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Solution of nonlinear variational problems for generalized fractional viscoelastic models☆

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Dedicated to the memory of Professor Kumbakonam Ramamani Rajagopal

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ABSTRACT

Generalized fractional viscoelastic models (shortly GFV models) are considered based on the Caputo derivative of fractional order. Constitutive relations of GFV models are given by Volterra hereditary integrals with creep functions described by Prony series replaced with a Mittag-Leffler function. In a quasi-static situation, the time-dependent equilibrium problem is set in the variational formulation. For linear as well as nonlinear material responses, a solution to the GFV model is provided by an analytical formula of convolution with a solution of the corresponding elastic problem. The numerical example is presented in 1D for a creep test under isotropic expansion, which is analyzed with respect to fractional derivative order and non-linearity.

1. Introduction

We consider a quasi-static problem for generalized fractional viscoelastic models (GFV models). A constitutive relation of the GFV model is described as a function of the Cauchy stress with respect to the linearized strain by use of the Caputo fractional derivatives. This relation can be regarded as a kind of implicit constitutive relation. Although the GFV model may have already been introduced in the context of fractional generalizations of classical linear viscoelastic models in Mainardi (2022) and Caputo and Mainardi (1971), its mathematical treatment appears to be challenging. A main aim of this paper is to establish an existence theorem for a solution to a boundary value problem for the GFV model.

Kumbakonam R. Rajagopal introduced a framework for modeling complex materials by using implicit constitutive relations rather than the traditional explicit ones such as the Navier–Stokes fluid model, and the Hookean or neo-Hookean solid models, see Rajagopal (2003), Rajagopal (2020) and the references therein. In contrast with classical constitutive models in which the stress is given explicitly in terms of kinematical quantities, the implicit constitutive relation of elastic bodies introduced in Rajagopal (2003) between the Cauchy stress $\boldsymbol{\sigma}$ and the deformation gradient $\boldsymbol{F}$ is given in the form

$$f(\boldsymbol{\sigma}, \boldsymbol{F}) = \mathbf{0}$$

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and cannot generally express the stress as a function of the strain, and vice versa. A particular subclass of these constitutive relations is described in terms of $\boldsymbol{\sigma}$ and the linearized strain $\boldsymbol{\varepsilon}$ wherein the gradient of the displacement and consequently the strain is assumed to be small. The reason that many elastic models choose to express the stress in terms of the linearized strain is to achieve mathematical simplification with respect to the governing equations that must be solved, obtained by substituting the constitutive law for the stress into the linear momentum balance equation. Expressing the linearized strain in terms of the stress requires one to solve the balance equations as well as the constitutive relation simultaneously. Traditional linearized elasticity assumes small deformations and linear stress–strain relationships. However, this leads to inconsistency near crack tips because the strain can become mathematically infinite due to stress concentration at the crack tips. A nonlinear elasticity model with limiting small strain may have the possibility to avoid these singularities, that is, the model allows the strain to be finite even if stress concentration occurs. One can say this is one of advantages of the implicit constitutive theory. Crack problems with Signorini-type contact conditions, so-called non-penetration conditions meaning that the opposing faces of a crack can touch but not penetrate each other, in this framework, are investigated by Itou et al. (2017a, 2017b).

Viscoelastic materials have the property involving both viscous and elastic characteristics at the same time when undergoing deformation, and exhibit time-dependent response. Namely, they can store and dissipate energy, manifesting as phenomena such as stress relaxation and creep at the macroscopic scale. There are many examples of viscoelastic materials in a wide range of fields: engineering, biology, geophysics, etc., such as synthetic polymers, biological tissues and concrete. To describe the viscoelastic responses of materials, various types of constitutive relations such as differential type, rate type, integral type models and also linear and non-linear models have been proposed. The initial models describing nonlinear viscoelastic response of bodies by Green and Rivlin (1959) and Green et al. (1959) are expressions for the stress in terms of kinematic variables. Lockett (1965, 1972) and later (Pipkin & Rogers, 1968) developed models concerning the viscoelastic response of bodies. These models can yield expressions representing either the stress in terms of the history of the strain or the strain in terms of the history of the stress. Wineman (2009) overviewed nonlinear viscoelastic models to consider some technologically important problems. Most of the nonlinear viscoelastic models were essentially developed using analogies to extend the class of linear viscoelastic models, and moreover, researchers primarily developed models for the stress in terms of the history of the strain, and models for the strain in terms of the history of the stress were essentially obtained by inverting the representation for the stress, when such expressions were invertible. On the other hand, implicit constitutive relations for viscoelastic response had been introduced much earlier by Burgers (1939) and Oldroyd (1950). While the constitutive relation introduced by Maxwell (1867) relates the stress and the time rate of the stress to the symmetric part of the velocity gradient, it is not an implicit relation as the symmetric part of the velocity gradient can be expressed in terms of the stress and the time rate of the stress. Later, Průša and Rajagopal (2012) developed a more general theory which embeds both classes of models for viscoelastic response discussed above by considering an implicit functional relationship between the histories of the stress and strain.

In linear viscoelasticity, one can either express the stress in terms of the history of the linearized strain or the linearized strain in terms of the history of the stress. The response of a linear viscoelastic material to the sum of multiple inputs is obtained by superimposing the responses to the individual inputs. The linear viscoelastic model is applicable only when a body experiences a small displacement gradient. However, some viscoelastic materials may sometimes experience small displacement gradients, and linearized strain measurements may suffice to describe their deformation. Nevertheless, their response may fail to satisfy the conditions of proportionality and superposition required for linear viscoelasticity; in such cases, they must be recognized as nonlinear viscoelastic materials. In such a situation, it is reasonable to express the linearized strain $\boldsymbol{\varepsilon}$ as a function of the Cauchy stress $\boldsymbol{\sigma}$ in the form of function composition

$$\boldsymbol{\varepsilon}(t) = [\mathcal{I}(t) \circ \mathcal{F}]\boldsymbol{\sigma}, \quad t \in [0, T], \tag{1.1}$$

for fixed final time $T > 0$, where the time-dependent function $\mathcal{I}$ represents the creep owing to the viscous phenomena, and the function $\mathcal{F}$ stands for the material response. In general, both functions are supposed to be nonlinear. Furthermore, from the perspective of causality (see Truesdell, 1966), when a force is applied to a body causing deformation, that is, when the force/stress is the cause and the deformation/strain is the effect, it is far more reasonable to express the effect as the cause than the reverse. From a practical standpoint, conducting creep tests is relatively easier than relaxation tests, allowing the material parameters of the constitutive model, expressed in terms of the stress, to be readily calibrated from experimental data.

The first integral constitutive relation to describe viscoelastic response was developed by Boltzmann (1874) based on a superposition principle, and this was followed by integral constitutive relations due to Green and Rivlin (1957, 1959), Green et al. (1959), Lockett (1972), Pipkin and Rogers (1968) and others. Fung (1993) developed a one-dimensional approximation wherein an explicit expression is provided for the stress in terms of the stretch, referred to as a quasi-linear constitutive relation, to describe the viscoelastic behavior of biological tissues. It is not guaranteed that one can invert all such models and express the stretch in terms of the history of the stress. Muliana et al. (2013) (see also Rajagopal & Wineman, 2008 and Muliana et al., 2015) developed a quasi-linear viscoelastic model in three-dimensions, wherein the relationship between the stress and strain is nonlinear. Since the linearized strain is not frame-indifferent, quasi-linear constitutive relations for the strain limiting models in common with the classical linearized elastic model and the linear viscoelastic model, are not frame-indifferent. Constitutive relations in Rajagopal and Wineman (2008) and Muliana et al. (2013) have been extended to implicit constitutive relations (Bustamante et al., 2020a, 2020b), in the context of strain-limiting approach by Itou et al. (2018, 2019a). The quasi-linear viscoelastic model has been used to study the Boussinesq problem by Itou et al. (2020) and Itou et al. (2021a) for the indentation of half-space by a rigid punch with unknown contact zones. In (1.1), typical forms of $\mathcal{I}(t)$ are the following tensorial hereditary integral

$$[\mathcal{I}(t)]\epsilon = \boldsymbol{J}(t)\epsilon(0) + \int_0^t \boldsymbol{J}(t-s)\frac{\partial \epsilon(s)}{\partial s}\,ds, \tag{1.2}$$

or

$$[\mathcal{I}(t)]\epsilon = \boldsymbol{J}(0)\epsilon(t) + \int_0^t \left(\frac{\partial}{\partial s}\boldsymbol{J}(t-s)\right)\epsilon(s)\,ds, \tag{1.3}$$

with a tensorial creep function $\boldsymbol{J}$. The fourth order creep tensor $\boldsymbol{J} = (J_{ijkl})$ describes the multi axial creep behavior of viscoelastic materials. In isotropic materials, since $\boldsymbol{J}$ is independent of direction, typical and general representatives of $\boldsymbol{J} = J\mathbf{I}$ with the identity tensor $\mathbf{I}$ in (1.2) or (1.3) are given by the Prony series

$$J(t) = J_0 + \sum_{n=1}^{N} J_n \left(1 - \exp\left(-\frac{t}{\tau_n}\right)\right), \tag{1.4}$$

where $J_0, J_1, \dots, J_N$ and $\tau_1, \tau_2, \dots, \tau_N$ are creep parameters.

The variational theory was established well for non-smooth problems on mechanics arising in elasticity (Bach et al., 2000; Hintermüller et al., 2009; Itou et al., 2021; Khludnev et al., 2008), creep and viscoelasticity (de Castro Motta et al., 2025; Khludnev & Kovtunenko, 2000; Khludnev & Sokolowski, 1997), strain-limiting elasticity (Erbay & Şengül, 2025; Ghosh et al., 2026; Itou et al., 2018; Rajagopal & Rodriguez, 2023), for porous elastic solids with material moduli dependent on density (see Gou & Mallikarjunaiah, 2025 and Itou et al., 2025b), and for implicit constitutive relations with pressure-dependent moduli in which both stress and strain appear linearly (Itou et al., 2019b, 2021b, 2022, 2023, 2024b). We refer to the implicitly constituted models of quasi-linear viscoelastic bodies in Itou et al. (2019a), and forward and inverse problems for viscoelastic creep models studied in the previous works by Itou et al. (2024a, 2025a).

Fractional differential equations (see Bazhlekova, 2001; Dzherbashian & Nersesian, 2020; Mainardi, 2020, 2022 and Podlubny, 1999) utilize primarily the theory of Riemann–Liouville fractional integral and the Mittag-Leffler function. The Caputo derivative of fractional order $\alpha \in (0,1)$ is defined by the Riemann–Liouville integral implying convolution of $g_{1-\alpha} \in L^1(0,T)$ and the weak derivative of an absolutely continuous function $\xi \in \mathrm{AC}([0,T])$, $T > 0$:

$$\xi^{(\alpha)}(t) := [g_{1-\alpha} * \xi'](t) = \int_0^t g_{1-\alpha}(t-s)\xi'(s)\,ds, \quad t \in (0,T), \quad g_{1-\alpha}(t) := \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \tag{1.5}$$

where $\Gamma(\cdot)$ denotes the Gamma function, and the limiting cases $\alpha \to 0$ and $\alpha \to 1$:

$$\xi^{(0)}(t) = \xi(t) - \xi(0), \quad \xi^{(1)}(t) = \xi'(t) \tag{1.6}$$

can be proved under some more condition on ξ. The integral formula (1.5) determines a continuous mapping $W^{1,p}(0,T) \longrightarrow L^p(0,T)$ for arbitrary $p \in [1,\infty)$ by applying Young's inequality, see e.g. Bazhlekova (2001, Chap. 1):

$$\|\xi^{(\alpha)}\|_{L^p(0,T)} \le \|g_{1-\alpha}\|_{L^1(0,T)} \|\xi'\|_{L^p(0,T)}. \tag{1.7}$$

The Mittag-Leffler function appears as a generalization of the exponential function and is defined by

$$E_\alpha(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(1+\alpha k)}, \quad E_0(t) = \frac{1}{1-t}, \quad E_1(t) = e^t, \quad t \in \mathbb{R}, \tag{1.8}$$

since it solves the fractional eigenvalue problem, see e.g. Kilbas et al. (2006, Lem. 2.23) and Gallican and Brenner (2019, App. B3):

$$E_\alpha^{(\alpha)}(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha). \tag{1.9}$$

In the case of negative variables, one sees that $E_\alpha(-t) \in C^\infty(\mathbb{R}_+)$, $E_\alpha(0) = 1$, is completely monotone (see Mainardi, 2020), and can be estimated as follows Podlubny (1999, Thm. 1.6) and Liu et al. (2016, Lem. 2.1):

$$0 < E_\alpha(-t) \le \frac{C}{1+t}, \quad t \in \mathbb{R}_+, \quad C > 0. \tag{1.10}$$

Using the Caputo derivative of fractional order, see Caputo (1969), a theoretical approach to fractional evolution equations was developed, e.g., in the works by Gorenflo et al. (2015), Kian and Yamamoto (2017), Liu et al. (2016) and Ridha and Shukur (2025) and others. Based on the principles of hereditary solid mechanics, see Rabotnov (1980), the reader is referred to Gross (1947), Mainardi (2022) and Koeller (1984) for applications of fractional calculus to the theory of viscoelasticity. Within fractional order viscoelasticity, see the well-posedness results by Oparnica and Süli (2020) and Saedpanah (2014), variational and hemivariational inequalities by Han et al. (2017), and homogenization for the effective response by Gallican and Brenner (2019). The corresponding inverse problem was treated by determining kernels in Kaltenbacher et al. (2022), and flux terms in BenSalah et al. (2025). Some nonlinear fractional viscoelastic material models were analyzed by Müller et al. (2011), Zhang et al. (2020) and Muliana (2023).

In the current work we consider a generalized fractional viscoelastic model (GFV model) whose constitutive relation is described by the composition of a nonlinear response function and a Volterra hereditary integral with a creep function. We generalize within fractional differential calculus the hereditary kernel given by Prony series that is commonly used in viscoelasticity. In GFV model, we apply the Caputo derivative of fractional order $\alpha \in (0,1)$, and according to Mainardi (2022, Chap. 3) we utilize the Mittag-Leffler function E_α approximating a series representation of the exponential function as $\alpha \to 1$, that is,

$$\mathcal{K}_\alpha(t) := K_0 + K_+ \frac{t^\alpha}{\Gamma(1+\alpha)} + \sum_{n=1}^{N} K_n \left(1 - E_\alpha\left(-\left(\frac{t}{\tau_n}\right)^\alpha\right)\right), \quad t \in (0,T), \tag{1.11}$$

with an integer $N \geq 1$ and creep parameters $K_+, K_0, \ldots, K_N \geq 0$ not all equal to zero, $\tau_1, \ldots, \tau_N > 0$. Indeed, passing to the limit $\alpha \to 1$ in (1.11) yields in the Prony series that is commonly used for hereditary kernels:

$$\mathcal{K}_1(t) = K_0 + K_+ t + \sum_{n=1}^{N} K_n \left(1 - e^{-t/\tau_n}\right). \tag{1.12}$$

Since the Caputo fractional derivative with order α of t^α is $\Gamma(1+\alpha)$, applying the differentiation rule (1.9) with $\lambda = -1/\tau_n^\alpha$ we calculate the Caputo fractional derivative of $\mathcal{K}_\alpha(t)$ as follows:

$$\mathcal{K}_\alpha^{(\alpha)}(t) = K_+ + \sum_{n=1}^{N} \frac{K_n}{\tau_n^\alpha} E_\alpha\left(-(t/\tau_n)^\alpha\right), \quad \mathcal{K}_1^{(1)}(t) = K_+ + \sum_{n=1}^{N} \frac{K_n}{\tau_n} e^{-t/\tau_n}. \tag{1.13}$$

Then, we have the following constitutive relation of GFV model by replacing $\mathcal{I}(t)$ in (1.1) with convolution with $\mathcal{K}_\alpha^{(\alpha)}(t)$

$$\varepsilon(t) = K_0 \mathcal{F}(\sigma) + [\mathcal{K}_\alpha^{(\alpha)} * \mathcal{F}(\sigma)](t), \quad t \in [0, T]. \tag{1.14}$$

Formula (1.14) implies a nonlinear fractional viscoelastic model where the material response function $\mathcal{F}$ is nonlinear. Note here that the right-hand side of (1.14) means applying the hereditary integral of the form (1.3) to $\mathcal{F}(\sigma)$, where the kernel $\mathcal{K}_\alpha(t)$ is used and the usual differentiation is replaced by the Caputo fractional derivative with order α. In the case of linear $\mathcal{F}$, the constitutive relation (1.1) with the creep function $\mathcal{K}_\alpha(t)$ defined in (1.11) leads to the following fractional differential equation with positive constant coefficients m, a_k and b_k for $p, q \in \mathbb{N}$:

$$\sigma(t) + \sum_{k=1}^{p} a_k \left(\sigma(t)\right)^{(\alpha_k)} = m\varepsilon(t) + \sum_{k=1}^{q} b_k \left(\varepsilon(t)\right)^{(\alpha_k)} \tag{1.15}$$

under the condition

$$\alpha_k = k + \alpha - 1, \quad k \in \mathbb{N}, \ \alpha \in (0, 1).$$

One sees that (1.15) covers the fractional generalization of the basic linear viscoelastic models such as Kelvin–Voigt, Maxwell, Zener, Burgers and so on, refer to Mainardi (2022) and Caputo and Mainardi (1971).

The rest of the paper is organized as follows. In Section 2, we introduce a nonlinear boundary value problem based on the GFV model mentioned above. In Section 3, the existence of variational solution to the fractional viscoelastic problem is stated in the main Theorem 3.1. Section 4 simulates a creep test under isotropic expansion in the one-dimensional case, varying the non-linearity parameter $p > 1$ and the fractional derivative coefficient α. Finally, Section 5 concludes this study.

2. GFV model for nonlinear fractional viscoelasticity

In a solid body, the linearized strain ε and Cauchy stress σ are given by second-order symmetric tensors in $\mathbb{R}^{3\times3}_{\rm sym}$, which admit the orthogonal volumetric–deviatoric decomposition:

$$\begin{aligned} \varepsilon &= \frac{1}{3}(\mathrm{tr}\varepsilon)\mathbf{I} + \varepsilon^* \quad (\mathrm{tr}\varepsilon^* = 0), \\ \sigma &= \frac{1}{3}(\mathrm{tr}\sigma)\mathbf{I} + \sigma^* \quad (\mathrm{tr}\sigma^* = 0), \end{aligned} \tag{2.1}$$

where $\mathrm{tr}(\cdot)$ stands for the trace, and $\mathbf{I}$ denotes the identity transformation. For the fractional kernel $\mathcal{K}_\alpha$ defined by (1.11), we introduce a viscoelastic response (1.14) that is determined by the convolution:

$$\varepsilon(t) = \frac{1}{9K}\left(K_0 \mathcal{F}(\mathrm{tr}\sigma) + [\mathcal{K}_\alpha^{(\alpha)} * \mathcal{F}(\mathrm{tr}\sigma)](t)\right)\mathbf{I} + \frac{1}{2G}\left(K_0 \tilde{\mathcal{F}}(\sigma^*) + [\mathcal{K}_\alpha^{(\alpha)} * \tilde{\mathcal{F}}(\sigma^*)](t)\right), \quad t \in (0, T), \tag{2.2}$$

with the bulk modulus $K > 0$ and shear modulus $G > 0$. The functions $\mathcal{F} : \mathbb{R} \longrightarrow \mathbb{R}$ and $\tilde{\mathcal{F}} : \mathbb{R}^{3\times3}_{\rm sym} \longrightarrow \mathbb{R}^{3\times3}_{\rm sym}$ describe material response. Owing to (2.1), the constitutive law (2.2) implies two integral equations:

$$\begin{aligned} \mathrm{tr}\varepsilon(t) &= \frac{1}{3K}\left(K_0 \mathcal{F}(\mathrm{tr}\sigma) + \int_0^t \mathcal{K}_\alpha^{(\alpha)}(t-s)[\mathcal{F}(\mathrm{tr}\sigma)](s)\,ds\right), \\ \varepsilon^*(t) &= \frac{1}{2G}\left(K_0 \tilde{\mathcal{F}}(\sigma^*) + \int_0^t \mathcal{K}_\alpha^{(\alpha)}(t-s)[\tilde{\mathcal{F}}(\sigma^*)](s)\,ds\right). \end{aligned} \tag{2.3}$$

For example, the linear isotropic response confirms $\mathcal{F}_2(\mathrm{tr}\sigma) := \mathrm{tr}\sigma$, $\tilde{\mathcal{F}}_2(\sigma^*) := \sigma^*$. Following Rajagopal (2014), its proper extension to nonlinear response reads:

$$\mathcal{F}_p(\mathrm{tr}\sigma) := \frac{\mathrm{tr}\sigma}{(1 + \lambda|\mathrm{tr}\sigma|)^{2-p}}, \quad \tilde{\mathcal{F}}_p(\sigma^*) := \frac{\sigma^*}{(1 + \tilde{\lambda}\|\sigma^*\|)^{2-p}} \tag{2.4}$$

with fitting parameters $\lambda, \tilde{\lambda} \geq 0$, $p > 1$ and the Frobenius norm $\|\sigma^*\|^2 = \mathrm{tr}((\sigma^*)^2)$, see the power-law and limiting small strains in Itou et al. (2018) and Itou et al. (2019a), implicit models in which stress and strain appear linearly in Itou et al. (2021b, 2023) and Itou et al. (2024b).

Here we note that the property (1.10) of $E_\alpha(-t)$ implies straightforwardly the result below.

Lemma 2.1. *The hereditary kernel* $\mathcal{K}_\alpha^{(\alpha)} \in C^\infty([0,T])$ *given by* (1.8) *and* (1.13) *is positive and uniformly bounded:*

$$0 < \mathcal{K}_\alpha^{(\alpha)}(t) \le C_\mathcal{K}, \quad t \in [0,T], \quad C_\mathcal{K} > 0. \tag{2.5}$$

Let the body occupy an open bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz continuous boundary $\partial\Omega$ and outward unit normal vector $\mathbf{n}$. We assume that $\partial\Omega$ consists of two disjoint parts Γ_D and Γ_N satisfying $|\Gamma_\mathrm{D}| \neq 0$. Let us denote the time-space cylinder $Q^T := (0,T)\times\Omega$ and its side portions $\Gamma_\mathrm{D}^T := (0,T)\times\Gamma_\mathrm{D}$, $\Gamma_\mathrm{N}^T := (0,T)\times\Gamma_\mathrm{N}$. Consider displacement vector $\mathbf{u}(t,\mathbf{x})$ of points $\mathbf{x} \in \overline{\Omega}$ at times $t \in [0,T)$. The linearized strain is determined by the symmetric part of the displacement gradient:

$$\varepsilon(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right), \tag{2.6}$$

where the superscript $\top$ denotes the transposition. For the prescribed volume force $\mathbf{f}(t,\mathbf{x})$ and boundary force $\mathbf{g}(t,\mathbf{x})$, the equilibrium equation in the quasi-static state is

$$-\nabla\cdot\boldsymbol{\sigma} = \mathbf{f} \quad \text{in } Q^T, \tag{2.7}$$

where the dot stands for scalar product, is supported by the homogeneous Dirichlet condition:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_\mathrm{D}^T, \tag{2.8}$$

and the Neumann boundary condition:

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_\mathrm{N}^T. \tag{2.9}$$

In the following we look for a solution to the nonlinear boundary value problem (2.2), (2.6)–(2.9).

3. Variational solution to the fractional viscoelastic problem

We give a variational formulation of the problem in function spaces. Consider the response functions:

$$\mathcal{F} : L^p(\Omega) \longrightarrow L^q(\Omega), \quad \tilde{\mathcal{F}} : L^p(\Omega;\mathbb{R}^{3\times3}_\mathrm{sym}) \longrightarrow L^q(\Omega;\mathbb{R}^{3\times3}_\mathrm{sym}), \quad \mathrm{tr}(\tilde{\mathcal{F}}(\boldsymbol{\sigma}^*)) = 0 \quad \left(p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1\right). \tag{3.1}$$

For the forces $\mathbf{f} \in L^\infty(0,T;L^p(\Omega))^3$, $\mathbf{g} \in L^\infty(0,T;L^p(\Gamma_\mathrm{N}))^3$, we look for a solution:

$$\mathbf{u} \in L^\infty(0,T;W^{1,q}(\Omega)^3) \text{ with } \mathbf{u} = \mathbf{0} \text{ on } \Gamma_\mathrm{D}^T, \quad \boldsymbol{\sigma} \in L^\infty(0,T;L^p(\Omega;\mathbb{R}^{3\times3}_\mathrm{sym})) \tag{3.2}$$

satisfying the variational equation, which is obtained after integration by parts of the equilibrium Eq. (2.7) multiplied by $\mathbf{v}$ and using the boundary conditions (2.8), (2.9):

$$\int_\Omega \boldsymbol{\sigma}(t)\cdot\varepsilon(\mathbf{v})\,d\mathbf{v} = \int_\Omega \mathbf{f}(t)\cdot\mathbf{v}\,d\mathbf{x} + \int_{\Gamma_\mathrm{N}} \mathbf{g}(t)\cdot\mathbf{v}\,d\mathbf{S}, \quad t \in (0,T), \tag{3.3}$$

for all test functions $\mathbf{v} \in W^{1,q}(\Omega)^3$ with $\mathbf{v} = \mathbf{0}$ on Γ_D, and the viscoelastic constitutive law for the linearized strain according to (2.2), (2.6):

$$[\varepsilon(\mathbf{u})](t) = K_0\left(\frac{1}{9K}\mathcal{F}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} + \frac{1}{2G}\tilde{\mathcal{F}}(\boldsymbol{\sigma}^*)\right) + \left[\mathcal{K}_\alpha^{(\alpha)} * \left(\frac{1}{9K}\mathcal{F}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} + \frac{1}{2G}\tilde{\mathcal{F}}(\boldsymbol{\sigma}^*)\right)\right](t), \quad t \in (0,T). \tag{3.4}$$

To show existence of a solution to (3.1)–(3.4), we introduce an auxiliary function $\overline{\mathbf{u}}$:

$$\mathbf{u}(t) = K_0\overline{\mathbf{u}} + [\mathcal{K}_\alpha^{(\alpha)} * \overline{\mathbf{u}}](t), \quad t \in (0,T). \tag{3.5}$$

This implies that the displacement together with the stress, such that

$$\overline{\mathbf{u}} \in L^\infty(0,T;W^{1,q}(\Omega)^3) \text{ with } \overline{\mathbf{u}} = \mathbf{0} \text{ on } \Gamma_\mathrm{D}^T, \quad \boldsymbol{\sigma} \in L^\infty(0,T;L^p(\Omega;\mathbb{R}^{3\times3}_\mathrm{sym})), \tag{3.6}$$

satisfy the variational Eq. (3.3) and the nonlinear elastic constitutive law:

$$[\varepsilon(\overline{\mathbf{u}})](t) = \left(\frac{1}{9K}\mathcal{F}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} + \frac{1}{2G}\tilde{\mathcal{F}}(\boldsymbol{\sigma}^*)\right)(t), \quad t \in (0,T). \tag{3.7}$$

Proposition 3.1. *Let the response functions in* (3.1) *be bounded:*

$$\|\mathcal{F}(\mathrm{tr}\boldsymbol{\sigma})\|^q_{L^q(\Omega)} \le M_0 + M_1\|\mathrm{tr}\boldsymbol{\sigma}\|^p_{L^p(\Omega)}, \quad \|\tilde{\mathcal{F}}(\boldsymbol{\sigma}^*)\|^q_{L^q(\Omega)} \le \tilde{M}_0 + \tilde{M}_1\|\boldsymbol{\sigma}^*\|^p_{L^p(\Omega)}, \tag{3.8}$$

with uniform constants $M_0, \tilde{M}_0 \ge 0$ *and* $M_1, \tilde{M}_1 > 0$*; monotone:*

$$\int_\Omega \big(\mathcal{F}(\mathrm{tr}\boldsymbol{\sigma}) - \mathcal{F}(\mathrm{tr}\overline{\boldsymbol{\sigma}})\big)(\mathrm{tr}\boldsymbol{\sigma} - \mathrm{tr}\overline{\boldsymbol{\sigma}})\,d\mathbf{x} \ge 0, \quad \int_\Omega \big(\tilde{\mathcal{F}}(\boldsymbol{\sigma}^*) - \tilde{\mathcal{F}}(\overline{\boldsymbol{\sigma}}^*)\big)\cdot(\boldsymbol{\sigma}^* - \overline{\boldsymbol{\sigma}}^*)\,d\mathbf{x} \ge 0, \tag{3.9}$$

for arbitrary $\boldsymbol{\sigma}, \overline{\boldsymbol{\sigma}} \in L^p(\Omega;\mathbb{R}^{3\times3}_\mathrm{sym})$*; semi-continuous:*

$$\int_\Omega \mathcal{F}\big(\mathrm{tr}(\boldsymbol{\sigma} + \lambda\overline{\boldsymbol{\sigma}})\big)\mathrm{tr}\overline{\boldsymbol{\sigma}}\,d\mathbf{x} \to \int_\Omega \mathcal{F}(\mathrm{tr}\boldsymbol{\sigma})\mathrm{tr}\overline{\boldsymbol{\sigma}}\,d\mathbf{x}, \quad \int_\Omega \tilde{\mathcal{F}}(\boldsymbol{\sigma}^* + \lambda\overline{\boldsymbol{\sigma}}^*)\cdot\overline{\boldsymbol{\sigma}}^*\,d\mathbf{x} \to \int_\Omega \tilde{\mathcal{F}}(\boldsymbol{\sigma}^*)\cdot\overline{\boldsymbol{\sigma}}^*\,d\mathbf{x} \tag{3.10}$$

as $\lambda \to 0$; and coercive:

$$\int_{\Omega} \mathcal{F}(\mathrm{tr}\boldsymbol{\sigma})\mathrm{tr}\boldsymbol{\sigma}\, d\mathbf{x} \geq M_4 \|\mathrm{tr}\boldsymbol{\sigma}\|^p_{L^p(\Omega)} - M_3, \quad \int_{\Omega} \tilde{\mathcal{F}}(\boldsymbol{\sigma}^*) \cdot \boldsymbol{\sigma}^*\, d\mathbf{x} \geq \tilde{M}_4 \|\boldsymbol{\sigma}^*\|^p_{L^p(\Omega)} - \tilde{M}_3, \tag{3.11}$$

with uniform constants $M_3, \tilde{M}_3 \geq 0$ and $M_4, \tilde{M}_4 > 0$. Then, there exists a solution (3.6) to the nonlinear elastic problem (3.3), (3.7). If the monotonicity property (3.9) is strict, then the solution is unique.

The proof of Proposition 3.1 is given in Itou et al. (2019a) and Itou et al. (2024a). The example functions $\mathcal{F}_p$, $\tilde{\mathcal{F}}_p$ in (2.4) satisfy the properties (3.8)–(3.11) with the following bounds, see Itou et al. (2019a, Appendix A):

$$\begin{cases} M_0 = 0, \quad M_1 = \dfrac{1}{\lambda^{2-p}}, & \text{if } 1 < p \leq 2, \\ M_0 = \dfrac{p-q}{p} C_{1,p}|\Omega|, \quad M_1 = \left(\lambda^{p-q} + \dfrac{q}{p}\right) C_{1,p}, & \text{if } p > 2, \end{cases}$$

$$\begin{cases} \tilde{M}_0 = 0, \quad \tilde{M}_1 = \dfrac{3^{2p}}{\tilde{\lambda}^{2-p}}, & \text{if } 1 < p \leq 2, \\ \tilde{M}_0 = \dfrac{3^2(p-q)}{p} C_{1,p}|\Omega|, \quad \tilde{M}_1 = \left(\tilde{\lambda}^{p-q} 3^{2p} + \dfrac{q}{p}\right) C_{1,p}, & \text{if } p > 2, \end{cases}$$

where $C_{1,p} := \max\{1, 2^{p-q-1}\}$,

$$\begin{cases} M_3 = \dfrac{|\Omega|}{\lambda^2 p C_{2,p}}, \quad M_4 = \dfrac{1}{\lambda^{2-p} p C_{2,p}}, \quad \tilde{M}_3 = \dfrac{|\Omega|}{\tilde{\lambda}^2 p C_{2,p}}, \quad \tilde{M}_4 = \dfrac{1}{3^2 \tilde{\lambda}^{2-p} p C_{2,p}}, & \text{if } 1 < p < 2, \\ M_3 = 0, \quad M_4 = \lambda^{p-2}, \quad \tilde{M}_3 = 0, \quad \tilde{M}_4 = \dfrac{\tilde{\lambda}^{p-2}}{3^2}, & \text{if } p \geq 2, \end{cases}$$

where $C_{2,p} := \left((2-p)2^{p-1}\right)^{(2-p)/p}$.

Under the assumption of Proposition 3.1 we derive the main existence result from Lemma 2.1.

Theorem 3.1. *If there exists a solution $(\overline{\mathbf{u}}, \boldsymbol{\sigma})$ to the nonlinear elasticity problem (3.3), (3.6), (3.7), then $(\mathbf{u}, \boldsymbol{\sigma})$ solves the viscoelastic problem (3.2)–(3.4) with the displacement $\mathbf{u}$ given by the convolution (3.5).*

Proof. The time evolution operator $K_0 + \mathcal{K}_\alpha^{(\alpha)} *$ commutes with the spatial strain operator (2.6):

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}(K_0\overline{\mathbf{u}} + \mathcal{K}_\alpha^{(\alpha)} * \overline{\mathbf{u}}) = K_0 \boldsymbol{\varepsilon}(\overline{\mathbf{u}}) + \mathcal{K}_\alpha^{(\alpha)} * \boldsymbol{\varepsilon}(\overline{\mathbf{u}}). \tag{3.12}$$

If the elastic relations (3.3), (3.6), (3.7) hold, then applying K_0 plus the convolution of $\mathcal{K}_\alpha^{(\alpha)}$ to both sides of the constitutive Eq. (3.7), with the help of (3.12) we arrive at the viscoelastic response (3.4):

$$\boldsymbol{\varepsilon}(\mathbf{u}) = K_0 \boldsymbol{\varepsilon}(\overline{\mathbf{u}}) + \mathcal{K}_\alpha^{(\alpha)} * \boldsymbol{\varepsilon}(\overline{\mathbf{u}}) = K_0 \left(\frac{1}{9K}\mathcal{F}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} + \frac{1}{2G}\tilde{\mathcal{F}}(\boldsymbol{\sigma}^*)\right) + \mathcal{K}_\alpha^{(\alpha)} * \left(\frac{1}{9K}\mathcal{F}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} + \frac{1}{2G}\tilde{\mathcal{F}}(\boldsymbol{\sigma}^*)\right) \quad \text{in } Q^T,$$

and the homogeneous Dirichlet condition (2.8):

$$\mathbf{u} = K_0 \overline{\mathbf{u}} + \mathcal{K}_\alpha^{(\alpha)} * \overline{\mathbf{u}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}^T. \tag{3.13}$$

The inclusion concerning $\mathbf{u}$ in (3.2) holds owing to the properties (2.5) of $\mathcal{K}_\alpha^{(\alpha)}$. The proof is completed.

4. Creep test under isotropic expansion

We simulate creep under isotropic expansion with a prescribed pressure $-\sigma/3$ such that

$$\boldsymbol{\varepsilon} = \frac{1}{3}\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \quad \boldsymbol{\sigma} = \frac{1}{3}\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix},$$

$$\sigma(t) = \begin{cases} t & \text{for } t \in [0,1], \\ 1 & \text{for } t \in (1,2], \\ 0 & \text{for } t \in (2,3]. \end{cases} \tag{4.1}$$

The creep test undergoes three stages in (4.1): linear loading for $t \in [0,1]$, maintaining the constant pressure for $t \in (1,2]$ (h), which is immediately removed for $t > 2$. We choose the material parameters (E: Young's modulus, ν: Poisson's ratio) as follows:

$$E = 30\,(\mathrm{GPa}), \quad \nu = 0.2, \quad \frac{1}{3K} = \frac{1-2\nu}{E} = 0.02, \quad K_0 = 0.1, \quad K_+ = 0, \quad K_1 = 1, \quad \tau_1 = 2, \quad \lambda = 1,$$

and vary the power $p \geq 1$ in the response function $\mathcal{F}_p$ in (2.4), and the order $\alpha \in (0,1]$ of the fractional Caputo derivative $\mathcal{K}_\alpha^{(\alpha)}$ in GFV model defined by the term (1.13) for $N = 1$. The dilatation $\tilde{\varepsilon}$ is computed from the nonlinear elastic law (3.7), and ε from the viscoelastic constitutive law (3.4), namely:

$$\tilde{\varepsilon}(t) = \frac{1}{3K}\mathcal{F}_p(\sigma(t)), \quad \mathcal{F}_p(\sigma(t)) = \frac{\sigma(t)}{(1+\lambda|\sigma(t)|)^{2-p}}, \quad \varepsilon(t) = K_0\tilde{\varepsilon}(t) + \int_0^t \frac{K_1}{\tau_1^\alpha} E_\alpha\left(-((t-s)/\tau_1)^\alpha\right)\tilde{\varepsilon}(s)\, ds. \tag{4.2}$$

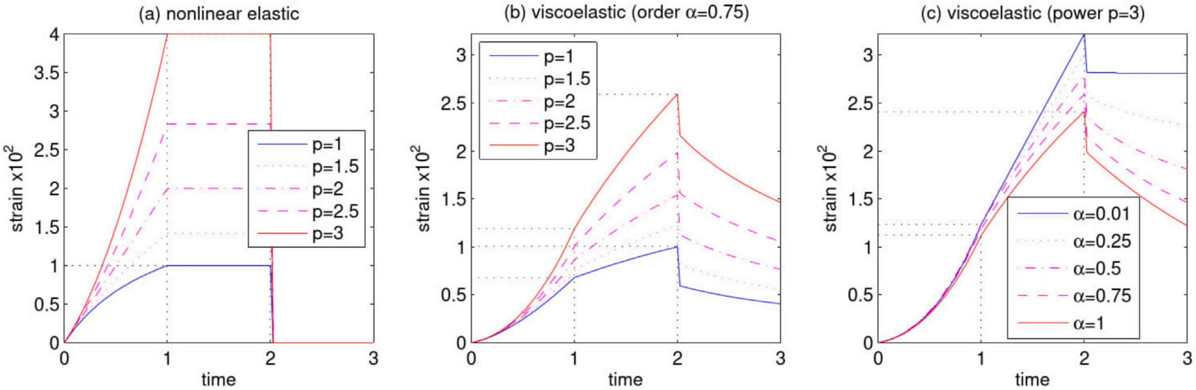

Fig. 1. Creep test of the GFV model for (4.1) and $K_0 = 0.1$, (a): elastic strain $\tilde{\varepsilon}$ and (b), (c): viscoelastic strain ε versus time t.

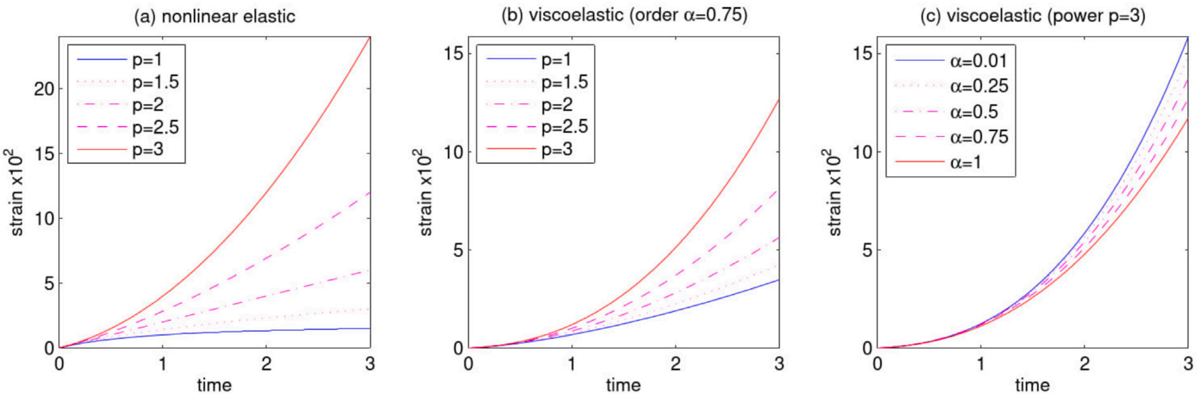

Fig. 2. Creep test of the GFV model for (4.3) and $K_0 = 0.1$, (a): elastic strain $\tilde{\varepsilon}$ and (b), (c): viscoelastic strain ε versus time t.

Using standard Matlab codes, with the step-size $\Delta t = 0.025$ the strains from (4.2) are depicted in Fig. 1. These figures are classical viscoelastic creep tests. The left plot (a) presents curves of the nonlinear elastic strain $\tilde{\varepsilon}(t)$ versus time $t \in [0, 3]$ for various powers $p \in \{1, 1.5, 2, 2.5, 3\}$, where $p = 2$ corresponds to the linear material response. The corresponding curves of the viscoelastic strain $\varepsilon(t)$ are shown in the middle plot (b) for the fixed fractional order $\alpha = 0.75$. The strain is higher for higher powers p. For fixed $p = 3$ and various orders of the fractional derivative $\alpha \in \{0.01, 0.25, 0.5, 0.75, 1\}$ super-linear curves of the viscoelastic strain $\varepsilon(t)$ are presented in the right plot (c) of Fig. 1. In contrast to the plot (b), high α causes a drop of the strain. The numerical results demonstrate viscoelastic behavior during the three subsequent stages: loading — maintaining — release of the creep test.

Next, to check the accuracy of our model we test another different stress history:

$$\sigma(t) = t \quad \text{for } t \in [0, 3]. \tag{4.3}$$

Similar to the previous case, the strains are shown in Fig. 2 via (4.2). We can observe the creep phenomenon in Fig. 1, whereas Fig. 2 illustrates that the strain increases monotonically with the stress in (4.3).

Finally, let compare the case where $K_0 = 0$ and all other settings are the same. Figs. 3 and 4 show the cases (4.1) and (4.3), respectively. Especially, since $K_0 = 0$ eliminates the effect of the instantaneous elastic response of a solid, Fig. 3 shows that the stress is discontinuous while the viscoelastic strain remains continuous, in contrast with Fig. 1.

5. Conclusions

In this paper, generalized fractional viscoelastic (GFV) models are introduced, which are an approximation to generalized classical linear viscoelastic models based on the Caputo fractional derivatives as proposed in Mainardi (2022, Chap. 3). The constitutive relations in GFV models (i.e. (1.14) or (2.2)) are expressed by convolution with the material response using Volterra hereditary integrals with creep functions represented by the Mittag-Leffler function instead of conventional Prony series. The study addresses both linear and nonlinear material responses, providing analytical solutions for GFV models through convolution with solutions of corresponding elastic problems. A key contribution is the proof of an existence theorem for solutions to boundary value problems within the GFV framework. In the proof, utilizing commutative property of two operators (the convolution of $\mathcal{K}_\alpha^{(\alpha)}$ and the spatial

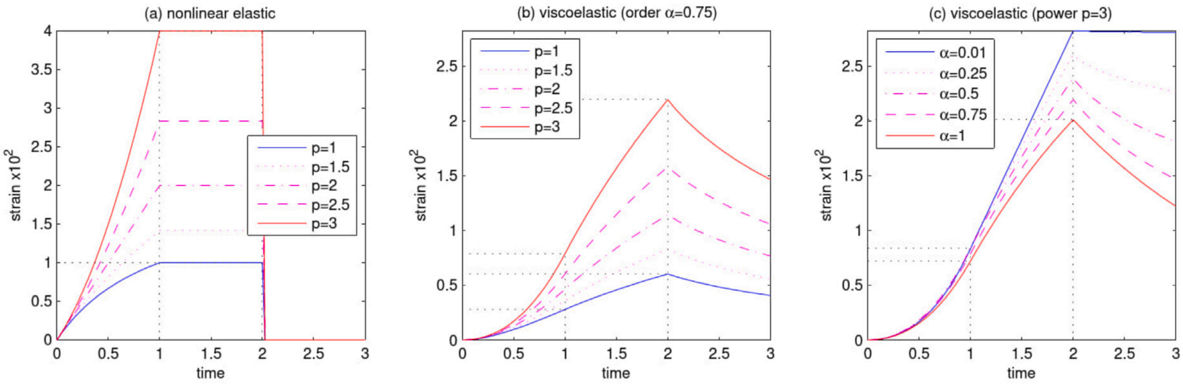

Fig. 3. Creep test of the GFV model for (4.1) and $K_0 = 0$, (a): elastic strain $\tilde{\varepsilon}$ and (b), (c): viscoelastic strain ε versus time t.

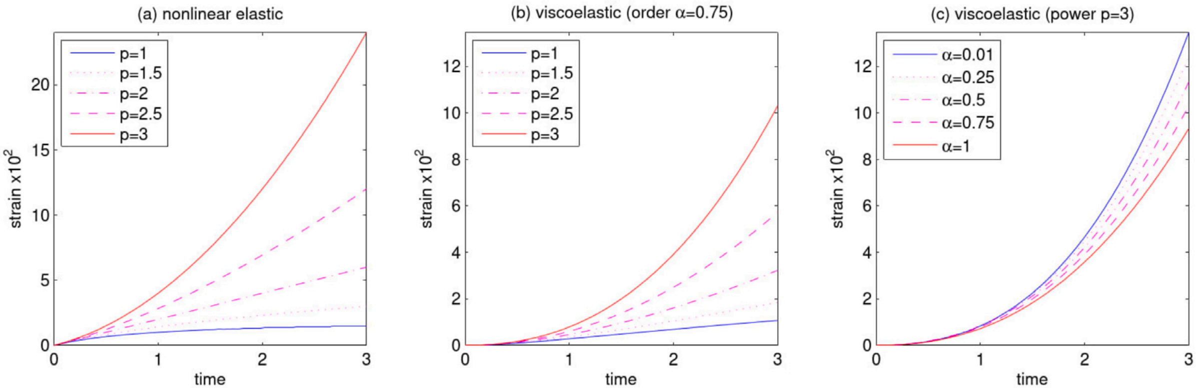

Fig. 4. Creep test of the GFV model for (4.3) and $K_0 = 0$, (a): elastic strain $\tilde{\varepsilon}$ and (b), (c): viscoelastic strain ε versus time t.

strain operator ε), the solution can be expressed in the form of convolution with a solution of the corresponding nonlinear elastic problem. This method evokes the correspondence principle which states that solutions to linear viscoelastic problems can be obtained from the corresponding solutions to linearized elastic problems via the Laplace transform. Then, the linearity of the response function is crucial for applying the correspondence principle, and its conditions for applicability and inapplicability are discussed in Rajagopal and Wineman (1980), Wineman and Rajagopal (2000) and also Itou et al. (2021a). Therefore, it is found that the present method can link solutions for viscoelastic and elastic problems even when the response function is nonlinear in the GFV model.

Moreover, we also perform a numerical simulation for a one-dimensional creep test under isotropic expansion, analyzing the effects of the fractional derivative order and non-linearity on material behavior in each of the three stages: loading, maintaining, and release stages of the creep test.

CRediT authorship contribution statement

Hiromichi Itou: Writing – review & editing, Writing – original draft, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Victor A. Kovtunenko:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis. **Masahiro Yamamoto:** Writing – review & editing, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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