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Primal–dual methods of shape sensitivity analysis for curvilinear cracks with nonpenetration

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Based on a level-set description of a crack moving with a given velocity, the problem of shape perturbation of the crack is considered. Nonpenetration conditions are imposed between opposite crack surfaces which result in a constrained minimization problem describing equilibrium of a solid with the crack. We suggest a minimax formulation of the state problem thus allowing curvilinear (nonplanar) cracks for the consideration. Utilizing primal–dual methods of shape sensitivity analysis we obtain the general formula for a shape derivative of the potential energy, which describes an energy-release rate for the curvilinear cracks. The conditions sufficient to rewrite it in the form of a path-independent integral (J-integral) are derived.

Keywords: curvilinear crack; nonpenetration; energy release; constrained minimization; shape sensitivity analysis; primal–dual methods.

1. Introduction

Problems on cracks and their evolutions have the important application to engineering and mechanical sciences, see Stavroulakis (2001). In fracture mechanics, there are adopted various measures to give an appropriate description of the crack propagation, for its account see Cherepanov (1979), Morozov (1984), Argatov & Nazarov (2002) and Obrezanova *et al.* (2002). The fracture criterion of Griffith is widely used here, which involves an energy-release rate at the end-points of a crack. In the particular case of a rectilinear crack in an isotropic solid in plane deformation, this can be expressed by a path-independent contour integral surrounding the crack tip by the well-known Cherepanov–Rice formula. In general case, it was observed that the energy-release rate implies a shape derivative of the potential energy in direction of a velocity vector field given a priori.

Recently, there were applied methods of the shape sensitivity analysis of cracks to obtain a formula for the shape derivative in a series of works, Khludnev & Sokolowski (1999), Khludnev & Kovtunenko (2000), Khludnev *et al.* (2002), Kovtunenko (2002, 2003), etc. The idea behind this approach is due to the specific class of cost functionals describing potential energies. To derive the shape differentiability of the potential energy functional it is not necessary to employ the shape sensitivity of a solution of the respective crack problem. Otherwise, to obtain the so-called material (directional) derivatives of the solution with respect to the crack perturbation, one needs a polyhedric property of the tangent sets in a topology class of singular domains with cracks. If this property holds true, the sensitivity technique developed in Sokolowski & Zolesio (1992) will provide the required differentiability of metric projections onto positive cones.

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Another specialty of the approach suggested in Khludnev & Sokolowski (1999) and Khludnev & Kovtunenko (2000) concerns nonpenetration conditions imposed between opposite crack surfaces. In comparison with classic linear conditions of the stress-free faces, the nonpenetration setting of the crack problem results in a constrained variational problem. Here appears also the difference when the shape sensitivity analysis is applied. For constrained crack problems with nonpenetration conditions, our consideration was restricted only to rectilinear (planar) cracks. This fact is due to pure primal methods of the shape sensitivity analysis which were utilized. In the present paper, we involve the primal–dual methods thus allowing us to obtain the shape derivative for curvilinear (nonplanar) cracks, too.

To clarify the matter, let us formulate the shape perturbation problem in a general form. For a fixed increment $s \geqslant 0$ of a (shape) parameter, the perturbed state problem can be expressed as a constrained minimization with respect to the primal variable $u^s \in K_s$:

$$\Pi^s(u^s) = \min_{v} \Pi^s(v) \quad \text{for all } v \in K_s, \tag{1.1}$$

where $\Pi^s\colon H_s \to \mathbb{R}$ is a (quadratic) convex, coercive, lower semi-continuous and differentiable functional in the Hilbert space H_s. The cone K_s including a constraint is given by

$$K_s = \{v \in H_s\colon B^s v \geqslant 0\}, \tag{1.2}$$

with a linear, continuous and bounded operator B^s mapping H_s onto the Hilbert space HB_s. We look for the (shape) derivative of the reduced functional Π with respect to s:

$$\Pi' := \lim_{s\to 0} s^{-1}(\Pi^s(u^s) - \Pi^0(u^0)). \tag{1.3}$$

To expand (1.1) as $s \to 0$ one needs an invertible map Ψ with one-to-one correspondence property related to the reference state of the system as $s = 0$, namely

$$v \in K_s \Rightarrow v \circ \Psi \in K_0, \quad u \in K_0 \Rightarrow u \circ \Psi^{-1} \in K_s. \tag{1.4}$$

Accounting (1.2) relations (1.4) require the identity

$$B^s \circ \Psi = B^0. \tag{1.5}$$

In the case of nonpenetration conditions B^s implies the normal vector at the crack, hence (1.5) is satisfied for constant normal vectors to planar cracks.

To avoid condition (1.5) we utilize primal–dual methods. We suggest a minimax formulation of the state problem (1.1) in the form:

$$L^s(u^s, \lambda^s) = \min_{v} \max_{\mu} L^s(v, \mu) \quad \text{for all } (v, \mu) \in H_s \times M_s, \tag{1.6}$$

where

$$L^s(v, \mu) := \Pi^s(v) + \langle \mu, B^s v\rangle,$$

with the duality pairing between HB_s and its dual space $HB_s^\star$. The dual variable $\lambda^s \in M_s$ as a Lagrange multiplier from the dual cone

$$M_s = \{\mu \in HB_s^\star\colon \mu \leqslant 0\} \tag{1.7}$$

can be defined from the optimality condition

$$\langle \Pi^s_{,u}(u^s), v\rangle + \langle \lambda^s, B^s v\rangle = 0 \quad \text{for all } v \in H_s. \tag{1.8}$$

Therefore, (1.6) and (1.1) are equivalent since (1.8) follows the identity

$$\min_{v\in K_s} \Pi^s(v) = \Pi^s(u^s) = L^s(u^s, \lambda^s) = \min_{v\in H_s} L^s(v, \lambda^s).$$

Thus, the primal constraint $B^s v \geqslant 0$ accounted in (1.2) can be omitted by setting (1.6) which replace it with the dual constraint in (1.7). Now (1.7) does not involve the operator B^s. By constructing the one-to-one mapping Ψ such that

$$\begin{aligned} &(v, \mu) \in H_s \times M_s \Rightarrow (v \circ \Psi, \mu \circ \Psi) \in H_0 \times M_0,\\ &(u, \lambda) \in H_0 \times M_0 \Rightarrow (u \circ \Psi^{-1}, \lambda \circ \Psi^{-1}) \in H_s \times M_s \end{aligned} \tag{1.9}$$

instead of (1.4), we expand (1.6) as $s \to 0$ and then calculate the (shape) derivative (1.3) from the equivalent representation

$$\Pi' = \lim_{s\to 0} s^{-1}(L^s(u^s, \lambda^s) - L^0(u^0, \lambda^0)) \tag{1.10}$$

in view of the identity $L^s(u^s, \lambda^s) = \Pi^s(u^s)$.

Of course, in comparison with (1.3) formula (1.10) employs additionally an expansion with respect to the Lagrange multiplier. This feature makes the principal difference of the constrained problems from the crack problems in linear setting with $\lambda^s = 0$ ($s \geqslant 0$). Note that for crack problems, λ^s implies a constrained component of the stress vector which describes the contact force at the crack. The integral formula representing an energy-release rate in (1.10) (respectively, (1.3)) describes the linear models, too, which are included here as the particular case $\lambda^s = 0$.

For the numerical treatment of curved and nonplanar cracks by mesh-free and level-set methods we refer to the works by Gravouil *et al.* (2002) and Stazi *et al.* (2003).

2. Kinematic description of cracks by level sets

First, we give an appropriate description of a geometric variable, here it is the crack shape, which is a necessary ingredient for shape sensitivity analysis later. We describe a crack Γ_t as a zero-level set of a 'nonnegative' distance function $\rho(t)$ in dependence of the time parameter t in a global way, i.e. for an arbitrary t on a fixed interval $[0, T]$ with $T < \infty$. This formulation is determined by a chosen velocity vector field $V(t)$ and an initial crack Γ_0 at $t = 0$. The generalized method of characteristics provides us with an equivalent description of Γ_t and Γ_0 by means of a one-to-one coordinate transformation $\Phi(t)$ and its inverse $\Phi^{-1}(t)$, mapping them to each other.

For a bounded domain Ω in $\mathbb{R}^N$, where $N = 2, 3$, let Γ_0 denote a fixed initial crack inside Ω as a Lipschitz $(N-1)$-dimensional manifold. We suppose a fixed $T > 0$ such that a velocity vector field $V(t)$ is supported strictly inside Ω for all $t \in [0, T]$ and

$$\begin{aligned} V(t, y) = (V_1, \dots, V_N)^\top \in C([0, T]; W^{1,\infty}(\mathbb{R}^N))^N,\\ V_{,y} \in C([0, T]; W^{1,\infty}(\mathbb{R}^N))^{N\times N}, \end{aligned} \tag{2.1}$$

with the spatial variable $y = (y_1, \ldots, y_N)^\top \in \mathbb{R}^N$. There are used the convention that the subscript ',' means the partial derivative with respect to a variable following after, and the notation

$$\xi_{,y} = (\xi_{,j})_{j=1}^N = (\nabla \xi)^\top \quad \text{for } \xi \in W^{1,\infty}(\mathbb{R}^N),$$
$$\xi_{,y} = (\xi_{i,j})_{i,j=1}^N \quad \text{for } \xi \in W^{1,\infty}(\mathbb{R}^N)^N.$$

First, we describe the initial crack $\Gamma_0 \subset \Omega$ as level set

$$\begin{aligned} \Gamma_0 &= \{x = (x_1, \ldots, x_N)^\top \in \mathbb{R}^N \colon \rho_0(x) = 0\}, \\ \mathbb{R}^N \backslash \Gamma_0 &= \{x \in \mathbb{R}^N \colon \rho_0(x) > 0\}, \end{aligned} \tag{2.2}$$

with a nonnegative level-set function

$$0 \leqslant \rho_0(x) \in W^{1,\infty}(\mathbb{R}^N). \tag{2.3}$$

It can be chosen, e.g. as the isotropic distance: $\rho_0(x) = \min |x - \bar{x}|$, for all $\bar{x} \in \Gamma_0$, which is uniformly Lipschitz continuous. We define a (moving) crack $\Gamma_t \subset \Omega$ at 'time' $t \in [0, T]$ similarly as level set

$$\begin{aligned} \Gamma_t &= \{y \in \mathbb{R}^N \colon \rho(t, y) = 0\}, \\ \mathbb{R}^N \backslash \Gamma_t &= \{y \in \mathbb{R}^N \colon \rho(t, y) > 0\}. \end{aligned} \tag{2.4}$$

A nonnegative level-set function can be taken as the generalized solution

$$0 \leqslant \rho(t, y) \in W^{1,\infty}((0, T) \times \mathbb{R}^N) \tag{2.5}$$

to the Cauchy problem for the transport equation

$$\rho_{,t} + V^\top \nabla \rho = 0 \ \text{ a.e. } (0, T) \times \mathbb{R}^N, \quad \rho(0) = \rho_0 \tag{2.6}$$

with the velocity field (2.1) and the initial data (2.3). Existence and uniqueness of such solutions (2.5) are provided by the fact that $\rho = \rho_0$ along characteristics of (2.6), see Kovtunenko *et al.* (2004). The inclusion $\Gamma_t \subset \Omega$ is guaranteed by the compactness of $V(t)$ in Ω due to the local structure of solution to (2.6).

The method of characteristics used to the transport equation employs a flow map $\Phi(t, x)$ related to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi = V(t, \Phi) \ \text{ for } t > 0, \quad \Phi(0) = x. \tag{2.7}$$

In view of the smoothness of V assumed in (2.1), the global Cauchy theorem guarantees the unique classical solution to (2.7) with the regularity:

$$\begin{aligned} \Phi &= (\Phi_1, \ldots, \Phi_N)^\top \in C^1([0, T]; W^{1,\infty}(\mathbb{R}^N))^N, \\ \Phi_{,x} &\in C([0, T]; W^{1,\infty}(\mathbb{R}^N))^{N \times N}. \end{aligned} \tag{2.8}$$

This map is uniquely invertible (Delfour & Zolesio, 2001, Theorem 4.1, Chapter 7). Then, we can construct its inverse again as the generalized solution

$$\Phi^{-1} = (\Phi_1^{-1}, \dots, \Phi_N^{-1})^\top \in W^{1,\infty}((0,T) \times \mathbb{R}^N)^N \tag{2.9}$$

to the system of transport equations

$$\Phi_{,t}^{-1} + \Phi_{,y}^{-1} V = 0 \ \text{ a.e. } (0,T) \times \mathbb{R}^N, \quad \Phi^{-1}(0) = y. \tag{2.10}$$

Differentiating with respect to y the identity

$$y = \Phi(t, \Phi^{-1}(t,y)) \quad \text{for } (t,y) \in [0,T] \times \mathbb{R}^N \tag{2.11}$$

implies that

$$\Phi_{,y}^{-1}(t,y) = \Phi_{,x}(t, \Phi^{-1}(t,y))^{-1}, \tag{2.12}$$

and due to (2.8), (2.9), by the Rademacher theorem we observe the additional spatial regularity

$$\Phi_{,y}^{-1} \in C([0,T]; W^{1,\infty}(\mathbb{R}^N))^{N \times N}. \tag{2.13}$$

Now, we consider the transformation of coordinates:

$$\begin{aligned} y &= \Phi(t,x) \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^N, \\ x &= \Phi^{-1}(t,y) \quad \text{for } (t,y) \in [0,T] \times \mathbb{R}^N, \end{aligned} \tag{2.14}$$

applied to the cracks Γ_0 and Γ_t. It follows the exact representation of the solution to (2.6):

$$\rho(t,y) = \rho_0(\Phi^{-1}(t,y)) \quad \text{for } (t,y) \in [0,T] \times \mathbb{R}^N. \tag{2.15}$$

The one-to-one correspondence property of (2.14) and (2.15) provides the equivalent expression of level sets in (2.2) and (2.4) via coordinate transformations:

$$\begin{aligned} \Gamma_t &= \{y \in \mathbb{R}^N \colon y = \Phi(t,x) \text{ for all } x \in \Gamma_0\}, \\ \Gamma_0 &= \{x \in \mathbb{R}^N \colon x = \Phi^{-1}(t,y) \text{ for all } y \in \Gamma_t\}. \end{aligned} \tag{2.16}$$

The spatial regularity assumed for the velocity field in (2.1) leads to the corresponding additional smoothness in (2.8) and (2.13). It ensures a uniform $C^{1,1}$-regularity of a crack and related Lipschitz continuity of the normal at the crack, which is required for the dual variable in a minimax formulation later.

In fact, let Γ_0 be the manifold of uniform $C^{1,1}$-class in $\mathbb{R}^N$ possessing a smooth closed extension Σ in Ω. We suppose Σ splitting Ω into two sub-domains Ω^+ and Ω^-. Take a neighborhood $\mathcal{O}$ of Σ inside Ω and $\mathcal{O}^+ = \mathcal{O} \cap \Omega^+$. According to (Delfour & Zolesio, 2001, Theorems 4.1–4.2, Chapter 2) there exist the neighborhood $\mathcal{O}^+$ and the level-set function ρ_0 in (2.2) such that

$$\rho_0 \in C^{1,1}(\overline{\mathcal{O}^+}). \tag{2.17}$$

We construct the level-set function ρ according to (2.15). Due to (2.9) and (2.13), inclusion (2.17) implies that

$$\rho \in C^{1,1}(\overline{\mathcal{O}_t^+}), \quad \mathcal{O}_t^+ = \{y \in \mathbb{R}^N \colon y = \Phi(t,x) \text{ for } x \in \mathcal{O}^+\}, \tag{2.18}$$

hence Γ_t is of the uniform $C^{1,1}$-class, too.

Let $\nu^0 = (\nu_1^0, \dots, \nu_N^0)^\top$ and $\nu^t = (\nu_1^t, \dots, \nu_N^t)^\top$ denote the unit normal vectors chosen at Γ_0 and Γ_t, respectively. According to (Sokolowski & Zolesio, 1992, Proposition 2.48), for the transformation (2.14) the following representation holds:

$$\nu^t = (\Phi_{,y}^{-1})^\top (\nu^0 \circ \Phi^{-1}) |(\Phi_{,y}^{-1})^\top (\nu^0 \circ \Phi^{-1})|^{-1}, \tag{2.19}$$

where we use the notation

$$\eta(y) \circ \Phi = \eta(\Phi(t,x)), \quad \xi(x) \circ \Phi^{-1} = \xi(\Phi^{-1}(t,y)).$$

From (2.17) and (2.18) it follows that ν^0 and ν^t are uniformly Lipschitz continuous:

$$\nu^t \in W^{1,\infty}(\Gamma_t)^N \quad \text{for } t \in [0,T]. \tag{2.20}$$

Note that $\nabla\rho/|\nabla\rho|$ can be viewed as an extension of ν^t from Γ_t into $\mathbb{R}^N$.

For an analytically given family of cracks Γ_t with $t \in [0,T]$, the level-set formalism allows us also to construct a velocity vector field $V(t)$ related to the crack. In fact, determining the level-set function ρ which describes Γ_t, we can treat (2.6) as an algebraic equation for unknown V. In such a way, we get analytic velocities in examples presented below, see Kovtunenko *et al.* (2004).

EXAMPLE 2.1 Curvilinear cracks in $\mathbb{R}^2$. For given $\psi \in C^2(\mathbb{R})$, $\psi'' \in W^{1,\infty}(\mathbb{R})$ and $a, b \in C^1([0,\infty))$ with $a < b$, consider a family of curvilinear cracks defined as

$$\Gamma_t = \{y = (y_1, y_2)^\top \colon a(t) < y_1 < b(t), y_2 = \psi(y_1)\}, \tag{2.21}$$

as it is illustrated in Fig. 1(a). We assume a and b such that Γ_t remains located inside a bounded domain $\Omega \subset \mathbb{R}^2$ for all $0 \leqslant t \leqslant T$ with $T > 0$. The crack Γ_t can be described as level set (2.4) for ρ satisfying the transport equation (2.6) with the velocity vector field

$$V = (-\chi^a(t,y)a'(t) + \chi^b(t,y)b'(t))(1, \psi'(y_1))^\top, \tag{2.22}$$

which is composed with the help of cut-off functions at crack tips. More precisely, we suppose that the cut-off function

$$\chi^a \in C([0,T]; W^{1,\infty}(\mathbb{R}^2)), \quad \nabla\chi^a \in C([0,T]; W^{1,\infty}(\mathbb{R}^2))^2$$

is supported in a neighborhood $B_0^a \subset \Omega$ with $\chi^a = 1$ in B_1^a such that $(a(t), \psi(a(t)))^\top \in B_1^a \subset B_0^a$. These neighborhoods may depend on t or not. Similarly, the cut-off function

$$\chi^b \in C([0,T]; W^{1,\infty}(\mathbb{R}^2)), \quad \nabla\chi^b \in C([0,T]; W^{1,\infty}(\mathbb{R}^2))^2$$

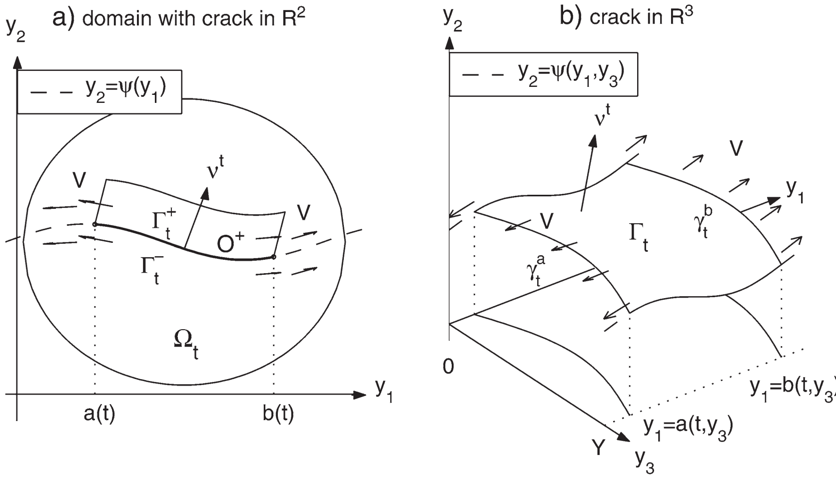

FIG. 1. The examples of curvilinear cracks.

is supported in a neighborhood $B_0^b \subset \Omega$ with $\chi^b = 1$ in B_1^b such that $(b(t), \psi(b(t)))^\top \in B_1^b \subset B_0^b$. The supports of χ^a and χ^b must not intersect each other. Locally, the level-set function ρ satisfying (2.6) with V from (2.22) can be taken, e.g. as the anisotropic distance:

$$\rho(t, y) = \max(0, a(t) - y_1) + |y_2 - \psi(y_1)| \quad \text{in } B_1^a,$$
$$\rho(t, y) = \max(0, y_1 - b(t)) + |y_2 - \psi(y_1)| \quad \text{in } B_1^b.$$

Globally, we cannot construct analytically the solutions to (2.6) and (2.7) or (2.10) due to the presence of the cut-off functions.

EXAMPLE 2.2 Nonplanar cracks with curvilinear fronts in $\mathbb{R}^3$. For given $\psi(y_1, y_3) \in C(\mathbb{R}^2)$, $\nabla\psi_{,1} \in W^{1,\infty}(\mathbb{R}^2)^2$ and $a, b(t, y_3) \in C^1([0, \infty); W^{1,\infty}(\mathbb{R}))$, $a_{,t3}, b_{,t3} \in C([0, \infty); W^{1,\infty}(\mathbb{R}))$ with $a < b$, we suppose the crack

$$\Gamma_t = \{y = (y_1, y_2, y_3)^\top : a(t, y_3) < y_1 < b(t, y_3), y_2 = \psi(y_1, y_3), 0 < y_3 < Y\} \tag{2.23}$$

located inside $\Omega \subset \mathbb{R}^3$ for $t \in [0, T]$, as it is illustrated in Fig. 1(b). This crack can be described by the velocity:

$$V = (-\chi^a(t, y)a_{,t}(t, y_3) + \chi^b(t, y)b_{,t}(t, y_3))(1, \psi_{,1}(y_1, y_3), 0)^\top. \tag{2.24}$$

The cut-off functions:

$$\chi^a, \chi^b \in C([0, T]; W^{1,\infty}(\mathbb{R}^3)), \quad \nabla\chi^a, \nabla\chi^b \in C([0, T]; W^{1,\infty}(\mathbb{R}^3))^3$$

are supported as before in separated neighborhood $B_1^a \subset B_0^a \subset \Omega$ and $B_1^b \subset B_0^b \subset \Omega$ of the moving fronts γ_t^a and γ_t^b of the crack Γ_t, which are assumed to be disconnected:

$$\gamma_t^a = \{y : y_1 = a(t, y_3), y_2 = \psi(y_1, y_3), 0 < y_3 < Y\} \subset B_1^a,$$
$$\gamma_t^b = \{y : y_1 = b(t, y_3), y_2 = \psi(y_1, y_3), 0 < y_3 < Y\} \subset B_1^b.$$

The level-set function can be constructed locally again as

$$\rho(t,y) = \max(0, a(t,y_3) - y_1) + |y_2 - \psi(y_1,y_3)| + \max(0,-y_3) + \max(0, y_3 - Y) \quad \text{in } B_1^a,$$

$$\rho(t,y) = \max(0, y_1 - b(t,y_3)) + |y_2 - \psi(y_1,y_3)| + \max(0,-y_3) + \max(0, y_3 - Y) \quad \text{in } B_1^b.$$

For the geometrical construction of movements of a connected crack front for nonplanar cracks in $\mathbb{R}^3$ see Kovtunenko (2002).

3. Minimax formulation of constrained crack problems with nonpenetration conditions

The problem of equilibrium of an elastic solid occupying the domain with crack $\Omega_t = \Omega \setminus \Gamma_t$ is considered, subject to nonpenetration conditions which prevent a mutual interpenetration between the opposite crack surfaces. Using the Lagrange multiplier arguments, we give an equivalent minimax formulation of the constrained problem with the crack Γ_t.

For a displacement vector $u = (u_1, \dots, u_N)^\top(y)$, we suppose the linear strain tensor

$$\varepsilon_{ij}(u) = 0.5(u_{i,j} + u_{j,i}), \quad i,j = 1,\dots,N, \tag{3.1}$$

and the symmetric stress tensor

$$\sigma_{ij}(u) = c_{ijkl}(y)\varepsilon_{kl}(u), \quad i,j = 1,\dots,N. \tag{3.2}$$

The summation convention over repeated indices is used. The elasticity coefficients are assumed to be symmetric and elliptic as it is usually adopted in the linear elasticity, and satisfying the smoothness requirement $c_{ijkl} \in C^1(\mathbb{R}^N)$ for $i,j,k,l = 1,\dots,N$.

Let $\Omega \subset \mathbb{R}^N$ have the Lipschitz boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$ with the outward normal vector $n = (n_1, \dots, n_N)^\top$, and $\Gamma_D \neq \emptyset$. We assume a clamping condition at Γ_D. The normal vector ν^t at Γ_t distinguishes the opposite crack surface Γ_t^+ and Γ_t^-. We suppose a nonpenetration between them, which is expressed by the inequality condition (see Khludnev & Kovtunenko, 2000):

$$0 \leqslant (\nu^t)^\top [\![u]\!] \quad \text{a.e. } \Gamma_t, \tag{3.3}$$

with the jump $[\![u]\!] = u|_{\Gamma_t^+} - u|_{\Gamma_t^-}$. For a given surface traction $g = (g_1, \dots, g_N)^\top$ at Γ_N and zeroth volume load, the equilibrium of the solid with crack is described by the following nonlinear boundary value problem:

$$-\sigma_{ij,j}(u) = 0, \quad i = 1,\dots,N, \ \text{ in } \Omega_t; \tag{3.4a}$$

$$u_i = 0, \quad i = 1,\dots,N, \ \text{ on } \Gamma_D; \tag{3.4b}$$

$$\sigma_{ij}(u)n_j = g_i, \quad i = 1,\dots,N, \ \text{ on } \Gamma_N; \tag{3.4c}$$

$$(\sigma_{\tau^t}(u))_i = 0, \quad i = 1,\dots,N, \ \text{ on } \Gamma_t^\pm; \tag{3.4d}$$

$$[\![\sigma_{\nu^t}(u)]\!] = 0, \quad \sigma_{\nu^t}(u) \leqslant 0, \quad (\nu^t)^\top [\![u]\!] \geqslant 0, \quad \sigma_{\nu^t}(u)((\nu^t)^\top [\![u]\!]) = 0 \quad \text{on } \Gamma_t. \tag{3.4e}$$

For the sake of simplicity, we used the notation of normal and tangential stress vectors at Γ_t according to the decomposition:

$$\begin{gathered}(\sigma_{1j}(u)\nu_j^t, \sigma_{2j}(u)\nu_j^t, \sigma_{3j}(u)\nu_j^t)^\top = \sigma_{\nu^t}(u)\nu^t + \sigma_{\tau^t}(u),\\ \sigma_{\nu^t}(u) = \sigma_{ij}(u)\nu_j^t\nu_i^t,\end{gathered} \tag{3.5}$$

with the tangential vector $\sigma_{\tau^t}(u) = ((\sigma_{\tau^t}(u))_1, \dots, (\sigma_{\tau^t}(u))_N)^\top$ satisfying $(\nu^t)^\top \sigma_{\tau^t}(u) = 0$.

Now, we give a variational formulation to (3.4). Accounting the homogeneous Dirichlet condition (3.4b) the displacement vector can be defined as an element of the Sobolev space

$$H(\Omega_t) = \{u \in H^1(\Omega_t)^N \colon u = 0 \text{ on } \Gamma_D\}.$$

Let the surface traction be given by a vector function $g \in L^2(\Gamma_N)^N$. The potential energy of the solid with crack is expressed by a quadratic functional in $H(\Omega_t)$:

$$\Pi(u; \Omega_t) = \frac{1}{2} \int_{\Omega_t} \sigma_{ij}(u)\varepsilon_{ij}(u)\mathrm{d}y - \int_{\Gamma_N} g^\top u \, \mathrm{d}s. \tag{3.6}$$

For the cone of admissible displacements

$$K(\Omega_t) = \{u \in H(\Omega_t) \colon u \text{ satisfies (3.3)}\},$$

the equilibrium problem (3.4) is equivalent to the constrained minimization of (3.6) with respect to the primal variable u:

$$\Pi(u^t; \Omega_t) \leqslant \Pi(u; \Omega_t) \quad \text{for all } u \in K(\Omega_t). \tag{3.7}$$

If the Korn inequality is satisfied:

$$\int_{\Omega_t} \sigma_{ij}(u)\varepsilon_{ij}(u)\mathrm{d}y \geqslant c_0 \|u\|^2_{H(\Omega_t)} \quad \text{for } u \in H(\Omega_t), \tag{3.8}$$

then there exists a unique solution $u^t \in K(\Omega_t)$ to (3.7). In view of the regularity of the normal vector ν^t presented in (2.20), we can characterize the solution to (3.7) at Γ_t via dual elements:

$$(\nu^t)^\top [\![u^t]\!] \in H^{1/2}_{00}(\Gamma_t), \quad \sigma_{\nu^t}(u^t) \in H^{1/2}_{00}(\Gamma_t)^\star, \tag{3.9}$$

and hence to fulfill relations (3.4e) in the dual sense. There $H^{1/2}_{00}(\Gamma_t)$ is the space of functions from $H^{1/2}(\Gamma_t)$ which admit a continuation by zero onto the smooth closed extension Σ in Ω_t, and $H^{1/2}_{00}(\Gamma_t)^\star$ is its dual space.

The fact presented in (3.9) allows us to employ Lagrange arguments following next. We define the dual cone

$$M(\Gamma_t) = \{\lambda \in H^{1/2}_{00}(\Gamma_t)^\star \colon \langle \lambda, \xi \rangle_{\Gamma_t} \leqslant 0 \text{ for all } 0 \leqslant \xi \in H^{1/2}_{00}(\Gamma_t)\},$$

where $\langle \cdot, \cdot \rangle_{\Gamma_t}$ denotes the pairing between $H^{1/2}_{00}(\Gamma_t)^\star$ and $H^{1/2}_{00}(\Gamma_t)$. It can be verified that the pair $(u^t, \lambda^t) \in H(\Omega_t) \times M(\Gamma_t)$ with the primal variable $u^t \in K(\Omega_t)$ being the solution to (3.7) and the dual variable

$$\lambda^t = \sigma_{\nu^t}(u^t) \tag{3.10}$$

is the unique solution to an equivalent problem in the primal–dual form:

$$\int_{\Omega_t} \sigma_{ij}(u^t)\varepsilon_{ij}(u)\mathrm{d}y + \langle \lambda^t, (\nu^t)^\top [\![u]\!] \rangle_{\Gamma_t} = \int_{\Gamma_N} g^\top u \, \mathrm{d}s \quad \text{for all } u \in H(\Omega_t), \tag{3.11a}$$

$$\langle \lambda - \lambda^t, (\nu^t)^\top [\![u^t]\!] \rangle_{\Gamma_t} \leqslant 0 \quad \text{for all } \lambda \in M(\Gamma_t). \tag{3.11b}$$

If we introduce a Lagrangian

$$L(u, \lambda; \Omega_t) := \Pi(u; \Omega_t) + \langle \lambda, (\nu^t)^\top [\![u]\!] \rangle_{\Gamma_t}, \tag{3.12}$$

then (3.11) is the optimality system for the constrained minimax problem

$$L(u^t, \lambda; \Omega_t) \leqslant L(u^t, \lambda^t; \Omega_t) \leqslant L(u, \lambda^t; \Omega_t) \quad \text{for all } (u, \lambda) \in H(\Omega_t) \times M(\Gamma_t). \tag{3.13}$$

Note that due to (3.11b), from (3.12) we have evidently the identity

$$L(u^t, \lambda^t; \Omega_t) = \Pi(u^t; \Omega_t). \tag{3.14}$$

In comparison to the pure primal setting (3.7) of the crack problem, the advantage of the minimax formulation consists in the fact that we neglect the constraint on primal variable in $K(\Omega_t)$, which involves the normal vector ν^t as a coefficient, by means of the dual constraint on the Lagrange multiplier in $M(\Gamma_t)$ without coefficients. This matter allows us to extend methods of the shape sensitivity analysis to a wide class of constraints involving nonidentity operators B^s in (1.2). For numerical treatment of such problems see Hintermüller *et al.* (2004).

4. Shape differentiability of the constrained crack problem

Let us define from (3.6) and (3.7) a reduced functional of the potential energy in dependence of the crack Γ_t

$$P(\Gamma_t) := \Pi(u^t; \Omega_t). \tag{4.1}$$

Our aim is to find the shape derivative of P in direction V:

$$P'_V(\Gamma_t) := \lim_{s \to 0} s^{-1}(P(\Gamma_{t+s}) - P(\Gamma_t)). \tag{4.2}$$

In view of (3.14), for this purpose, it is sufficient to investigate the shape differentiability of the Lagrangian instead of (4.1).

Fix $t \geqslant 0$. For the increment s, we consider the state of the system at $t + s$ as a perturbation of the problem formulated in Section 3 and expand it as $s \to 0$. For this reason, we utilize the results of Section 2. By setting

$$\Psi(s, y) := \Phi(t + s, \Phi^{-1}(t, y)), \quad \Psi^{-1}(s, z) := \Phi(t, \Phi^{-1}(t + s, z)) \tag{4.3}$$

and using (2.14), from (2.16) we get the one-to-one coordinate transformation:

$$\begin{aligned} z &= \Psi(s, y) \colon \Omega_t \to \Omega_{t+s},\ \Gamma_t \to \Gamma_{t+s}, \\ y &= \Psi^{-1}(s, z) \colon \Omega_{t+s} \to \Omega_t,\ \Gamma_{t+s} \to \Gamma_t. \end{aligned} \tag{4.4}$$

The functions in (4.3) satisfy relations similar to (2.7) and (2.10):

$$\frac{\mathrm{d}}{\mathrm{d}s} \Psi(s, y) = V(t + s, \Psi(s, y)), \quad \Psi(0, y) = y, \tag{4.5a}$$

$$\Psi^{-1}_{,s}(s, z) + \Psi^{-1}_{,z}(s, z) V(t + s, z) = 0, \quad \Psi^{-1}(0, z) = z. \tag{4.5b}$$

By (4.3) and (4.5a) we can decompose Ψ with respect to s as follows:

$$\Psi(s, y) = y + sV(t, y) + r(s), \quad \|r(s)\|_{W^{1,\infty}(\mathbb{R}^N)^N} = o(s). \tag{4.6}$$

Hence, (4.6) leads to expansions of the functional matrices and Jacobian of transformation (4.4) in Ω:

$$\begin{aligned} \Psi_{,y} &= I + sV_{,y} + r_1(s), \quad \|r_1(s)\|_{L^\infty(\mathbb{R}^N)^{N\times N}} = o(s), \\ \det(\Psi_{,y}) &= 1 + s\operatorname{div}(V) + r_2(s), \quad \|r_2(s)\|_{L^\infty(\mathbb{R}^N)} = o(s), \\ \Psi_{,y}^{-1} &= I - sV_{,y} + r_3(s), \quad \|r_3(s)\|_{L^\infty(\mathbb{R}^N)^{N\times N}} = o(s), \end{aligned} \tag{4.7}$$

with the identity operator I. Transformation (4.4) applied to the crack Γ_t as the boundary of Ω^+ implies the Jacobian (Sokolowski & Zolesio, 1992, Proposition 2.47):

$$\omega := \det(\Psi_{,y})|(\Psi_{,y}^{-1})^\top \nu^t|. \tag{4.8}$$

Due to (4.7) a decomposition of (4.8) with respect to s is derived as

$$\omega(s, y) = 1 + s(\operatorname{div}(V) - (\nu^t)^\top V_{,y}\nu^t)(t, y) + o(s). \tag{4.9}$$

Thus, ω is strictly positive for small s. For further consideration, we need also an expansion of the transformed normal vector $\nu^{t+s} \circ \Psi$. Using (2.12) from (2.19) we obtain that

$$\nu^{t+s} \circ \Psi = (\Psi_{,y}^{-1})^\top \nu^t |(\Psi_{,y}^{-1})^\top \nu^t|^{-1}, \tag{4.10}$$

and again due to (4.7), from (4.10) we arrive at the asymptotic formula

$$\nu^{t+s} \circ \Psi = \nu^t + s(((\nu^t)^\top V_{,y}\nu^t)\nu^t - V_{,y}^\top \nu^t) + o(s). \tag{4.11}$$

For the elasticity coefficients, the standard Taylor expansion and (4.6) yield

$$c_{ijkl} \circ \Psi = c_{ijkl} + sV^\top \nabla c_{ijkl} + o(s), \quad i, j, k, l = 1, \ldots, N. \tag{4.12}$$

Consider the perturbed minimax problem

$$\begin{aligned} L(u^{t+s}, \mu; \Omega_{t+s}) \leqslant L(u^{t+s}, \lambda^{t+s}; \Omega_{t+s}) \leqslant L(v, \lambda^{t+s}; \Omega_{t+s}) \\ \text{for all } (v, \mu) \in H(\Omega_{t+s}) \times M(\Gamma_{t+s}). \end{aligned} \tag{4.13}$$

First, we map (4.13) onto the reference domain Ω_t. For this reason, one needs a one-to-one correspondence between the involved functional sets, i.e.

$$\begin{aligned} (v, \mu) \in H(\Omega_{t+s}) \times M(\Gamma_{t+s}) &\Rightarrow (v \circ \Psi, \mu \circ \Psi) \in H(\Omega_t) \times M(\Gamma_t), \\ (u, \lambda) \in H(\Omega_t) \times M(\Gamma_t) &\Rightarrow (u \circ \Psi^{-1}, \lambda \circ \Psi^{-1}) \in H(\Omega_{t+s}) \times M(\Gamma_{t+s}). \end{aligned} \tag{4.14}$$

Transformations of λ and μ in (4.14) are defined in the dual sense:

$$\begin{aligned} \langle \mu \circ \Psi, \xi \rangle_{\Gamma_t} &:= \langle \mu, (\omega^{-1}\xi) \circ \Psi^{-1} \rangle_{\Gamma_{t+s}} \quad \text{for } \xi \in H^{1/2}_{00}(\Gamma_t), \\ \langle \lambda \circ \Psi^{-1}, \eta \rangle_{\Gamma_{t+s}} &:= \langle \lambda, \omega(\eta \circ \Psi) \rangle_{\Gamma_t} \quad \text{for } \eta \in H^{1/2}_{00}(\Gamma_{t+s}), \end{aligned} \tag{4.15}$$

with the Jacobian ω from (4.8). In our case, (4.14) is provided by the one-to-one coordinate transformation (4.4) with the Lipschitz continuous functions Ψ and Ψ^{-1}. Due to $v_{,z} = (v \circ \Psi)_{,y} \Psi^{-1}_{,y}$ for $v \in H(\Omega_{t+s})$, we can transport the differential operators in (3.1) and (3.2) with the help of (4.4) and arrive at the generalized strain tensor

$$E_{ij}(\Psi^{-1}_{,y}; u) := 0.5(u_{i,k}(\Psi^{-1}_{,y})_{kj} + u_{j,k}(\Psi^{-1}_{,y})_{ki}), \quad i, j = 1, \ldots, N, \tag{4.16}$$

for $u \in H(\Omega_t)$. Note that from (4.7) it admits a decomposition of (4.16) as $s \to 0$ as follows:

$$E_{ij}(\Psi^{-1}_{,y}; u) = \varepsilon_{ij}(u) - sE_{ij}(V_{,y}; u) + \mathrm{o}(s), \quad i, j = 1, \ldots, N. \tag{4.17}$$

Thus, using (3.6), (3.12), (4.15) and (4.16) gets transformation of the Lagrangian for $(v, \mu) \in H(\Omega_{t+s}) \times M(\Gamma_{t+s})$:

$$L(v, \mu; \Omega_{t+s}) = L_s(v \circ \Psi, \mu \circ \Psi; \Omega_t), \tag{4.18}$$

where

$$\begin{aligned} L_s(u, \lambda; \Omega_t) &:= \Pi_s(u; \Omega_t) + \langle \lambda, \omega(\nu^{t+s} \circ \Psi)^\top [\![u]\!] \rangle_{\Gamma_t}, \\ \Pi_s(u; \Omega_t) &:= \frac{1}{2} \int_{\Omega_t} \det(\Psi_{,y})(c_{ijkl} \circ \Psi) E_{kl}(\Psi^{-1}_{,y}; u) E_{ij}(\Psi^{-1}_{,y}; u) \mathrm{d}y \\ &\quad - \int_{\Gamma_N} g^\top u \, \mathrm{d}s \quad \text{for } (u, \lambda) \in H(\Omega_t) \times M(\Gamma_t). \end{aligned} \tag{4.19}$$

For arbitrary $(u, \lambda) \in H(\Omega_t) \times M(\Gamma_t)$ it is easy to deduce from (4.13) and (4.18) that

$$\begin{aligned} L_s(u^{t+s} \circ \Psi, \lambda; \Omega_t) &= L(u^{t+s}, \lambda \circ \Psi^{-1}; \Omega_{t+s}) \leqslant L(u^{t+s}, \lambda^{t+s}; \Omega_{t+s}) \\ &\leqslant L(u \circ \Psi^{-1}, \lambda^{t+s}; \Omega_{t+s}) = L_s(u, \lambda^{t+s} \circ \Psi; \Omega_t). \end{aligned}$$

Henceforth, $(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi) \in H(\Omega_t) \times M(\Gamma_t)$ satisfies the following minimax problem formulated in the reference domain Ω_t:

$$\begin{aligned} L_s(u^{t+s} \circ \Psi, \lambda; \Omega_t) &\leqslant L_s(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) \\ &\leqslant L_s(u, \lambda^{t+s} \circ \Psi; \Omega_t) \quad \text{for all } (u, \lambda) \in H(\Omega_t) \times M(\Gamma_t), \end{aligned} \tag{4.20}$$

and the related optimality conditions hold:

$$\begin{aligned} &\int_{\Omega_t} \det(\Psi_{,y})(c_{ijkl} \circ \Psi) E_{kl}(\Psi^{-1}_{,y}; u^{t+s} \circ \Psi) E_{ij}(\Psi^{-1}_{,y}; u) \mathrm{d}y \\ &\quad + \langle \lambda^{t+s} \circ \Psi, \omega(\nu^{t+s} \circ \Psi)^\top [\![u]\!] \rangle_{\Gamma_t} = \int_{\Gamma_N} g^\top u \, \mathrm{d}s \quad \text{for all } u \in H(\Omega_t), \end{aligned} \tag{4.21a}$$

$$\langle \lambda - (\lambda^{t+s} \circ \Psi), \omega(\nu^{t+s} \circ \Psi)^\top [\![u^{t+s} \circ \Psi]\!] \rangle_{\Gamma_t} \leqslant 0 \quad \text{for all } \lambda \in M(\Gamma_t). \tag{4.21b}$$

Our next aim is to evaluate the minimax problem (4.20) with respect to s. Accounting expansions (4.7), (4.9), (4.11), (4.12) and (4.17) we get the decomposition of L_s as

$$L_s(u, \lambda; \Omega_t) = L(u, \lambda; \Omega_t) + s L'_V(u, \lambda; \Omega_t) + \mathrm{o}(s), \tag{4.22}$$

for fixed $(u, \lambda) \in H(\Omega_t) \times M(\Gamma_t)$, with the first asymptotic term

$$L'_V(u, \lambda; \Omega_t) := \int_{\Omega_t} \left(\frac{1}{2} \mathrm{div}(V c_{ijkl}) \varepsilon_{kl}(u) \varepsilon_{ij}(u) - \sigma_{ij}(u) E_{ij}(V_{,y}; u) \right) \mathrm{d}y$$
$$+ \langle \lambda, (\mathrm{div}(V) \nu^t - V_{,y}^\top \nu^t)^\top [\![u]\!] \rangle_{\Gamma_t}. \tag{4.23}$$

In view of (4.22), the substitution of $u = 0$ into (4.20) yields

$$\frac{1}{2} \int_{\Omega_t} \sigma_{ij}(u^{t+s} \circ \Psi) \varepsilon_{ij}(u^{t+s} \circ \Psi) \mathrm{d}y \leqslant \int_{\Gamma_N} g^\top (u^{t+s} \circ \Psi) \mathrm{d}s + s R_1(u^{t+s} \circ \Psi),$$

and hence for small s it provides the uniform estimation of $u^{t+s} \circ \Psi$ in the $H^1(\Omega_t)$-vector norm due to the Korn inequality (3.8). Then from (4.21a), we can estimate $\lambda^{t+s} \circ \Psi$ in the $H_{00}^{1/2}(\Gamma_t)^\star$-norm and we arrive at

$$\| u^{t+s} \circ \Psi \|_{H(\Omega_t)} + \| \lambda^{t+s} \circ \Psi \|_{M(\Gamma_t)} \leqslant \text{const}. \tag{4.24}$$

Therefore, there exists a subsequence of the solutions such that as $s \to 0$:

$$(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi) \to (\bar{u}, \bar{\lambda}) \quad \text{weakly in } H(\Omega_t) \times M(\Gamma_t). \tag{4.25}$$

Since the quadratic functional Π is weakly lower semi-continuous, in view of (4.20), (4.22), (4.24) and (4.25) we have for arbitrary $u \in H(\Omega_t)$:

$$\begin{aligned}
L(u, \bar{\lambda}; \Omega_t) &= \lim_{s \to 0} L_s(u, \lambda^{t+s} \circ \Psi; \Omega_t) \\
&\geqslant \liminf_{s \to 0} L_s(u^{t+s} \circ \Phi, \lambda^{t+s} \circ \Psi; \Omega_t) \\
&\geqslant \liminf_{s \to 0} L_s(u^{t+s} \circ \Psi, \bar{\lambda}; \Omega_t) \\
&\geqslant \liminf_{s \to 0} (L(u^{t+s} \circ \Psi, \bar{\lambda}; \Omega_t) - s|R_3(u^{t+s} \circ \Psi, \bar{\lambda})|) \\
&= \liminf_{s \to 0} L(u^{t+s} \circ \Psi, \bar{\lambda}; \Omega_t) \geqslant L(\bar{u}, \bar{\lambda}; \Omega_t),
\end{aligned}$$

and for $\lambda \in M(\Gamma_t)$:

$$\begin{aligned} L(\bar{u}, \lambda; \Omega_t) &\leqslant \limsup_{s\to 0} L(u^{t+s} \circ \Psi, \lambda; \Omega_t) \\ &\leqslant \limsup_{s\to 0} (L_s(u^{t+s} \circ \Phi, \lambda; \Omega_t) + s|R_4(u^{t+s} \circ \Psi, \lambda)|) \\ &= \limsup_{s\to 0} L_s(u^{t+s} \circ \Psi, \lambda; \Omega_t) \\ &\leqslant \limsup_{s\to 0} L_s(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) \\ &\leqslant \limsup_{s\to 0} L_s(\bar{u}, \lambda^{t+s} \circ \Psi; \Omega_t) = L(\bar{u}, \bar{\lambda}; \Omega_t). \end{aligned}$$

Thus, obtaining (3.13) implies $(\bar{u}, \bar{\lambda}) = (u^t, \lambda^t)$.

Substituting $u = u^t$ into (4.20), using (4.22), (4.24) and the weak convergence $(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi) \to (u^t, \lambda^t)$ it follows the estimate

$$\begin{aligned} &\frac{1}{2} \int_{\Omega_t} \sigma_{ij}(u^{t+s} \circ \Psi - u^t)\varepsilon_{ij}(u^{t+s} \circ \Psi - u^t)\mathrm{d}y \\ &= -\int_{\Omega_t} \sigma_{ij}(u^t)\varepsilon_{ij}(u^{t+s} \circ \Psi - u^t)\mathrm{d}y + \frac{1}{2} \int_{\Omega_t} \sigma_{ij}(u^{t+s} \circ \Psi)\varepsilon_{ij}(u^{t+s} \circ \Psi)\mathrm{d}y \\ &\quad - \frac{1}{2} \int_{\Omega_t} \sigma_{ij}(u^t)\varepsilon_{ij}(u^t)\mathrm{d}y \\ &= -\int_{\Omega_t} \sigma_{ij}(u^t)\varepsilon_{ij}(u^{t+s} \circ \Psi - u^t)\mathrm{d}y + \int_{\Gamma_N} g^\top (u^{t+s} \circ \Psi - u^t)\mathrm{d}s \\ &\quad + L_s(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) - sR_5(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi) \\ &\quad - L_s(u^t, \lambda^{t+s} \circ \Psi; \Omega_t) + sR_6(u^t, \lambda^{t+s} \circ \Psi) \\ &\leqslant -\int_{\Omega_t} \sigma_{ij}(u^t)\varepsilon_{ij}(u^{t+s} \circ \Psi - u^t)\mathrm{d}y + \int_{\Gamma_N} g^\top (u^{t+s} \circ \Psi - u^t)\mathrm{d}s + \mathrm{O}(s) \end{aligned}$$

and, consequently, due to (3.8) this implies the strong convergence of $u^{t+s} \circ \Psi$ to u^t as $s \to 0$. The subtraction of (3.11a) from (4.21a):

$$\begin{aligned} \langle \lambda^{t+s} \circ \Psi - \lambda^t, (\nu^t)^\top [\![u]\!] \rangle_{\Gamma_t} &= -\int_{\Omega_t} \sigma_{ij}(u^{t+s} \circ \Psi - u^t)\varepsilon_{ij}(u)\mathrm{d}y \\ &\quad - \langle \lambda^{t+s} \circ \Psi, (\nu^{t+s} \circ \Psi - \nu^t)^\top [\![u]\!] \rangle_{\Gamma_t} + sR_7(u^{t+s} \circ \Psi) \end{aligned}$$

provides then the strong convergence of $\lambda^{t+s} \circ \Psi$ to λ^t. Thus, we derive that

$$(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi) \to (u^t, \lambda^t) \quad \text{strongly in } H(\Omega_t) \times M(\Gamma_t). \tag{4.26}$$

Finally, we find the shape derivative (4.2) of the reduced energy functional (4.1). Let us substitute $u = u^t$ into (4.20) and $\lambda = \lambda^{t+s} \circ \Psi$ into (3.13), with the help of (4.22) we obtain the upper bound

$$\begin{aligned} L(u^{t+s}, \lambda^{t+s}; \Omega_{t+s}) - L(u^t, \lambda^t; \Omega_t) &= L_s(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) - L(u^t, \lambda^t; \Omega_t) \\ &\leqslant L_s(u^t, \lambda^{t+s} \circ \Psi; \Omega_t) - L(u^t, \lambda^t; \Omega_t) \\ &= L(u^t, \lambda^{t+s} \circ \Psi; \Omega_t) - L(u^t, \lambda^t; \Omega_t) \\ &\quad + sL'_V(u^t, \lambda^{t+s} \circ \Psi; \Omega_t) + \mathrm{o}(s) \\ &\leqslant sL'_V(u^t, \lambda^{t+s} \circ \Psi; \Omega_t) + \mathrm{o}(s). \end{aligned}$$

Similarly, the substitution of $u = u^{t+s} \circ \Psi$ into (3.13) and $\lambda = \lambda^t$ into (4.20) leads to the lower bound

$$\begin{aligned} L(u^{t+s}, \lambda^{t+s}; \Omega_{t+s}) - L(u^t, \lambda^t; \Omega_t) &= L_s(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) - L(u^t, \lambda^t; \Omega_t) \\ &\geqslant L_s(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) - L(u^{t+s} \circ \Psi, \lambda^t; \Omega_t) \\ &= L_s(u^{t+s} \circ \Psi, \lambda^{t+s} \circ \Psi; \Omega_t) - L_s(u^{t+s} \circ \Psi, \lambda^t; \Omega_t) \\ &\quad + sL'_V(u^{t+s} \circ \Psi, \lambda^t; \Omega_t) + \mathrm{o}(s) \\ &\geqslant sL'_V(u^{t+s} \circ \Psi, \lambda^t; \Omega_t) + \mathrm{o}(s). \end{aligned}$$

Dividing the last two inequalities by s and passing to the limit $s \to 0$ in view of (4.26) we conclude that

$$\lim_{s \to 0} s^{-1}(L(u^{t+s}, \lambda^{t+s}; \Omega_{t+s}) - L(u^t, \lambda^t; \Omega_t)) = L'_V(u^t, \lambda^t; \Omega_t). \tag{4.27}$$

Recalling (3.10) and (3.14) for t and $t + s$, from (4.27) we arrive at the formula

$$\lim_{s \to 0} s^{-1}(\Pi(u^{t+s}; \Omega_{t+s}) - \Pi(u^t; \Omega_t)) = \Pi'_V(u^t; \Omega_t), \tag{4.28}$$

where Π'_V is derived according to (4.23):

$$\begin{aligned} \Pi'_V(u^t; \Omega_t) := \int_{\Omega_t} \left(\frac{1}{2} \mathrm{div}(Vc_{ijkl}) \varepsilon_{kl}(u^t) \varepsilon_{ij}(u^t) - \sigma_{ij}(u^t) E_{ij}(V_{,y}; u^t) \right) \mathrm{d}y \\ + \langle \sigma_{\nu^t}(u^t), (\mathrm{div}(V)\nu^t - V_{,y}^{\top}\nu^t)^{\top} [\![u^t]\!] \rangle_{\Gamma_t}. \end{aligned} \tag{4.29}$$

Now, we rewrite the boundary term in (4.29) using properties of the solution u^t. Let us introduce the notation of tangential derivatives at Γ_t as

$$\nabla_{\tau^t} \xi := \nabla \xi - ((\nu^t)^{\top} \nabla \xi)\nu^t \quad \text{for } \xi \in W^{1,\infty}(\mathbb{R}^N),$$

$$\mathrm{div}_{\tau^t} \xi := \mathrm{div}\xi - (\nu^t)^{\top} \xi_{,y} \nu^t \quad \text{for } \xi \in W^{1,\infty}(\mathbb{R}^N)^N,$$

and mark the useful property

$$\begin{aligned} \nabla_{\tau^t}(\xi^{\top}\eta) = (\xi_{,y}^{\top}\eta)_{\tau^t} + (\eta_{,y}^{\top}\xi)_{\tau^t} \quad \text{for } \xi, \eta \in W^{1,\infty}(\mathbb{R}^N)^N, \\ (\xi_{,y}^{\top}\eta)_{\tau^t} := \xi_{,y}^{\top}\eta - ((\nu^t)^{\top}\xi_{,y}^{\top}\eta)\nu^t. \end{aligned} \tag{4.30}$$

The boundary term over $\varGamma_t$ in (4.29):

$$\langle \sigma_{\nu^t}(u^t), (\mathrm{div}(V)\nu^t - V_{,y}^\top \nu^t)^\top [\![u^t]\!]\rangle_{\varGamma_t} := I_1 \tag{4.31}$$

admits the equivalent representation

$$I_1 = \langle \sigma_{\nu^t}(u^t), ((\mathrm{div}_{\tau^t} V - V^\top \nu_{,y}^t \nu^t)\nu^t - \nabla_{\tau^t}((\nu^t)^\top V) + (\nu_{,y}^t)^\top V)^\top [\![u^t]\!]\rangle_{\varGamma_t}.$$

Note that for arbitrary scalar function h which is uniformly Lipschitz continuous, in view of (3.11b) there is fulfilled the identity:

$$\langle \sigma_{\nu^t}(u^t), h(\nu^t)^\top [\![u^t]\!]\rangle_{\varGamma_t} = 0. \tag{4.32}$$

In fact, taking a small constant $s > 0$ such that $1 \pm sh \geqslant 0$ at $\varGamma_t$ provides $(1 \pm sh)\lambda^t \in M(\varGamma_t)$, hence the substitution of $\lambda = (1 \pm sh)\lambda^t$ into (3.11b) leads to (4.32), when λ^t is defined in (3.10). With the help of (4.32), accounting (4.31) and (4.30), for the tangential vectors

$$((\nu_{,y}^t)^\top V)_{\tau^t} := (\nu_{,y}^t)^\top V - ((\nu^t)^\top (\nu_{,y}^t)^\top V)\nu^t,$$

$$[\![(u^t)_{\tau^t}]\!] := [\![u^t]\!] - ((\nu^t)^\top [\![u^t]\!])\nu^t,$$

from (4.29) we arrive at the final representation

$$\begin{aligned}\varPi_V'(u^t; \varOmega_t) = \int_{\varOmega_t} \left(\frac{1}{2}\mathrm{div}(V c_{ijkl})\varepsilon_{kl}(u^t)\varepsilon_{ij}(u^t) - \sigma_{ij}(u^t)E_{ij}(V_{,y}; u^t)\right)\mathrm{d}y \\ + \langle \sigma_{\nu^t}(u^t), (((\nu_{,y}^t)^\top V)_{\tau^t} - \nabla_{\tau^t}((\nu^t)^\top V))^\top [\![(u^t)_{\tau^t}]\!]\rangle_{\varGamma_t}.\end{aligned} \tag{4.33}$$

THEOREM 4.1 For a given velocity $V(t)$ from (2.1), for every $t \in [0, T]$ there exists the shape derivative (4.28) of the reduced potential energy functional (4.1) in the direction V, which is represented by formula (4.33).

For comparison, the crack problem in a classical formulation is linear and implies

$$\sigma_{\nu^t}(u^t) = 0 \quad \text{on } \varGamma_t^{\pm}$$

instead of the nonpenetration conditions (3.4e). We have then the boundary term over the crack $\varGamma_t$ in (4.33) equal to zero. This fact makes a principal difference of the constrained crack problem (3.4) in comparison with linear crack problems assuming the stress-free crack surfaces.

On the other hand, if the jump of the tangential components $[\![(u^t)_{\tau^t}]\!]$ is equal to zero on the support of V at $\varGamma_t$, then the boundary term I_1 in (4.33) is zeroth, too. The case $I_1 = 0$ is realized also for constant ν^t and velocities V such that $(\nu^t)^\top V = 0$, which describe tangential perturbations of rectilinear and planar cracks investigated in the previous works.

5. The shape derivative and path-independent integrals of energy

We relate the shape derivative obtained in Section 4 to invariants of energy in the form of path-independent contour integrals (or J-integrals) as it is adopted in the literature on mechanics of cracks, see Knowles & Sternberg (1972), Budiansky & Rice (1973), Nazarov & Polyakova (1995) and Kovtunenko (2003).

For the sake of simplicity there we assume the constant elasticity coefficients c_{ijkl}. In fracture mechanics, time-evolution problems on a crack propagation suppose that the previous in time shape of the crack is preserved. Accounting this interest, we consider the velocity vector V tangential to the crack, i.e.

$$(\nu^t)^\top V = 0 \quad \text{on } \Gamma_t. \tag{5.1}$$

Following the examples of Section 2, we suppose the velocity field of the form

$$V = \chi\Lambda, \quad \chi = \begin{cases} 1, & \text{inside } B_1, \\ 0, & \text{outside } B_0, \end{cases} \quad B_1 \subset B_0 \subset \Omega. \tag{5.2}$$

This structure involves a standard cut-off function χ and a vector Λ which can be seen as an extension of V onto $\mathbb{R}^N$ (or, conversely, say V is a restriction of Λ into $B_0 \subset \mathbb{R}^N$). Since regularity of the solution to the crack problem (3.4) is connected with the geometrical singularity of boundary (crack tip, crack front), we assume it to be located inside B_1 thus allowing outside an additional smoothness of u^t, namely

$$u^t \in H^2((B_0 \backslash B_1) \cap \Omega_t)^N. \tag{5.3}$$

Property (5.3) can be derived in a standard way with the help of shift technique for smooth data possessing an additional spatial $C^{2,1}$-regularity, see Khludnev & Kovtunenko (2000).

Applying these assumptions we rewrite the shape derivative. Due to (5.1) and (5.2) we have the following expression of (4.31):

$$I_1 = \langle \sigma_{\nu^t}(u^t), ((\nu^t_{,y})^\top V)^\top_{\tau^t} [\![(u^t)_{\tau^t}]\!] \rangle_{B_0 \cap \Gamma_t}. \tag{5.4}$$

We represent the domain integral in (4.29) as a sum of three integrals over $\Omega_t \backslash B_0$, $(B_0 \backslash B_1) \cap \Omega_t$ and $B_1 \cap \Omega_t$. These geometric constructions in $\mathbb{R}^2$ are illustrated in Fig. 2. Due to (5.2) the first integral is zero, and $V = \Lambda$ in $B_1 \cap \Omega_t$. In view of (5.3), we can integrate by the parts in $(B_0 \backslash B_1) \cap \Omega_t$. Let $q = (q_1, \dots, q_N)^\top$ denote a unit normal vector at ∂B_1 which is outward to B_1. Hence, we obtain

$$\int_{\Omega_t} \sigma_{ij}(u^t) \left(\frac{1}{2} \mathrm{div} V \varepsilon_{ij}(u^t) - E_{ij}(V_{,y}; u^t) \right) \mathrm{d}y = -I_\Lambda(u^t; \partial B_1) + I_2 + I_3 + I_4, \tag{5.5}$$

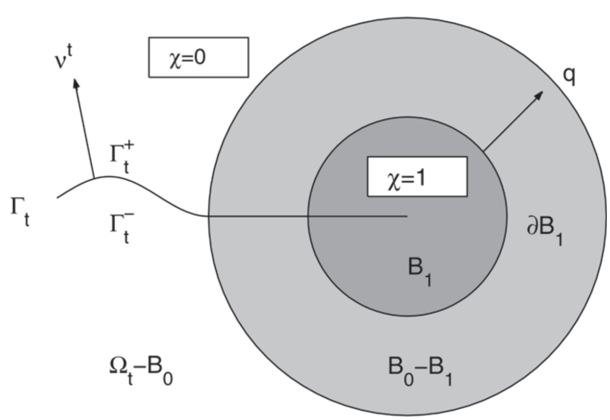

FIG. 2. Neighborhood of a crack tip in $\mathbb{R}^2$.

with the following terms:

$$I_\Lambda(u^t;\partial B_1) := \int_{\partial B_1} \sigma_{ij}(u^t)\left(\frac{1}{2}(q^\top \Lambda)\varepsilon_{ij}(u^t) - q_j(\Lambda^\top \nabla u_i^t)\right)\mathrm{d}s, \tag{5.6}$$

$$I_2 := -\int_{(B_0\setminus B_1)\cap\Gamma_t} \left(\frac{1}{2}((\nu^t)^\top V)[\![\sigma_{ij}(u^t)\varepsilon_{ij}(u^t)]\!] - [\![\sigma_{ij}(u^t)\nu_j^t(V^\top\nabla u_i^t)]\!]\right)\mathrm{d}s,$$

$$I_3 := \int_{B_1\cap\Omega_t} \sigma_{ij}(u^t)\left(\frac{1}{2}\mathrm{div}\Lambda\varepsilon_{ij}(u^t) - E_{ij}(\Lambda_{,y};u^t)\right)\mathrm{d}y, \tag{5.7}$$

and

$$I_4 := \int_{(B_0\setminus B_1)\cap\Omega_t} \sigma_{ij,j}(u^t)(V^\top\nabla u_i^t)\mathrm{d}y = 0$$

due to (3.4a).

Now we express I_2. Accounting (5.1) and using decomposition (3.5), it follows that

$$I_2 = \int_{(B_0\setminus B_1)\cap\Gamma_t} [\![\sigma_{\nu^t}(u^t)((\nu^t)^\top u^t_{,y}V) + \sigma^\top_{\tau^t}(u^t)(u^t_{,y}V - ((\nu^t)^\top u^t_{,y}V)\nu^t)]\!]\mathrm{d}s.$$

Henceforth, applying (3.4d) and (3.4e) there results in formula

$$I_2 = \int_{(B_0\setminus B_1)\cap\Gamma_t} \sigma_{\nu^t}(u^t)(V^\top\nabla_{\tau^t}((\nu^t)^\top[\![u^t]\!])$$
$$+ ((\nu^t)^\top V)(\nu^t)^\top\nabla((\nu^t)^\top[\![u^t]\!]) - V^\top(\nu^t_{,y})^\top[\![u^t]\!])\mathrm{d}s.$$

In view of the solution regularity assumed in (5.3), from (3.4e) we derive that $(B_0\setminus B_1)\cap\Gamma_t$ is splitting into the sets such that either

$$\sigma_{\nu^t}(u^t) = 0 \quad \text{or} \quad (\nu^t)^\top[\![u^t]\!] = 0 \quad \text{on } (B_0\setminus B_1)\cap\Gamma_t. \tag{5.8}$$

Moreover, integrating along the crack (5.8) implies also

$$\sigma_{\nu^t}(u^t) = 0 \quad \text{or} \quad \nabla_{\tau^t}((\nu^t)^\top[\![u^t]\!]) = 0 \quad \text{on } (B_0\setminus B_1)\cap\Gamma_t. \tag{5.9}$$

Applying (5.9) and (5.1) to I_2 results in formula

$$I_2 = -\int_{(B_0\setminus B_1)\cap\Gamma_t} \sigma_{\nu^t}(u^t)V^\top(\nu^t_{,y})^\top[\![u^t]\!]\mathrm{d}s.$$

Note that vector $\nu^t_{,y}V$ is tangential to Γ_t, while $(\nu^t)^\top\nu^t_{,y}V = 0$ due to $(\nu^t)^\top\nu^t_{,y} = 0$ following from the fact that $(\nu^t)^\top\nu^t = 1$. Henceforth

$$I_2 = -\int_{(B_0\setminus B_1)\cap\Gamma_t} \sigma_{\nu^t}(u^t)(\nu^t_{,y}V)^\top[\![(u^t)_{\tau^t}]\!]\mathrm{d}s. \tag{5.10}$$

The summation of (4.31) and (5.5) gets finally representation of the shape derivative from (4.29) in the form

$$\Pi'_V(u^t;\Omega_t) = -I_\Lambda(u^t;\partial B_1) + I_1 + I_2 + I_3, \tag{5.11}$$

with the terms given in (5.6), (5.4), (5.10) and (5.7), respectively.

THEOREM 5.1 Under the assumptions (5.1)–(5.3), let the velocity field V, normal ν^t and constant elasticity coefficients c_{ijkl} satisfy the following relations:

$$((\nu^t_{,y})^\top \Lambda)_{\tau^t} = 0 \quad \text{on } B_1 \cap \Gamma_t, \tag{5.12a}$$

$$(\nu^t_{,y})^\top V)_{\tau^t} - \nu^t_{,y} V = 0 \quad \text{on } (B_0 \backslash B_1) \cap \Gamma_t, \tag{5.12b}$$

$$c_{ijkl}\xi_{kl}\left(\frac{1}{2}\mathrm{div}\Lambda\xi_{ij} - \xi_{im}\Lambda_{m,j}\right) = 0 \quad \text{for } \xi_{ij} = \xi_{ji} \in \mathbb{R}^{N\times N} \quad \text{in } B_1 \cap \Omega_t. \tag{5.12c}$$

Then the shape derivative is represented equivalently by the contour integral

$$\Pi'_V(u^t;\Omega_t) = -I_\Lambda(u^t;\partial B_1) \tag{5.13}$$

given in (5.6).

Proof. Evidently, condition (5.12c) is sufficient for $I_3 = 0$. If (5.12a) holds, then due to (5.3) from (5.4) we get

$$I_1 = \int_{(B_0\backslash B_1)\cap\Gamma_t} \sigma_{\nu^t}(u^t)((\nu^t_{,y})^\top V)^\top_{\tau^t}[\![(u^t)_{\tau^t}]\!]\mathrm{d}s. \tag{5.14}$$

Adding (5.14) to (5.10) and using (5.12b), we arrive at the identity $I_1 + I_2 = 0$. Henceforth, (5.13) follows from (5.11) thus proving the theorem. □

Note that the form (5.6) is standard in fracture mechanics to express the energy-release rate at a crack tip as an invariant integral of energy. There is a well-known path-independent Cherepanov–Rice contour integral for the rectilinear crack in $\mathbb{R}^2$ with stress-free faces ($\sigma_{\nu^t}(u^t) = 0$). It is involved as the particular case in formula (5.13), too.

REMARK 5.1 If $\sigma_{\nu^t}(u^t) = 0$ or $[\![(u^t)_{\tau^t}]\!] = 0$ at $B_0 \cap \Gamma_t$, then (5.12c) is enough to fulfill the assertion of Theorem 5.1.

REMARK 5.2 If $[\![u^t]\!] = 0$ at $B_0 \cap \Gamma_t$ and the solution u^t is H^2-regular in $B_1 \cap \Gamma_t$, then integration by parts of integral in (4.29) over all $B_0 \cap \Omega_t$ similar to the lines between (5.5) and (5.10) provides us with the fact $\Pi'_V(u^t;\Omega_t) = 0$.

We illustrate conditions (5.12) on the examples from Section 2.

EXAMPLE 5.2 Curvilinear cracks in $\mathbb{R}^2$. Consider the curvilinear crack Γ_t presented in (2.21). Since the velocity V from (2.22) has the local structure at every of two crack end-points, the shape derivative is splitting, too, into two integrals over neighborhoods of every of the crack tips. Therefore, it is enough to consider only one crack tip, let us restrict our attention to the right end-point $(b(t), \psi(b(t)))^\top$.

Accounting the notation (5.2), we have in this case

$$V = \chi^b \Lambda, \quad \Lambda(t, y) = b'(t)(1, \psi'(y_1))^\top. \tag{5.15}$$

The normal to Γ_t, vector ν^t, is given by

$$\nu^t = Z^{-1/2}(y)(-\psi'(y_1), 1)^\top, \quad Z := 1 + (\psi')^2. \tag{5.16}$$

From (5.15) and (5.16) we conclude that Λ (respectively, V) is tangential to Γ_t, thus (5.1) holds true. Smoothness assumed in (5.3) is provided by the $C^{2,1}$-regularity of ψ.

With the matrix of derivatives

$$\nu^t_{,y} = Z^{-3/2} \begin{bmatrix} -\psi'' & 0 \\ -\psi'\psi'' & 0 \end{bmatrix}, \quad \Lambda_{,y} = b' \begin{bmatrix} 0 & 0 \\ \psi'' & 0 \end{bmatrix},$$

calculation of the terms in (5.12) gets

$$\begin{aligned} ((\nu^t_{,y})^\top \Lambda)_{\tau^t} &= -b' Z^{-3/2} \psi''(1, \psi')^\top = 0, \\ c_{ijkl}\xi_{kl} \left(\frac{1}{2} \operatorname{div} \Lambda \xi_{ij} - \xi_{im} \Lambda_{m,j} \right) &= -b'\psi'' c_{i1kl}\xi_{kl}\xi_{i2} = 0, \end{aligned} \tag{5.17}$$

and (5.12b) is fulfilled. Conditions (5.17) can be satisfied only for $\psi' = \text{const}$ in B^b_0 implying a crack which is rectilinear at the vicinity of the moving crack tip.

EXAMPLE 5.3 Nonplanar cracks with curvilinear fronts in $\mathbb{R}^3$. Consider the crack (2.23) and its front γ^b_t. The velocity vector in (2.24) and the normal to Γ_t vector are given by

$$\begin{aligned} V &= \chi^b \Lambda, \quad \Lambda(t, y) = b_{,t}(t, y_3) \begin{bmatrix} 1 \\ \psi_{,1}(y_1, y_3) \\ 0 \end{bmatrix}, \\ \nu^t &= Z^{-1/2}(y) \begin{bmatrix} -\psi_{,1}(y_1, y_3) \\ 1 \\ -\psi_{,3}(y_1, y_3) \end{bmatrix}, \quad Z := 1 + \psi_{,1}^2 + \psi_{,3}^2, \end{aligned} \tag{5.18}$$

and they are orthogonal to each other thus fulfilling (5.1). From (5.18) we calculate

$$\nu^t_{,y} = Z^{-3/2} \begin{bmatrix} \psi_{,1}\psi_{,3}\psi_{,13} - (1 + \psi_{,3}^2)\psi_{,11} & 0 & \psi_{,1}\psi_{,3}\psi_{,33} - (1 + \psi_{,3}^2)\psi_{,13} \\ -\psi_{,1}\psi_{,11} - \psi_{,3}\psi_{,13} & 0 & -\psi_{,1}\psi_{,13} - \psi_{,3}\psi_{,33} \\ \psi_{,1}\psi_{,3}\psi_{,11} - (1 + \psi_{,1}^2)\psi_{,13} & 0 & \psi_{,1}\psi_{,3}\psi_{,13} - (1 + \psi_{,1}^2)\psi_{,33} \end{bmatrix},$$

$$\Lambda_{,y} = b_{,t} \begin{bmatrix} 0 & 0 & 0 \\ \psi_{,11} & 0 & \psi_{,13} \\ 0 & 0 & 0 \end{bmatrix}$$

and derive conditions (5.12) in the following form:

$$((\nu^t_{,y})^\top \Lambda)_{\tau^t} = b_{,t}\nu^t_{,1} = 0,$$
$$c_{ijkl}\xi_{kl}\left(\frac{1}{2}\mathrm{div}\Lambda\xi_{ij} - \xi_{im}\Lambda_{m,j}\right) = -b_{,t}c_{ijkl}\xi_{kl}\xi_{i2}\psi_{,1j} = 0, \tag{5.19}$$

while (5.12b) holds true. It is possible to satisfy (5.19) only for $\psi_{,1} = \mathrm{const}$ in B^b_0, which implies that a crack is rectilinear with respect to direction of the crack movement in a neighborhood of the moving crack front.

6. Discussion of the results in application to fracture mechanics

In this section, we fix some velocity vector field V which is tangential to Γ_t and vanishes at $\partial\Omega$, namely in the form of (5.1) and (5.2). Following the Griffith hypothesis, the criterion of fracture of the solid with fixed crack Γ_t should be written in terms of the energy-release rate $G = -\Pi'_V$ (in direction of V) and its critical value G_c as follows:

$$\Pi'_V(u^t;\Omega_t) + G_c = 0, \tag{6.1}$$

where constant $G_c > 0$ is assumed to be given. That is why we need a formula to calculate Π'_V in (6.1).

Firstly, let us consider a variational solution $\bar{u}^t \in H(\Omega_t)$ of the crack problem in the linear setting:

$$-\sigma_{ij,j}(\bar{u}^t) = 0, \quad i = 1,\dots,N, \ \text{ in } \Omega_t; \tag{6.2a}$$

$$\bar{u}^t_i = 0, \quad i = 1,\dots,N, \ \text{ on } \Gamma_D; \tag{6.2b}$$

$$\sigma_{ij}(\bar{u}^t)n_j = g_i, \quad i = 1,\dots,N, \ \text{ on } \Gamma_N; \tag{6.2c}$$

$$\sigma_{\nu^t}(\bar{u}^t) = 0, \quad (\sigma_{\tau^t}(\bar{u}^t))_i = 0, \quad i = 1,\dots,N, \quad \text{on } \Gamma^{\pm}_t, \tag{6.2d}$$

as the particular case of (3.4) without nonpenetration conditions (3.4e). Employing (4.33) results in formula

$$\Pi'_V(\bar{u}^t;\Omega_t) = \int_{\Omega_t}\left(\frac{1}{2}\mathrm{div}(Vc_{ijkl})\varepsilon_{kl}(\bar{u}^t)\varepsilon_{ij}(\bar{u}^t) - \sigma_{ij}(\bar{u}^t)E_{ij}(V_{,y};\bar{u}^t)\right)\mathrm{d}y. \tag{6.3}$$

Integrating by parts (6.3) under the assumption that $\bar{u}^t \in H^2((B_0\backslash B_1)\cap\Omega_t)^N$ and $c_{ijkl} = \mathrm{const}$, we derive

$$\Pi'_V(\bar{u}^t;\Omega_t) = -I_\Lambda(\bar{u}^t;\partial B_1), \tag{6.4}$$

with the (generalized Cherepanov–Rice) contour integral

$$I_\Lambda(\bar{u}^t;\partial B_1) := \int_{\partial B_1}\sigma_{ij}(\bar{u}^t)\left(\frac{1}{2}(q^\top\Lambda)\varepsilon_{ij}(\bar{u}^t) - q_j(\Lambda^\top\nabla\bar{u}^t_i)\right)\mathrm{d}s, \tag{6.5}$$

where we used the notation of Section 5. Thus, from (6.1) and (6.4) we arrive at the following fracture criterion:

$$I_\Lambda(\bar{u}^t;\partial B_1) = G_c \tag{6.6}$$

formulated in terms of the path-independent integral I_Λ, as it is usually adopted in fracture mechanics. For the specific cases of the linear crack problem (6.2), these results are well-known.

Secondly, we assume that the crack Γ_t is rectilinear (planar) with $\nu^t = \text{const}$. Employing a variational solution $u^t \in K(\Omega_t)$ of (3.4) with the nonpenetration conditions (3.4e), the results of Khludnev *et al.* (2002) and Kovtunenko (2003) provide the expressions:

$$\Pi'_V(u^t;\Omega_t) = \int_{\Omega_t} \left(\frac{1}{2} \mathrm{div}(V c_{ijkl}) \varepsilon_{kl}(u^t) \varepsilon_{ij}(u^t) - \sigma_{ij}(u^t) E_{ij}(V_{,y}; u^t) \right) \mathrm{d}y, \tag{6.7a}$$

$$\Pi'_V(u^t;\Omega_t) = -I_\Lambda(u^t; \partial B_1), \tag{6.7b}$$

$$I_\Lambda(u^t; \partial B_1) := \int_{\partial B_1} \sigma_{ij}(u^t) \left(\frac{1}{2} (q^\top \Lambda) \varepsilon_{ij}(u^t) - q_j (\Lambda^\top \nabla u_i^t) \right) \mathrm{d}s. \tag{6.7c}$$

In view of (6.7b), the fracture criterion (6.1) can be expressed equivalently with the help of the path-independent contour integral I_Λ as

$$I_\Lambda(u^t; \partial B_1) = G_c. \tag{6.8}$$

Thus, we observe the full similarity between (6.3) and (6.7a), between (6.4) and (6.7b), between (6.5) and (6.7c) and between (6.6) and (6.8).

This is not the case of curvilinear (nonplanar) cracks. The representation in (6.7a) is no more true, and Theorem 4.1 provides a new (more general than (6.7a)) formula (4.33) for calculation of the energy-release rate $G = -\Pi'_V$ in the Griffith fracture criterion (6.1). The equality in (6.7b) holds also not true, thus I_Λ is not more path-independent, and Theorem 5.1 provides the sufficient conditions to fulfill (6.7b). Finally, (6.8) should be replaced with the following fracture criterion:

$$I_\Lambda(u^t; \partial B_1) - I_1 - I_2 - I_3 = G_c,$$

with the notation of Section 5.

Thus, in the present paper we obtained a new formula (4.33) for calculation of the energy-release rate for curvilinear (nonplanar) cracks and the respective Griffith fracture criterion, which generalizes the results known for rectilinear (planar) cracks and for the crack problems in the linear setting.

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