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Poroelastic medium with non-penetrating crack driven by hydraulic fracture: Variational inequality and its semidiscretization ☆

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A new class of unilateral variational models appearing in the theory of poroelasticity is introduced and studied. A poroelastic medium consists of solid phase and pores saturated with a Newtonian fluid. The medium contains a fluid-driven crack, which is subjected to non-penetration between the opposite crack faces. The fully coupled poroelastic system includes elliptic–parabolic governing equations under the unilateral constraint. Well-posedness of the corresponding variational inequality is established based on the Rothe semi-discretization in time, after subsequent passing time step to zero. The NLCP-formulation of non-penetration conditions is given which is useful for a semi-smooth Newton solution strategy.

1. Introduction

In the paper we introduce mathematical modeling for a new class of variational inequalities motivated by hydrofracking. A poroelastic body consisting of solid phase and pores saturated with a Newtonian fluid is considered, which contains a fluid-driven crack. The crack associates a hydraulic fracture created by pumping fracturing fluid, as it is used to stimulate production of oil and natural gas in mining. The novelty consists in the fact that we endow the crack with a non-penetration condition between its opposite faces (the fracture walls), thus allowing compressive pressure at which the crack might close. As the result, the poroelastic problem is described by a coupled system of governing equations and inequalities for unknown solid phase displacement, pore pressure, and contact force. The model is endowed with the fluid pressure prescribed to be inhomogeneous and different at the fracture walls.

The variational theory of solids with non-penetrating cracks was developed in the works by Khludnev and Kovtunenko [1], Khludnev and Sokołowski [2] and co-authors. For physical issues of fracture modeling we refer to [3], and cite [4]

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for geometrical description of crack-tip/ crack-front singularities. Physical consistency will allow contact between the opposite crack faces, that imposes a condition of non-penetration on the jump of crack displacement. The non-penetration approach was extended for dissipative contact phenomena at the crack owing to friction [5], cohesion [6], and the limiting small strain [7]. Recent studies treat a class of variational problems for anti-cracks and inclusions (see [8–10]). In the current paper we formulate and study well-posedness for a new class of poroelastic problems with non-penetrating cracks.

For numerical solution, the nonlinear complementarity problem (NLCP) formulation of the problem is presented, which is useful for primal–dual active set strategies based on a semi-smooth Newton method (see [11]). Taking into account frictional and adhesive contact phenomena at the crack, an Uzawa type projection algorithm was suggested in [12]. To avoid interpenetration between the crack faces, a poroelastic model was investigated numerically in [13] by utilizing a penalty. In comparison, the semi-smooth Newton method is advantageous because obeys a super-linear rate of convergence. Alternatively to the sharp interface formulation, a phase-field approximation of cracks was developed by Amor et al. [14], Borden et al. [15]. We note that advantages and disadvantages of these approaches are well-known from numerical tests presented e.g. in the cited works. However, practical simulations coupling solid and fluid models are rather involved (see [16]).

The theory of poroelasticity was established well by Biot [17], Terzaghi [18] and further developed by Barenblatt et al. [19], Coussy [20], Meirmanov [21], and others. We refer to [22,23] for modeling of a two-phase medium consisted of solid phase and pores with interfacial jumps, and cite [24] for multi-scale analysis of related interface problems. Our model is motivated by application to hydraulic fractures in oil and gas well-bores that is a challenging issue in modern geophysical technologies. We utilize the mathematical formulation of poroelastic constitutive relations from [25,26], by this accounting for positive and negative pressure phases in fracturing fluid during pumping cycle. The unilateral contact conditions are imposed which guarantee a non-negative width between fracture walls. It is worth noting the fact that the fracture width/ aperture is used to compute the fracture permeability (as a quadratic function) and the fracture transmissivity (as a cubic function). This means that coefficients describing conduction properties of fractures might become negative for a negative fracture aperture. Thus, the unilateral setting is advantageous compared to other models without this constraint.

From a mathematical point of view, the poroelastic constitutive equations are similar to those for models in thermoelasticity. The problem of thermoelastic contact was solved first in [27] and then extended to nonpenetrating cracks in thermoelastic plates by Hoffmann and Khludnev [28], Hömberg and Khludnev [29]. In a fixed point approach, applying a compactness argument to successive problems for the temperature when the elastic field is known and vice versa, only the initial condition for the heat equation was allowed to prescribe a-priori, and the initial elastic state can be derived from the time-limit of the quasi-static equilibrium equation. Moreover, the coefficient of thermal expansion should be assumed sufficiently small. Our approach by semi-discretization solves the problem in a general form, when both the initial temperature and the elastic field are given arbitrarily, thus including the earlier formulation as a particular case.

The current paper is organized as follows. In Section 2, the coupled elliptic–parabolic system describing poroelastic problem with a fluid-driven crack is introduced, and non-penetration conditions for the crack are stated in Section 2.1. In Section 3 we endow the model with a variational formulation and establish its well-posedness based on Rothe's method of semi-discretization in time. The rigorous proof is presented in Section 3.1 supported by a-priori estimates.

2. Problem modeling

We start with geometric description of a poroelastic medium containing inside a fluid-driven crack.

In the Euclidean space of spatial points $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $d = 2, 3$, let Ω be a domain with the Lipschitz continuous boundary $\partial\Omega$ and outward normal vector $\mathbf{n} = (n_1, \dots, n_d)$. Let $\partial\Omega = \Gamma_{\rm D} \cup \Gamma_{\rm N}$. We assume an oriented manifold of co-dimension one Σ which splits Ω into two sub-domains $\Omega^\pm$ with Lipschitz continuous boundaries $\partial\Omega^\pm$ such that

$$\partial\Omega^+ \cap \partial\Omega^- = \Sigma, \quad \Omega = \Omega^+ \cup \Omega^- \cup \Sigma. \tag{2.1}$$

The normal vector $\mathbf{n}$ at Σ is chosen outward to Ω^-, thus inward to Ω^+. A part $\Gamma_{\rm c}$ of the interface with two faces $\Gamma_{\rm c}^\pm$ and its complement to Ω are defined as

$$\Gamma_{\rm c} \subset \Sigma, \quad \Gamma_{\rm c}^+ \subset \Sigma^+, \quad \Gamma_{\rm c}^- \subset \Sigma^-, \quad \Omega_{\rm c} = \Omega \setminus \overline{\Gamma_{\rm c}}. \tag{2.2}$$

Physically, $\Gamma_{\rm c}$ is associated with the crack (fracture filled with a Newtonian fluid), whereas the complement $\Omega_{\rm c}$ represents a reservoir filled of solid phase and pores saturated with the same fluid. In time $t \in [0, T]$, $T > 0$, this determines the time–space geometry as follows (see 2d illustration in Fig. 1):

$$\Omega_{\rm c}^T = (0, T) \times \Omega_{\rm c}, \quad \partial\Omega^T = (0, T) \times \partial\Omega, \quad \Gamma_\gamma^T = (0, T) \times \Gamma_\gamma, \ \gamma \in \{{\rm c, D, N}\}. \tag{2.3}$$

A poroelastic medium occupying $\Omega_{\rm c}^T$ according to (2.1)–(2.3) is described by the solid phase displacement $\mathbf{u} = (u_1, \dots, u_d)(t, \mathbf{x})$ and the pore pressure $p(t, \mathbf{x})$, which are governed by the poroelastic relations following [25,26].

For the linear elastic solid phase, the second-order d-by-d symmetric tensor of linearized strain $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}_{i,j=1}^d(t, \mathbf{x})$ is defined by the symmetric gradient of the displacement vector as

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i, j = 1, \dots, d. \tag{2.4}$$

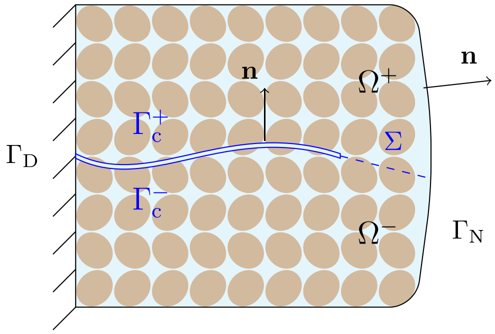

Fig. 1. The example geometry for a poroelastic medium with a crack in 2d.

The second-order symmetric tensor of Cauchy stress $\boldsymbol{\sigma} = \{\sigma_{ij}\}_{i,j=1}^d(t, \mathbf{x})$ is given corresponding to the matrix multiplication of the strain

$$\boldsymbol{\sigma} = \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\tau}^0 \tag{2.5}$$

subjected to a prestress $\boldsymbol{\tau}^0 = \{\tau_{ij}^0\}_{i,j=1}^d(\mathbf{x})$, by the fourth-order symmetric tensor of elastic coefficients $\mathbf{A} = \{A_{ijkl}\}_{i,j,k,l=1}^d(\mathbf{x})$ such that

$$A_{ijkl} = A_{jikl} = A_{klij}, \quad i, j, k, l = 1, \dots, d.$$

Accounting for the pore pressure, the effective stress is introduced as

$$\boldsymbol{\tau} = \boldsymbol{\sigma} - \alpha p\mathbf{I}, \tag{2.6}$$

where constant $\alpha \in (0, 1]$ is the Biot coefficient, and $\mathbf{I} \in \mathbb{R}^{d\times d}$ is the identity tensor. Omitting inertia terms, the quasi-static equilibrium equation reads

$$-(\mathrm{div}\boldsymbol{\tau})_i := -\sum_{j=1}^d \frac{\partial \tau_{ij}}{\partial x_j} = 0, \quad i = 1, \dots, d, \quad \text{in } \Omega_c^T. \tag{2.7}$$

The fluid content in pores $\zeta(t, \mathbf{x})$ is described by the Fick's diffusion law

$$\frac{\partial \zeta}{\partial t} = -\mathrm{div}\mathbf{q} := -\sum_{i=1}^d \frac{\partial q_i}{\partial x_i} \quad \text{in } \Omega_c^T, \tag{2.8}$$

with the flow velocity $\mathbf{q} = (q_1, \dots, q_d)(t, \mathbf{x})$ subjected to the Darcy flow

$$\mathbf{q} = -\kappa \nabla p := -\kappa\Big(\frac{\partial p}{\partial x_1}, \dots, \frac{\partial p}{\partial x_d}\Big), \tag{2.9}$$

with $\kappa = k_r/\eta_r$, where $k_r(\mathbf{x})$ is the permeability and constant η_r stands for the effective viscosity such that

$$0 < \underline{\kappa} \le \kappa(\mathbf{x}) \le \overline{\kappa}.$$

The system is completed with the constitutive law connecting p, ζ and the dilatation $\mathrm{tr}\boldsymbol{\varepsilon}$ as

$$Sp = \zeta - \alpha\,\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}), \quad \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}) := \mathrm{div}\mathbf{u}, \tag{2.10}$$

where constant $S > 0$ is the storativity.

The poroelastic Eqs. (2.4)–(2.10) are endowed with initial conditions

$$\mathbf{u}(0) = \mathbf{u}^0, \quad p(0) = p^0 \quad \text{in } \Omega_c \tag{2.11}$$

for the undrained state given by $\mathbf{u}^0 = (u_1^0, \dots, u_d^0)(\mathbf{x})$ and $p^0(\mathbf{x})$, and by mixed boundary conditions prescribed on the outer boundary

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D^T, \quad \boldsymbol{\tau}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_N^T, \quad p = p^\infty \quad \text{on } \partial\Omega^T \tag{2.12}$$

for the given traction $\mathbf{g} = (g_1, \dots, g_d)(t, \mathbf{x})$ and pressure $p^\infty(t, \mathbf{x})$, which conform (2.11) at $t = 0$. At the boundary we decompose the displacement and the effective boundary stress vectors into its normal and tangential components:

$$\mathbf{u} = (\mathbf{u}\cdot\mathbf{n})\mathbf{n} + \big(\mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\big), \quad \boldsymbol{\tau}\mathbf{n} = (\boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n})\mathbf{n} + \big(\boldsymbol{\tau}\mathbf{n} - (\boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n})\mathbf{n}\big),$$

where $\boldsymbol{\tau}\mathbf{n} = (\sum_{j=1}^d \tau_{1j}n_j, \dots, \sum_{j=1}^d \tau_{dj}n_j)$ implies the matrix–vector multiplication, and dot stands for the scalar product of vectors, thus $\mathbf{u}\cdot\mathbf{n} = \sum_{i=1}^d u_i n_i$, $\boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n} = \sum_{i,j=1}^d \tau_{ij}n_i n_j$. Across the crack, the unknowns are discontinuous in general and allow jumps:

$$[\![\mathbf{u}]\!] := \mathbf{u}|_{\Gamma_c^+} - \mathbf{u}|_{\Gamma_c^-}, \quad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}|_{\Gamma_c^+} - \boldsymbol{\tau}|_{\Gamma_c^-}, \quad [\![p]\!] := p|_{\Gamma_c^+} - p|_{\Gamma_c^-}.$$

We suggest no effective tangential stress at the crack faces

$$\boldsymbol{\tau}\mathbf{n} - (\boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n})\mathbf{n} = \mathbf{0} \quad \text{on } (0,T)\times\Gamma_c^{\pm}, \tag{2.13}$$

and continuity of the fluid pressure over the fracture wall

$$p = p_f^{\pm} \quad \text{on } (0,T)\times\Gamma_c^{\pm} \tag{2.14}$$

for known fluid pressure on the opposite fracture walls $p_f^+(t,\mathbf{x})$ and $p_f^-(t,\mathbf{x})$, which can be different and coincide at the crack tip/ crack front. To calculate p_f in the fracture, we refer to the approach based on lubrication theory equations for aperture (see [30]).

Let us remark that in numerical simulations as well as theoretical analysis in the cited literature, symmetry of the problem with respect to the direct crack Γ_c is often assumed for simplicity. In this case, we do not need to consider the opposite faces, rather to set

$$p_f^- = -p_f^+, \quad \mathbf{u}|_{\Gamma_c^-} = -\mathbf{u}|_{\Gamma_c^+}, \quad \boldsymbol{\tau}|_{\Gamma_c^-} = -\boldsymbol{\tau}|_{\Gamma_c^+}, \quad p|_{\Gamma_c^-} = -p|_{\Gamma_c^+}$$

here and in the following formula.

2.1. Non-penetration conditions

In the normal direction, the standard boundary condition is

$$\boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n} = -p_f^{\pm} \quad \text{on } (0,T)\times\Gamma_c^{\pm}. \tag{2.15}$$

For few heuristic approaches generalizing (2.15) within unilateral conditions we cite [31,32]. In our approach, to prevent penetration between the opposite crack faces $[\![\mathbf{u}]\!]\cdot\mathbf{n} < 0$, the unilateral contact conditions are set in the complementarity form (see [1]):

$$\begin{gathered}(\boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n} + p_f)|_{(0,T)\times\Gamma_c^+} = (\boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n} + p_f)|_{(0,T)\times\Gamma_c^-} =: \boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n} + p_f \le 0,\\ [\![\mathbf{u}]\!]\cdot\mathbf{n} \ge 0, \quad ([\![\mathbf{u}]\!]\cdot\mathbf{n})(\boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n} + p_f) = 0 \quad \text{on } (0,T)\times\Gamma_c^{\pm},\end{gathered} \tag{2.16}$$

reminding that $p_f|_{\Gamma_c^{\pm}} = p_f^{\pm}$. Physically, relations (2.16) imply a compressive contact stress, thus confining the pressure at which the hydraulic fracture closes. It is worth noting that (2.16) implies (2.15) as a particular case when the crack is fully open, i.e. $[\![\mathbf{u}]\!]\cdot\mathbf{n} > 0$.

Reducing variables $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$, ζ, $\mathbf{q}$ from the system, the governing Eqs. (2.5)–(2.10) turn into the following two equations for unknown $\mathbf{u}$ and p:

$$-\operatorname{div}(\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\tau}^0) + \alpha\nabla p = 0 \quad \text{in } \Omega_c^T, \tag{2.17}$$

$$\frac{\partial}{\partial t}\big(Sp + \alpha\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u})\big) - \operatorname{div}(\kappa\nabla p) = 0 \quad \text{in } \Omega_c^T \tag{2.18}$$

endowed with initial conditions (2.11). Using calculation at the boundary

$$\boldsymbol{\tau}\mathbf{n} = \boldsymbol{\sigma}\mathbf{n} - \alpha p\mathbf{n}, \quad \boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n} = \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} - \alpha p, \quad \boldsymbol{\tau}\mathbf{n} - (\boldsymbol{\tau}\mathbf{n}\cdot\mathbf{n})\mathbf{n} = \boldsymbol{\sigma}\mathbf{n} - (\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n})\mathbf{n},$$

the boundary conditions (2.12)–(2.14) and (2.16) reduce to

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D^T, \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_N^T, \quad p = p^{\infty} \quad \text{on } \partial\Omega^T, \tag{2.19}$$

$$\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - (\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}\cdot\mathbf{n})\mathbf{n} = \mathbf{0}, \quad p = p_f^{\pm} \quad \text{on } (0,T)\times\Gamma_c^{\pm}, \tag{2.20}$$

$$\begin{gathered}[\![\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}\cdot\mathbf{n} + (1-\alpha)p_f]\!] = 0, \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}\cdot\mathbf{n} + (1-\alpha)p_f \le 0,\\ [\![\mathbf{u}]\!]\cdot\mathbf{n} \ge 0, \quad ([\![\mathbf{u}]\!]\cdot\mathbf{n})\big(\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}\cdot\mathbf{n} + (1-\alpha)p_f\big) = 0 \quad \text{on } (0,T)\times\Gamma_c^{\pm}.\end{gathered} \tag{2.21}$$

where we use in (2.19)–(2.21) the notation $\boldsymbol{\sigma}(\mathbf{u}) := \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\tau}^0$ for short.

We note that the complementarity conditions (2.21) can be represented equivalently by splitting Γ_c^T into a coincidence set $\mathcal{C}^T$ where contact occurs

$$[\![\mathbf{u}]\!]\cdot\mathbf{n} = 0, \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}\cdot\mathbf{n} + (1-\alpha)p_f \le 0 \quad \text{on } \mathcal{C}^T, \tag{2.22}$$

and its complement $\Gamma_c^T \setminus \mathcal{C}^T$ as follows

$$[\![\mathbf{u}]\!] \cdot \mathbf{n} > 0, \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \cdot \mathbf{n} + (1-\alpha)p_f = 0 \quad \text{on } \Gamma_c^T \setminus \mathcal{C}^T. \tag{2.23}$$

Since $\mathcal{C}^T$ is unknown a-priori, relations (2.22) and (2.23) imply a free boundary condition. On the other hand, (2.21) can be expressed with the help of nonlinear complementarity problem (NLCP) functions, e.g. min-based function, as

$$\min\big\{0, \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \cdot \mathbf{n} + (1-\alpha)p_f + c[\![\mathbf{u}]\!] \cdot \mathbf{n}\big\} = \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \cdot \mathbf{n} + (1-\alpha)p_f \quad \text{on } \Gamma_c^T \tag{2.24}$$

for arbitrary constant $c > 0$. The nonlinear equation (2.24) is equivalent to

$$[\![\mathbf{u}]\!] \cdot \mathbf{n} = 0 \quad \text{on } \mathcal{A}^T = \{(t, \mathbf{x}) \in \Gamma_c^T \,|\, \big(\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \cdot \mathbf{n} + (1-\alpha)p_f + c[\![\mathbf{u}]\!] \cdot \mathbf{n}\big)(t, \mathbf{x}) < 0\} \tag{2.25}$$

over the active set $\mathcal{A}^T$, where the constraint is active, and

$$\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \cdot \mathbf{n} = 0 \quad \text{on } \mathcal{I}^T = \{(t, \mathbf{x}) \in \Gamma_c^T \,|\, \big(\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \cdot \mathbf{n} + (1-\alpha)p_f + c[\![\mathbf{u}]\!] \cdot \mathbf{n}\big)(t, \mathbf{x}) \geq 0\} \tag{2.26}$$

over its complementary inactive set $\mathcal{I}^T = \Gamma_c^T \setminus \mathcal{A}^T$. The mixed formulation (2.25) and (2.26) is advantageous for numerical implementation of solution strategies, see the semi-smooth Newton method and its primal–dual active set realization in [11].

To provide well-posedness analysis of the coupled system (2.17)–(2.21), we present the following important observations.

- Formally, the governing Eqs. (2.17) and (2.18) coincide with the thermoelastic equations when p stands for temperature.
- From the point of view of partial differential equations, the system (2.17) and (2.18) is degenerate since of elliptic–parabolic type.

In this sense, from the literature on thermoelasticity there are known existence results, which utilize the pseudo-monotone theory over a compact feasible set (see [27] and [1, Section 3.3]). They justify a variational solution to problem (3.1) and (3.2), however, restricted to small Biot coefficients α. For arbitrary α, differentiating the elliptic equation (2.17) with respect to time, it turns into a pure parabolic problem. Its solvability is provided by applying the theory of accretive operators for implicit evolution equations (see [33,34]). However, the parabolic problem is not well conforming to the unilateral conditions (2.21). Instead, in our approach we apply to the parabolic equation (2.18) a discrete integration with respect to time and reduce the system to a pure elliptic problem. Following the Rothe method for the incremental formulation as described in [35, Section 8.2], we prove well-posedness of the poroelastic problem.

In Section 3 we endow the problem with a variational formulation and state the existence result. Section 3.1 is devoted to the rigorous proof of existence theorem and obtaining a-priori estimates.

3. Variational theory

Let $\mathbf{v} = (v_1, \dots, v_d)(t, \mathbf{x})$ be a smooth test function such that $\mathbf{v} = \mathbf{0}$ on Γ_D^T and $[\![\mathbf{v}]\!] \cdot \mathbf{n} \geq 0$ on Γ_c^T. We multiply the equilibrium equation (2.17) by $\mathbf{v}-\mathbf{u}$, integrate it by parts over Ω_c, use the notation of strain (2.4) and $\mathrm{div}(\mathbf{v}-\mathbf{u}) = \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})$ such that

$$\begin{aligned} 0 &= -\int_{\Omega_c} \big(\mathrm{div}(\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\tau}^0) - \alpha\nabla p\big) \cdot (\mathbf{v}-\mathbf{u})\, d\mathbf{x} \\ &= \int_{\Omega_c} \big((\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\tau}^0) \cdot \boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}) - \alpha p\, \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\big)\, d\mathbf{x} - \int_{\partial\Omega_c} (\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n}) \cdot (\mathbf{v}-\mathbf{u})\, dS_{\mathbf{x}}, \end{aligned}$$

where dot denotes the scalar product of tensors, in particular $\boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) = \sum_{i,j=1}^d \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})$. From boundary conditions (2.19)–(2.21) we obtain

$$\begin{aligned} \int_{\partial\Omega_c} (\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n}) \cdot (\mathbf{v}-\mathbf{u})\, dS_{\mathbf{x}} &= \int_{\Gamma_N} (\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n}) \cdot (\mathbf{v}-\mathbf{u})\, dS_{\mathbf{x}} \\ &- \int_{\Gamma_c} [\![\big(\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - (\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \cdot \mathbf{n})\mathbf{n} + (\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \cdot \mathbf{n} - \alpha p_f)\mathbf{n}\big) \cdot (\mathbf{v}-\mathbf{u})]\!]\, dS_{\mathbf{x}}. \end{aligned}$$

Integrating the result over time provides us with the variational inequality

$$\int_{\Omega_c^T} \big((\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\tau}^0) \cdot \boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}) - \alpha p\, \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\big)\, d\mathbf{x}dt \geq \int_{\Gamma_N^T} \mathbf{g} \cdot (\mathbf{v}-\mathbf{u})\, dS_{\mathbf{x}}dt + \int_{\Gamma_c^T} [\![p_f(\mathbf{v}-\mathbf{u})]\!] \cdot \mathbf{n}\, dS_{\mathbf{x}}dt. \tag{3.1}$$

For a smooth test function $q(t, \mathbf{x})$ such that $q = 0$ on $\partial\Omega^T \cup ((0, T) \times \Gamma_c^{\pm})$, multiplying the diffusion equation (2.18) by q and integrating it by parts over Ω_c^T with respect to the spatial divergence operator, we derive the variational equation

$$\int_{\Omega_c^T} \Big(\frac{\partial}{\partial t}(Sp + \alpha\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}))q + \kappa\nabla p \cdot \nabla q\Big)\, d\mathbf{x}dt = 0. \tag{3.2}$$

Now we give a function setting of the problem. Further $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ denotes d-by-d symmetric tensors. Let the initial data be given in Lebesgue spaces:

$$\boldsymbol{\tau}^0 \in L^2(\Omega_{\mathrm{c}}; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \quad \mathbf{u}^0 \in L^2(\Omega_{\mathrm{c}}; \mathbb{R}^d), \quad p^0 \in L^2(\Omega_{\mathrm{c}}; \mathbb{R}),$$

and the boundary data be expressed in Bochner–Lebesgue spaces:

$$\mathbf{g} \in H^1(0,T; L^2(\Gamma_{\mathrm{N}}; \mathbb{R}^d)).$$

Also we assume existence of such a function

$$p_{\mathrm{r}}(t,\mathbf{x}) \in H^1(0,T; H^1(\Omega_{\mathrm{c}}; \mathbb{R}))$$

that is conforming to the data in the reservoir:

$$p_{\mathrm{r}}(0) = p^0 \text{ in } \Omega_{\mathrm{c}}, \quad p_{\mathrm{r}} = p^\infty \text{ on } \partial\Omega^T, \quad p_{\mathrm{r}} = p_{\mathrm{f}}^\pm \text{ on } (0,T)\times\Gamma_{\mathrm{c}}^\pm. \tag{3.3}$$

The function p_{r} implies an extension into the cracked domain of the initial and boundary data prescribed by (3.3). It has a non-zero jump across Γ_{c}^T except for the crack tip/ crack front, and can be specified, e.g., as a solution to a heat equation in Ω_{c}^T under these data.

Applying Korn and Poincaré inequalities, the elasticity coefficients are assumed to be elliptic and bounded: there exists $0 < \underline{a} \le \overline{a}$ such that

$$\begin{aligned}
&\int_{\Omega_{\mathrm{c}}} \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{u})\,d\mathbf{x} \ge \underline{a}\|\mathbf{u}\|^2_{H^1(\Omega_{\mathrm{c}})} \quad \text{for } \mathbf{u}=\mathbf{0} \text{ on } \Gamma_{\mathrm{D}},\\
&\Big|\int_{\Omega_{\mathrm{c}}} \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{v})\,d\mathbf{x}\Big| \le \overline{a}\|\mathbf{u}\|_{H^1(\Omega_{\mathrm{c}})}\|\mathbf{v}\|_{H^1(\Omega_{\mathrm{c}})}.
\end{aligned} \tag{3.4}$$

The trace inequality is used expressed in the form

$$\|\mathbf{u}\|^2_{L^2(\partial\Omega\cup\Gamma_{\mathrm{c}}^+\cup\Gamma_{\mathrm{c}}^-)} \le K_{\mathrm{tr}}\|\mathbf{u}\|^2_{H^1(\Omega_{\mathrm{c}})}, \quad K_{\mathrm{tr}} > 0. \tag{3.5}$$

The set of trial functions subjected to initial conditions (2.11), boundary conditions in (2.19), (2.20), and the non-penetration condition in (2.21) builds the convex closed cone

$$\begin{aligned}
\mathcal{K}_{\mathrm{trial}} = \Big\{ &\mathbf{u} \in H^1(0,T; H^1(\Omega_{\mathrm{c}}; \mathbb{R}^d)), \quad p \in H^1(0,T; L^2(\Omega_{\mathrm{c}}; \mathbb{R})) \cap L^2(0,T; H^1(\Omega_{\mathrm{c}}; \mathbb{R}))|\\
&\mathbf{u}(0) = \mathbf{u}^0,\ p(0) = p^0 \text{ in } \Omega_{\mathrm{c}}, \quad \mathbf{u} = \mathbf{0} \text{ on } \Gamma_{\mathrm{D}}^T, \quad p = p^\infty \text{ on } \partial\Omega^T,\\
&[\![\mathbf{u}]\!]\cdot\mathbf{n} \ge 0 \text{ on } \Gamma_{\mathrm{c}}^T, \quad p = p_{\mathrm{f}}^\pm \text{ on } (0,T)\times\Gamma_{\mathrm{c}}^\pm \Big\}.
\end{aligned}$$

Whereas the corresponding test functions satisfy homogeneous boundary conditions and the non-penetration within the set:

$$\begin{aligned}
\mathcal{K}_{\mathrm{test}} = \big\{&\mathbf{v} \in L^2(0,T; H^1(\Omega_{\mathrm{c}}; \mathbb{R}^d)),\ q \in L^2(0,T; H^1_0(\Omega_{\mathrm{c}}; \mathbb{R}))|\\
&\mathbf{v} = \mathbf{0} \text{ on } \Gamma_{\mathrm{D}}^T, \quad [\![\mathbf{v}]\!]\cdot\mathbf{n} \ge 0 \text{ on } \Gamma_{\mathrm{c}}^T\big\}.
\end{aligned}$$

Theorem 3.1. *There exists a unique pair $(\mathbf{u}, p) \in \mathcal{K}_{\mathrm{trial}}$ solving the variational inequality (3.1) and the variational equation (3.2) for all test functions $(\mathbf{v}, q) \in \mathcal{K}_{\mathrm{test}}$. The a-priori estimates hold for the time derivatives:*

$$\begin{aligned}
&\frac{\underline{a}}{4}\Big\|\frac{\partial\mathbf{u}}{\partial t}\Big\|^2_{L^2(0,T;H^1(\Omega_{\mathrm{c}}))} + S\Big\|\frac{\partial p}{\partial t}\Big\|^2_{L^2(\Omega_{\mathrm{c}}^T)}\\
&\le \frac{\overline{\kappa}}{2}\|\nabla p^0\|^2_{L^2(\Omega_{\mathrm{c}})} + \frac{2K_{\mathrm{tr}}}{\underline{a}}\sum_{\pm}\Big\|\frac{\partial p_{\mathrm{f}}^\pm}{\partial t}\Big\|^2_{L^2(0,T;L^2(\Gamma_{\mathrm{c}}^\pm))} + \frac{K_{\mathrm{tr}}}{\underline{a}}\Big\|\frac{\partial g}{\partial t}\Big\|^2_{L^2(\Gamma_{\mathrm{N}}^T)} + \frac{\alpha^2 d}{\underline{a}}\Big\|\frac{\partial p_{\mathrm{r}}}{\partial t}\Big\|^2_{L^2(\Omega_{\mathrm{c}}^T)} + \frac{\overline{\kappa}T}{2}\Big\|\frac{\partial\nabla p_{\mathrm{r}}}{\partial t}\Big\|^2_{L^2(\Omega_{\mathrm{c}}^T)},
\end{aligned} \tag{3.6}$$

and for the solution pair:

$$
\begin{aligned}
&\underline{a}\|\mathbf{u}\|^2_{C(0,T;H^1(\Omega_c))} + S\|p\|^2_{C(0,T;L^2(\Omega_c))} + \underline{\kappa}\|\nabla p\|^2_{L^2(\Omega_c^T)} \\
&\le 2\sum_{\pm} \|p_f^\pm\|^2_{L^2(0,T;L^2(\Gamma_c^\pm))} + \|g\|^2_{L^2(\Gamma_N^T)} + T\|\boldsymbol{\tau}^0\|^2_{L^2(\Omega_c)} + \overline{a}\|\mathbf{u}^0\|^2_{H^1(\Omega_c)} + S\|p^0\|^2_{L^2(\Omega_c)} \\
&+ (\alpha+S)\|p_r\|^2_{L^2(\Omega_c^T)} + \overline{\kappa}\|\nabla p_r\|^2_{L^2(\Omega_c^T)} + S\left\|\frac{\partial p}{\partial t}\right\|^2_{L^2(\Omega_c^T)} + (1+\alpha d + 2K_{tr})\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|^2_{L^2(0,T;H^1(\Omega_c))},
\end{aligned}
\tag{3.7}
$$

where the constant $\underline{a}$, $\overline{a}$ and K_{tr} are from inequalities (3.4) and *(3.5).*

3.1. Proof of Theorem 3.1

In the proof we approximate (3.1) and (3.2) by an incremental problem using Rothe's semi-discretization in time, and then pass it to the limit as the time step decreases.

For integer N and time step $\delta = T/N > 0$, we set the equidistant mesh points

$$
t_0 = 0,\ t_1 = \delta,\ \dots,\ t_k = k\delta,\ \dots,\ t_N = N\delta = T.
$$

The final time T is fixed, and $\delta \to 0$ when $N \to \infty$. At the moment we fix N, thus δ. Since the data from the Sobolev space $H^1(0,T)$ are continuous in time, it holds

$$
\begin{aligned}
&(p_r)_k^\delta := p_r(t_k) \in H^1(\Omega_c;\mathbb{R}), \quad (p_f^\pm)_k^\delta := p_f^\pm(t_k) \in L^2(\Gamma_c^\pm;\mathbb{R}), \\
&\mathbf{g}_k^\delta := \mathbf{g}(t_k) \in L^2(\Gamma_N;\mathbb{R}^d), \quad (p^\infty)_k^\delta := p^\infty(t_k) \in L^2(\partial\Omega;\mathbb{R}) \quad \text{for } k = 1,\dots,N.
\end{aligned}
$$

We initialize $\mathbf{u}_0^\delta(\mathbf{x}) = \mathbf{u}^0$, $p_0^\delta(\mathbf{x}) = p^0$ according to initial conditions (2.11), and look for unknown $\mathbf{u}_k^\delta(\mathbf{x})$, $p_k^\delta(\mathbf{x})$ for $k = 1,\dots,N$. Semi-discretizing by finite differences in time the reference variational relations (3.1) and (3.2), for p_r satisfying (3.3) and the feasible set

$$
\mathcal{K} = \left\{\mathbf{v} \in H^1(\Omega_c;\mathbb{R}^d) \,\middle|\, \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, \quad [\![\mathbf{v}]\!]\cdot\mathbf{n} \ge 0 \text{ on } \Gamma_c\right\},
$$

functions $\mathbf{u}_k^\delta \in \mathcal{K}$ and $p_k^\delta - (p_r)_k^\delta \in H_0^1(\Omega_c;\mathbb{R})$ solve the recursion relations

$$
\begin{aligned}
&\int_{\Omega_c} \big(\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta)\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}_k^\delta) - \alpha p_k^\delta \operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}_k^\delta)\big)\,d\mathbf{x} \ge \int_{\Omega_c} \boldsymbol{\tau}^0\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}_k^\delta)\,d\mathbf{x} \\
&+ \int_{\Gamma_N} \mathbf{g}_k^\delta\cdot(\mathbf{v}-\mathbf{u}_k^\delta)\,dS_{\mathbf{x}} + \int_{\Gamma_c} [\![(p_f)_k^\delta(\mathbf{v}-\mathbf{u}_k^\delta)]\!]\cdot\mathbf{n}\,dS_{\mathbf{x}},
\end{aligned}
\tag{3.8}
$$

$$
\int_{\Omega_c} \big((Sp_k^\delta + \alpha\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta))q + \delta\kappa\nabla p_k^\delta\cdot\nabla q\big)\,d\mathbf{x} = \int_{\Omega_c} (Sp_{k-1}^\delta + \alpha\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}_{k-1}^\delta))q\,d\mathbf{x}
\tag{3.9}
$$

for all test functions $\mathbf{v} \in \mathcal{K}$ and $q \in H_0^1(\Omega_c;\mathbb{R})$.

Existence of incremental solution. The left-hand side of (3.8) and (3.9) after summation establishes a bilinear form in $H^1(\Omega_c;\mathbb{R}^d) \times H^1(\Omega_c;\mathbb{R})$:

$$
\int_{\Omega_c} \big(\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta)\cdot\boldsymbol{\varepsilon}(\mathbf{v}) - \alpha p_k^\delta\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{v}) + (Sp_k^\delta + \alpha\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta))q + \delta\kappa\nabla p_k^\delta\cdot\nabla q\big)\,d\mathbf{x}.
$$

It is bounded and coercive, when substitute $\mathbf{v} = \mathbf{u}_k^\delta$ and $q = p_k^\delta$ here

$$
\int_{\Omega_c} \big(\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta)\cdot\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta) + S(p_k^\delta)^2 + \delta\kappa|\nabla p_k^\delta|^2\big)\,d\mathbf{x} \ge \underline{a}\|\mathbf{u}_k^\delta\|^2_{H^1(\Omega_c)} + S\|p_k^\delta\|^2_{L^2(\Omega_c)} + \delta\underline{\kappa}\|\nabla p_k^\delta\|^2_{L^2(\Omega_c)}
\tag{3.10}
$$

such that the terms involving $\operatorname{tr}\boldsymbol{\varepsilon}$ are shortened, due to the assumption on elasticity coefficients (3.4) and recalling $0 < \underline{\kappa} \le \kappa(\mathbf{x}) \le \overline{\kappa}$. Therefore, by the Lions–Stampacchia theorem the unique solution to (3.8) and (3.9) exists for every $k = 1,\dots,N$.

Uniform estimate of time derivatives. Inserting $\mathbf{v} = \mathbf{u}^\delta_{k-1} \in \mathcal{K}$ into (3.8):

$$\int_{\Omega_c} \big(\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}^\delta_k)\cdot\boldsymbol{\varepsilon}(\mathbf{u}^\delta_{k-1}-\mathbf{u}^\delta_k) - \alpha p^\delta_k \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}^\delta_{k-1}-\mathbf{u}^\delta_k)\big)\,d\mathbf{x} \ge \int_{\Omega_c} \boldsymbol{\tau}^0\cdot\boldsymbol{\varepsilon}(\mathbf{u}^\delta_{k-1}-\mathbf{u}^\delta_k)\,d\mathbf{x}$$
$$+\int_{\Gamma_N} \mathbf{g}^\delta_k\cdot(\mathbf{u}^\delta_{k-1}-\mathbf{u}^\delta_k)\,dS_{\mathbf{x}} + \int_{\Gamma_c} [\![(p_f)^\delta_k(\mathbf{u}^\delta_{k-1}-\mathbf{u}^\delta_k)]\!]\cdot\mathbf{n}\,dS_{\mathbf{x}}, \tag{3.11}$$

and $\mathbf{v} = \mathbf{u}^\delta_k \in \mathcal{K}$ into (3.8) at $t = t_{k-1}$:

$$\int_{\Omega_c} \big(\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}^\delta_{k-1})\cdot\boldsymbol{\varepsilon}(\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}) - \alpha p^\delta_{k-1} \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1})\big)\,d\mathbf{x} \ge \int_{\Omega_c} \boldsymbol{\tau}^0\cdot\boldsymbol{\varepsilon}(\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1})\,d\mathbf{x}$$
$$+\int_{\Gamma_N} \mathbf{g}^\delta_{k-1}\cdot(\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1})\,dS_{\mathbf{x}} + \int_{\Gamma_c} [\![(p_f)^\delta_{k-1}(\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1})]\!]\cdot\mathbf{n}\,dS_{\mathbf{x}},$$

after its summation and division by δ^2 yields the inequality

$$\int_{\Omega_c} \Big\{\mathbf{A}\boldsymbol{\varepsilon}\Big(\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big)\cdot\boldsymbol{\varepsilon}\Big(\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big) - \alpha\frac{p^\delta_k-p^\delta_{k-1}}{\delta}\mathrm{tr}\boldsymbol{\varepsilon}\Big(\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big)\Big\}\,d\mathbf{x}$$
$$\le I_1 + I_2, \tag{3.12}$$

where the integrals I_1 and I_2 are defined as follows

$$I_1 := \int_{\Gamma_c} \Big[\!\!\Big[\frac{(p_f)^\delta_k-(p_f)^\delta_{k-1}}{\delta}\,\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big]\!\!\Big]\cdot\mathbf{n}\,dS_{\mathbf{x}},$$
$$I_2 := \int_{\Gamma_N} \frac{\mathbf{g}^\delta_k-\mathbf{g}^\delta_{k-1}}{\delta}\cdot\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\,dS_{\mathbf{x}}.$$

Dividing (3.9) by δ^2 and testing it with $q = p^\delta_k - p^\delta_{k-1} - (p_r)^\delta_k + (p_r)^\delta_{k-1} \in H^1_0(\Omega_c;\mathbb{R})$ we have

$$\int_{\Omega_c} \Big\{\Big[S\frac{p^\delta_k-p^\delta_{k-1}}{\delta} + \alpha\mathrm{tr}\boldsymbol{\varepsilon}\Big(\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big)\Big]\frac{p^\delta_k-p^\delta_{k-1}-(p_r)^\delta_k+(p_r)^\delta_{k-1}}{\delta}$$
$$+\kappa\nabla p^\delta_k\cdot\nabla\frac{p^\delta_k-p^\delta_{k-1}-(p_r)^\delta_k+(p_r)^\delta_{k-1}}{\delta}\Big\}\,d\mathbf{x} = 0. \tag{3.13}$$

The sum of (3.12) and (3.13) after shortening the term $\alpha(p^\delta_k-p^\delta_{k-1})/\delta\cdot\mathrm{tr}\boldsymbol{\varepsilon}((\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1})/\delta)$ and using the lower bound akin to (3.10) gives the inequality

$$\underline{a}\Big\|\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big\|^2_{H^1(\Omega_c)} + S\Big\|\frac{p^\delta_k-p^\delta_{k-1}}{\delta}\Big\|^2_{L^2(\Omega_c)} + \frac{1}{\delta}\|\sqrt{\kappa}\nabla p^\delta_k\|^2_{L^2(\Omega_c)}$$
$$\le I_1 + I_2 + I_3 + I_4, \tag{3.14}$$

where the integrals I_3 and I_4 are

$$I_3 := \int_{\Omega_c} \alpha\mathrm{tr}\boldsymbol{\varepsilon}\Big(\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big)\frac{(p_r)^\delta_k-(p_r)^\delta_{k-1}}{\delta}\,d\mathbf{x},$$
$$I_4 := \int_{\Omega_c} \Big(\frac{\kappa}{\delta}\nabla p^\delta_k\cdot\nabla p^\delta_{k-1} + \kappa\nabla p^\delta_k\cdot\nabla\frac{(p_r)^\delta_k-(p_r)^\delta_{k-1}}{\delta}\Big)\,d\mathbf{x}.$$

Applying weighted Young's and the trace (3.5) inequalities we estimate

$$|I_1| \le \sum_{\pm}\Big\|\frac{(p_f^\pm)^\delta_k-(p_f^\pm)^\delta_{k-1}}{\delta}\Big\|_{L^2(\Gamma_c^\pm)}\Big\|\Big(\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big)^\pm\cdot\mathbf{n}\Big\|_{L^2(\Gamma_c^\pm)}$$
$$\le \frac{\underline{a}}{4}\Big\|\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big\|^2_{H^1(\Omega_c)} + \frac{2K_{tr}}{\underline{a}}\sum_{\pm}\Big\|\frac{(p_f^\pm)^\delta_k-(p_f^\pm)^\delta_{k-1}}{\delta}\Big\|^2_{L^2(\Gamma_c^\pm)},$$

$$|I_2| \le \frac{\underline{a}}{4}\Big\|\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big\|^2_{H^1(\Omega_c)} + \frac{K_{tr}}{\underline{a}}\Big\|\frac{\mathbf{g}^\delta_k-\mathbf{g}^\delta_{k-1}}{\delta}\Big\|^2_{L^2(\Gamma_N)},$$

and using $\mathrm{tr}^2\boldsymbol{\varepsilon}(\mathbf{u}) \le d\boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{u}) = d\sum_{i,j=1}^d(\partial u_i/\partial x_j)^2$ we proceed

$$|I_3| \le \frac{\underline{a}}{4}\Big\|\frac{\mathbf{u}^\delta_k-\mathbf{u}^\delta_{k-1}}{\delta}\Big\|^2_{H^1(\Omega_c)} + \frac{\alpha^2 d}{\underline{a}}\Big\|\frac{(p_r)^\delta_k-(p_r)^\delta_{k-1}}{\delta}\Big\|^2_{L^2(\Omega_c)},$$

$$|I_4| \le \frac{1}{2\delta}\|\sqrt{\kappa}\nabla p_k^\delta\|^2_{L^2(\Omega_c)} + \frac{1}{2\delta}\|\sqrt{\kappa}\nabla p_{k-1}^\delta\|^2_{L^2(\Omega_c)} + \frac{1}{2\delta N}\|\sqrt{\kappa}\nabla p_k^\delta\|^2_{L^2(\Omega_c)} + \frac{\delta N}{2}\Big\|\sqrt{\kappa}\nabla\frac{(p_r)_k^\delta-(p_r)_{k-1}^\delta}{\delta}\Big\|^2_{L^2(\Omega_c)},$$

where $\delta N = T$. Inserting these estimates into (3.14), gathering the same terms, and taking the maximum over $k \in [1, N]$ for the third term in the upper bound of $|I_4|$ gives us

$$\begin{aligned}&\frac{\underline{a}}{4}\Big\|\frac{\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta}{\delta}\Big\|^2_{H^1(\Omega_c)} + S\Big\|\frac{p_k^\delta-p_{k-1}^\delta}{\delta}\Big\|^2_{L^2(\Omega_c)} + \frac{1}{2\delta}\|\sqrt{\kappa}\nabla p_k^\delta\|^2_{L^2(\Omega_c)}\\ &\le \frac{1}{2\delta}\|\sqrt{\kappa}\nabla p_{k-1}^\delta\|^2_{L^2(\Omega_c)} + \frac{1}{2\delta N}\max_{k\in[1,N]}\|\sqrt{\kappa}\nabla p_k^\delta\|^2_{L^2(\Omega_c)} + R_k^\delta\end{aligned}\tag{3.15}$$

with the notation for short

$$\begin{aligned}R_k^\delta &:= \frac{2K_{\mathrm{tr}}}{\underline{a}}\sum_{\pm}\Big\|\frac{(p_f^\pm)_k^\delta-(p_f^\pm)_{k-1}^\delta}{\delta}\Big\|^2_{L^2(\Gamma_c^\pm)} + \frac{K_{\mathrm{tr}}}{\underline{a}}\Big\|\frac{\mathbf{g}_k^\delta-\mathbf{g}_{k-1}^\delta}{\delta}\Big\|^2_{L^2(\Gamma_N)}\\ &+ \frac{\alpha^2 d}{\underline{a}}\Big\|\frac{(p_r)_k^\delta-(p_r)_{k-1}^\delta}{\delta}\Big\|^2_{L^2(\Omega_c)} + \frac{\overline{\kappa}T}{2}\Big\|\nabla\frac{(p_r)_k^\delta-(p_r)_{k-1}^\delta}{\delta}\Big\|^2_{L^2(\Omega_c)}.\end{aligned}\tag{3.16}$$

After summation of (3.15) over $k = 1, \dots, m$ for integer m and using the telescope sum such that

$$\begin{aligned}&\frac{\underline{a}}{4}\sum_{k=1}^m\Big\|\frac{\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta}{\delta}\Big\|^2_{H^1(\Omega_c)} + S\sum_{k=1}^m\Big\|\frac{p_k^\delta-p_{k-1}^\delta}{\delta}\Big\|^2_{L^2(\Omega_c)} + \frac{1}{2\delta}\|\sqrt{\kappa}\nabla p_m^\delta\|^2_{L^2(\Omega_c)}\\ &\le \frac{1}{2\delta}\|\sqrt{\kappa}\nabla p_0^\delta\|^2_{L^2(\Omega_c)} + \frac{m}{2\delta N}\max_{k\in[1,N]}\|\sqrt{\kappa}\nabla p_k^\delta\|^2_{L^2(\Omega_c)} + \sum_{k=1}^m R_k^\delta,\end{aligned}$$

taking its maximum over $m \in [1, N]$ and multiplying by δ concludes with

$$\begin{aligned}&\frac{\underline{a}\delta}{4}\sum_{k=1}^N\Big\|\frac{\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta}{\delta}\Big\|^2_{H^1(\Omega_c)} + S\delta\sum_{k=1}^N\Big\|\frac{p_k^\delta-p_{k-1}^\delta}{\delta}\Big\|^2_{L^2(\Omega_c)}\\ &\le \frac{\overline{\kappa}}{2}\|\nabla p^0\|^2_{L^2(\Omega_c)} + \delta\sum_{k=1}^N R_k^\delta.\end{aligned}\tag{3.17}$$

Uniform estimate of solutions. We test (3.9) with $q = p_k^\delta - (p_r)_k^\delta \in H_0^1(\Omega_c;\mathbb{R})$:

$$\int_{\Omega_c}\Big(\big(S(p_k^\delta-p_{k-1}^\delta)+\alpha\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta)\big)(p_k^\delta-(p_r)_k^\delta) + \delta\kappa\nabla p_k^\delta\cdot\nabla(p_k^\delta-(p_r)_k^\delta)\Big)\,d\mathbf{x} = 0$$

and subtract (3.11) such that the term $\alpha\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta)p_k^\delta$ is shortened, that results in

$$\int_{\Omega_c}\big(\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta)\cdot\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta) + S(p_k^\delta)^2 + \delta\kappa|\nabla p_k^\delta|^2\big)\,d\mathbf{x} \le I_5+I_6+I_7+I_8,\tag{3.18}$$

where the integrals are

$$\begin{aligned}I_5 &:= \int_{\Gamma_c}[\![(p_f)_k^\delta(\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta)]\!]\cdot\mathbf{n}\,dS_{\mathbf{x}},\\ I_6 &:= \int_{\Gamma_N}\mathbf{g}_k^\delta\cdot(\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta)\,dS_{\mathbf{x}},\\ I_7 &:= \int_{\Omega_c}\big(\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta)\cdot\boldsymbol{\varepsilon}(\mathbf{u}_{k-1}^\delta) - \boldsymbol{\tau}^0\cdot\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta) + \alpha\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta)(p_r)_k^\delta\big)\,d\mathbf{x},\\ I_8 &:= \int_{\Omega_c}\big(Sp_k^\delta p_{k-1}^\delta + \delta\kappa\nabla p_k^\delta\cdot\nabla(p_r)_k^\delta + S(p_k^\delta-p_{k-1}^\delta)(p_r)_k^\delta\big)\,d\mathbf{x}.\end{aligned}$$

Applying weighted Young's and the trace (3.5) inequalities such that

$$\begin{aligned}|I_5| &\le \frac{K_{\mathrm{tr}}\delta}{2}\Big\|\frac{\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta}{\delta}\Big\|^2_{H^1(\Omega_c)} + \delta\sum_{\pm}\|(p_f^\pm)_k^\delta\|^2_{L^2(\Gamma_c^\pm)},\\ |I_6| &\le \frac{K_{\mathrm{tr}}\delta}{2}\Big\|\frac{\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta}{\delta}\Big\|^2_{H^1(\Omega_c)} + \frac{\delta}{2}\|\mathbf{g}_k^\delta\|^2_{L^2(\Gamma_N)},\end{aligned}$$

and for domain integrals it follows that

$$|I_7| \le \frac{1}{2}\int_{\Omega_c}\big(\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta)\cdot\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta) + \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_{k-1}^\delta)\cdot\boldsymbol{\varepsilon}(\mathbf{u}_{k-1}^\delta)\big)\,d\mathbf{x}$$

$$+\frac{\delta}{2}(1+\alpha d)\Big\|\frac{\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta}{\delta}\Big\|^2_{H^1(\Omega_c)}+\frac{\delta}{2}\|\boldsymbol{\tau}^0\|^2_{L^2(\Omega_c)}+\frac{\alpha\delta}{2}\|(p_r)_k^\delta\|^2_{L^2(\Omega_c)},$$

$$|I_8|\le\frac{S}{2}\|p_k^\delta\|^2_{L^2(\Omega_c)}+\frac{S}{2}\|p_{k-1}^\delta\|^2_{L^2(\Omega_c)}+\frac{\delta}{2}\|\sqrt{\kappa}\nabla p_k^\delta\|^2_{L^2(\Omega_c)}$$
$$+\frac{\overline{\kappa}\delta}{2}\|\nabla(p_r)_k^\delta\|^2_{L^2(\Omega_c)}+\frac{S\delta}{2}\Big\|\frac{p_k^\delta-p_{k-1}^\delta}{\delta}\Big\|^2_{L^2(\Omega_c)}+\frac{S\delta}{2}\|(p_r)_k^\delta\|^2_{L^2(\Omega_c)}.$$

Inserting these estimates into (3.18) and gathering the same terms, after multiplication by 2 and remembering $0<\underline{\kappa}\le\kappa(\mathbf{x})\le\overline{\kappa}$ yields

$$\int_{\Omega_c}\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta)\cdot\boldsymbol{\varepsilon}(\mathbf{u}_k^\delta)\,d\mathbf{x}+S\|p_k^\delta\|^2_{L^2(\Omega_c)}+\underline{\kappa}\delta\|\nabla p_k^\delta\|^2_{L^2(\Omega_c)}$$
$$\le\int_{\Omega_c}\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_{k-1}^\delta)\cdot\boldsymbol{\varepsilon}(\mathbf{u}_{k-1}^\delta)\,d\mathbf{x}+S\|p_{k-1}^\delta\|^2_{L^2(\Omega_c)}+\delta S_k^\delta \tag{3.19}$$

with the notation introduced for short

$$S_k^\delta:=2\sum_{\pm}\|(p_f^\pm)_k^\delta\|^2_{L^2(\Gamma_c^\pm)}+\|\mathbf{g}_k^\delta\|^2_{L^2(\Gamma_N)}+\|\boldsymbol{\tau}^0\|^2_{L^2(\Omega_c)}$$
$$+(\alpha+S)\|(p_r)_k^\delta\|^2_{L^2(\Omega_c)}+\overline{\kappa}\|\nabla(p_r)_k^\delta\|^2_{L^2(\Omega_c)}+S\Big\|\frac{p_k^\delta-p_{k-1}^\delta}{\delta}\Big\|^2_{L^2(\Omega_c)}+(1+\alpha d+2K_{tr})\Big\|\frac{\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta}{\delta}\Big\|^2_{H^1(\Omega_c)}. \tag{3.20}$$

We sum up (3.19) over $k=1,\dots,m$, use the telescope sum, take maximum over $m\in[1,N]$, and use the lower and upper bounds of $\mathbf{A}$ from (3.4), which together follows the estimate of the solution as

$$\underline{a}\max_{k\in[1,N]}\|\mathbf{u}_k^\delta\|^2_{H^1(\Omega_c)}+S\max_{k\in[1,N]}\|p_k^\delta\|^2_{L^2(\Omega_c)}+\underline{\kappa}\delta\sum_{k=1}^N\|\nabla p_k^\delta\|^2_{L^2(\Omega_c)}$$
$$\le\overline{a}\|\mathbf{u}^0\|^2_{H^1(\Omega_c)}+S\|p^0\|^2_{L^2(\Omega_c)}+\delta\sum_{k=1}^N S_k^\delta. \tag{3.21}$$

Convergence as $\delta\to0$. We introduce interpolants: piecewise-affine

$$\mathbf{u}^\delta(t)=\frac{t-(k-1)\delta}{\delta}\mathbf{u}_k^\delta+\frac{k\delta-t}{\delta}\mathbf{u}_{k-1}^\delta,\quad p^\delta(t)=\frac{t-(k-1)\delta}{\delta}p_k^\delta+\frac{k\delta-t}{\delta}p_{k-1}^\delta$$

as $t\in((k-1)\delta,k\delta]$, and piecewise-constant for time derivatives

$$\frac{\partial\mathbf{u}^\delta}{\partial t}(t)=\frac{\mathbf{u}_k^\delta-\mathbf{u}_{k-1}^\delta}{\delta},\quad\frac{\partial p^\delta}{\partial t}(t)=\frac{p_k^\delta-p_{k-1}^\delta}{\delta}\quad\text{as } t\in((k-1)\delta,k\delta]$$

for $k=1,\dots,N$. The interpolants $\mathbf{g}^\delta(t)$, $(p_f^\pm)^\delta(t)$, $p_r^\delta(t)$ and their time derivatives for $t\in(0,T]$ are defined similarly. Then formulas (3.16) and (3.17) read

$$\frac{\underline{a}}{4}\Big\|\frac{\partial\mathbf{u}^\delta}{\partial t}\Big\|^2_{L^2(0,T;H^1(\Omega_c))}+S\Big\|\frac{\partial p^\delta}{\partial t}\Big\|^2_{L^2(\Omega_c^T)}$$
$$\le\frac{\overline{\kappa}}{2}\|\nabla p^0\|^2_{L^2(\Omega_c)}+\frac{2K_{tr}}{\underline{a}}\sum_{\pm}\Big\|\frac{\partial(p_f^\pm)^\delta}{\partial t}\Big\|^2_{L^2(0,T;L^2(\Gamma_c^\pm))}+\frac{K_{tr}}{\underline{a}}\Big\|\frac{\partial g^\delta}{\partial t}\Big\|^2_{L^2(\Gamma_N^T)}$$
$$+\frac{\alpha^2d}{\underline{a}}\Big\|\frac{\partial p_r^\delta}{\partial t}\Big\|^2_{L^2(\Omega_c^T)}+\frac{\overline{\kappa}T}{2}\Big\|\frac{\partial\nabla p_r^\delta}{\partial t}\Big\|^2_{L^2(\Omega_c^T)}, \tag{3.22}$$

and from (3.20) and (3.21) it follows

$$\underline{a}\|\mathbf{u}^\delta\|^2_{C(0,T;H^1(\Omega_c))}+S\|p^\delta\|^2_{C(0,T;L^2(\Omega_c))}+\underline{\kappa}\|\nabla p^\delta\|^2_{L^2(\Omega_c^T)}$$
$$\le2\sum_{\pm}\|(p_f^\pm)^\delta\|^2_{L^2(0,T;L^2(\Gamma_c^\pm))}+\|g^\delta\|^2_{L^2(\Gamma_N^T)}+T\|\boldsymbol{\tau}^0\|^2_{L^2(\Omega_c)}+\overline{a}\|\mathbf{u}^0\|^2_{H^1(\Omega_c)}$$
$$+S\|p^0\|^2_{L^2(\Omega_c)}+(\alpha+S)\|p_r^\delta\|^2_{L^2(\Omega_c^T)}+\overline{\kappa}\|\nabla p_r^\delta\|^2_{L^2(\Omega_c^T)}+S\Big\|\frac{\partial p^\delta}{\partial t}\Big\|^2_{L^2(\Omega_c^T)}+(1+\alpha d+2K_{tr})\Big\|\frac{\partial\mathbf{u}^\delta}{\partial t}\Big\|^2_{L^2(0,T;H^1(\Omega_c))}. \tag{3.23}$$

Using the convergence of interpolants (see [35, Lemma 8.7 and Remark 8.10])

$$\mathbf{g}^\delta \to \mathbf{g},\ \frac{\partial \mathbf{g}^\delta}{\partial t} \to \frac{\partial \mathbf{g}}{\partial t},\ p_{\mathrm{r}}^\delta \to p_{\mathrm{r}},\ \frac{\partial p_{\mathrm{r}}^\delta}{\partial t} \to \frac{\partial p_{\mathrm{r}}}{\partial t},\ (p_{\mathrm{f}}^\pm)^\delta \to p_{\mathrm{f}}^\pm,\ \frac{\partial (p_{\mathrm{f}}^\pm)^\delta}{\partial t} \to \frac{\partial p_{\mathrm{f}}^\pm}{\partial t}$$
$$\text{strongly in } L^2(0,T) \text{ as } \delta \to 0, \tag{3.24}$$

from (3.22) and (3.23) we conclude with the uniform bound

$$\|\mathbf{u}^\delta\|^2_{C(0,T;H^1(\Omega_{\mathrm{c}}))} + \|p^\delta\|^2_{C(0,T;L^2(\Omega_{\mathrm{c}}))} + \|\nabla p^\delta\|^2_{L^2(\Omega_{\mathrm{c}}^T)} + \left\|\frac{\partial \mathbf{u}^\delta}{\partial t}\right\|^2_{L^2(0,T;H^1(\Omega_{\mathrm{c}}))} + \left\|\frac{\partial p^\delta}{\partial t}\right\|^2_{L^2(\Omega_{\mathrm{c}}^T)}$$
$$\le \text{const.}$$

Therefore, there exists a convergent subsequence (still denoted by δ for short) and an accumulation point $(\mathbf{u}, p)$ such that the convergences as $\delta \to 0$ hold

$$\mathbf{u}^\delta \to \mathbf{u} \quad \text{weakly in } H^1(0,T;H^1(\Omega_{\mathrm{c}})), \quad \text{strongly in } L^2(0,T;L^2(\Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{c}}^+ \cup \Gamma_{\mathrm{c}}^-)), \tag{3.25}$$

where the strong convergence takes place by compactness, and

$$p^\delta \to p \quad \text{weakly in } H^1(0,T;L^2(\Omega_{\mathrm{c}})) \cap L^2(0,T;H^1(\Omega_{\mathrm{c}})), \quad \text{strongly in } L^2(0,T;H^{-1}(\Omega_{\mathrm{c}})), \tag{3.26}$$

with the strong convergence due to Aubin–Lions lemma.

With the help of convergences (3.24)–(3.26) we pass the incremental relations (3.8) and (3.9) to the limit as $\delta \to 0$ and get the solution $(\mathbf{u}, p) \in \mathcal{K}_{\mathrm{trial}}$ to the variational inequality (3.1) and the variational equation (3.2). Passing (3.22) and (3.23) to the limit justifies the a-priori estimates (3.6) and (3.7). This finishes the proof of Theorem 3.1.

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