

Research Article

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Inverse problem of shape identification from boundary measurement for Stokes equations: Shape differentiability of Lagrangian

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Abstract: For Stokes equations under divergence-free and mixed boundary conditions, the inverse problem of shape identification from boundary measurement is investigated. Taking the least-square misfit as an objective function, the state-constrained optimization is treated by using an adjoint state within the Lagrange approach. The directional differentiability of a Lagrangian function with respect to shape variations is proved within the velocity method, and a Hadamard representation of the shape derivative by boundary integrals is derived explicitly. The application to gradient descent methods of iterative optimization is discussed.

Keywords: Stokes flow, incompressibility, state-constrained optimization, Lagrangian, saddle-point problem, adjoint state, inverse identification problem, shape derivative

MSC 2010: 35Q30, 49Q10, 49Q12, 90C47

Dedicated to M. V. Klibanov 70th birthday [17]

1 Introduction

In the present paper, we prove the shape derivative for optimal value of a Lagrange function, which describes the inverse Stokes problem of shape identification by a least-square misfit from boundary measurements.

In a broad scope, optimization of shapes is a specific class of inverse problems; look at the survey [22]. The shape optimization is ill-posed in general because objectives have typically many local minima when varying shapes. For the theory of coefficient inverse problems, see [4], its applications in mathematical physics [36], and proper regularization in [23]. We cite [3] for the least-square method, [1, 7, 32] for the use of least squares in inverse scattering by obstacles, and [16] for coefficient identification in variational inequalities. Our special interest concerns variational fracture models for nonpenetrating cracks in solids [26] and their differentiability with respect to crack perturbations [27, 31], which are useful for optimal control [37, 38, 47], overdetermined [34] and inverse problems [19, 25]. The Stokes equations under consideration can be interpreted within the incompressible elasticity in solid mechanics; in this sense, the shape derivative is linked to Griffith's fracture criteria (see [2, 40]).

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The shape optimization in fluid mechanics was developed in [42, 44]. We refer to [35, 41] for the mathematical theory of incompressible flows described by Stokes and Navier–Stokes equations, to [28, 29] for flow in channels and thin layers, and [5, 39] for mixed variational formulations provided by boundary conditions. The overdetermined problems were studied in [11] with respect to well-posedness; the optimality condition under mixed control-state constraints was obtained in [10]. In the optimization context, Bernoulli-type free boundary problems [6, 24] and the coefficient identification [18] were based on least squares, while the reconstruction of obstacles immersed in a fluid by boundary measurements in [45] utilized the enclosure method.

A shape derivative is of the first importance in shape optimization because determining the first-order optimality condition with respect to perturbation of geometry. The variational approach involving shape derivatives was developed in [48] and further extended to constrained PDE models; see [12, 20, 21] and other works. The geometry-dependent function space formulation needs a bijective change of coordinates to transform a shape-perturbed problem to the reference geometry. The bijectivity is, however, not always the case when dealing with feasible sets due to constraints such as contact, incompressibility (divergence-free), etc. For comparison, preserving the divergence, in [43], the Piola transform was suggested to treat unsteady problems in generalized Bochner spaces on moving non-cylindrical domains. In a general case, the abstract formalism of directional derivatives of a minimax function [8] was successful to justify the shape differentiability for constrained problems within the Lagrange approach [9].

In [30], the Lagrange method was applied first to derive the shape derivative of strain energy (the Griffith formula) for curvilinear cracks constrained by the nonpenetration inequality, and for breaking-line identification under state constraints [13]. Recently, we studied Stokes and Brinkman–Stokes equations subject to the divergence-free equality with respect to the shape differentiability of its energy function [14, 33]. For objective functions given in a general form, the Hadamard formula of shape derivatives by boundary integrals was formally used in [46]. In the current paper, we prove rigorously the shape differentiability of the least-square objective using the equivalent Lagrange formulation for Stokes equations and its adjoint state. The obtained analytical expression of the shape derivative and the respective Hadamard representation are advantageous for gradient descent algorithms solving the inverse problem of shape identification from boundary measurements (see Corollary 5.2).

In Section 2, we set well-posedness for a geometry-dependent forward Stokes problem under mixed boundary condition (see Proposition 2.1), existence and non-uniqueness for the inverse identification problem (see Proposition 2.2). In Section 3, an equivalent saddle-point formulation with the adjoint Stokes problem is presented in Theorem 3.1 and Corollaries 3.2 and 3.3. Based on Traits 1–4, the main theorem (Theorem 4.1) on shape derivative of the corresponding optimal value Lagrange function is proved in Section 4 and Appendix A. The Hadamard formula is established in Theorem 5.1 in Section 5, which provides an identification strategy based on the descent gradient method.

2 Inverse Stokes problem of shape identification

We start with the description of a family of parameter-dependent geometries

$$[t \mapsto \Omega_t]: (t_0, t_1) \mapsto \mathbb{R}^d, \quad d = 2, 3, \quad \Omega_t \subset D, \quad (2.1)$$

where $D \subset \mathbb{R}^d$ is a bounded hold-all set. For every fixed time parameter t , let Ω_t be a domain with the Lipschitz continuous boundary $\partial\Omega_t$ and the unit normal vector $n^t = (n_1^t, \dots, n_d^t)^\top$ which is outward to Ω_t . Here and in what follows, the upper script \top swaps between rows and columns. We assume that $\partial\Omega_t$ consists of two nonempty, mutually disjoint sets Γ_t^D and Γ_t^N .

For a given stationary force $f(x) = (f_1, \dots, f_d)^\top \in H^1(D)^d$ and the fluid viscosity $\mu > 0$, we consider the *forward Stokes problem* under mixed Dirichlet–Neumann boundary conditions: find a flow velocity vector $u^t(x) = (u_1^t, \dots, u_d^t)^\top$ and a pressure $p_t(x)$ satisfying the following relations (see [5, 18]):

$$-\mu\Delta u^t + \nabla p_t = f, \quad \operatorname{div}(u^t) = 0 \quad \text{in } \Omega_t, \quad (2.2a)$$

$$\varepsilon(u^t) := \frac{1}{2}(\nabla u^t + (\nabla u^t)^\top), \quad (2.2b)$$

$$u^t = 0 \quad \text{on } \Gamma_t^D, \quad 2\mu\varepsilon(u^t)n^t - p_t n^t = 0 \quad \text{on } \Gamma_t^N. \quad (2.2c)$$

Here the notation stands for the gradient vector $\nabla := (\partial/\partial x_1, \dots, \partial/\partial x_d)^\top$, the divergence $\operatorname{div} := (\nabla \cdot)$, and the Laplace operator Δ . The gradient of a vector is defined as $\nabla u^t = (\partial u_i^t / \partial x_j)_{i,j=1}^d$, the linearized strain $\varepsilon(u^t)$ in (2.2b) is a d -by- d symmetric matrix, and $\varepsilon(u^t)n^t$ in (2.2c) implies a matrix-vector multiplication.

Taking into account the no-slip boundary condition in (2.2c), we introduce the Sobolev space of admissible velocity vectors

$$V(\Omega_t) := \{w = (w_1, \dots, w_d)^\top \in H^1(\Omega_t)^d \mid w = 0 \text{ a.e. } \Gamma_t^D\}. \quad (2.3)$$

The incompressibility condition in (2.2a) is determined well by the mapping

$$[w \mapsto \operatorname{div}]: V(\Omega_t) \mapsto L^2(\Omega_t). \quad (2.4)$$

Proposition 2.1. *There exists the unique solution pair $(u^t, p_t) \in V(\Omega_t) \times L^2(\Omega_t)$ satisfying the forward Stokes problem (2.2) in a mixed variational form*

$$\int_{\Omega_t} (2\mu\varepsilon(u^t) \cdot \varepsilon(w) - p_t \operatorname{div}(w) - f^\top w) dx = 0 \quad \text{for all } w \in V(\Omega_t), \quad (2.5a)$$

$$\int_{\Omega_t} \lambda \operatorname{div}(u^t) dx = 0 \quad \text{for all } \lambda \in L^2(\Omega_t), \quad (2.5b)$$

where dot (\cdot) in (2.5a) implies the scalar product of second order tensors.

Proof. The equilibrium equation in (2.2a) can be rewritten equivalently using

$$-\mu\Delta u^t = -\mu(\Delta u^t + \nabla \operatorname{div}(u^t)) = -2\mu \operatorname{div} \varepsilon(u^t)$$

by virtue of incompressibility and (2.2b). Then formulation (2.5) can be derived by the standard variational technique multiplying equations (2.2a) with the corresponding test functions w, λ and integrating them over Ω_t , with the subsequent integration of the first equation by part using boundary conditions (2.2c).

The quadratic term in (2.5a) determines a bounded, symmetric, bilinear quadratic form, which is strongly elliptic by the Korn–Poincaré inequality,

$$\int_{\Omega_t} \varepsilon(w) \cdot \varepsilon(w) dx \geq K_{\text{KP}} \|w\|_{H^1(\Omega_t)^d}^2 \quad \text{for } w \in V(\Omega_t), \quad K_{\text{KP}} > 0. \quad (2.6)$$

Since $\Gamma_t^N \neq \emptyset$, the inf-sup (LBB) condition holds for $\lambda \in L^2(\Omega_t)$ (see [35]),

$$\sup_{w \in V(\Omega_t), w \neq 0} \frac{1}{\|w\|_{H^1(\Omega_t)^d}} \int_{\Omega_t} \lambda \operatorname{div}(w) dx \geq K_{\text{LBB}} \|\lambda\|_{L^2(\Omega_t)}, \quad K_{\text{LBB}} > 0. \quad (2.7)$$

Then the mapping in (2.4) is surjective, and the Ladyzhenskaya–Babuška–Brezzi–Nečas theorem follows that there exists the unique solution to (2.5). \square

Our long-term aim is to identify Ω_t by the shape optimization approach as described in [13]. Let

$$z(x) = (z_1, \dots, z_d)^\top \in L^2(\Gamma_t^O)^d$$

be an observation given at a part $\Gamma_t^O \subset \Gamma_t^N$ of the boundary $\partial\Omega_t$. We introduce a least-square L^2 -misfit from the observation as the geometry-dependent objective $[w \mapsto \mathcal{J}]: V(\Omega_t) \mapsto \mathbb{R}$,

$$\mathcal{J}(w; \Omega_t) := \frac{1}{2} \int_{\Gamma_t^O} |w - z|^2 dS_x, \quad (2.8)$$

where $|w| = \sqrt{w^\top w}$ denotes the Euclidean norm of vectors. In the hold-all domain D with a fixed part $D^D \subset D$, admissible geometries form a set

$$\mathfrak{G} = \{\Omega_t \subset D \mid \Gamma_t^D \subset D^D\}. \quad (2.9)$$

For variable shapes from (2.9) parameterized by t , the *inverse Stokes problem* consists in a state-constrained (MPEC) optimization problem: find Ω_t such that

$$\inf_{\Omega_t \subset \mathcal{G}} \mathcal{J}(u^t; \Omega_t), \quad \text{where } (u^t, p_t) \text{ solves (2.5) in } \Omega_t. \quad (2.10)$$

Let $\Omega \subset \mathbb{R}^d$ be a domain with the Lipschitz continuous boundary $\partial\Omega$ consisting of two nonempty, mutually disjoint sets Γ^D and Γ^N such that $\Omega \subset D$. We say that Ω is a *feasible geometry* if there exists a pair $(z, p_z) \in V(\Omega) \times L^2(\Omega)$ which satisfies the variational equations from Proposition 2.1 stated in Ω ,

$$\begin{aligned} \int_{\Omega} (2\mu\varepsilon(z) \cdot \varepsilon(w) - p_z \operatorname{div}(w) - f^\top w) dx &= 0 \quad \text{for all } w \in V(\Omega), \\ \int_{\Omega} \lambda \operatorname{div}(z) dx &= 0 \quad \text{for all } \lambda \in L^2(\Omega). \end{aligned} \quad (2.11)$$

For $\Gamma^0 \subset \Gamma^N$, we call the trace at Γ^0 of z satisfying (2.11) a *feasible measurement*.

Proposition 2.2. *For every feasible measurement z , there exists a solution to the inverse Stokes problem (2.10), which is non-unique in general.*

Proof. If z is feasible, thus (z, p_z) satisfy (2.11), then $\mathcal{J}(u^t; \Omega_t)$ in (2.10) attains the minimal value zero as $\Omega_t = \Omega$ and $(u^t, p_t) = (z, p_z)$.

Supposing uniqueness, we present the following counter-example. Let $f = 0$ outside Ω , and let z solving (2.11) be zero at $\Gamma^N \setminus \Gamma^0 \neq \emptyset$. Extending (z, p_z) with zero into larger Ω_t such that $\Omega \subset \Omega_t \subset D$ and preserving the observation boundary $\Gamma_t^0 = \Gamma^0$, we get an extension $(\bar{z}, \bar{p}_z) \in V(\Omega_t) \times L^2(\Omega_t)$. The zero-extension satisfies the variational equations (2.11) restated in Ω_t . Henceforth, $\mathcal{J}(\bar{z}; \Omega_t) = \mathcal{J}(z; \Omega) = 0$, and those Ω_t solve (2.10) as well as Ω . \square

3 Lagrange formulation using adjoint state

To express the state-constrained optimization problem (2.10) in a form suitable for analysis and numerical solution, we apply the Lagrange approach.

Based on the objective \mathcal{J} from (2.8) and state equations (2.5), the *Lagrangian function* $\mathcal{L} : U(\Omega_t)^2 \mapsto \mathbb{R}$ over $U(\Omega_t) := V(\Omega_t) \times L^2(\Omega_t)$ is defined by

$$\mathcal{L}(u, p, v, q; \Omega_t) := \frac{1}{2} \int_{\Gamma_t^0} |u - z|^2 dS_x - \int_{\Omega_t} (2\mu\varepsilon(u) \cdot \varepsilon(v) - p \operatorname{div}(v) - f^\top v - q \operatorname{div}(u)) dx. \quad (3.1)$$

We formulate the corresponding *saddle-point (minimax) problem*: find a solution quadruple

$$(u^t, p_t, v^t, q_t) \in U(\Omega_t)^2$$

such that

$$\mathcal{L}(u^t, p_t, v, q; \Omega_t) \leq \mathcal{L}(u^t, p_t, v^t, q_t; \Omega_t) \leq \mathcal{L}(u, p, v^t, q_t; \Omega_t) \quad (3.2)$$

for all test functions $(u, p, v, q) \in U(\Omega_t)^2$.

Theorem 3.1. *There exists the unique saddle point (u^t, p_t, v^t, q_t) satisfying (3.2):*

$$\sup_{(v, q) \in \dot{U}(\Omega_t)} \inf_{(u, p) \in U(\Omega_t)} \mathcal{L}(u, p, v, q; \Omega_t) = \mathcal{L}(u^t, p_t, v^t, q_t; \Omega_t) = \inf_{(u, p) \in U(\Omega_t)} \sup_{(v, q) \in \dot{U}(\Omega_t)} \mathcal{L}(u, p, v, q; \Omega_t). \quad (3.3)$$

The primal state $(u^t, p_t) \in U(\Omega_t)$ solves the forward Stokes problem (2.5). The adjoint state $(v^t, q_t) \in U(\Omega_t)$ is the unique solution to Stokes equations (see [46]),

$$\int_{\Omega_t} (2\mu\varepsilon(w) \cdot \varepsilon(v^t) - q_t \operatorname{div}(w)) dx = \int_{\Gamma_t^0} (u^t - z)^\top w dS_x, \quad (3.4a)$$

$$\int_{\Omega_t} \lambda \operatorname{div}(v^t) dx = 0 \quad \text{for all } (w, \lambda) \in U(\Omega_t), \quad (3.4b)$$

which describes the boundary-value relations

$$-\mu\Delta v^t + \nabla q_t = 0, \quad \operatorname{div}(v^t) = 0 \quad \text{in } \Omega_t, \quad (3.5a)$$

$$\varepsilon(v^t) := \frac{1}{2}(\nabla v^t + (\nabla v^t)^\top), \quad (3.5b)$$

$$v^t = 0 \quad \text{on } \Gamma_t^D, \quad 2\mu\varepsilon(v^t)n^t - q_t n^t = \begin{cases} u^t - z & \text{on } \Gamma_t^O, \\ 0 & \text{on } \Gamma_t^N \setminus \Gamma_t^O. \end{cases} \quad (3.5c)$$

Proof. The former maximization problem in (3.2) after shortening reads

$$\begin{aligned} & - \int_{\Omega_t} (2\mu\varepsilon(u^t) \cdot \varepsilon(v) - p_t \operatorname{div}(v) - f^\top v - q \operatorname{div}(u^t)) \, dx \\ & \leq - \int_{\Omega_t} (2\mu\varepsilon(u^t) \cdot \varepsilon(v^t) - p_t \operatorname{div}(v^t) - f^\top v^t - q_t \operatorname{div}(u^t)) \, dx \end{aligned} \quad (3.6)$$

for all test functions $(v, q) \in U(\Omega_t)$. Testing (3.6) with $v = v^t \pm w$ and $q = q_t$, we get equality (2.5a); inserting $v = v^t$ and $q = q_t \pm \lambda$ leads to (2.5b). Conversely, from (2.5a) with $w = v - v^t$ and $\operatorname{div}(u^t) = 0$ according to (2.5b), we arrive at (3.6) which hold with the equality sign.

The latter minimization problem in (3.2) after shortening reads

$$\begin{aligned} & \frac{1}{2} \int_{\Gamma_t^O} |u^t - z|^2 \, dS_x - \int_{\Omega_t} (2\mu\varepsilon(u^t) \cdot \varepsilon(v^t) - p_t \operatorname{div}(v^t) - q_t \operatorname{div}(u^t)) \, dx \\ & \leq \frac{1}{2} \int_{\Gamma_t^O} |u - z|^2 \, dS_x - \int_{\Omega_t} (2\mu\varepsilon(u) \cdot \varepsilon(v^t) - p \operatorname{div}(v^t) - q_t \operatorname{div}(u)) \, dx \end{aligned} \quad (3.7)$$

for all test functions $(u, p) \in U(\Omega_t)$. Substituting in (3.7) $u = u^t \pm sw$ and $p = p_t$ such that

$$\frac{1}{2} \int_{\Gamma_t^O} |u^t - z \pm sw|^2 \, dS_x - \frac{1}{2} \int_{\Gamma_t^O} |u^t - z|^2 \, dS_x \geq \int_{\Omega_t} (2\mu\varepsilon(\pm sw) \cdot \varepsilon(v^t) + q_t \operatorname{div}(\pm sw)) \, dx,$$

after dividing it with $s \neq 0$ and then passing the parameter $s \rightarrow 0$, it follows equality (3.4a). The substitution of $u = u^t$ and $p = p_t \pm \lambda$ in (3.7) follows (3.4b). Conversely, testing (3.4a) with $w = u - u^t$ such that

$$\int_{\Omega_t} (2\mu\varepsilon(u - u^t) \cdot \varepsilon(v^t) - q_t \operatorname{div}(u - u^t)) \, dx = \int_{\Gamma_t^O} (u^t - z)^\top (u - u^t) \, dS_x, \quad (3.8)$$

using the incompressibility $\operatorname{div}(v^t) = 0$ in (3.4b) and the convexity of \mathcal{J} ,

$$\int_{\Gamma_t^O} (u^t - z)^\top (u - u^t) \, dS_x \leq \frac{1}{2} \int_{\Gamma_t^O} |u - z|^2 \, dS_x - \frac{1}{2} \int_{\Gamma_t^O} |u^t - z|^2 \, dS_x, \quad (3.9)$$

from (3.8) and (3.9), we conclude with (3.7).

The unique solvability to the adjoint variational equations (3.4) follows from Proposition 2.1. This finishes the proof. \square

Based on the well-posedness assertion proved in Theorem 3.1, as a corollary, we claim the equivalence below.

Corollary 3.2. *An equivalent formulation of the shape optimization problem (2.10) using adjoint state reads: find Ω_t such that*

$$\inf_{\Omega_t \in \mathfrak{G}} \mathcal{L}(u^t, p_t, v^t, q_t; \Omega_t), \quad \text{where } (u^t, p_t, v^t, q_t) \text{ solves (3.2) in } \Omega_t. \quad (3.10)$$

Proof. For solutions (u^t, p_t) to the reference problem (2.5) and (u^t, p_t, v^t, q_t) to the minimax problem (3.2), the optimal value function $\ell: (t_0, t_1) \mapsto \mathbb{R}$ defined as $\ell(t) := \mathcal{J}(u^t; \Omega_t)$ allows an equivalent representation

$$\ell(t) := \mathcal{J}(u^t; \Omega_t) = \mathcal{L}(u^t, p_t, v^t, q_t; \Omega_t), \quad (3.11)$$

derived straightforwardly from (2.8) and (3.1). This proves the assertion. \square

For a small perturbation parameter $s \in (0, t_1 - t)$, we consider a perturbed according to (2.1) geometry $\Omega_{t+s} \subset D$ with the Dirichlet, observation and Neumann boundaries $\Gamma_{t+s}^D, \Gamma_{t+s}^O \subset \Gamma_{t+s}^N$. Recalling the notation $U(\Omega_{t+s}) = V(\Omega_{t+s}) \times L^2(\Omega_{t+s})$, the space $V(\Omega_{t+s})$ is defined in accordance with (2.3) as

$$V(\Omega_{t+s}) := \{w = (w_1, \dots, w_d)^\top \in H^1(\Omega_{t+s})^d \mid w = 0 \text{ a.e. } \Gamma_{t+s}^D\}.$$

The perturbed Stokes problem (2.5) reads: find $(u^{t+s}, p_{t+s}) \in U(\Omega_{t+s})$ such that

$$\begin{aligned} \int_{\Omega_{t+s}} (2\mu\varepsilon(u^{t+s}) \cdot \varepsilon(w) - p_{t+s} \operatorname{div}(w) - f^\top w) dx &= 0, \\ \int_{\Omega_{t+s}} \lambda \operatorname{div}(u^{t+s}) dx &= 0 \quad \text{for all } (w, \lambda) \in U(\Omega_{t+s}). \end{aligned}$$

The corresponding $(v^{t+s}, q_{t+s}) \in U(\Omega_{t+s})$ solves the perturbed adjoint equations

$$\begin{aligned} \int_{\Omega_{t+s}} (2\mu\varepsilon(w) \cdot \varepsilon(v^{t+s}) - q_{t+s} \operatorname{div}(w)) dx &= \int_{\Gamma_{t+s}^O} (u^{t+s} - z)^\top w dS_x, \\ \int_{\Omega_{t+s}} \lambda \operatorname{div}(v^{t+s}) dx &= 0 \quad \text{for all } (w, \lambda) \in U(\Omega_{t+s}). \end{aligned}$$

To attain a minimum in (2.10), we look for a decreasing optimal value function

$$\ell(t+s) = \mathcal{J}(u^{t+s}; \Omega_{t+s}) = \mathcal{L}(u^{t+s}, p_{t+s}, v^{t+s}, q_{t+s}; \Omega_{t+s}) < \ell(t).$$

If the asymptotic expansion $\ell(t+s) = \ell(t) + s\partial_t \mathcal{J}(u^t; \Omega_t) + o(s)$ holds as $s \rightarrow 0^+$, then we aim at the descent direction as common for gradient numerical methods,

$$\partial_t \mathcal{J}(u^t; \Omega_t) < 0. \quad (3.12)$$

Corollary 3.3. *If the one-sided limit $\partial_t \ell(t) := \lim_{s \rightarrow 0^+} (\ell(t+s) - \ell(t))/s$ exists, then the right derivative has two equivalent formulations*

$$\begin{aligned} \lim_{s \rightarrow 0^+} \frac{\mathcal{J}(u^{t+s}; \Omega_{t+s}) - \mathcal{J}(u^t; \Omega_t)}{s} &=: \partial_t \mathcal{J}(u^t; \Omega_t) = \partial_t \ell(t) = \partial_t \mathcal{L}(u^t, p_t, v^t, q_t; \Omega_t) \\ &:= \lim_{s \rightarrow 0^+} \frac{\mathcal{L}(u^{t+s}, p_{t+s}, v^{t+s}, q_{t+s}; \Omega_{t+s}) - \mathcal{L}(u^t, p_t, v^t, q_t; \Omega_t)}{s}. \end{aligned} \quad (3.13)$$

Proof. Indeed, assertion (3.13) follows straightforwardly from equation (3.11) after its formal differentiation with respect to t . \square

Our task is to provide the limit in (3.13) called a *shape derivative*.

4 Shape differentiability of the Lagrangian

Since the optimal value function in (3.11) is shape-dependent, we utilize a coordinate transformation to fixed geometry. For $t \in (t_0, t_1)$ fixed, let flows

$$[(s, x) \mapsto \phi_s(x)], [(s, y) \mapsto \phi_s^{-1}(y)] \in C^1([0, t_1 - t]; W^{1, \infty}(D))^d, \quad \phi_s|_{\partial D} = \phi_s^{-1}|_{\partial D} = 0, \quad (4.1)$$

describe a coordinate transformation $y = \phi_s(x)$ and the inverse $x = \phi_s^{-1}(y)$ such that their composition is $[\phi_s^{-1} \circ \phi_s](x) = x$ and $[\phi_s \circ \phi_s^{-1}](y) = y$. Here and in what follows, we associate space points $x = (x_1, \dots, x_d)^\top$ to the fixed geometry Ω_t , and $y = (y_1, \dots, y_d)^\top$ to a perturbed one Ω_{t+s} . Then (4.1) forms a diffeomorphism

$$\phi_s: \Omega_t \mapsto \Omega_{t+s}, \quad x \mapsto y, \quad \phi_s^{-1}: \Omega_{t+s} \mapsto \Omega_t, \quad y \mapsto x. \quad (4.2)$$

The transformation determines the kinematic velocity Λ by the implicit formula

$$\Lambda(t + s, y) := \frac{d}{ds} \phi_s(\phi_s^{-1}(y)).$$

Conversely, given explicitly a velocity vector $\Lambda = (\Lambda_1, \dots, \Lambda_d)^\top$ such that

$$\Lambda(t, x) \in C([t_0, t_1]; W^{1,\infty}(D))^d, \quad \Lambda|_{\partial D} = 0, \quad (4.3)$$

where $\Lambda = 0$ at ∂D preserves the hold-all domain D , it determines flows in (4.1) as the solution vector $\phi_s = ((\phi_s)_1, \dots, (\phi_s)_d)^\top$ to the non-autonomous ODE system

$$\frac{d}{ds} \phi_s = \Lambda(t + s, \phi_s) \quad \text{for } s \in (0, t_1 - t), \quad \phi_s = x \quad \text{as } s = 0, \quad (4.4a)$$

and the solution vector $\phi_s^{-1}(y) = ((\phi_s^{-1})_1, \dots, (\phi_s^{-1})_d)^\top$ to transport equations

$$\frac{\partial}{\partial s} \phi_s^{-1} + (\nabla_y \phi_s^{-1}) \Lambda|_{t+s} = 0 \quad \text{in } (0, t_1 - t) \times D, \quad \phi_s^{-1} = y \quad \text{as } s = 0, \quad (4.4b)$$

where the second-order tensor is $\nabla_y \phi_s^{-1} = (\partial(\phi_s^{-1})_i / \partial y_j)_{i,j=1}^d$ and $\Lambda|_{t+s}$ denotes $\Lambda(t + s, y)$ for short. For the details of (4.1)–(4.4), see [20].

The following traits are required to prove the shape differentiability.

Trait 1. *The function spaces constitute a bijective map*

$$(w, \lambda) \mapsto (w \circ \phi_s, \lambda \circ \phi_s): U(\Omega_{t+s}) \mapsto U(\Omega_t). \quad (4.5)$$

Proof. By the construction, coordinate transformation (4.1) builds a diffeomorphism, thus preserving integrable functions and first derivatives that form L^2 and H^1 spaces entering (4.5). \square

Based on Trait 1, after transformation to the reference geometry Ω_t , perturbed objective function

$$\tilde{\mathcal{J}}(s, \tilde{u}): (0, t_1 - t) \times V(\Omega_t) \mapsto \mathbb{R}$$

and perturbed Lagrangian function $\tilde{\mathcal{L}}(s, \tilde{u}, \tilde{p}, \tilde{v}, \tilde{q}): (0, t_1 - t) \times U(\Omega_t)^2 \mapsto \mathbb{R}$ are well defined by

$$\tilde{\mathcal{J}}(s, w \circ \phi_s; \Omega_t) = \mathcal{J}(w; \Omega_{t+s}) \quad \text{for } w \in V(\Omega_{t+s}), \quad (4.6a)$$

$$\tilde{\mathcal{L}}(s, u \circ \phi_s, p \circ \phi_s, v \circ \phi_s, q \circ \phi_s; \Omega_t) = \mathcal{L}(u, p, v, q; \Omega_{t+s}) \quad (4.6b)$$

for all $(u, p, v, q) \in U(\Omega_{t+s})^2$. At $s = 0$, relations (4.6) imply that

$$\tilde{\mathcal{J}}(0, w; \Omega_t) = \mathcal{J}(w; \Omega_t) \quad \text{for } w \in V(\Omega_t),$$

$$\tilde{\mathcal{L}}(0, u, p, v, q; \Omega_t) = \mathcal{L}(u, p, v, q; \Omega_{t+s}) \quad \text{for } (u, p, v, q) \in U(\Omega_t)^2.$$

We formulate corresponding to (4.6b) a perturbed saddle-point problem: find a solution quadruple

$$(\tilde{u}^{t+s}, \tilde{p}_{t+s}, \tilde{v}^{t+s}, \tilde{q}_{t+s}) \in U(\Omega_t)^2$$

such that

$$\begin{aligned} \tilde{\mathcal{L}}(s, \tilde{u}^{t+s}, \tilde{p}_{t+s}, v, q; \Omega_t) &\leq \tilde{\mathcal{L}}(s, \tilde{u}^{t+s}, \tilde{p}_{t+s}, \tilde{v}^{t+s}, \tilde{q}_{t+s}; \Omega_t) \\ &\leq \tilde{\mathcal{L}}(s, u, p, \tilde{v}^{t+s}, \tilde{q}_{t+s}; \Omega_t) \quad \text{for all } (u, p, v, q) \in U(\Omega_t)^2. \end{aligned} \quad (4.7)$$

Trait 2. *The set of saddle points $(\tilde{u}^{t+s}, \tilde{p}_{t+s}, \tilde{v}^{t+s}, \tilde{q}_{t+s})$ satisfying (4.7), that is,*

$$\begin{aligned} \sup_{(v,q) \in U(\Omega_t)} \inf_{(u,p) \in U(\Omega_t)} \tilde{\mathcal{L}}(s, u, p, v, q; \Omega_t) &= \tilde{\mathcal{L}}(s, \tilde{u}^{t+s}, \tilde{p}_{t+s}, \tilde{v}^{t+s}, \tilde{q}_{t+s}; \Omega_t) \\ &= \inf_{(u,p) \in U(\Omega_t)} \sup_{(v,q) \in U(\Omega_t)} \tilde{\mathcal{L}}(s, u, p, v, q; \Omega_t), \end{aligned} \quad (4.8)$$

is nonempty for all $s \in [0, t_1 - t]$.

Proof. According to Theorem 3.1, in the perturbed domain Ω_{t+s} , there exists the unique saddle point

$$(u^{t+s}, p_{t+s}, v^{t+s}, q_{t+s}) \in U(\Omega_{t+s})^2$$

such that

$$\begin{aligned} \mathcal{L}(u^{t+s}, p_{t+s}, v, q; \Omega_{t+s}) &\leq \mathcal{L}(u^{t+s}, p_{t+s}, v^{t+s}, q_{t+s}; \Omega_{t+s}) \\ &\leq \mathcal{L}(u, p, v^{t+s}, q_{t+s}; \Omega_{t+s}) \quad \text{for all } (u, p, v, q) \in U(\Omega_{t+s})^2. \end{aligned} \quad (4.9)$$

By virtue of (4.6b) applied to (4.9), we derive the transformed saddle point

$$(\tilde{u}^{t+s}, \tilde{p}_{t+s}, \tilde{v}^{t+s}, \tilde{q}_{t+s}) := (u^{t+s} \circ \phi_s, p_{t+s} \circ \phi_s, v^{t+s} \circ \phi_s, q_{t+s} \circ \phi_s),$$

which satisfies perturbed minimax problem (4.7) in the reference space Ω_t . \square

Applying coordinate transformation (4.2) to \mathcal{J} and \mathcal{L} in accordance with (4.6), we derive from (2.8) explicit expressions for the perturbed objective

$$\tilde{\mathcal{J}}(s, w; \Omega_t) = \frac{1}{2} \int_{\Gamma_t^0} |w - z \circ \phi_s|^2 \omega_s dS_x, \quad (4.10a)$$

and from (3.1) for the perturbed Lagrange function

$$\begin{aligned} \tilde{\mathcal{L}}(s, u, p, v, q; \Omega_t) &= \tilde{\mathcal{J}}(s, u; \Omega_t) - \int_{\Omega_t} (2\mu E(\nabla \phi_s^{-\top} \circ \phi_s, u) \cdot E(\nabla \phi_s^{-\top} \circ \phi_s, v) - p \operatorname{tr}((\nabla \phi_s^{-\top} \circ \phi_s) \nabla v) \\ &\quad - (f \circ \phi_s)^\top v - q \operatorname{tr}((\nabla \phi_s^{-\top} \circ \phi_s) \nabla u)) J_s dx. \end{aligned} \quad (4.10b)$$

For the derivation, we have used the chain rule $\nabla_y u = (\nabla \phi_s^{-\top} \circ \phi_s) \nabla(u \circ \phi_s)$ with the transpose of the inverse $^{-\top}$, the convention $\operatorname{div}(u) = \operatorname{tr}(\nabla u)$ using the trace of matrix, and the Jacobian determinant

$$J_s := \det(\nabla \phi_s) \quad \text{in } \Omega_t, \quad \omega_s := |(\nabla \phi_s^{-\top} \circ \phi_s) n^t| J_s \quad \text{at } \partial \Omega_t. \quad (4.11)$$

The notation in (4.10b) implies a generalized strain tensor

$$E(M, w) := \frac{1}{2} (M \nabla w + \nabla w^\top M^\top), \quad M \in \mathbb{R}^{d \times d},$$

such that $E(I, w) = \varepsilon(w)$ for the $d \times d$ identity matrix I . For more details, see [30, 31, 34].

Trait 3. Let $z \in H^2(\Omega)^d$. The asymptotic expansions in the first argument of the objective $\tilde{\mathcal{J}}$ and the Lagrange function $\tilde{\mathcal{L}}$ from (4.10),

$$\tilde{\mathcal{J}}(s, w; \Omega_t) = \mathcal{J}(w; \Omega_t) + \mathcal{O}(s), \quad (4.12a)$$

$$\tilde{\mathcal{L}}(s, u, p, v, q; \Omega_t) = \mathcal{L}(u, p, v, q; \Omega_t) + s \frac{\partial}{\partial s} \tilde{\mathcal{L}}(0, u, p, v, q; \Omega_t) + \mathcal{O}(s), \quad (4.12b)$$

hold as $s \rightarrow 0^+$. The partial derivative $\partial \tilde{\mathcal{L}} / \partial s: (t_0 - t, t_1 - t) \times U(\Omega_t)^2 \mapsto \mathbb{R}$ in (4.12b) yields a continuous function given analytically as

$$\begin{aligned} \frac{\partial}{\partial s} \tilde{\mathcal{L}}(s, u, p, v, q; \Omega_t) &:= \int_{\Gamma_t^0} \left(\frac{1}{2} \operatorname{div}_{\tau^t} \Lambda|_{t+s} |u - z|^2 - (\nabla z \Lambda|_{t+s})^\top (u - z) \right) dS_x \\ &\quad - \int_{\Omega_t} (2\mu (\operatorname{div} \Lambda|_{t+s} \varepsilon(u) \cdot \varepsilon(v) - \varepsilon(u) \cdot E(\nabla \Lambda|_{t+s}^\top, v) - \varepsilon(v) \cdot E(\nabla \Lambda|_{t+s}^\top, u)) \\ &\quad + p \operatorname{tr}(\nabla \Lambda|_{t+s}^\top \nabla v) - (\operatorname{div} \Lambda|_{t+s} f + \nabla f \Lambda|_{t+s})^\top v + q \operatorname{tr}(\nabla \Lambda|_{t+s}^\top \nabla u)) dx, \end{aligned} \quad (4.13)$$

where we have used the notation $\Lambda|_{t+s} = \Lambda(t + s, x)$ and the tangential divergence $\operatorname{div}_{\tau^t} \Lambda = \operatorname{div} \Lambda - (\nabla \Lambda n^t)^\top n^t$ at $\partial \Omega_t$.

Proof. As $s \rightarrow 0^+$, the following asymptotic formula holds (see [48, Chapter 2]):

$$\begin{aligned} f \circ \phi_s &= f + s \nabla f \Lambda + \mathcal{O}(s), & \nabla \phi_s^{-1} \circ \phi_s &= I - s \nabla \Lambda + \mathcal{O}(s), \\ J_s &= 1 + s \operatorname{div} \Lambda + \mathcal{O}(s), & \omega_s &= 1 + s \operatorname{div}_{\tau^t} \Lambda + \mathcal{O}(s). \end{aligned} \quad (4.14)$$

With the help of (4.14), we expand the terms in (4.10), (4.11) and derive directly expansion (4.12) with the first asymptotic term given in (4.13). \square

Trait 4. For the saddle points in (3.3) and (4.8), there exists a subsequence s_k for $k \rightarrow \infty$ such that, as $s_k \rightarrow 0^+$,

$$(\tilde{u}^{t+s}, \tilde{p}_{t+s}, \tilde{v}^{t+s}, \tilde{q}_{t+s}) \rightarrow (u^t, p_t, v^t, q_t) \quad \text{strongly in } U(\Omega_t)^2. \quad (4.15)$$

The proof of Trait 4 is rather technical and given in Appendix A.

Based on Traits 1–4, we establish the following theorem on differentiability.

Theorem 4.1. The shape derivative $\partial_t \ell(t)$ in (3.13) exists, expressed explicitly by the partial derivative from (4.13) as

$$\partial_t \partial(u^t; \Omega_t) = \partial_t \mathcal{L}(u^t, p_t, v^t, q_t; \Omega_t) = \frac{\partial}{\partial s} \tilde{\mathcal{L}}(0, u^t, p_t, v^t, q_t; \Omega_t), \quad (4.16)$$

where $(u^t, p_t, v^t, q_t) \in U(\Omega_t)^2$ is the saddle point defined in (3.3).

Proof. All assumptions in [9, Chapter 10, Theorem 5.1] specifying the abstract result on directional differentiability for shape optimization problems are satisfied by (4.8), (4.12) and (4.15) (see details in [33]). \square

5 Hadamard formula

Provided by a smooth solution to the Stokes problem (see [15]), a Hadamard representation of the shape derivative $\partial_t \ell(t)$ by boundary integrals is presented next.

Theorem 5.1. Assume that the primal and adjoint solutions of (2.5) and (3.4) are smooth such that

$$(u^t, p_t, v^t, q_t) \in (H^2(O)^d \times H^1(O))^2 \quad \text{in } O \subset \Omega_t.$$

If Λ is constant outside some domain $O_t \subset O$ with $C^{2,0}$ -smooth boundary ∂O_t and outward normal vector n^t , then an equivalent expression of the shape derivative (4.16) holds,

$$\begin{aligned} \frac{\partial}{\partial s} \tilde{\mathcal{L}}(0, u^t, p_t, v^t, q_t; \Omega_t) &= J_{\partial O_t}(\Lambda) + J_{\Gamma_t^0}(\Lambda), \\ J_{\Gamma_t^0}(\Lambda) &= \int_{\Gamma_t^0} ((\Lambda^\top n^t) \mathcal{D}_1(u^t) + \Lambda^\top \mathcal{D}^2(u^t)) dS_x + \begin{cases} (\Lambda^\top \tau^t) m(u^t)|_{\partial \Gamma_t^0} & \text{in } 2d, \\ \int_{\partial \Gamma_t^0} (\Lambda^\top b^t) m(u^t) dL_x & \text{in } 3d, \end{cases} \\ J_{\partial O_t}(\Lambda) &:= \int_{\partial O_t} ((\Lambda^\top n^t) \mathcal{D}_3(u^t, v^t) + \Lambda^\top \mathcal{D}^4(u^t, p_t, v^t, q_t)) dS_x, \end{aligned} \quad (5.1)$$

where τ^t is a tangential vector at the boundary positive oriented to n^t in 2d, and $b^t = \tau^t \times n^t$ is a binomial vector within the moving frame at the respective boundary in 3d. The notation $m(u)$, the scalar \mathcal{D}_1 , \mathcal{D}_3 and vector-valued $\mathcal{D}^2 = (\mathcal{D}_1^2, \dots, \mathcal{D}_d^2)^\top$, $\mathcal{D}^4 = (\mathcal{D}_1^4, \dots, \mathcal{D}_d^4)^\top$ terms in (5.1) are defined as follows:

$$\begin{aligned} \mathcal{D}_1(u) &:= \kappa_t m(u) + \nabla m(u)^\top n^t, \quad m(u) := \frac{1}{2} |u - z|^2, \\ \mathcal{D}^2(u) &:= \nabla u^\top (u - z), \\ \mathcal{D}_3(u, v) &:= f^\top v - 2\mu \varepsilon(u) \cdot \varepsilon(v), \\ \mathcal{D}^4(u, p, v, q) &:= \nabla u^\top (2\mu \varepsilon(v) n^t - q n^t) + \nabla v^\top (2\mu \varepsilon(u) n^t - p n^t), \end{aligned} \quad (5.2)$$

and the curvature is $\kappa_t := \operatorname{div}_{\tau^t} n^t$.

Proof. Since $\nabla \Lambda = 0$ in $\Omega_t \setminus O_t$ and the solution (u^t, p_t, v^t, q_t) is smooth in O_t , the shape derivative from Theorem 4.1 in accordance with formula (4.13) reads

$$\begin{aligned} \frac{\partial}{\partial s} \tilde{\mathcal{L}}(s, u^t, p_t, v^t, q_t; \Omega_t) &= I_1 + I_2 + I_3 + J_{\Gamma_t^0}(\Lambda), \\ J_{\Gamma_t^0}(\Lambda) &:= \int_{\Gamma_t^0} \left(\frac{1}{2} \operatorname{div}_{\tau^t} \Lambda |u^t - z|^2 - (\nabla z \Lambda)^\top (u^t - z) \right) dS_x. \end{aligned}$$

Integrating the terms entering (4.13) by parts in O_t , we have

$$\begin{aligned} I_1 &:= - \int_{O_t} 2\mu(\operatorname{div} \Lambda \varepsilon(u^t) \cdot \varepsilon(v^t) - \varepsilon(u^t) \cdot E(\nabla \Lambda, v^t) - \varepsilon(v^t) \cdot E(\nabla \Lambda, u^t)) dx \\ &= - \int_{O_t} \Lambda^\top (\nabla(u^t)^\top 2\mu \operatorname{div} \varepsilon(v^t) + \nabla(v^t)^\top 2\mu \operatorname{div} \varepsilon(u^t)) dx \\ &\quad - \int_{\partial O_t} 2\mu((\Lambda^\top n^t) \varepsilon(u^t) \cdot \varepsilon(v^t) - \Lambda^\top (\nabla(u^t)^\top \varepsilon(v^t) + \nabla(v^t)^\top \varepsilon(u^t)) n^t) dS_x, \end{aligned}$$

and using the incompressibility $\operatorname{div}(u^t) = \operatorname{div}(v^t) = 0$,

$$\begin{aligned} I_2 &:= - \int_{O_t} (p_t \operatorname{tr}(\nabla \Lambda^\top \nabla v^t) + q_t \operatorname{tr}(\nabla \Lambda^\top \nabla u^t)) dx \\ &= \int_{O_t} \Lambda^\top (\nabla(v^t)^\top \nabla p_t + \nabla(u^t)^\top \nabla q_t) dx - \int_{\partial O_t} \Lambda^\top (\nabla(v^t)^\top p_t + \nabla(u^t)^\top q_t) n^t dS_x, \end{aligned}$$

whereas the integral

$$I_3 := \int_{O_t} (\operatorname{div} \Lambda f + \nabla f \Lambda)^\top v^t dx.$$

After summation, employing the equilibrium equations (2.2a), (3.5a) and the identity

$$\operatorname{div}(\Lambda(f^\top v^t)) = \operatorname{div} \Lambda(f^\top v^t) + \Lambda^\top \nabla(f^\top v^t),$$

gathering the like terms yields

$$\begin{aligned} I_1 + I_2 + I_3 &= \int_{O_t} \operatorname{div}(\Lambda(f^\top v^t)) dx \\ &\quad - \int_{\partial O_t} ((\Lambda^\top n^t) 2\mu \varepsilon(u^t) \cdot \varepsilon(v^t) - \Lambda^\top (\nabla(u^t)^\top (2\mu \varepsilon(v^t) - q_t) + \nabla(v^t)^\top (2\mu \varepsilon(u^t) - p_t)) n^t) dS_x \\ &= J_{\partial O_t}(\Lambda). \end{aligned}$$

Applying the divergence theorem to the first integral over O_t in the right-hand side, we arrive at formulas for \mathcal{D}_3 and \mathcal{D}^4 in (5.1) and (5.2).

Integration along the boundary Γ_t^0 is given by the following formula (see [48, equation (2.125)]):

$$\int_{\Gamma_t^0} (\operatorname{div}_{\tau^t} \Lambda m(u) + \Lambda^\top \nabla m(u)) dS_x = \int_{\Gamma_t^0} (\Lambda^\top n^t) (\chi_t m(u) + \nabla m(u)^\top n^t) dS_x + \begin{cases} (\Lambda^\top \tau^t) m(u) |_{\partial \Gamma_t^0} & \text{in } 2d, \\ \int_{\partial \Gamma_t^0} (\Lambda^\top b^t) m(u) dL_x & \text{in } 3d. \end{cases}$$

Together with the identity $\nabla m(u) = \nabla u^\top (u - z) - \nabla z^\top (u - z)$, this leads to the expression of $J_{\partial O_t}(\Lambda)$ involving \mathcal{D}_1 , \mathcal{D}^2 and $m(u)$ in (5.1) and (5.2). \square

The important corollary deals with the inverse problem of shape identification (3.10) and guarantees the descent direction for optimization as suggested in (3.12). In the following consideration, we decompose the vectors into orthogonal, normal and tangential components at the boundary,

$$\Lambda = (\Lambda^\top n^t) n^t + \Lambda_{\tau^t}, \quad \mathcal{D}^i = ((\mathcal{D}^i)^\top n^t) n^t + \mathcal{D}_{\tau^t}^i, \quad i = 2, 4.$$

Corollary 5.2. *A descent direction for $\partial_t \mathcal{J}(u^t; \Omega_t) < 0$ in (3.12) is provided by the velocity Λ ,*

$$\begin{aligned} \Lambda^\top n^t &= -k_1 (\mathcal{D}_1(u^t) + \mathcal{D}^2(u^t)^\top n^t), & \Lambda_{\tau^t} &= -k_2 \mathcal{D}^2(u^t)_{\tau^t} & \text{at } \Gamma_t^0, \\ \Lambda^\top \tau^t &= -k_5 m(u^t) \quad \text{in } 2d, & \Lambda^\top b^t &= -k_5 m(u^t) \quad \text{in } 3d & \text{at } \partial \Gamma_t^0, \\ \Lambda^\top n^t &= -k_3 (\mathcal{D}_3(u^t, v^t) + \mathcal{D}^4(u^t, p_t, v^t, q_t)^\top n^t), & \Lambda_{\tau^t} &= -k_4 \mathcal{D}^4(u^t, p_t, v^t, q_t)_{\tau^t} & \text{at } \partial O_t, \end{aligned} \quad (5.3)$$

with not all $k_i \geq 0$, $i = 1, \dots, 5$, simultaneously equal to zero.

Proof. The direct substitution into (5.1) of Λ from (5.3) yields

$$\begin{aligned} \frac{\partial}{\partial s} \tilde{\mathcal{L}}(0, u^t, p_t, v^t, q_t; \Omega_t) = & - \int_{\Gamma_t^0} (k_1(\mathcal{D}_1(u^t) + \mathcal{D}^2(u^t)^\top n^t)^2 + k_2 \mathcal{D}^2(u^t)_{\tau^t}^2) dS_x \\ & - \int_{\partial O_t} (k_3(\mathcal{D}_3(u^t, v^t) + \mathcal{D}^4(u^t, p_t, v^t, q_t)^\top n^t)^2 + k_4 \mathcal{D}^4(u^t, p_t, v^t, q_t)_{\tau^t}^2) dS_x \\ & - \begin{cases} k_5 m(u^t)_{\partial \Gamma_t^0}^2 & \text{in 2d,} \\ \int_{\partial \Gamma_t^0} k_5 m(u^t)^2 dL_x & \text{in 3d,} \end{cases} \end{aligned}$$

thus provides the decrease $\partial \tilde{\mathcal{L}}(0, u^t, p_t, v^t, q_t; \Omega_t) / \partial s < 0$. \square

Corollary 5.2 gives practical formulas for numerical simulation of the inverse Stokes problem by gradient methods.

A Proof of Trait 4

We split the proof in three blocks: uniform estimate, weak convergence and strong convergence.

Uniform estimate. By virtue of representation (4.10) of $\tilde{\mathcal{L}}$, the former, primal maximization problem in (4.7) implies the optimality conditions

$$\int_{\Omega_t} (2\mu E(\nabla \phi_s^{-\top} \circ \phi_s, \tilde{u}^{t+s}) \cdot E(\nabla \phi_s^{-\top} \circ \phi_s, v) - \tilde{p}_{t+s} \operatorname{tr}((\nabla \phi_s^{-\top} \circ \phi_s) \nabla v) - (f \circ \phi_s)^\top v) J_s dx = 0, \quad (\text{A.1a})$$

$$\int_{\Omega_t} q \operatorname{tr}((\nabla \phi_s^{-\top} \circ \phi_s) \nabla \tilde{u}^{t+s}) J_s dx = 0 \quad \text{for all } (v, q) \in U(\Omega_t). \quad (\text{A.1b})$$

We expand (A.1a) due to asymptotic formulas (4.14) as $s \rightarrow 0^+$ and apply the Cauchy–Schwarz inequality to derive the uniform estimate

$$\left| \int_{\Omega_t} (2\mu \varepsilon(\tilde{u}^{t+s}) \cdot \varepsilon(v) - \tilde{p}_{t+s} \operatorname{div}(v) - f^\top v) dx \right| \leq s C_0 \|v\|_{H^1(\Omega_t)^d} \quad (\text{A.2})$$

with constant $C_0 > 0$. From $\operatorname{tr}((\nabla \phi_s^{-\top} \circ \phi_s) \nabla \tilde{u}^{t+s}) J_s = 0$ due to (A.1b), the incompressibility condition holds in the asymptotic sense $\operatorname{div}(\tilde{u}^{t+s}) = O(s \nabla \tilde{u}^{t+s})$. Therefore, testing (A.2) with $v = \tilde{u}^{t+s}$, it follows the inequality

$$\left| \int_{\Omega_t} (2\mu \varepsilon(\tilde{u}^{t+s}) \cdot \varepsilon(\tilde{u}^{t+s}) - f^\top \tilde{u}^{t+s}) dx \right| \leq s C_1 \|\tilde{u}^{t+s}\|_{H^1(\Omega_t)^d}, \quad C_1 > 0. \quad (\text{A.3})$$

We apply here the Korn–Poincaré inequality (2.6) to get the lower bound

$$\|\tilde{u}^{t+s}\|_{H^1(\Omega_t)^d} \leq C_u := \frac{1}{2\mu K_{\text{KP}}} (\|f\|_{L^2(\Omega_t)^d} + C_1 s). \quad (\text{A.4})$$

Henceforth, (A.1a) implies a linear bounded functional such that

$$\begin{aligned} \left| \int_{\Omega_t} \tilde{p}_{t+s} \operatorname{tr}((\nabla \phi_s^{-\top} \circ \phi_s) \nabla v) J_s dx \right| &= \left| \int_{\Omega_t} (2\mu E(\nabla \phi_s^{-\top} \circ \phi_s, \tilde{u}^{t+s}) \cdot E(\nabla \phi_s^{-\top} \circ \phi_s, v) - (f \circ \phi_s)^\top v) J_s dx \right| \\ &\leq C_2 \|v\|_{H^1(\Omega_t)^d} \end{aligned}$$

with constant $C_2 := C_3(C_u + \|f\|_{L^2(\Omega_t)^d})$ and $C_3 > 0$. Dividing this inequality with the norm of v , using asymptotic formulas (4.14) as $s \rightarrow 0^+$, it follows

$$\frac{1}{\|v\|_{H^1(\Omega_t)^d}} \left| \int_{\Omega_t} \tilde{p}_{t+s} \operatorname{div}(v) dx \right| \leq C_2 + s C_4 \|\tilde{p}_{t+s}\|_{L^2(\Omega_t)}, \quad C_4 > 0.$$

For sufficiently small $s \leq s_1 < K_{\text{LBB}}/C_4$, from LBB condition (2.7), we derive

$$\|\tilde{p}_{t+s}\|_{L^2(\Omega_t)} \leq C_p := \frac{C_2}{K_{\text{LBB}} - s_1 C_4}. \quad (\text{A.5})$$

The latter, dual minimization problem in (4.7) implies the adjoint system

$$\begin{aligned} \int_{\Omega_t} (2\mu E(\nabla \phi_s^{-\top} \circ \phi_s, u) \cdot E(\nabla \phi_s^{-\top} \circ \phi_s, \tilde{v}^{t+s}) \\ - \tilde{q}_{t+s} \operatorname{tr}((\nabla \phi_s^{-\top} \circ \phi_s) \nabla u)) J_s \, dx = \int_{\Gamma_t^0} (\tilde{u}^{t+s} - z \circ \phi_s)^\top u \omega_s \, dS_x, \\ \int_{\Omega_t} p \operatorname{tr}((\nabla \phi_s^{-\top} \circ \phi_s) \nabla \tilde{v}^{t+s}) J_s \, dx = 0 \quad \text{for all } (u, p) \in U(\Omega_t), \end{aligned} \quad (\text{A.6})$$

which is of type (A.1), hence admits similar to (A.4) and (A.5) estimates of \tilde{v}^{t+s} and \tilde{q}_{t+s} . Therefore, we conclude with existence of $C > 0$ such that

$$\|\tilde{u}^{t+s}\|_{H^1(\Omega_t)^d} + \|\tilde{p}_{t+s}\|_{L^2(\Omega_t)} + \|\tilde{v}^{t+s}\|_{H^1(\Omega_t)^d} + \|\tilde{q}_{t+s}\|_{L^2(\Omega_t)} \leq C. \quad (\text{A.7})$$

Weak convergence. Based on uniform estimate (A.7), a subsequence $s_k \rightarrow 0^+$ as $k \rightarrow \infty$ and an accumulation point $(\bar{u}, \bar{p}, \bar{v}, \bar{q}) \in U(\Omega_t)^2$ exist such that

$$(\tilde{u}^{t+s_k}, \tilde{p}_{t+s_k}, \tilde{v}^{t+s_k}, \tilde{q}_{t+s_k}) \rightharpoonup (\bar{u}, \bar{p}, \bar{v}, \bar{q}) \quad \text{weakly in } U(\Omega_t)^2. \quad (\text{A.8})$$

On taking the limit in (A.1) and (A.6) as $s_k \rightarrow 0$, we arrive at the Stokes problem (2.5) and adjoint system (3.4). Therefore, $(\bar{u}, \bar{p}, \bar{v}, \bar{q}) = (u^t, p_t, v^t, q_t)$.

Strong convergence. With the help of the algebraic formula $(a - b)^2 = a^2 - b^2 - 2(a - b)b$, we rearrange the terms in (A.3) and (2.5a) tested with $w = u^t$ as follows:

$$\int_{\Omega_t} 2\mu \varepsilon(\tilde{u}^{t+s_k} - u^t) \cdot \varepsilon(\tilde{u}^{t+s_k} - u^t) \, dx \leq \int_{\Omega_t} (f^\top (\tilde{u}^{t+s_k} - u^t) - 4\mu \varepsilon(\tilde{u}^{t+s_k} - u^t) \cdot \varepsilon(u^t)) \, dx + s_k C_1 \|\tilde{u}^{t+s_k}\|_{H^1(\Omega_t)^d}.$$

The application of limit as $s_k \rightarrow 0^+$ due to weak convergences (A.8) and the Korn–Poincaré inequality (2.6) provide the strong convergence by means of

$$\limsup_{s_k \rightarrow 0^+} \|\tilde{u}^{t+s_k} - u^t\|_{H^1(\Omega_t)^d}^2 \leq 0. \quad (\text{A.9})$$

On the other hand, subtracting (2.5a) with $w = u$ from (A.2), we get the inequality

$$\left| \int_{\Omega_t} ((\tilde{p}_{t+s_k} - p_t) \operatorname{div}(v) - 2\mu \varepsilon(\tilde{u}^{t+s_k} - u^t) \cdot \varepsilon(v)) \, dx \right| \leq s_k C_0 \|v\|_{H^1(\Omega_t)^d}.$$

Dividing it with the norm of v such that

$$\frac{1}{\|v\|_{H^1(\Omega_t)^d}} \left| \int_{\Omega_t} (\tilde{p}_{t+s_k} - p_t) \operatorname{div}(v) \, dx \right| \leq 2\mu \|\tilde{u}^{t+s_k} - u^t\|_{H^1(\Omega_t)^d} + s_k C_0$$

and passing to the limit as $s_k \rightarrow 0^+$ due to (A.8) and (A.9), LBB condition (2.7) leads to the upper bound

$$\limsup_{s_k \rightarrow 0^+} \|\tilde{p}_{t+s_k} - p_t\|_{L^2(\Omega_t)} \leq 0. \quad (\text{A.10})$$

Since adjoint systems (3.4), (A.6) are of the same type as (2.5), (A.1), then the estimates of type (A.9) and (A.10) hold also true for the adjoint state,

$$\limsup_{s_k \rightarrow 0^+} \|\tilde{v}^{t+s_k} - v^t\|_{H^1(\Omega_t)^d}^2 + \limsup_{s_k \rightarrow 0^+} \|\tilde{q}_{t+s_k} - q_t\|_{L^2(\Omega_t)} \leq 0,$$

which finishes the proof of the strong convergence (4.15).

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