\Omega_{(r,\phi)}), \tag{25}$$

where the quadratic functional $u \mapsto \Pi : H(\Omega_{(r,\phi)}) \mapsto \boldsymbol{R}$ presents the potential energy

$$\Pi(u;\Omega_{(r,\phi)}) = \frac{1}{2} \int\limits_{\Omega_{(r,\phi)}} \sigma_{ij}(u)\varepsilon_{ij}(u)\,dx - \int\limits_{\Omega_{(r,\phi)}} f_i u_i\,dx - \int\limits_{\Gamma_N} g_i u_i\,ds. \tag{26}$$

The properties of coercivity and weakly lower semicontinuity of $u \mapsto \Pi$ and (18a) provide the existence of unique solution to (25). The solution is characterized equivalently by a variational inequality

$$\frac{\partial}{\partial u}\Pi\big(u^{(r,\phi)}, v - u^{(r,\phi)}; \Omega_{(r,\phi)}\big) \geq 0 \quad \text{for all } v \in K_{\nu^\phi}(\Omega_{(r,\phi)}), \tag{27}$$

with the Gâteaux derivative of $u \mapsto \Pi$

$$\frac{\partial}{\partial u}\Pi(u,v;\Omega_{(r,\phi)}) = \int\limits_{\Omega_{(r,\phi)}} \sigma_{ij}(u)\varepsilon_{ij}(v)\,dx - \int\limits_{\Omega_{(r,\phi)}} f_i v_i\,dx - \int\limits_{\Gamma_N} g_i v_i\,ds.$$

If the solution of the variational inequality (27) is H^2-smooth, then it satisfies relations (20) in the pointwise almost everywhere sense. Otherwise, see [**15**] for a detailed description of the weak formulation of the boundary conditions (20e) at the crack.

Substituting the solution of (25) into (26) we define the potential energy as a function of two variables

$$(r,\phi) \mapsto P(r,\phi) = \Pi\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big) : \quad \overline{\omega} \mapsto \boldsymbol{R}, \tag{28}$$

which is given in an implicit way via shape parameters r and ϕ. In the following we aim to investigate continuity and differentiability properties of the function (28) with the help of the preliminaries of Section 2.

3.1. Continuity properties of the energy function.

Based on Lemma 1 and Lemma 2, from (23) and (24) we conclude with the following auxiliary result.

LEMMA 3. *The mapping in* (17) *is bijective between the sets* K:

$$\begin{aligned} &\text{if } u \in K_{\nu^\phi}(\Omega_{(r,\phi)}), \quad \text{then } u \circ \Phi_V^{-1}(t) \in K_{\nu^\phi}(\Omega_{(re^t,\phi)});\\ &\text{if } u \in K_{\nu^\phi}(\Omega_{(re^t,\phi)}), \quad \text{then } u \circ \Phi_V(t) \in K_{\nu^\phi}(\Omega_{(r,\phi)}). \end{aligned} \tag{29}$$

Meanwhile, the maps in (11) *result in the following property between the sets* K:

$$\begin{aligned} &\text{if } u \in K_{\nu^\phi}(\Omega_{(r,\phi)}),\\ &\text{then } u \circ \Phi_W^{-1}(t) \in K_{\nu^\phi}(\Omega_{(r_\phi(t),\phi+t)}) \neq K_{\nu^{\phi+t}}(\Omega_{(r_\phi(t),\phi+t)});\\ &\text{if } u \in K_{\nu^{\phi+t}}(\Omega_{(r_\phi(t),\phi+t)}),\\ &\text{then } u \circ \Phi_W(t) \in K_{\nu^{\phi+t}}(\Omega_{(r,\phi)}) \neq K_{\nu^\phi}(\Omega_{(r,\phi)}). \end{aligned} \tag{30}$$

Properties (29) and (30) follow due to (18b), the $W^{1,\infty}$-regularity of the velocity fields V and W, and due the fact that the vector ν^ϕ (given by a piecewise-constant function) is not changed after coordinate transformations in (17) and (11). The relation $K_{\nu^{\phi+t}} \neq K_{\nu^\phi}$ in (30) is the main difficulty in proving of the following theorem.

Theorem 1. *The energy function in* (28) *possesses the properties*:

$$\begin{aligned} &r \mapsto P(r,\phi) : [0, R(\phi)] \mapsto \boldsymbol{R} \text{ is a nonincreasing function}\\ &\text{for fixed } \phi \in [\phi_0, \phi_1]; \end{aligned} \tag{31a}$$

$$P \text{ is uniformly bounded in } \overline{\omega}; \tag{31b}$$

$$P \text{ is lower semicontinuous in } \overline{\omega}; \tag{31c}$$

$$P \in C(\overline{\omega}_1 \setminus \{\phi = 0, r > 0\}); \tag{31d}$$

$$\phi \mapsto P(R(\phi),\phi) \in C([\phi_0,\phi_1] \setminus \{\phi = 0\}); \tag{31e}$$

$$r \mapsto P(r,0) \in C([0,R(0)] \cap \omega_1) \quad \text{at } \phi = 0. \tag{31f}$$

Proof. First, for fixed $\phi \in [\phi_0, \phi_1]$, due to the fact that

$$K_{\nu^\phi}(\Omega_{(0,\phi)}) \subseteq K_{\nu^\phi}(\Omega_{(r,\phi)}) \subseteq K_{\nu^\phi}(\Omega_{(R(\phi),\phi)}), \tag{32}$$

we infer

$$\begin{aligned} P(0,\phi) &= \min_{v \in K_{\nu^\phi}(\Omega_{(0,\phi)})} \Pi(v; \Omega_{(0,\phi)}) \geq P(r,\phi)\\ &= \min_{v \in K_{\nu^\phi}(\Omega_{(r,\phi)})} \Pi(v; \Omega_{(r,\phi)}) \geq P(R(\phi),\phi)\\ &= \min_{v \in K_{\nu^\phi}(\Omega_{(R(\phi),\phi)})} \Pi(v; \Omega_{(R(\phi),\phi)}), \end{aligned}$$

thus (31a). This property provides the uniform estimation

$$\mathrm{c}_0 = P(0,\phi) \geq P(r,\phi) \geq \min_{\phi\in[\phi_0,\phi_1]} P(R(\phi),\phi) = \mathrm{c}_1 \quad \text{for } (r,\phi) \in \overline{\omega}$$

and assertion (31b).

Second, we prove (31c). Let us fix $(r,\phi) \in \overline{\omega}$ and consider a convergent sequence

$$\overline{\omega} \ni (r_\varepsilon, \phi_\varepsilon) \to (r,\phi) \quad \text{in } \boldsymbol{R}^2 \text{ as } \varepsilon \to 0. \tag{33}$$

Our goal is to show that

$$\liminf_{\varepsilon\to 0} P(r_\varepsilon,\phi_\varepsilon) = \liminf_{\varepsilon\to 0} \Pi(u^{(r_\varepsilon,\phi_\varepsilon)};\Omega_{(r_\varepsilon,\phi_\varepsilon)}) \geq P(r,\phi). \tag{34}$$

To do this, we map all cracks $\Gamma_{(r_\varepsilon,\phi_\varepsilon)}$ to ones along the reference direction ϕ with the help of the rotation from Lemma 1. From (11) we find the values

$$t_\varepsilon = \phi_\varepsilon - \phi, \quad \hat{r}_\varepsilon = r_\varepsilon R(\phi)/R(\phi_\varepsilon),$$

such that $t_\varepsilon \to 0$, $\hat{r}_\varepsilon \to r$ as $\varepsilon \to 0$, and

$$\begin{aligned} y = \Phi_W(t_\varepsilon, x) :\ & \Omega_{(\hat{r}_\varepsilon,\phi)} \mapsto \Omega_{(r_\varepsilon,\phi_\varepsilon)}, \\ x = \Phi_W^{-1}(t_\varepsilon, y) :\ & \Omega_{(r_\varepsilon,\phi_\varepsilon)} \mapsto \Omega_{(\hat{r}_\varepsilon,\phi)}. \end{aligned} \tag{35}$$

Using (30) we conclude with the properties:

$$\begin{aligned} &\text{if } u \in K_{\nu^\phi}(\Omega_{(\hat{r}_\varepsilon,\phi)}), \\ &\text{then } u \circ \Phi_W^{-1}(t_\varepsilon) \in K_{\nu^\phi}(\Omega_{(r_\varepsilon,\phi_\varepsilon)}) \neq K_{\nu^{\phi_\varepsilon}}(\Omega_{(r_\varepsilon,\phi_\varepsilon)}); \\ &\text{if } u \in K_{\nu^{\phi_\varepsilon}}(\Omega_{(r_\varepsilon,\phi_\varepsilon)}), \\ &\text{then } u \circ \Phi_W(t_\varepsilon) \in K_{\nu^{\phi_\varepsilon}}(\Omega_{(\hat{r}_\varepsilon,\phi)}) \neq K_{\nu^\phi}(\Omega_{(\hat{r}_\varepsilon,\phi)}). \end{aligned} \tag{36}$$

Next we can apply the coordinate transformations in (35) to the integrals in Π from (26) and derive for $u \in H(\Omega_{(r_\varepsilon,\phi_\varepsilon)})$ that

$$\Pi(u;\Omega_{(r_\varepsilon,\phi_\varepsilon)}) = \Pi \circ \Phi_W(t_\varepsilon)(u \circ \Phi_W(t_\varepsilon);\Omega_{(\hat{r}_\varepsilon,\phi)}), \tag{37}$$

where, for $u \in H(\Omega_{(\hat{r}_\varepsilon,\phi)})$, the transformed functional is given by

$$\begin{aligned} &\Pi \circ \Phi_W(t_\varepsilon)(u;\Omega_{(\hat{r}_\varepsilon,\phi)}) \\ &\quad = \frac{1}{2} \int\limits_{\Omega_{(\hat{r}_\varepsilon,\phi)}} J_W(t_\varepsilon)\Sigma_{ij}\big(\Phi_W^{-1}(t_\varepsilon);u\big)E_{ij}\big(\Phi_W^{-1}(t_\varepsilon);u\big)\,dx \\ &\quad\quad - \int\limits_{\Omega_{(\hat{r}_\varepsilon,\phi)}} J_W(t_\varepsilon)(f_i \circ \Phi_W(t_\varepsilon))u_i\,dx - \int\limits_{\Gamma_N} j_W(t_\varepsilon)(g_i \circ \Phi_W(t_\varepsilon))u_i\,ds. \end{aligned} \tag{38}$$

Representation (38) involves the generalized stress and strain tensors related to (19):

$$\begin{aligned} \Sigma_{ij}(v;u) &= c_{ijkl}E_{kl}(v;u), \quad i,j=1,2, \\ E_{ij}(v;u) &= 0.5(u_{i,k}v_{k,j} + u_{j,k}v_{k,i}), \end{aligned} \tag{39}$$

and the Jacobian in the domain and at the boundary:

$$J_W(t) = \det\left(\frac{\partial}{\partial x}\Phi_W(t)\right), \quad \text{where } \frac{\partial}{\partial x}v = \{v_{i,j}\}_{i,j=1}^2, \tag{40a}$$

$$j_W(t) = J_W(t)\left|\left(\frac{\partial}{\partial x}\Phi_W^{-1}(t)\right)^{\top} n\right|. \tag{40b}$$

For the further use we get an asymptotic expansion with respect to $t \to 0$ in (40) and (39) (I denotes the identity operator):

$$\begin{aligned} &\Phi_W(t,x) = x + tW(x) + \mathrm{Res}_t, \quad \Phi_W^{-1}(t,y) = y - tW(y) + \mathrm{Res}_t, \\ &\|\mathrm{Res}_t\|_{W^{1,\infty}_{loc}(\boldsymbol{R}^2)^2} = o(t); \end{aligned} \tag{41a}$$

$$\begin{aligned} &\frac{\partial}{\partial x}\Phi_W(t) = I + t\frac{\partial}{\partial x}W + \mathrm{Res}_t, \quad \frac{\partial}{\partial y}\Phi_W^{-1}(t) = I - t\frac{\partial}{\partial y}W + \mathrm{Res}_t, \\ &\|\mathrm{Res}_t\|_{L^\infty_{loc}(\boldsymbol{R}^2)^{2\times 2}} = o(t); \end{aligned} \tag{41b}$$

$$J_W(t) = 1 + t\,\mathrm{div}(W) + \mathrm{Res}_t, \quad \|\mathrm{Res}_t\|_{L^\infty_{loc}(\boldsymbol{R}^2)} = o(t); \tag{41c}$$

$$j_W(t) = 1 + t(\mathrm{div}(W) - W_{i,j}n_jn_i) + \mathrm{Res}_t, \quad \|\mathrm{Res}_t\|_{L^\infty(\Gamma_N)} = o(t); \tag{41d}$$

$$\begin{aligned} &E_{ij}\big(\Phi_W^{-1}(t);u\big) = \varepsilon_{ij}(u) - tE_{ij}(W;u) + \mathrm{Res}_t(u), \quad i,j=1,2, \\ &\|\mathrm{Res}_t(u)\|_{L^2(\Omega_{(\hat{r}_\varepsilon,\phi)})} \le \mathrm{c}(t)\,\|u\|_{H(\Omega_{(\hat{r}_\varepsilon,\phi)})}, \quad 0 \le \mathrm{c}(t) = o(t); \end{aligned} \tag{41e}$$

and representation of the perturbed load:

$$\begin{aligned} &f_i \circ \Phi_W(t) = f_i + t f_{i,j} W_j + \mathrm{Res}_t, \quad i = 1,2, \\ &\|\mathrm{Res}_t\|_{L^2(\Omega)} = o(t); \end{aligned} \tag{42a}$$

$$\begin{aligned} &g_i \circ \Phi_W(t) = g_i + t g_{i,j} W_j + \mathrm{Res}_t, \quad i = 1,2, \\ &\|\mathrm{Res}_t\|_{L^2(\Gamma_N)} = o(t). \end{aligned} \tag{42b}$$

The expansion (41d) is argued by the additional smoothness of W in (10) at Γ_N. The other expansions in (41) and (42) are provided by the C^1-smoothness of the decomposed functions with respect to t.

Using (41) and (42) yields the representation of (38) in the form

$$\begin{aligned} &\Pi \circ \Phi_W(t_\varepsilon)(u; \Omega_{(\hat{r}_\varepsilon,\phi)}) = \Pi(u; \Omega_{(\hat{r}_\varepsilon,\phi)}) + t_\varepsilon \Pi^1_W(u; \Omega_{(\hat{r}_\varepsilon,\phi)}) + \mathrm{Res}_{t_\varepsilon}(u), \\ &|\mathrm{Res}_{t_\varepsilon}(u)| \le \mathrm{c}(t_\varepsilon)\big(\|u\|^2_{H(\Omega_{(\hat{r}_\varepsilon,\phi)})} + \mathrm{const}\big), \quad 0 \le \mathrm{c}(t_\varepsilon) = o(t_\varepsilon), \end{aligned} \tag{43}$$

with the first asymptotic term

$$\begin{aligned} \Pi^1_W(u; \Omega_{(\hat{r}_\varepsilon,\phi)}) = \frac{1}{2} \int\limits_{\Omega_{(\hat{r}_\varepsilon,\phi)}} &\big(\mathrm{div}(W)\sigma_{ij}(u) - 2\Sigma_{ij}(W;u)\big)\varepsilon_{ij}(u)\,dx \\ - \int\limits_{\Omega_{(\hat{r}_\varepsilon,\phi)}} &\mathrm{div}(W f_i) u_i\,dx - \int\limits_{\Gamma_N} \big(\mathrm{div}(W g_i) - W_{k,j} n_j n_k g_i\big) u_i\,ds. \end{aligned} \tag{44}$$

The constant term in (43) includes the norms of f and g.

We consider problem (25) at $(r_\varepsilon, \phi_\varepsilon)$,

$$\Pi\big(u^{(r_\varepsilon,\phi_\varepsilon)}; \Omega_{(r_\varepsilon,\phi_\varepsilon)}\big) \le \Pi\big(v; \Omega_{(r_\varepsilon,\phi_\varepsilon)}\big) \quad \text{for all } v \in K_{\nu^{\phi_\varepsilon}}(\Omega_{(r_\varepsilon,\phi_\varepsilon)}), \tag{45}$$

apply here transformations (35) and use (37) to derive

$$\begin{aligned} &\Pi \circ \Phi_W(t_\varepsilon)\big(u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon); \Omega_{(\hat{r}_\varepsilon,\phi)}\big) \\ &\quad \le \Pi \circ \Phi_W(t_\varepsilon)\big(v \circ \Phi_W(t_\varepsilon); \Omega_{(\hat{r}_\varepsilon,\phi)}\big) \quad \text{for all } v \in K_{\nu^{\phi_\varepsilon}}(\Omega_{(r_\varepsilon,\phi_\varepsilon)}). \end{aligned} \tag{46}$$

Denoting the solution to (25) at $r = 0$ (thus for the initial crack $\Gamma_0 = \Gamma_{(0,\phi)}$ in the domain $\Omega_0 = \Omega_{(0,\phi)}$) by $u^0 = u^{(0,\phi)}$, we observe that $u^0 \circ \Phi_W^{-1}(t_\varepsilon)$ can be substituted as a test function in (46). As the result of substitution, due to (43) we obtain the estimation

$$\|u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon)\|^2_{H(\Omega_{(\hat r_\varepsilon,\phi)})} \\ \leq \mathrm{c}_0 + \mathrm{c}_1 \|u^0\|^2_{H(\Omega_{(\hat r_\varepsilon,\phi)})} + \mathrm{C}(t_\varepsilon)\|u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon)\|^2_{H(\Omega_{(\hat r_\varepsilon,\phi)})}, \\ 0 \leq \mathrm{C}(t_\varepsilon) = O(t_\varepsilon).$$

Applying property (32) yields the inclusions

$$u^0,\ u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon) \in H(\Omega_{(R(\phi),\phi)}) \quad \text{for all } 0 \leq r_\varepsilon \leq R(\phi),$$

thus providing the following estimate, which is uniform in ε,

$$\|u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon)\|_{H(\Omega_{(R(\phi),\phi)})} \leq \text{const.} \tag{47}$$

Therefore, there exists a weak limit (of a subsequence) such that

$$u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon) \to u^\star \quad \text{weakly in } H(\Omega_{(R(\phi),\phi)}) \text{ as } \varepsilon \to 0. \tag{48}$$

Since $[\![u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon)]\!] = 0$ along the extension of crack $\Gamma_{(\hat r_\varepsilon,\phi)}$, it follows that $[\![u^\star]\!] = 0$ at the extension of $\Gamma_{(r,\phi)}$ (due to $\hat r_\varepsilon \to r$), thus $u^\star \in H(\Omega_{(r,\phi)})$. The explicit expression of the normal vector in (21) provides the following representation

$$\nu^{\phi_\varepsilon} = \nu^\phi + \begin{cases} (0,0)^\top & \text{on } \Gamma_0, \\ (\sin\phi - \sin\phi_\varepsilon, \cos\phi_\varepsilon - \cos\phi)^\top & \text{on } OC^{(R(\phi),\phi)}. \end{cases} \tag{49}$$

Due to (49), $\nu^{\phi_\varepsilon} \to \nu^\phi$ as $\varepsilon \to 0$, hence $u^\star \in K_{\nu^\phi}(\Omega_{(r,\phi)})$. Using the weakly lower semicontinuity property of the functional $u \mapsto \Pi(u)$, from (43), (47), (48), and (25) we infer directly

$$\liminf_{\varepsilon\to 0} \Pi \circ \Phi_W(t_\varepsilon)\big(u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon); \Omega_{(\hat r_\varepsilon,\phi)}\big) \\ \geq \liminf_{\varepsilon\to 0} \Pi\big(u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon); \Omega_{(\hat r_\varepsilon,\phi)}\big) \\ \geq \Pi(u^\star; \Omega_{(R(\phi),\phi)}) \geq \Pi\big(u^{(r,\phi)}; \Omega_{(r,\phi)}\big).$$

Together with (37) it follows (34), thus proving (31c).

To prove assertion (31d) we need to construct a strongly convergent sequence in the sets $K_{\nu^{\phi_\varepsilon}}(\Omega_{(r_\varepsilon,\phi_\varepsilon)})$. Let us fix $(r,\phi) \in \overline{\omega}_1 \setminus \{\phi = 0, r > 0\}$. For small ε and $r > 0$ we have $\phi_\varepsilon \neq 0$, and the following sequence can be defined for arbitrary $v \in K_{\nu^\phi}(\Omega_{(r,\phi)})$:

$$\hat{v} = G^\varepsilon \widetilde{v} = \begin{cases} (\widetilde{v}_1, \widetilde{v}_2)^\top & \text{for } r = 0, \\ \left(\widetilde{v}_1 \frac{\sin\phi}{\sin\phi_\varepsilon}, \widetilde{v}_2 \frac{\cos\phi}{\cos\phi_\varepsilon}\right)^\top & \text{for } r > 0,\ \phi \neq 0,\ \ \pm\pi/2, \\ (\widetilde{v}_1 + \widetilde{v}_2 \cot\phi_\varepsilon, \widetilde{v}_2)^\top & \text{for } r > 0,\ \phi = \pm\pi/2, \end{cases} \tag{50}$$

$$\widetilde{v} = \begin{cases} v & \text{for } r = 0, \\ v \circ \Phi_V^{-1}(s_\varepsilon) & \text{for } r > 0, \end{cases} \qquad s_\varepsilon = \ln(\hat{r}_\varepsilon / r).$$

The transformation of extension Φ_V^{-1} is determined from (15), (16), and it compensates the change of crack length r_ε to $\hat{r}_\varepsilon$ by the rotation in (35). For $r > 0$, using the mapping (17) in Lemma 2 with $t = s_\varepsilon$ yields $v \circ \Phi_V^{-1}(s_\varepsilon) \in H(\Omega_{(\hat{r}_\varepsilon,\phi)})$. Due to (29) in Lemma 3 and the subsequent expansion (55a) we have $v \circ \Phi_V^{-1}(s_\varepsilon) \in K_{\nu^\phi}(\Omega_{(\hat{r}_\varepsilon,\phi)})$, and

$$v \circ \Phi_V^{-1}(s_\varepsilon) \to v \quad \text{strongly in } H(\Omega_{(R(\phi),\phi)}) \text{ as } \varepsilon \to 0. \tag{51}$$

Thus, $\hat{v} \in H(\Omega_{(\hat{r}_\varepsilon,\phi)})$. Substituting (50) into the non-penetration condition and accounting representation of the normal vector (21), we calculate directly $[\![\hat{v}_i]\!]\nu_i^{\phi_\varepsilon} \geq 0$ at $\Gamma_{(\hat{r}_\varepsilon,\phi)}$. Hence, we conclude with the following properties:

$$\hat{v} \in K_{\nu^{\phi_\varepsilon}}(\Omega_{(\hat{r}_\varepsilon,\phi)}), \tag{52a}$$

$$\hat{v} \to v \quad \text{strongly in } H(\Omega_{(R(\phi),\phi)}) \text{ as } \varepsilon \to 0. \tag{52b}$$

Due to (52a), the auxiliary function

$$\hat{v} \circ \Phi_W^{-1}(t_\varepsilon) \in K_{\nu^{\phi_\varepsilon}}(\Omega_{(r_\varepsilon,\phi_\varepsilon)})$$

can be substituted as a test function in inequality (46). As the result of substitution, with the use of (43) and (47) we evaluate $\Pi \circ \Phi_W(t_\varepsilon)$ at the perturbed solution:

$$\begin{aligned} &\Pi \circ \Phi_W(t_\varepsilon)\big(u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon); \Omega_{(\hat{r}_\varepsilon,\phi)}\big) \\ &\quad \leq \Pi \circ \Phi_W(t_\varepsilon)(\hat{v}; \Omega_{(\hat{r}_\varepsilon,\phi)}) \leq \Pi(\hat{v}; \Omega_{(R(\phi),\phi)}) + \mathrm{C}(t_\varepsilon), \end{aligned}$$

$$\Pi\circ\Phi_W(t_\varepsilon)\big(u^{(r_\varepsilon,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(\hat r_\varepsilon,\phi)}\big)$$
$$\ge\Pi\big(u^{(r_\varepsilon,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(R(\phi),\phi)}\big)-\mathrm{C}(t_\varepsilon),\quad 0\le\mathrm{C}(t_\varepsilon)=O(t_\varepsilon),$$

and derive the estimate

$$\Pi\big(u^{(r_\varepsilon,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(R(\phi),\phi)}\big)\le\Pi(\hat v;\Omega_{(R(\phi),\phi)})+O(\varepsilon).$$

Passing here to the limit as $\varepsilon\to 0$ due to the convergences (48) and (52b) provides

$$\Pi(u^\star;\Omega_{(r,\phi)})\le\Pi(v;\Omega_{(r,\phi)})\quad\text{for all } v\in K_{\nu^\phi}(\Omega_{(r,\phi)}),$$

i.e., $u^\star=u^{(r,\phi)}$ is the solution to problem (25).

It remains to prove the strong convergence in (48) for $(r,\phi)\in\overline{\omega}_1\setminus\{\phi=0,r>0\}$. For this reason, we apply construction (50) for $v=u^{(r,\phi)}\in K_{\nu^\phi}(\Omega_{(r,\phi)})$ and obtain the sequence

$$K_{\nu^{\phi_\varepsilon}}(\Omega_{(\hat r_\varepsilon,\phi)})\ni\hat u^{(r,\phi)}\to u^{(r,\phi)}\text{ strongly in } H(\Omega_{(R(\phi),\phi)})\text{ as }\varepsilon\to 0. \tag{53}$$

Denoting the difference $\delta u=u^{(r_\varepsilon,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon)-\hat u^{(r,\phi)}$, we estimate its norm

$$\mathrm{c}\|\delta u\|^2_{H(\Omega_{(R(\phi),\phi)})}\le\frac{1}{2}\int\limits_{\Omega_{(R(\phi),\phi)}}\sigma_{ij}(\delta u)\varepsilon_{ij}(\delta u)\,dx$$
$$=\int\limits_{\Omega_{(R(\phi),\phi)}}\big(-\sigma_{ij}(\hat u^{(r,\phi)})\varepsilon_{ij}(\delta u)+f_i(\delta u)_i\big)\,dx+\int\limits_{\Gamma_N}g_i(\delta u)_i\,ds$$
$$+\Pi\big(u^{(r_\varepsilon,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(R(\phi),\phi)}\big)-\Pi\big(\hat u^{(r,\phi)};\Omega_{(R(\phi),\phi)}\big),$$

where the latter two terms can be evaluated as

$$\Pi\big(u^{(r_\varepsilon,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(R(\phi),\phi)}\big)-\Pi\big(\hat u^{(r,\phi)};\Omega_{(R(\phi),\phi)}\big)$$
$$\le\Pi\circ\Phi_W(t_\varepsilon)\big(u^{(r_\varepsilon,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(\hat r_\varepsilon,\phi)}\big)-\Pi\circ\Phi_W(t_\varepsilon)\big(\hat u^{(r,\phi)};\Omega_{(\hat r_\varepsilon,\phi)}\big)+O(\varepsilon).$$

Observing that due to (46)

$$\Pi\circ\Phi_W(t_\varepsilon)\big(u^{(r_\varepsilon,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(\hat r_\varepsilon,\phi)}\big)\le\Pi\circ\Phi_W(t_\varepsilon)\big(\hat u^{(r,\phi)};\Omega_{(\hat r_\varepsilon,\phi)}\big),$$

by (48) and (53) we establish $\delta u \to 0$ strongly and

$$u^{(r_\varepsilon,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon) \to u^{(r,\phi)} \quad \text{strongly in } H(\Omega_{(R(\phi),\phi)}) \text{ as } \varepsilon \to 0. \tag{54}$$

Passing to the limit as $\varepsilon \to 0$ in

$$P(r_\varepsilon,\phi_\varepsilon) = \Pi\big(u^{(r_\varepsilon,\phi_\varepsilon)};\Omega_{(r_\varepsilon,\phi_\varepsilon)}\big) = \Pi\circ\Phi_W(t_\varepsilon)\big(u^{(r_\varepsilon,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(R(\phi),\phi)}\big),$$

with the help of (43) and (54) we deduce (31d).

This construction (without the extension Φ_V^{-1}) can be repeated for the particular case of $(R(\phi_\varepsilon),\phi_\varepsilon) \to (R(\phi),\phi)$, $\phi \neq 0$, as $\varepsilon \to 0$, thus providing (31e).

The assertion (31f) for the other specific case of parameters $\phi_\varepsilon = \phi = 0$, thus implying rectilinear cracks since $\nu^{\phi_\varepsilon} = \nu^{\phi} = (0,1)^\top$, follows from the results of earlier works [**17**], [**18**]. Note that (31d) and (31f) imply the continuity of P at $(r,\phi) = (0,0)$. $\square$

For the further use we write an asymptotic expansion as $t \to 0$, similar to (41)–(44), with respect to (17):

$$\begin{aligned}&\Phi_V(t,x) = x + tV(x) + \mathrm{Res}_t, \quad \Phi_V^{-1}(t,y) = y - tV(y) + \mathrm{Res}_t,\\ &\|\mathrm{Res}_t\|_{W^{1,\infty}_{loc}(\boldsymbol{R}^2)^2} = o(t);\end{aligned} \tag{55a}$$

$$\begin{aligned}&\frac{\partial}{\partial x}\Phi_V(t) = I + t\frac{\partial}{\partial x}V + \mathrm{Res}_t, \quad \frac{\partial}{\partial y}\Phi_V^{-1}(t) = I - t\frac{\partial}{\partial y}V + \mathrm{Res}_t,\\ &\|\mathrm{Res}_t\|_{L^\infty_{loc}(\boldsymbol{R}^2)^{2\times 2}} = o(t);\end{aligned} \tag{55b}$$

$$\begin{aligned}&J_V(t) = \det\left(\frac{\partial}{\partial x}\Phi_V(t)\right) = 1 + t\mathrm{div}(V) + \mathrm{Res}_t,\\ &\|\mathrm{Res}_t\|_{L^\infty_{loc}(\boldsymbol{R}^2)} = o(t);\end{aligned} \tag{55c}$$

$$\begin{aligned}&E_{ij}\big(\Phi_V^{-1}(t);u\big) = \varepsilon_{ij}(u) - tE_{ij}(V;u) + \mathrm{Res}_t(u), \quad i,j = 1,2,\\ &\|\mathrm{Res}_t(u)\|_{L^2(\Omega_{(r,\phi)})} \le \mathrm{c}(t)\,\|u\|_{H(\Omega_{(r,\phi)})}, \quad 0 \le \mathrm{c}(t) = o(t);\end{aligned} \tag{55d}$$

$$\begin{aligned}&f_i\circ\Phi_V(t) = f_i + tf_{i,j}V_j + \mathrm{Res}_t, \quad i = 1,2,\\ &\|\mathrm{Res}_t\|_{L^2(\Omega)} = o(t);\end{aligned} \tag{55e}$$

and asymptotic representation of the energy functional after applying coordinate transformations (17), for $u \in H(\Omega_{(r,\phi)})$,

$$\begin{aligned}&\Pi\circ\Phi_V(t)(u;\Omega_{(r,\phi)})=\Pi(u;\Omega_{(r,\phi)})+t\Pi_V^1(u;\Omega_{(r,\phi)})+\mathrm{Res}_t(u),\\&|\mathrm{Res}_t(u)|\le \mathrm{c}(t)\big(\|u\|^2_{H(\Omega_{(r,\phi)})}+\mathrm{const}\big),\quad 0\le \mathrm{c}(t)=o(t),\end{aligned}\tag{56}$$

with the first asymptotic term

$$\Pi_V^1(u;\Omega_{(r,\phi)})=\frac{1}{2}\int\limits_{\Omega_{(r,\phi)}}\big(\mathrm{div}(V)\sigma_{ij}(u)-2\Sigma_{ij}(V;u)\big)\varepsilon_{ij}(u)\,dx-\int\limits_{\Omega_{(r,\phi)}}\mathrm{div}(Vf_i)u_i\,dx.\tag{57}$$

Note that, in comparison with Π_W^1 from (44), expression (57) does not contain the boundary integral over Γ_N, since $V=0$ near Γ_N in (16).

3.2. Differentiability properties of the energy function.

Based on Theorem 1 we derive the following results.

Theorem 2. *The energy function in* (28) *possesses the properties*:

(*a*) $\phi\mapsto P(r,\phi)$ *is differentiable for* $(r,\phi)\in\overline{\omega}_1\setminus\{\phi=0,r>0\}$ *with the derivative*

$$\frac{\partial}{\partial\phi}P(r,\phi)=\mathscr{H}(r)\bigg(\Pi^1_{W-\frac{R'(\phi)}{R(\phi)}V}\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)+\frac{\partial}{\partial u}\Pi\big(u^{(r,\phi)},G^1_\phi u^{(r,\phi)};\Omega_{(r,\phi)}\big)\bigg),\tag{58}$$

where $\mathscr{H}(r)=0$ *for* $r=0$, *otherwise* $\mathscr{H}(r)=1$, *and*

$$G^1_\phi u=\begin{cases}(-u_1\cot\phi,u_2\tan\phi)^\top & \text{for } \phi\neq 0,\ \pm\pi/2,\\(-u_2,0)^\top & \text{for } \phi=\pm\pi/2;\end{cases}$$

(*b*) $\phi\mapsto P(R(\phi),\phi)\in C^1([\phi_0,\phi_1]\setminus\{0\})$ *with the derivative*

$$\frac{\partial}{\partial\phi}P(R(\phi),\phi)=\Pi^1_W\big(u^{(R(\phi),\phi)};\Omega_{(R(\phi),\phi)}\big)+\frac{\partial}{\partial u}\Pi\big(u^{(R(\phi),\phi)},G^1_\phi u^{(R(\phi),\phi)};\Omega_{(R(\phi),\phi)}\big);\tag{59}$$

(*c*) $r\mapsto P(r,\phi)$ *is continuously differentiable for* $(r,\phi)\in\overline{\omega}_1\setminus\{r=0\}$ *with the derivative*

$$0\ge\frac{\partial}{\partial r}P(r,\phi)=\Pi^1_{V/r}\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big);\tag{60}$$

(*d*) $r \mapsto P(r,0)$ *is differentiable at* $r=0$ *with the derivative*

$$0 \geq \frac{\partial}{\partial r} P(r,0) = \Pi^1_U\big(u^{(r,0)}; \Omega_{(r,0)}\big), \quad U = \eta(1,0)^\top, \tag{61}$$

where

$$\Pi^1_U(u; \Omega_{(r,0)}) = \frac{1}{2} \int\limits_{\Omega_{(r,0)}} \big(\mathrm{div}(U)\sigma_{ij}(u) - 2\Sigma_{ij}(U;u)\big)\varepsilon_{ij}(u)\,dx - \int\limits_{\Omega_{(r,0)}} \mathrm{div}(Uf_i)u_i\,dx.$$

PROOF. We start with the assertion (a), which deals with the particular case of $r_\varepsilon = r$ in (33), i.e.,

$$(r, \phi_\varepsilon) \to (r,\phi) \in \overline{\omega}_1 \setminus \{\phi = 0, r > 0\}, \quad 0 \neq t_\varepsilon = \phi_\varepsilon - \phi \to 0,$$
$$\hat{r}_\varepsilon = rR(\phi)/R(\phi_\varepsilon) \to r \quad \text{as } \varepsilon \to 0.$$

For the sequence $\hat{u}^{(r,\phi)} \in K_{\nu^{\phi_\varepsilon}}(\Omega_{(\hat{r}_\varepsilon,\phi)})$ constructed in (53), due to (50) and expansions

$$\frac{\sin\phi}{\sin\phi_\varepsilon} = 1 - t_\varepsilon \cot\phi + o(t_\varepsilon), \quad \frac{\cos\phi}{\cos\phi_\varepsilon} = 1 + t_\varepsilon \tan\phi + o(t_\varepsilon),$$
$$\cot\phi_\varepsilon|_{\phi=\pm\pi/2} = -t_\varepsilon + o(t_\varepsilon),$$

we derive representation

$$\begin{aligned} &\hat{u}^{(r,\phi)} = \widetilde{u}^{(r,\phi)} + t_\varepsilon \mathscr{H}(r) G^1_\phi \widetilde{u}^{(r,\phi)} + \mathscr{H}(r)\mathrm{Res}_{t_\varepsilon}(\widetilde{u}^{(r,\phi)}), \\ &\|\mathrm{Res}_{t_\varepsilon}(\widetilde{u}^{(r,\phi)})\|_{H(\Omega_{(\hat{r}_\varepsilon,\phi)})} \leq \mathrm{c}(t_\varepsilon)\,\|\widetilde{u}^{(r,\phi)}\|_{H(\Omega_{(\hat{r}_\varepsilon,\phi)})}, \quad 0 \leq \mathrm{c}(t_\varepsilon) = o(t_\varepsilon). \end{aligned} \tag{62}$$

Substituting $\hat{u}^{(r,\phi)} \circ \Phi_W^{-1}(t_\varepsilon) \in K_{\nu^{\phi_\varepsilon}}(\Omega_{(r,\phi_\varepsilon)})$ into (46) with $r_\varepsilon = r$, decompositions (43) and (62) provide an upper estimate of the finite difference:

$$\begin{aligned} D_{\phi_\varepsilon} &= \frac{P(r,\phi_\varepsilon) - P(r,\phi)}{|\phi_\varepsilon - \phi|} = \frac{1}{|t_\varepsilon|}\big(\Pi(u^{(r,\phi_\varepsilon)}; \Omega_{(r,\phi_\varepsilon)}) - \Pi(u^{(r,\phi)}; \Omega_{(r,\phi)})\big) \\ &= \frac{1}{|t_\varepsilon|}\big(\Pi \circ \Phi_W(t_\varepsilon)(u^{(r,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon); \Omega_{(\hat{r}_\varepsilon,\phi)}) - \Pi(u^{(r,\phi)}; \Omega_{(r,\phi)})\big) \end{aligned}$$

$$\leq \frac{1}{|t_\varepsilon|}\big(\Pi(\hat{u}^{(r,\phi)};\Omega_{(\hat{r}_\varepsilon,\phi)}) - \Pi(u^{(r,\phi)};\Omega_{(r,\phi)})\big) + \frac{t_\varepsilon}{|t_\varepsilon|}\Pi^1_W\big(\hat{u}^{(r,\phi)};\Omega_{(\hat{r}_\varepsilon,\phi)}\big)$$
$$+ \frac{1}{|t_\varepsilon|}\mathrm{Res}_{t_\varepsilon}(\hat{u}^{(r,\phi)})$$
$$\leq \frac{1}{|t_\varepsilon|}\big(\Pi(\widetilde{u}^{(r,\phi)};\Omega_{(\hat{r}_\varepsilon,\phi)}) - \Pi(u^{(r,\phi)};\Omega_{(r,\phi)})\big) + \frac{t_\varepsilon}{|t_\varepsilon|}\Pi^1_W\big(\hat{u}^{(r,\phi)};\Omega_{(\hat{r}_\varepsilon,\phi)}\big)$$
$$+ \frac{t_\varepsilon}{|t_\varepsilon|}\mathscr{H}(r)\frac{\partial}{\partial u}\Pi\big(\widetilde{u}^{(r,\phi)}, G^1_\phi\widetilde{u}^{(r,\phi)};\Omega_{(\hat{r}_\varepsilon,\phi)}\big) + \frac{1}{|t_\varepsilon|}\mathrm{Res}_{t_\varepsilon}(\hat{u}^{(r,\phi)}).$$

Here all the residual terms of the expansions are collected in $\mathrm{Res}_{t_\varepsilon}$. For $r = 0$, we have $\widetilde{u}^{(r,\phi)} = u^{(r,\phi)}$, and

$$\Pi\big(\widetilde{u}^{(r,\phi)};\Omega_{(\hat{r}_\varepsilon,\phi)}\big) - \Pi\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big) = 0.$$

For $r > 0$, thus $\widetilde{u}^{(r,\phi)} = u^{(r,\phi)} \circ \Phi_V^{-1}(s_\varepsilon)$, we apply the coordinate transformation $\Phi_V(s_\varepsilon)$ and derive from (56) that

$$\begin{aligned}\Pi\big(\widetilde{u}^{(r,\phi)};\Omega_{(\hat{r}_\varepsilon,\phi)}\big) - \Pi\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big) &= \Pi\circ\Phi_V(s_\varepsilon)\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big) - \Pi\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\\ &= s_\varepsilon\Pi^1_V\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big) + \mathrm{Res}_{s_\varepsilon}(u^{(r,\phi)}).\end{aligned}$$

Using representation $s_\varepsilon = -t_\varepsilon R'(\phi)/R(\phi) + o(t_\varepsilon)$ and passing to the upper limit due to the strong convergences (51) and (54), we obtain two inequalities

$$\begin{aligned}\limsup_{\substack{t_\varepsilon\to 0\\ \pm t_\varepsilon>0}} D_{\phi_\varepsilon} \leq S\Big\{&\Pi^1_W\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\\ &+ \mathscr{H}(r)\Big(\frac{\partial}{\partial u}\Pi\big(u^{(r,\phi)}, G^1_\phi u^{(r,\phi)};\Omega_{(r,\phi)}\big) - \frac{R'(\phi)}{R(\phi)}\Pi^1_V\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\Big)\Big\},\end{aligned} \tag{63}$$

where $S = 1$ for the subsequence $\{t_\varepsilon \to 0 : t_\varepsilon > 0\}$, and $S = -1$ for the residual subsequence $\{t_\varepsilon \to 0 : t_\varepsilon < 0\}$.

Conversely, we construct the sequence

$$\begin{aligned}&\hat{u}^{(r,\phi_\varepsilon)} = F^\varepsilon\widetilde{u}^{(r,\phi_\varepsilon)},\\ &\widetilde{u}^{(r,\phi_\varepsilon)} = \begin{cases} u^{(r,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon) & \text{for } r = 0,\\ (u^{(r,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon))\circ\Phi_V(s_\varepsilon) & \text{for } r > 0,\end{cases}\end{aligned} \tag{64}$$

where

$$F^\varepsilon u = \begin{cases} u & \text{for } r = 0, \\ \left(u_1 \frac{\sin\phi_\varepsilon}{\sin\phi}, u_2 \frac{\cos\phi_\varepsilon}{\cos\phi}\right)^\top & \text{for } r > 0,\ \phi \neq 0,\ \ \pm\pi/2, \\ \left(u_1 \frac{\sin\phi_\varepsilon}{\sin\phi} - u_2 \frac{\cos\phi_\varepsilon}{\sin\phi}, u_2\right)^\top & \text{for } r > 0,\ \phi = \pm\pi/2. \end{cases}$$

This sequence satisfies the following properties:

$$\hat{u}^{(r,\phi_\varepsilon)} \in K_{\nu^\phi}(\Omega_{(r,\phi)}), \tag{65a}$$

$$\hat{u}^{(r,\phi_\varepsilon)} \to u^{(r,\phi)} \quad \text{strongly in } H(\Omega_{(R(\phi),\phi)}) \text{ as } \varepsilon \to 0. \tag{65b}$$

The inclusion (65a) can be checked directly, (65b) follows from (54) and (55a). With the help of (65a) we evaluate D_{ϕ_ε} from below:

$$\begin{aligned} D_{\phi_\varepsilon} &\geq \frac{1}{|t_\varepsilon|}\big(\Pi \circ \Phi_W(t_\varepsilon)(u^{(r,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon); \Omega_{(\hat{r}_\varepsilon,\phi)}) - \Pi(\hat{u}^{(r,\phi_\varepsilon)}; \Omega_{(r,\phi)})\big) \\ &= \frac{1}{|t_\varepsilon|}\big(\Pi(u^{(r,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon); \Omega_{(\hat{r}_\varepsilon,\phi)}) - \Pi(\hat{u}^{(r,\phi_\varepsilon)}; \Omega_{(r,\phi)})\big) \\ &\quad + \frac{t_\varepsilon}{|t_\varepsilon|}\Pi^1_W\big(u^{(r,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon); \Omega_{(\hat{r}_\varepsilon,\phi)}\big) + \frac{1}{|t_\varepsilon|}\mathrm{Res}_{t_\varepsilon}(u^{(r,\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon)). \end{aligned}$$

Using expansions

$$\begin{aligned} &\frac{\sin\phi_\varepsilon}{\sin\phi} = 1 + t_\varepsilon \cot\phi + o(t_\varepsilon), \quad \frac{\cos\phi_\varepsilon}{\cos\phi} = 1 - t_\varepsilon \tan\phi + o(t_\varepsilon), \\ &\frac{\sin\phi_\varepsilon}{\sin\phi}\bigg|_{\phi=\pm\pi/2} = 1 + o(t_\varepsilon), \quad \frac{\cos\phi_\varepsilon}{\sin\phi}\bigg|_{\phi=\pm\pi/2} = -t_\varepsilon + o(t_\varepsilon), \end{aligned}$$

similar to (62) we decompose

$$\hat{u}^{(r,\phi_\varepsilon)} = \widetilde{u}^{(r,\phi_\varepsilon)} - t_\varepsilon \mathscr{H}(r) G^1_\phi \widetilde{u}^{(r,\phi_\varepsilon)} + \mathscr{H}(r)\mathrm{Res}_{t_\varepsilon}(\widetilde{u}^{(r,\phi_\varepsilon)}).$$

For $r = 0$, the difference of the first two terms is zero in the right-hand side of the above evaluation of D_{ϕ_ε}. For $r > 0$, we apply the coordinate transformation $\Phi_V^{-1}(s_\varepsilon)$ and derive

$$
\begin{aligned}
&\Pi\big(u^{(r,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(\hat r_\varepsilon,\phi)}\big)-\Pi\big(\widetilde u^{(r,\phi_\varepsilon)};\Omega_{(r,\phi)}\big)\\
&\quad=\Pi\big(u^{(r,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(\hat r_\varepsilon,\phi)}\big)-\Pi\circ\Phi_V^{-1}(s_\varepsilon)\big(u^{(r,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(\hat r_\varepsilon,\phi)}\big)\\
&\quad=s_\varepsilon\Pi_V^1\big(u^{(r,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon);\Omega_{(\hat r_\varepsilon,\phi)}\big)+\mathrm{Res}_{s_\varepsilon}(u^{(r,\phi_\varepsilon)}\circ\Phi_W(t_\varepsilon)).
\end{aligned}
$$

Passing to the lower limit as $\varepsilon\to 0$ due to (54) and (65b) provides

$$
\begin{aligned}
\liminf_{\substack{t_\varepsilon\to 0\\ \pm t_\varepsilon>0}} D_{\phi_\varepsilon}\ge S\Big\{&\Pi_W^1\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\\
&+\mathscr{H}(r)\Big(\frac{\partial}{\partial u}\Pi\big(u^{(r,\phi)},G_\phi^1 u^{(r,\phi)};\Omega_{(r,\phi)}\big)-\frac{R'(\phi)}{R(\phi)}\Pi_V^1\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\Big)\Big\}.
\end{aligned}
\tag{66}
$$

From opposite inequalities (63) and (66) we conclude with the equality

$$
\begin{aligned}
\frac{\partial}{\partial\phi}P(r,\phi)&=\lim_{\phi_\varepsilon\to\phi}\frac{P(r,\phi_\varepsilon)-P(r,\phi)}{\phi_\varepsilon-\phi}\\
&=\Pi_W^1\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\\
&\quad+\mathscr{H}(r)\Big(\frac{\partial}{\partial u}\Pi\big(u^{(r,\phi)},G_\phi^1 u^{(r,\phi)};\Omega_{(r,\phi)}\big)-\frac{R'(\phi)}{R(\phi)}\Pi_V^1\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\Big).
\end{aligned}
\tag{67}
$$

Note that, $\Omega_{(0,\phi)}=\Omega_{(0,\widetilde\phi)}=\Omega_0$, hence $u^{(0,\phi)}=u^{(0,\widetilde\phi)}=u^0$ for all $\phi,\widetilde\phi\in[\phi_0,\phi_1]$, and $P(0,\phi)=\text{const}$. Therefore, $\frac{\partial}{\partial\phi}P(0,\phi)=0$ and $\Pi_W^1(u^0;\Omega_0)=0$ in (67) at $r=0$. Using the linearity of mapping $W\mapsto\Pi_W^1$, from (67) we arrive at formula (58).

To provide the assertion (b), the above consideration (without extension) is repeated for parameters

$$
(R(\phi_\varepsilon),\phi_\varepsilon)\to(R(\phi),\phi)\quad\text{as }\varepsilon\to 0,\quad \phi\neq 0.
$$

In this case, instead of (62) we construct:

$$
\hat u^{(R(\phi),\phi)}=G^\varepsilon u^{(R(\phi),\phi)}\in K_{\nu^{\phi_\varepsilon}}(\Omega_{(R(\phi),\phi)}),\tag{68a}
$$

$$
\hat u^{(R(\phi),\phi)}=u^{(R(\phi),\phi)}+t_\varepsilon G_\phi^1 u^{(R(\phi),\phi)}+\mathrm{Res}_{t_\varepsilon}(u^{(R(\phi),\phi)}),\tag{68b}
$$

$$
\hat u^{(R(\phi),\phi)}\to u^{(R(\phi),\phi)}\quad\text{strongly in }H(\Omega_{(R(\phi),\phi)})\text{ as }\varepsilon\to 0,\tag{68c}
$$

where G^ε is given by (50); instead of (64) we set:

$$\hat{u}^{(R(\phi_\varepsilon),\phi_\varepsilon)} = F^\varepsilon u^{(R(\phi_\varepsilon),\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon) \in K_{\nu^\phi}(\Omega_{(R(\phi),\phi)}), \tag{69a}$$

$$\begin{aligned}\hat{u}^{(R(\phi_\varepsilon),\phi_\varepsilon)} &= u^{(R(\phi_\varepsilon),\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon) - t_\varepsilon G^1_\phi u^{(R(\phi_\varepsilon),\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon) \\ &\quad + \mathrm{Res}_{t_\varepsilon}(u^{(R(\phi_\varepsilon),\phi_\varepsilon)} \circ \Phi_W(t_\varepsilon)),\end{aligned} \tag{69b}$$

$$\hat{u}^{(R(\phi_\varepsilon),\phi_\varepsilon)} \to u^{(R(\phi),\phi)} \quad \text{strongly in } H(\Omega_{(R(\phi),\phi)}) \text{ as } \varepsilon \to 0. \tag{69c}$$

As the result, similarly to (67) we obtain (59).

To find the derivative of P with respect to $r > 0$, we fix ϕ and consider

$$(r_\varepsilon, \phi) \to (r, \phi) \in \overline{\omega}_1 \setminus \{r = 0\} \quad \text{as } \varepsilon \to 0.$$

Using the one-to-one correspondence property (29) in Lemma 3, we estimate the corresponding finite difference from above:

$$\begin{aligned}D_{r_\varepsilon} &= \frac{P(r_\varepsilon,\phi) - P(r,\phi)}{|r_\varepsilon - r|} = \frac{\Pi(u^{(r_\varepsilon,\phi)};\Omega_{(r_\varepsilon,\phi)}) - \Pi(u^{(r,\phi)};\Omega_{(r,\phi)})}{|r_\varepsilon - r|} \\ &= \frac{1}{|r_\varepsilon - r|}\big(\Pi \circ \Phi_V(s_\varepsilon)(u^{(r_\varepsilon,\phi)} \circ \Phi_V(s_\varepsilon);\Omega_{(r,\phi)}) - \Pi(u^{(r,\phi)};\Omega_{(r,\phi)})\big) \\ &\le \frac{1}{|r_\varepsilon - r|}\big(\Pi \circ \Phi_V(s_\varepsilon)(u^{(r,\phi)};\Omega_{(r,\phi)}) - \Pi(u^{(r,\phi)};\Omega_{(r,\phi)})\big) \\ &= \frac{s_\varepsilon}{|r_\varepsilon - r|}\Pi^1_V(u^{(r,\phi)};\Omega_{(r,\phi)}) + \frac{1}{|r_\varepsilon - r|}\mathrm{Res}_{s_\varepsilon}(u^{(r,\phi)}), \quad s_\varepsilon = \ln(r_\varepsilon/r);\end{aligned}$$

and from below:

$$\begin{aligned}D_{r_\varepsilon} &\ge \frac{1}{|r_\varepsilon - r|}\big(\Pi \circ \Phi_V(s_\varepsilon)(u^{(r_\varepsilon,\phi)} \circ \Phi_V(s_\varepsilon);\Omega_{(r,\phi)}) - \Pi(u^{(r_\varepsilon,\phi)} \circ \Phi_V(s_\varepsilon);\Omega_{(r,\phi)})\big) \\ &= \frac{s_\varepsilon}{|r_\varepsilon - r|}\Pi^1_V(u^{(r_\varepsilon,\phi)} \circ \Phi_V(s_\varepsilon);\Omega_{(r,\phi)}) + \frac{1}{|r_\varepsilon - r|}\mathrm{Res}_{s_\varepsilon}(u^{(r_\varepsilon,\phi)} \circ \Phi_V(s_\varepsilon)).\end{aligned}$$

Expanding $s_\varepsilon = \frac{r_\varepsilon - r}{r} + o(r_\varepsilon - r)$, we conclude with formula (60) and the assertion (c), where the non-positivity property of the derivatives follows from (31a).

For the specific case of $\phi = 0$, using the transformation of tangential shift with the velocity $U = \eta(x)(1,0)^\top$ along the rectilinear crack $\Gamma_{(r,0)}$, formula (61) for the derivative $\frac{\partial}{\partial r}P(r,0)$ follows from the earlier works [**17**], [**18**]. Note that, in these references, a perturbation of the identity operator was applied for the sensitivity analysis. □

Remark 1. The last term in expression (58) (respectively, in (59)) can be rewritten equivalently by excluding the vectors at which the equality is attained in (27). For example, for $\phi \neq 0$, $\pm\pi/2$, substitution into (27) of the test elements

$$v = \big(u_1^{(r,\phi)} - u_2^{(r,\phi)} \cot\phi, 0\big)^\top, \quad v = \big(u_1^{(r,\phi)} + u_2^{(r,\phi)} \cot\phi, 2u_2^{(r,\phi)}\big)^\top$$

yields

$$\frac{\partial}{\partial u}\Pi\big(u^{(r,\phi)}, (u_2^{(r,\phi)} \cot\phi, u_2^{(r,\phi)})^\top; \Omega_{(r,\phi)}\big) = 0.$$

Multiplying this identity with $-\tan\phi$ and adding the resulting zeroth term to (58), we obtain

$$\frac{\partial}{\partial u}\Pi\big(u^{(r,\phi)}, G_\phi^1 u^{(r,\phi)}; \Omega_{(r,\phi)}\big) = \frac{\partial}{\partial u}\Pi\big(u^{(r,\phi)}, \widetilde{G}_\phi^1 u^{(r,\phi)}; \Omega_{(r,\phi)}\big),$$

where

$$\widetilde{G}_\phi^1 u = (-u_1 \cot\phi - u_2, 0)^\top \quad \text{for } \phi \neq 0.$$

In comparison to G_ϕ^1, the coefficients in $\widetilde{G}_\phi^1$ are continuous with respect to $\phi \to \pm\pi/2$.

Remark 2. For $A = O$, the differentiability properties (a) and (b) of Theorem 2 can be extended to $\phi = 0$, with the derivative

$$\frac{\partial}{\partial\phi}P(r,0) = \mathscr{H}(r)\bigg(\Pi^1_{W-\frac{R'(0)}{R(0)}V}(u^{(r,0)}; \Omega_{(r,0)}) + \frac{\partial}{\partial u}\Pi\big(u^{(r,0)}, (0, u_1^{(r,0)})^\top; \Omega_{(r,0)}\big)\bigg).$$

This expression follows by construction in (50) of the function

$$G^\varepsilon \widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2 + \widetilde{v}_1 \tan\phi_\varepsilon)^\top \quad \text{for } r > 0,\ \phi = 0.$$

Remark 3. For the crack problem in the linearized setting, omitting the non-penetration condition, i.e.,

$$u^{(r,\phi)} \in H(\Omega_{(r,\phi)}) \text{ minimizes } \Pi(v; \Omega_{(r,\phi)}) \quad \text{over all } v \in H(\Omega_{(r,\phi)}),$$

the terms with $\frac{\partial}{\partial u}\Pi$ disappear in the expressions (58) and (59), since in this case

$$\frac{\partial}{\partial u}\Pi\big(u^{(r,\phi)}, v; \Omega_{(r,\phi)}\big) = 0 \quad \text{for all } v \in H(\Omega_{(r,\phi)}).$$

Notice also, for the linearized crack problem without non-penetration, properties (31d), (31e) and (58), (59) can be extended to the case of $\phi = 0$.

4. Evolution problem for the crack with kink.

Let the initial crack (at $t = 0$) be given by the segment $AO = \Gamma_0$ of the length $l_0 \geq 0$. For a time-like loading parameter $t \geq 0$, we suppose a linear loading applied to the solid with crack by means of $f(t) = tf$ and $g(t) = tg$ in the equilibrium relations (20a) and (20c). Note, it follows that $u(t) = tu$ is a solution to (20) in this case. For such loading, we look for a propagating crack $\Gamma_{(r(t),\phi^\star)} \subset \Omega$ with a kink at the origin O and unknown shape parameters of the crack length $l_0 + r(t)$ and the kink angle $\phi^\star \in [\phi_0, \phi_1]$. We suggest a shape optimization approach, which is based on the Griffith hypothesis, and we use the crack sensitivity results of Section 3.

Following the Griffith hypothesis, for $t \geq 0$ we define a function of the total potential energy

$$\overline{\omega} \ni (r, \phi) \mapsto T(r, \phi)(t) = 2\gamma(l_0 + r) + t^2 P(r, \phi). \tag{70}$$

The first term $2\gamma(l_0 + r)$ presents the surface energy distributed uniformly at the two crack surfaces with a constant density $\gamma > 0$ (the given material parameter). The second term in (70) represents the potential energy, which is quadratic in t:

$$P(r, \phi)(t) = \Pi\big(u^{(r,\phi)}(t); \Omega_{(r,\phi)}\big) = \Pi\big(tu^{(r,\phi)}; \Omega_{(r,\phi)}\big) = t^2\Pi\big(u^{(r,\phi)}; \Omega_{(r,\phi)}\big).$$

Obviously, for each fixed $t < \infty$, the function $(r, \phi) \mapsto T$ obeys the same continuity and differentiability properties as established in Theorem 1 and Theorem 2 for $(r, \phi) \mapsto P$, excepting the non-increase property (31a), because $r \mapsto 2\gamma(l_0 + r)$ is a strictly increasing function.

Evolution of the crack is described with the help of the two-parametric shape optimization problem: $r(0) = 0$; for $t > 0$, find parameters $(r(t), \phi(t)) \in \overline{\omega}$ that

$$\text{minimize } T(r, \phi)(t) \quad \text{over } (r, \phi) \in \overline{\omega} \tag{71a}$$

$$\text{subject to } \phi \in \bigcap_{s<t} \{\phi(s)\}. \tag{71b}$$

The latter constraint (71b) allows us to preserve the shape of a kinking crack during its evolution. To clarify this feature, first we investigate the minimization problem (71a) without constraint in the following lemma.

Lemma 4. *For arbitrary fixed $D \subseteq \omega$ and $0 \leq t < \infty$, there exists the solution $(r(t), \phi(t)) \in \overline{D}$ to minimization problem* (71a) *over all $(r, \phi) \in \overline{D}$, which is characterized by the following relations*:

$$r(0) = 0, \tag{72a}$$

$$r(s) \leq r(t) \quad \text{for } s \leq t, \tag{72b}$$

$$2\gamma(l_0 + r(t)) + t^2 P(r(t), \phi(t)) \leq 2\gamma(l_0 + r) + t^2 P(r, \phi) \quad \text{for all } (r, \phi) \in \overline{D}. \tag{72c}$$

The function $t \mapsto r(t)$ can be multi-valued, $r(t) \in [r^-(t), r^+(t)]$ with

$$\liminf_{\substack{s \to t \\ s < t}} r(s) = r^-(t) < r^+(t) = \limsup_{\substack{s \to t \\ s > t}} r(s)$$

at some point t, where it satisfies

$$T(r^-(t), \phi^-)(t) = T(r^+(t), \phi^+)(t), \quad \phi^\pm \in \{\phi(t)\}, \tag{73}$$

otherwise $r(t)$ is unique. The crack is defined by $\Gamma_{(r^+(t), \phi^+)}$.

Proof. Fix $t \geq 0$. We consider a minimizing sequence $(r^n, \phi^n) \in \overline{D}$ such that

$$T(r^n, \phi^n)(t) \to T_0(t) = \inf_{(r,\phi) \in \overline{D}} T(r, \phi)(t) \quad \text{as } n \to \infty.$$

Since $\overline{D}$ is bounded, there exists a convergent subsequence (still denoted by (r^n, ϕ^n)) such that

$$(r^n, \phi^n) \to (r(t), \phi(t)) \in \overline{D} \quad \text{as } n \to \infty.$$

Using the lower semicontinuity of $\overline{D} \ni (r, \phi) \mapsto T$, which follows from (31c) in Theorem 1, we estimate

$$T_0(t) = \liminf_{n \to \infty} T(r^n, \phi^n)(t) \geq T(r(t), \phi(t))(t) \geq T_0(t).$$

Thus, $(r(t),\phi(t)) \in \overline{D}$ solves (71a) for every $t \geq 0$, that is written in (72c).

For $t = 0$, evidently, (72c) follows (72a). For $t > 0$ and $s \leq t$, we have similarly to (72c) the inequality

$$2\gamma(l_0 + r(s)) + s^2 P(r(s),\phi(s)) \leq 2\gamma(l_0 + r) + s^2 P(r,\phi) \quad \text{for all } (r,\phi) \in \overline{D}. \tag{74}$$

Substituting $(r,\phi) = (r(s),\phi(s))$ into (72c), and $(r,\phi) = (r(t),\phi(t))$ into (74), summarizing these inequalities we infer

$$P(r(t),\phi(t)) \leq P(r(s),\phi(s)).$$

The assumption of $r(t) < r(s)$ leads to a contradiction to (74). Therefore, (72b) holds true.

If $r(t)$ is not unique, then from (72b) we conclude

$$r^-(t) \leq \inf r(t) < \sup r(t) \leq r^+(t).$$

For $s \to t$, there exist two subsequences (with the same notation) such that $(r(s),\phi(s)) \to (r^+(t),\phi^+) \in \overline{D}$ for $s > t$ and $(r(s),\phi(s)) \to (r^-(t),\phi^-) \in \overline{D}$ for $s < t$. Assertions (31b) and (31c) of Theorem 1 provide the lower semicontinuity property of $s \mapsto T(r(s),\phi(s))(s)$:

$$\begin{aligned}
&\liminf_{s\to t} T(r(s),\phi(s))(s) \\
&\quad = \liminf_{s\to t} \big(2\gamma(l_0 + r(s)) + (s^2 - t^2)P(r(s),\phi(s)) + t^2 P(r(s),\phi(s))\big) \\
&\quad \geq 2\gamma(l_0 + r(t)) + t^2 P(r(t),\phi(t)) = T(r(t),\phi(t))(t)
\end{aligned}$$

for both $s > t$ and $s < t$. Passing to the lower limit in (74) as $s \to t$, for $s > t$ we obtain

$$2\gamma(l_0 + r^+(t)) + t^2 P(r^+(t),\phi^+) \leq 2\gamma(l_0 + r) + t^2 P(r,\phi) \quad \text{for all } (r,\phi) \in \overline{D},$$

and for $s < t$ similarly

$$2\gamma(l_0 + r^-(t)) + t^2 P(r^-(t),\phi^-) \leq 2\gamma(l_0 + r) + t^2 P(r,\phi) \quad \text{for all } (r,\phi) \in \overline{D}.$$

Therefore, $\phi^\pm \in \{\phi(t)\}$,

$$\inf r(t) = r^-(t) = \min r(t), \quad \sup r(t) = r^+(t) = \max r(t),$$

and condition (73) holds true. □

From Lemma 4 we conclude that, generally, $\phi(t) \neq \phi(s)$ after solving the minimization problem (71a). Hence, (71a) cannot be used to describe the phenomenon of crack propagation preserving the crack shape by means of $\phi(t) = \phi(s) = \text{const}$. For this reason, we employ the constraint (71b) and prove the following result.

Theorem 3. *Excepting trivial solutions $r(t) = 0$ for all $t \geq 0$, there exist $0 \leq t_\star < \infty$, $\phi^\star \in [\phi_0, \phi_1]$, and $r(t) \in [0, R(\phi^\star)]$ such that parameters $(r(t), \phi(t)) \in \overline{\omega}$ given by*

$$\begin{cases} r(t) = 0, \ \phi(t) \text{ is arbitrary in } [\phi_0, \phi_1], & \text{for } 0 \leq t < t_\star, \ t_\star > 0, \\ \phi(t) = \phi^\star & \text{for } t \geq t_\star, \end{cases} \tag{75}$$

solve the minimization problem (71). *If there exists $c_* = const$ such that*

$$\liminf_{r \to 0} \frac{\partial}{\partial r} P(r, \phi) > c_* \quad \text{for } \phi \in [\phi_0, \phi_1], \tag{76}$$

then $t_\star > 0$.

At the time $t_\star$ of kink, the solution components $r^-(t_\star) = 0 \leq r^+(t_\star)$ and $\phi^\star$ are characterized by the necessary conditions:

$$T(r(t_\star), \phi^\star)(t_\star) \leq T(r, \phi)(t_\star) \quad \text{for all } (r, \phi) \in \overline{\omega}, \tag{77a}$$

$$T(0, \phi^\star)(t_\star) = T(r^+(t_\star), \phi^\star)(t_\star) \quad \text{for } r^+(t_\star) > 0. \tag{77b}$$

For $t > t_\star$, the solution is characterized by the following relations:

$$r(s) \leq r(t) \quad \text{for } s \leq t, \tag{78a}$$

$$T(r(t), \phi^\star)(t) \leq T(r, \phi^\star)(t) \quad \text{for all } r \in [0, R(\phi^\star)], \tag{78b}$$

$$T(r^-(t), \phi^\star)(t) = T(r^+(t), \phi^\star)(t) \quad \text{for } r^-(t) < r^+(t). \tag{78c}$$

Proof. Reminding that $P(0, \phi) = P(0) = \text{const}$ for all $\phi \in [\phi_0, \phi_1]$, let us denote by $T(0)(t) = 2\gamma l_0 + t^2 P(0)$. The trivial solution $r(t) = 0$ and arbitrary $\phi(t) \in [\phi_0, \phi_1]$ fulfill the strict inequality

$$T(0)(t) < T(r, \phi)(t) \quad \text{for all } (r, \phi) \in \overline{\omega} \setminus \{r = 0\}, \ t \geq 0.$$

Otherwise, there exists $0 < t < \infty$ such that

$$\widetilde{\phi} \in [\phi_0, \phi_1], \quad 0 < \widetilde{r} \le R(\widetilde{\phi}), \quad T(0)(t) \ge T(\widetilde{r}, \widetilde{\phi})(t).$$

The latter inequality can be reduced to the equality by decreasing t due to the continuity of $t \mapsto T(\widetilde{r}, \widetilde{\phi})(t)$ and $T(0)(0) < T(\widetilde{r}, \widetilde{\phi})(0)$. Therefore, there exists $0 < \widetilde{t} < \infty$ such that

$$\widetilde{\phi} \in [\phi_0, \phi_1], \quad 0 < \widetilde{r} \le R(\widetilde{\phi}), \quad T(0)(\widetilde{t}) = T(\widetilde{r}, \widetilde{\phi})(\widetilde{t}). \tag{79}$$

Let us define

$$t_\star = \inf\{\widetilde{t} : \ (79) \text{ is satisfied}\}.$$

By the assumption (76) we assert $t_\star > 0$ and prove this by contradiction arguments. If $t_\star = 0$, then there exists a minimizing sequence $\widetilde{t}_n \to 0$ as $n \to \infty$ satisfying similar to (79) relations:

$$\widetilde{\phi}_n \in [\phi_0, \phi_1], \quad 0 < \widetilde{r}_n \le R(\widetilde{\phi}_n), \tag{80a}$$

$$2\gamma l_0 + \widetilde{t}_n^2 \, P(0) = 2\gamma(l_0 + \widetilde{r}_n) + \widetilde{t}_n^2 \, P(\widetilde{r}_n, \widetilde{\phi}_n). \tag{80b}$$

We extract a convergent subsequence $(\widetilde{r}_n, \widetilde{\phi}_n) \to (\widetilde{r}, \widetilde{\phi}) \in \overline{\omega}$ as $n \to \infty$. Using the lower semicontinuity property (31c), from (80) it follows that

$$2\gamma l_0 = \lim_{n\to\infty} T(0)(\widetilde{t}_n) = \liminf_{n\to\infty} T(\widetilde{r}_n, \widetilde{\phi}_n)(\widetilde{t}_n) \ge T(\widetilde{r}, \widetilde{\phi})(0) = 2\gamma(l_0 + \widetilde{r}).$$

Therefore, $\widetilde{r} = 0$ and $\widetilde{r}_n \to 0$. Let us rewrite (80b) as

$$0 = 2\gamma + \widetilde{t}_n^2 \, \frac{P(\widetilde{r}_n, \widetilde{\phi}_n) - P(0)}{\widetilde{r}_n}.$$

For fixed $\widetilde{\phi}_n$, using the continuity of $r \mapsto P(r, \widetilde{\phi}_n)$ at $r \in [0, \widetilde{r}_n]$ and its differentiability at $r \in (0, \widetilde{r}_n)$ due to (c) of Theorem 2, we can apply the mean value theorem and derive

$$0 = 2\gamma + \widetilde{t}_n^2 \, \frac{\partial}{\partial r} P(\overline{r}_n, \widetilde{\phi}_n), \quad \overline{r}_n \in (0, \widetilde{r}_n).$$

Passing here to the lower limit as $n \to \infty$ due to (76), assertion $\widetilde{t}_n \to 0$ results in the contradiction $0 \geq 2\gamma$. Thus, $t_\star > 0$.

Applying Lemma 4 with $D = \omega$, we find the solution $(r(t_\star), \phi^\star) \in \overline{\omega}$ to minimization problem (71a) at $t_\star$, which satisfies (71b), too. This gives us the angle $\phi^\star$ of kink. The solution fulfills inequality (77a), and equality (77b) follows from (77a) in the case of a non-zero jump of $t \mapsto r(t)$ at $t_\star$, similarly to (73).

For $t > t_\star$, the constraint in (71b) becomes $\phi \in \cap_{s<t}\{\phi(s)\} = \{\phi^\star\}$. Therefore, Lemma 4 with $D = (0, R(\phi^\star)) \times \{\phi^\star\}$ guarantees existence of the solution to the resulting one-parametric shape optimization problem: For $t > t_\star$, find $r(t) \in [0, R(\phi^\star)]$ such that

$$r(t) \text{ minimizes } T(r, \phi^\star)(t) \quad \text{over all } r \in [0, R(\phi^\star)]. \tag{81}$$

Relations (78) follow directly from (72) and (73). □

Note that, if $\phi^\star = 0$ in (77), then the crack propagates without kink along the line $x_2 = 0$.

Theorem 3 provides existence of the optimal crack $\Gamma_{(r(t),\phi(t))}$ presented by the shape parameters $r(t)$ and $\phi(t)$ in the form (75). Using the differentiability properties of Section 3.2 for the necessary optimality conditions, the optimal parameters are determined on a set of extremal points. In fact, to solve (71a), we are to minimize $T(r, \phi)(t)$ over the following extremal points in $\overline{\omega}$:

$$\begin{aligned}
&(r, \phi_0) \quad \text{for } r \in (0, R(\phi_0)) \text{ such that } 2\gamma + t^2 \frac{\partial}{\partial r} P(r, \phi_0) = 0,\\
&(r, \phi_1) \quad \text{for } r \in (0, R(\phi_1)) \text{ such that } 2\gamma + t^2 \frac{\partial}{\partial r} P(r, \phi_1) = 0;
\end{aligned} \tag{82a}$$

$$\begin{aligned}
&(0, \phi) \quad \text{for } \phi \in [\phi_0, \phi_1],\\
&(r, 0) \quad \text{for } r \in (0, R(0)) \text{ such that } 2\gamma + t^2 \frac{\partial}{\partial r} P(r, 0) = 0;
\end{aligned} \tag{82b}$$

$$\begin{aligned}
&(R(\phi_0), \phi_0),\ (R(\phi_1), \phi_1),\ (R(0), 0),\\
&(R(\phi), \phi) \quad \text{for } \phi \in (\phi_0, \phi_1) \setminus \{0\} \text{ such that } \frac{\partial}{\partial \phi} P(R(\phi), \phi) = 0;
\end{aligned} \tag{82c}$$

$$\begin{aligned}
&(r, \phi) \in \omega \setminus \{\phi = 0\} \quad \text{such that}\\
&2\gamma + t^2 \frac{\partial}{\partial r} P(r, \phi) = 0, \quad \frac{\partial}{\partial \phi} P(r, \phi) = 0.
\end{aligned} \tag{82d}$$

Formulas for calculation of the derivatives in (82) are given by (58)–(61) in Theorem 2. Notice that this calculation is independent of t.

After finding the time $t_\star$ and the angle $\phi^\star$ of kink, the minimization problem (81) is realized over the extremal points along the line $\phi = \phi^\star$:

$$\begin{aligned} &(0,\phi^\star),\ (R(\phi^\star),\phi^\star),\ \text{and } (r,\phi^\star) \quad \text{for } r \in (0, R(\phi^\star)) \text{ such that} \\ &2\gamma + t^2 \frac{\partial}{\partial r} P(r,\phi^\star) = 0. \end{aligned} \tag{83}$$

From the fracture standpoint, it is important to note that equality $2\gamma + t^2 \frac{\partial}{\partial r} P(r,\phi^\star) = 0$ implies exactly the Griffith fracture criterion for the tangential propagation of crack along the pre-defined path (along the $\phi^\star$-direction in our case).

The problem formulation accounts all geometrically possible situations such as: unbounded time-interval of t; accumulation of $r(t)$ near some point as $t \to \infty$; non-uniqueness with respect to $\phi^\star$, and so on. Treating particular situations allows us to describe the solution in simplified way. For example, if we suggest that $r^+(t_\star) = R(\phi^\star)$, i.e., the crack kinking effects an immediate break of the solid, then $\phi^\star$ is determined only from (82c), which is independent of t.

5. Conclusion.

We emphasize the fact that the evolution problem describes time-propagation of a crack with kink in the nonlinear setting, which admits the condition of non-penetration between the opposite crack surfaces.

It is important to note that our consideration deals also with the case of $A = O$ thus describing the phenomenon of appearance of a crack in a homogeneous solid.

In contrast to the local asymptotic analysis as $r(t) \approx 0$, we describe arbitrary long-time evolution $r(t)$ of a crack with kink. From Theorem 3 we arrive at the conclusion, that the evolution of a crack can be stable for $r^+(t_\star) = 0$, as well as unstable with a non-zero jump $r^+(t_\star) > 0$ for the advanced crack. In the latter case, local asymptotic methods are not applicable. This feature presents the principal advantage of our optimization approach.

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