International Journal of Engineering Science 136 (2019) 17–25

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

Well-posedness of the problem of non-penetrating cracks in elastic bodies whose material moduli depend on the mean normal stress

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ARTICLE INFO

Article history:
Received 23 December 2018
Accepted 26 December 2018

MSC:
35Q74
49J52
74B20

Keywords:
Nonlinear elasticity
Neo-Hookean law
Implicit constitutive response
Inhomogeneous media
Concrete
Crack with non-penetration
Variational inequality

ABSTRACT

The well-posedness of the problem of non-penetrating crack in an elastic body whose material moduli depends on the mean normal stress is studied. The type of models considered are based on a new implicit theory for the response of elastic bodies. Bodies defined by these implicit theories are more general than Cauchy elastic bodies. The problem is studied within the context of a variational inequality.

1. Introduction

There is considerable evidence that properties which characterize the mechanical, optical, electrical response of materials depend on the mean normal stress (see Bridgman, 1931). Much of the work that pertains to the response of such materials appeal to models that do not have a sound basis. Ad hoc generalizations of classical models are used which often times contradict the tenets on which the classical models are based. In order to justify, for instance, models that contain material moduli that depend on both the invariants of the stress (such as its mean value) and invariants of appropriate kinematic variables (second principal invariant of the strain to describe shear softening or hardening in solids or the second invariant of the symmetric part of the velocity gradient to describe shear thinning or shear thickening in fluids) one needs to systematically develop implicit constitutive relations in which to imbed them. Within the context of elastic bodies, in order to

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https://doi.org/10.1016/j.ijengsci.2018.12.005
0020-7225/

describe in a rational manner properties the dependence of the Young's modulus or the shear modulus on the mean normal stress one needs to generalize the classical framework of elasticity due to Cauchy (1823, 1828) and Green (1838, 1839).

Recently, Rajagopal (2003, 2007) showed that the class of bodies that qualify to be called elastic are much larger than Cauchy elastic bodies (Green elastic bodies are a sub-class of Cauchy elastic bodies, see (Truesdell & Noll, 2004) for a detailed discussion of Cauchy and Green elasticity). Rajagopal and Srinivasa (2007, 2009) provided a thermodynamic basis for this enlarged class of elastic bodies. Unlike Cauchy elasticity where one assumes that the stress is a function of the deformation gradient, the starting point for the larger class of elastic bodies is the assumption that the stress and deformation gradient are related in the sense of "relations" in mathematics, that is

$$\mathbf{f}(\mathbf{T}, \mathbf{F}) = \mathbf{0}, \tag{1}$$

where $\mathbf{T}$ denotes the Cauchy stress and $\mathbf{F}$ the deformation gradient. If one assumes that the body is isotropic (see Rajagopal, 2015a for a discussion of the material symmetry of bodies defined through implicit constitutive relations), then one obtains

$$\mathbf{f}(\mathbf{T}, \mathbf{B}) = \mathbf{0}, \tag{2}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^\top$ is referred to as the Left Cauchy–Green tensor. If $\mathbf{f}$ is an isotropic function, then (see Spencer, 1971)

$$\alpha_0\mathbf{1} + \alpha_1\mathbf{T} + \alpha_2\mathbf{B} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{B}^2 + \alpha_5(\mathbf{TB} + \mathbf{BT}) + \alpha_6(\mathbf{T}^2\mathbf{B} + \mathbf{BT}^2) + \alpha_7(\mathbf{TB}^2 + \mathbf{B}^2\mathbf{T}) + \alpha_8(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2) = \mathbf{0}, \tag{3}$$

where $\mathbf{1}$ is the matrix of ones, and the material moduli α_i, $i = 0, \dots, 8$ depend upon the density ρ and the following traces

$$\mathrm{tr}\mathbf{T},\ \mathrm{tr}\mathbf{B},\ \mathrm{tr}\mathbf{T}^2,\ \mathrm{tr}\mathbf{B}^2,\ \mathrm{tr}\mathbf{T}^3,\ \mathrm{tr}\mathbf{B}^3,\ \mathrm{tr}(\mathbf{TB}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{B}),\ \mathrm{tr}(\mathbf{TB}^2),\ \mathrm{tr}(\mathbf{T}^2\mathbf{B}^2). \tag{4}$$

Two very important sub-classes of the above general class of implicit constitutive relations are

$$\mathbf{T} = \delta_1\mathbf{I} + \delta_2\mathbf{B} + \delta_3\mathbf{B}^2 \tag{5}$$

employing the identity matrix $\mathbf{I}$, where the material functions δ_i, $i = 1, 2, 3$ depend on ρ, $\mathrm{tr}\mathbf{B}$, $\mathrm{tr}\mathbf{B}^2$, $\mathrm{tr}\mathbf{B}^3$, namely the Cauchy elastic model, and

$$\mathbf{B} = \alpha_0^s\mathbf{1} + \alpha_1^s\mathbf{T} + \alpha_2^s\mathbf{T}^2, \tag{6}$$

where the α_i^s, $i = 0, 1, 2$ depend on $\mathrm{tr}\mathbf{T}$, $\mathrm{tr}\mathbf{T}^2$, $\mathrm{tr}\mathbf{T}^3$ and the density ρ. Eqs. (5) and (6) simply constitute the Hamilton–Cayley representation, for an isotropic tensor function (and hence do not require any additional assumptions). Since in virtue of the balance of mass the current density can be expressed in terms of the reference density and the determinant $\det\mathbf{F}$, the model (6) is yet an implicit model. If the α_i^s, $i = 0, 1, 2$ do not depend on the density ρ, then the model (6) becomes an explicit expression for the Cauchy–Green tensor in terms of the stress. The model might not necessarily be invertible, that is, one may not be able to express the stress in terms of $\mathbf{B}$. Truesdell and Moon (1975) have obtained necessary conditions for the semi-invertibility and invertibility of (5), but this is not relevant to the work being carried out here.

In this study we shall use a constitutive relation wherein the material moduli depend on the mechanical pressure, and here by pressure we mean the mean normal stress. For implicit models for fluids with pressure dependent material moduli, see (Bulîček, Màlek, & Rajagopal, 2009). Before getting into a discussion of our model, it would be instructive to make the following observation. Let us consider the following incompressible elastic material given by the constitutive relation

$$\mathbf{T} = -p\mathbf{I} + \mu(p)\mathbf{B}, \tag{7}$$

where p is the indeterminate part of the stress that enforces the constraint. Notice that is not the mean normal stress, that is $p \neq -\frac{\mathrm{tr}\mathbf{T}}{3}$ since $\mathrm{tr}\mathbf{B}$ is not necessarily zero in all motions (the constraint of incompressibility requires that $\det\mathbf{B} = 0$ in all motions). Unfortunately, one loosely and incorrectly uses the term pressure to refer to the Lagrange multiplier p (see Rajagopal, 2015b for a detailed discussion on the various uses, misuses and abuses of the terminology "pressure"). Thus, the dependence on the mean value of the stress and on the Lagrange multiplier are not the same and this is important to recognize in this study.

Rajagopal and Saccomandi (2009), in their study of the response of a class of polymeric solids, used the constitutive relation of the form

$$\mathbf{T} = \alpha(\mathrm{tr}\mathbf{T})\mathbf{I} + \beta(\mathrm{tr}\mathbf{T})\mathbf{B}. \tag{8}$$

The above model belongs to the class of bodies described by (6) as $\mathbf{B}$ can be expressed as

$$\mathbf{B} = \frac{\mathbf{T} - \alpha(\mathrm{tr}\mathbf{T})\mathbf{I}}{\beta(\mathrm{tr}\mathbf{T})}. \tag{9}$$

The model that we use in our study arises as a linearization of the above model. We first note that if one linearizes the expression (6) by assuming that the maximum of the Frobenius norm of the gradient of displacement $\mathbf{u}$ for all particles $\mathbf{X}$ belonging to the body in a stress-free reference configuration and for all time t is small in the sense

$$\max_{\mathbf{X},t}\left\|\frac{\partial\mathbf{u}}{\partial\mathbf{X}}\right\| \ll 1, \tag{10}$$

then one obtains

$$\boldsymbol{\varepsilon} = \hat{\alpha}_0 \mathbf{1} + \hat{\alpha}_1 \mathbf{T} + \hat{\alpha}_2 \mathbf{T}^2, \tag{11}$$

where the $\hat{\alpha}_i$, $i = 0, 1, 2$ depend on ρ, $\mathrm{tr}\mathbf{T}$, $\mathrm{tr}\mathbf{T}^2$, and $\mathrm{tr}\mathbf{T}^3$. It is possible that the material moduli $\hat{\alpha}_i$ depend linearly on the $\mathrm{tr}\boldsymbol{\varepsilon}$ in virtue of the fact that the material moduli could depend on the density in virtue of the balance of mass and the linearization $\det\mathbf{F} \approx 1 + \mathrm{tr}\boldsymbol{\varepsilon}$. The model considered by Rajagopal and Saccomandi (2009), namely (8) does not depend explicitly on the density but it is a compressible model nonetheless as the $\mathrm{tr}\boldsymbol{\varepsilon}$ is not necessarily zero in all motions. If we linearize (9) by assuming that (10) holds, then linearization of the constitutive expression (9) would give rise to a sub-class of (11) and take the form

$$\boldsymbol{\varepsilon} = \frac{\mathbf{T}}{\beta(\mathrm{tr}\mathbf{T})} - \frac{\alpha(\mathrm{tr}\mathbf{T})}{\beta(\mathrm{tr}\mathbf{T})}\mathbf{I} = \frac{\mathbf{T}}{\beta(\mathrm{tr}\mathbf{T})} + \gamma(\mathrm{tr}\mathbf{T})\mathbf{I}. \tag{12}$$

Expressing the stress in terms of its deviatoric and spherical parts by using

$$\mathbf{T} = \mathbf{T}^* + \frac{\mathrm{tr}\mathbf{T}}{3}\mathbf{I}, \tag{13}$$

the constitutive expression (12) can be expressed as

$$\boldsymbol{\varepsilon} = \frac{\mathbf{T}^*}{\beta(\mathrm{tr}\mathbf{T})} + \phi(\mathrm{tr}\mathbf{T})\mathbf{I}. \tag{14}$$

The model that will be studied in this paper is a special case of (14).

Recall that the classical linearized elastic solid is described by the constitutive expression

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\mathbf{T} - \frac{\nu}{E}(\mathrm{tr}\mathbf{T})\mathbf{I}, \tag{15}$$

where E and ν are the Young's modulus and Poisson's ratio, respectively, or equivalently as

$$\mathbf{T} = \lambda(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}, \tag{16}$$

where λ and μ are Lame' constants related to E and ν through

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{17}$$

Suppose we generalize the above constitutive expression in an ad hoc manner by assuming that the Young's modulus and Poisson's ratio depend on the stress (this is possible within the context of the constitutive relation (11)), then

$$\boldsymbol{\varepsilon} = \frac{1+\nu(\mathrm{tr}\mathbf{T})}{E(\mathrm{tr}\mathbf{T})}\mathbf{T} - \frac{\nu(\mathrm{tr}\mathbf{T})}{E(\mathrm{tr}\mathbf{T})}(\mathrm{tr}\mathbf{T})\mathbf{I}. \tag{18}$$

We immediately observe that (18) is the same as (12) with the following identification

$$\beta(\mathrm{tr}\mathbf{T}) = \frac{E(\mathrm{tr}\mathbf{T})}{1+\nu(\mathrm{tr}\mathbf{T})}, \quad \gamma(\mathrm{tr}\mathbf{T}) = -\frac{\nu(\mathrm{tr}\mathbf{T})}{E(\mathrm{tr}\mathbf{T})}\mathrm{tr}\mathbf{T}. \tag{19}$$

Also, it follows from (12) and (14) that

$$\phi(\mathrm{tr}\mathbf{T}) = \gamma(\mathrm{tr}\mathbf{T}) + \frac{\mathrm{tr}\mathbf{T}}{3\beta(\mathrm{tr}\mathbf{T})}. \tag{20}$$

Let us suppose that the Lame' constants λ, μ, and the bulk modulus K can be generalized to depend on the mean value of the stress (once again such a model is not possible within the Cauchy theory of elasticity but possible with the context of the implicit theory), and let us further suppose that these generalized material moduli obey the same relationship that K bears to λ and μ in the classical linearized theory of elasticity, that is,

$$K(\mathrm{tr}\mathbf{T}) = \frac{3\lambda(\mathrm{tr}\mathbf{T}) + 2\mu(\mathrm{tr}\mathbf{T})}{3}, \tag{21}$$

then the constitutive expression (14) can be re-written, by virtue of (19) and (20) as

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\mathbf{T}^* + \frac{\mathrm{tr}\mathbf{T}}{9K(\mathrm{tr}\mathbf{T})}\mathbf{I}. \tag{22}$$

We shall use this generalization of a Hookean model to study the state of stress and strain adjacent to a non-penetrating crack in a body described by the constitutive expression (22).

In the engineering literature, various heuristic relations of the elastic moduli on pressure are considered, linear as well as nonlinear ones, e.g. of the exponential type. To fit the experimental data of Paterson (1964), Rajagopal and Saccomandi (2009) suggested an inverse tangent function (see Formula (4.7) in Rajagopal & Saccomandi, 2009). Applying this

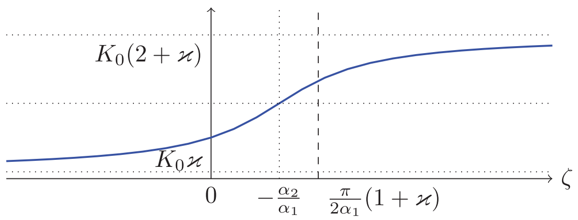

Fig. 1. The example function $K(\zeta)$ in (23) for $K_0 = 1, \varkappa = 0.1, \alpha_1 = 1, \alpha_2 = -1$.

ansatz to the generalization of the bulk modulus, namely the function K, with the convention that $\arctan\zeta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for a scalar parameter $\zeta \in \mathbb{R}$, we set a bounded function $K : \mathbb{R} \mapsto (K_0\varkappa, K_0(2+\varkappa))$ of the general form

$$K(\zeta) = K_0\Big(1 + \varkappa + \frac{2}{\pi}\arctan(\alpha_1\zeta + \alpha_2)\Big) \tag{23}$$

with positive parameters K_0, α_1, and parameter α_2 may be positive as well as negative. While $\varkappa > 0$ is important in (23) for the mathematical consistency: $\frac{1}{K(\zeta)} \in \Big(\frac{1}{K_0(2+\varkappa)}, \frac{1}{K_0\varkappa}\Big)$ is bounded, otherwise unbounded from above if $\varkappa = 0$. A function curve (23) is sketched in Fig. 1, where $\alpha_2 < 0$ and its critical value $-\frac{\pi}{2}(1+\varkappa)$ is from Lemma 1 in Appendix A.

Following (23), we can generalize also the shear modulus to a variable function $\mu : \mathbb{R} \mapsto (\mu_0\gamma, \mu_0(2+\gamma))$ of the same form

$$\mu(\zeta) = \mu_0\Big(1 + \gamma + \frac{2}{\pi}\arctan(\beta_1\zeta + \beta_2)\Big) \tag{24}$$

with parameters β_2 and μ_0, β_1, $\gamma > 0$, and consider the fully nonlinear response

$$\boldsymbol{\varepsilon} = \frac{\mathbf{T}^*}{2\mu(\|\mathbf{T}^*\|)} + \frac{\mathrm{tr}\mathbf{T}}{9K(\mathrm{tr}\mathbf{T})}\mathbf{I}, \tag{25}$$

which is a neo-Hookean model, where $\|\mathbf{T}^*\| = \sqrt{\mathrm{tr}(\mathbf{T}^*)^2}$. It is worth noting that the models (22) and (25) belong to implicit constitutive theories that describe non-dissipative response of solids introduced by Rajagopal, see (Rajagopal, 2007) and other works with co-authors.

The other specialty of our modeling concerns the possibility that the faces may be in contact, that is of especial importance in the context of fracture mechanics, see (Bratov, Morozov, & Petrov, 2009). For the engineering theory of heterogeneous materials with micro-structures and their effective properties we refer to Kachanov and Sevostianov (2018). The variational theory describing contacting crack faces in linear elastic solids and plates with the help of non-penetration conditions was established in Khludnev and Kovtunenko (2000), Khludnev, Kovtunenko, and Tani (2010), Khludnev, Ohtsuka, and Sokołowski (2002) and Kovtunenko and Leugering (2016) and other works by the authors. In particular, they took into consideration dissipative contact phenomena due to friction and cohesion in Itou, Kovtunenko, and Tani (2011) and Kovtunenko (2011)), orthotropic materials were considered in Hintermüller, Kovtunenko, and Kunisch (2007), and Timoshenko's plate in Lazarev and Rudoy (2014). Further development of the nonlinear crack problems adjacent to non-penetration was carried out in Itou, Kovtunenko, and Rajagopal (2017, 2018) for the limiting small strain model.

In Section 3 we prove the mathematical well-posedness for nonlinear elastic models of the type (22) and (25), when the solid possesses a crack subject to the non-penetration condition, based on the Browder–Minty theorem and penalty approximation. In the following Section 2 we start with the rigorous definition of geometry and functions used.

2. Theory

Let Ω be a bounded domain in the Euclidean space $\mathbb{R}^3$ with the Lipschitz continuous boundary $\partial\Omega$ and the unit normal vector $\mathbf{n} = (n_1, n_2, n_3)$ which is directed outward to Ω. Let $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_D}$ consist of two disjoint parts: the Neumann boundary Γ_N and the Dirichlet boundary Γ_D such that $\Gamma_D \neq \emptyset$ is nonempty. Let Γ_c be a crack inside Ω. We assume that $\Gamma_c \subseteq \Sigma$ is a part of the two-dimensional oriented open manifold Σ with a unit normal vector $\mathbf{n} = (n_1, n_2, n_3)$, and that Σ splits Ω into two sub-domains $\Omega^\pm$ with Lipschitz continuous boundaries $\partial\Omega^\pm$. Two opposite crack faces $\Gamma_c^\pm$ can be distinguished as the parts of $\partial\Omega^\pm$. We call the geometric set $\Omega_c = \Omega \setminus \overline{\Gamma_c}$ with the boundary $\partial\Omega_c = \partial\Omega \cup \overline{\Gamma_c^+} \cup \overline{\Gamma_c^-}$ the domain with the crack.

For spatial points $\mathbf{x} = (x_1, x_2, x_3)$ in the domain with crack and its boundary $\overline{\Omega_c} = \Omega_c \cup \partial\Omega_c$ we define a displacement vector $\mathbf{u} = (u_1, u_2, u_3)(\mathbf{x})$. The displacement obeys a jump across the crack faces, which is generally non-zero:

$$[\![\mathbf{u}]\!](\mathbf{x}) := \mathbf{u}\big|_{\mathbf{x}\in\Gamma_c^+} - \mathbf{u}\big|_{\mathbf{x}\in\Gamma_c^-}. \tag{26}$$

Following (Khludnev and Kovtunenko, 2000, Section 1.1.7) we interpret the non-penetration of the crack faces by means of non-negativeness of the normal jump as follows

$$[\![\mathbf{u}]\!] \cdot \mathbf{n} \ge 0 \quad \text{on } \Gamma_c, \tag{27}$$

where $[\![\mathbf{u}]\!]\cdot\mathbf{n} = [\![u_i]\!]n_i$ implies the scalar product of vectors under the convention of summation over the repeated index $i = 1, 2, 3$.

Given the body force $\mathbf{f} = (f_1, f_2, f_3)(\mathbf{x})$ for $\mathbf{x}\in\Omega_{\rm c}$ and the boundary traction $\mathbf{g} = (g_1, g_2, g_3)(\mathbf{x})$ for $\mathbf{x}\in\Gamma_{\rm N}$, we look for the displacement vector $\mathbf{u}(\mathbf{x})$, symmetric 3-by-3 tensors of the strain $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}_{i,j=1}^3(\mathbf{x})$ and the Cauchy stress $\mathbf{T} = \{T_{ij}\}_{i,j=1}^3(\mathbf{x})$ for $\mathbf{x}\in\overline{\Omega_{\rm c}}$ which satisfy: the equilibrium equation

$$-\frac{\partial}{\partial x_j}T_{ij} = f_i, \quad i = 1, 2, 3, \quad \text{in } \Omega_{\rm c}, \tag{28}$$

where the convention of summation over the repeated index $j = 1, 2, 3$ was used; the linearized strain is defined as

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3; \tag{29}$$

and the implicit constitutive response equation according to (25)

$$\varepsilon_{ij}(\mathbf{u}) = \frac{T^*_{ij}}{2\mu(\|\mathbf{T}^*\|)} + \frac{\mathrm{tr}\mathbf{T}}{9K(\mathrm{tr}\mathbf{T})}\delta_{ij}, \quad i, j = 1, 2, 3, \tag{30}$$

where $\mathrm{tr}\mathbf{T} = T_{ii}$, $\|\mathbf{T}^*\| = \sqrt{(T^*_{ij})^2}$ with the summation convention, and the Kronecker delta $\delta_{ij} = 1$ for $i = j$, otherwise $\delta_{ij} = 0$ for $i \neq j$.

The governing Eqs. (28)–(30) are augmented by the following boundary conditions: the Dirichlet condition for the clamp

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\rm D}; \tag{31}$$

the Neumann type condition for the traction

$$\mathbf{T}\cdot\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\rm N}; \tag{32}$$

and the complete system of conditions due to the non-penetration (27):

$$\mathbf{T}\cdot\mathbf{n} - \big((\mathbf{T}\cdot\mathbf{n})\cdot\mathbf{n}\big)\mathbf{n} = 0 \qquad \text{on } \Gamma_{\rm c}^{\pm}, \tag{33}$$

$$([\![\mathbf{T}]\!]\cdot\mathbf{n})\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_{\rm c}, \tag{34}$$

$$[\![\mathbf{u}]\!]\cdot\mathbf{n} \geq 0, \quad (\mathbf{T}\cdot\mathbf{n})\cdot\mathbf{n} \leq 0, \quad \big((\mathbf{T}\cdot\mathbf{n})\cdot\mathbf{n}\big)[\![\mathbf{u}]\!]\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_{\rm c}, \tag{35}$$

where the stress jump $[[\mathbf{T}]]$ is defined according to (26), the boundary stress vector $\mathbf{T}\cdot\mathbf{n} = \{T_{ij}n_j\}_{i=1,2,3}$ and $(\mathbf{T}\cdot\mathbf{n})\cdot\mathbf{n} = T_{ij}n_jn_i$ with the usual summation convention. In the system of conditions at the crack, (33) stands for zero tangential stress, (34) describes continuity of the normal stress $(\mathbf{T}\cdot\mathbf{n})\cdot\mathbf{n}$, and the complementarity conditions (35) imply that the crack faces are pointwise either in contact: $[\![\mathbf{u}]\!]\cdot\mathbf{n} = 0$ and $(\mathbf{T}\cdot\mathbf{n})\cdot\mathbf{n} \leq 0$, or open: $[[\mathbf{u}]]\cdot\mathbf{n} > 0$ and $(\mathbf{T}\cdot\mathbf{n})\cdot\mathbf{n} = 0$.

Now we give a variational formulation to the nonlinear boundary value problem (28)–(35). Let $\mathbf{f}\in L^2(\Omega_{\rm c};\mathbb{R}^3)$, $\mathbf{g}\in L^2(\Gamma_{\rm N};\mathbb{R}^3)$, and let $\mathbb{R}^{3\times3}_{\rm sym}$ denote 3-by-3 symmetric tensors. Find functions $\mathbf{u}\in H^1(\Omega_{\rm c};\mathbb{R}^3)$ and $\mathbf{T}\in L^2(\Omega_{\rm c};\mathbb{R}^{3\times3}_{\rm sym})$ such that they satisfy the non-penetration condition (27), the Dirichlet boundary condition (31), the implicit response Eq. (30), and the following variational inequality:

$$\int_{\Omega_{\rm c}}\mathbf{T}\cdot\big(\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})\big)\,d\mathbf{x} \geq \int_{\Omega_{\rm c}}\mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\,d\mathbf{x} + \int_{\Gamma_{\rm N}}\mathbf{g}\cdot(\mathbf{v}-\mathbf{u})\,dS_{\mathbf{x}} \tag{36}$$

for all admissible test functions $\mathbf{v}\in H^1(\Omega_{\rm c};\mathbb{R}^3)$ such that $[[\mathbf{v}]]\cdot\mathbf{n}\geq 0$ on $\Gamma_{\rm c}$ and $\mathbf{v} = \mathbf{0}$ at $\Gamma_{\rm D}$, where the linearized strain tensors $\boldsymbol{\varepsilon}(\mathbf{u})$ and $\boldsymbol{\varepsilon}(\mathbf{v})$ are defined according to formula (29). In the left hand side of (36), $\mathbf{T}\cdot\boldsymbol{\varepsilon} = T_{ij}\varepsilon_{ij}$ (in the usual summation convention) implies the scalar product of tensors. The variational inequality is obtained in a standard way after multiplication of the equilibrium Eq. (28) with $\mathbf{v}-\mathbf{u}$, integration by part over $\Omega_{\rm c}$ with the help of boundary conditions (32) and (33)–(35), for details see (Khludnev and Kovtunenko, 2000, Section 1.4.4).

Based on the examples (23) and (24), we generalize the elastic coefficient functions in (30) to $\mu:\mathbb{R}\mapsto(a_\mu, b_\mu)$ with $0 < a_\mu \leq b_\mu < \infty$ and $K:\mathbb{R}\mapsto(a_K, b_K)$ with $0 < a_K \leq b_K < \infty$, satisfying the uniform growth conditions with $0 < c_\mu \leq C_\mu < \infty$ and $0 < c_K \leq C_K < \infty$:

$$c_\mu \leq \frac{d}{d\zeta}\left(\frac{\zeta}{\mu(\zeta)}\right) \leq C_\mu, \quad c_K \leq \frac{d}{d\zeta}\left(\frac{\zeta}{K(\zeta)}\right) \leq C_K \quad \text{for } \zeta\in\mathbb{R}. \tag{37}$$

The properties (37) (see (A.3)) are proven for (23) and (24) in Lemma 1 in Appendix A.

3. The main result

Theorem 1 (Well-posedness)**.** *Under conditions (37) on the variable material moduli* μ *and K, there exists the unique solution* $\mathbf{u} \in H^1(\Omega_c; \mathbb{R}^3)$, $\mathbf{T} \in L^2(\Omega_c; \mathbb{R}^{3\times3}_{sym})$ *which satisfies the nonlinear inhomogeneous problem with the crack, namely: the non-penetration condition (27), the Dirichlet boundary condition (31), the implicit response equation (30), the variational inequality (36).*

That is the following a-priori estimates hold:

$$\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})} \le \max\left\{\frac{1}{2a_\mu}, \frac{1}{9a_K}\right\}\sqrt{2}\|\mathbf{T}\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})}, \tag{38}$$

$$\|\mathbf{T}\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})} \le C_f \frac{\sqrt{2}\max\{\frac{1}{2a_\mu}, \frac{1}{9a_K}\}}{\min\{\frac{1}{2b_\mu}, \frac{1}{9b_K}\}}, \tag{39}$$

where constant $C_f > 0$ *is related to the given forces that meets the estimate*

$$\left|\int_{\Omega_c} \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} + \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\,dS_\mathbf{x}\right| \le C_f\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})} \tag{40}$$

for $\mathbf{v} \in H^1(\Omega_c; \mathbb{R}^3)$ *such that* $\mathbf{v} = \mathbf{0}$ *at* Γ_D.

Proof. To deal with the variational inequality (36), penalization avoiding the non-penetration condition (27) is applied. Namely, for small penalization parameter $\delta > 0$, we set the approximate problem: Find functions $\mathbf{u}^\delta \in H^1(\Omega_c; \mathbb{R}^3)$ and $\mathbf{T}^\delta \in L^2(\Omega_c; \mathbb{R}^{3\times3}_{sym})$ which satisfy the Dirichlet boundary condition:

$$\mathbf{u}^\delta = \mathbf{0} \quad \text{on } \Gamma_D; \tag{41}$$

the implicit response equation from (30):

$$\boldsymbol{\varepsilon}(\mathbf{u}^\delta) = \mathcal{F}(\mathbf{T}^\delta), \quad \mathcal{F}(\mathbf{T}^\delta) := \frac{(\mathbf{T}^\delta)^*}{2\mu(\|(\mathbf{T}^\delta)^*\|)} + \frac{\mathrm{tr}\mathbf{T}^\delta}{9K(\mathrm{tr}\mathbf{T}^\delta)}\mathbf{I} \tag{42}$$

with $\boldsymbol{\varepsilon}(\mathbf{u}^\delta)$ defined according to (29), and the variational equation:

$$\int_{\Omega_c} \mathbf{T}^\delta\cdot\boldsymbol{\varepsilon}(\mathbf{v})\,d\mathbf{x} + \frac{1}{\delta}\int_{\Gamma_c} \min\{0, [\![\mathbf{u}^\delta]\!]\cdot\mathbf{n}\}([\![\mathbf{v}]\!]\cdot\mathbf{n})\,dS_\mathbf{x} = \int_{\Omega_c} \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} + \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\,dS_\mathbf{x} \tag{43}$$

for all admissible test functions $\mathbf{v} \in H^1(\Omega_c; \mathbb{R}^3)$ such that $\mathbf{v} = \mathbf{0}$ at Γ_D.

It is well-known that the penalty function $\Pi^\delta(\zeta) := \frac{1}{\delta}\min\{0, \zeta\}$ in the left-hand side of Eq. (43) is positive semidefinite: $\Pi^\delta(\zeta)\zeta \ge 0$, bounded: $|\Pi^\delta(\zeta)| \le \frac{1}{\delta}|\zeta|$, monotone: $(\Pi^\delta(\zeta) - \Pi^\delta(\eta))(\zeta - \eta) \ge 0$, and continuous: $|\Pi^\delta(\zeta) - \Pi^\delta(\eta)| \le \frac{1}{\delta}|\zeta - \eta|$ for $\zeta, \eta \in \mathbb{R}$, see e.g. (Khludnev and Kovtunenko, 2000, Section 1.3.2). Under the conditions (37), the nonlinear function $\mathcal{F}$ in the right-hand side of Eq. (42) is coercive, bounded, strongly monotone, and continuous as proved in Lemma 2 in Appendix A. Therefore, the Browder–Minty theorem provides a unique solution to the problem (41)–(43) (see Itou, Kovtunenko, & Rajagopal, 2019, Theorem 2.1 for the problem (2.6) here with the identity operator $\mathcal{I}$ and powers $p = p' = 2$).

Next, according to the boundedness property (A.7) of the response function $\mathcal{F}$ in Lemma 2 in Appendix A, from (42) it follows that there exists the upper bound

$$\|\boldsymbol{\varepsilon}(\mathbf{u}^\delta)\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})} \le \max\left\{\frac{1}{2a_\mu}, \frac{1}{9a_K}\right\}\sqrt{2}\|\mathbf{T}^\delta\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})}. \tag{44}$$

Then we test the variational Eq. (43) with $\mathbf{v} = \mathbf{u}^\delta$. The coercivity property (A.6) of $\mathcal{F}$ from Lemma 2 is applied to the left-hand side of (43), while its right-hand side is estimated with the help of (40) by virtue of the Poincaré–Korn and trace inequalities. As a result we get

$$\min\left\{\frac{1}{2b_\mu}, \frac{1}{9b_K}\right\}\|\mathbf{T}^\delta\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})} + \frac{1}{\delta}\|\min\{0, [\![\mathbf{u}^\delta]\!]\cdot\mathbf{n}\}\|^2_{L^2(\Gamma_c;\mathbb{R})} \le C_f\|\boldsymbol{\varepsilon}(\mathbf{u}^\delta)\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})},$$

which together with (44) leads to the following two upper bounds:

$$\frac{1}{\delta}\|\min\{0, [\![\mathbf{u}^\delta]\!]\cdot\mathbf{n}\}\|^2_{L^2(\Gamma_c;\mathbb{R})} \le C_f\max\left\{\frac{1}{2a_\mu}, \frac{1}{9a_K}\right\}\sqrt{2}\|\mathbf{T}^\delta\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})}, \tag{45}$$

$$\|\mathbf{T}^\delta\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})} \le C_f \frac{\sqrt{2}\max\{\frac{1}{2a_\mu}, \frac{1}{9a_K}\}}{\min\{\frac{1}{2b_\mu}, \frac{1}{9b_K}\}}. \tag{46}$$

By the compactness principle, the uniform estimates (44), (46) imply that there exist a subsequence $\mathbf{u}^{\delta_k}$, $\mathbf{T}^{\delta_k}$ weakly convergent as $\delta_k \to 0$ to an accumulation point $\mathbf{u} \in H^1(\Omega_c; \mathbb{R}^3)$, $\mathbf{T} \in L^2(\Omega_c; \mathbb{R}^{3\times3}_{sym})$, and $\min\{0, [\![\mathbf{u}]\!]\cdot\mathbf{n}\} = 0$ that is $[[\mathbf{u}]]\cdot\mathbf{n} \ge 0$

on Γ_c due to (45). Therefore, in the standard manner we pass to the limit as $\delta_k \to 0$ the relations (41), (42), and (43) which we express for the test function $\mathbf{v}$ such that $[[\mathbf{v}]]\cdot\mathbf{n}\geq 0$ on Γ_c by the chain of inequalities

$$\int_{\Omega_c}\big(\mathbf{T}^\delta\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})+\mathbf{T}\cdot\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^\delta)\big)\,d\mathbf{x}\geq\int_{\Omega_c}\mathbf{T}^\delta\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}^\delta)\,d\mathbf{x}$$
$$\geq\int_{\Omega_c}\mathbf{T}^\delta\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}^\delta)\,d\mathbf{x}+\frac{1}{\delta}\int_{\Gamma_c}\min\big\{0,[\![\mathbf{u}^\delta]\!]\cdot\mathbf{n}\big\}([\![\mathbf{v}-\mathbf{u}^\delta]\!]\cdot\mathbf{n})\,dS_{\mathbf{x}}$$
$$=\int_{\Omega_c}\mathbf{f}\cdot(\mathbf{v}-\mathbf{u}^\delta)\,d\mathbf{x}+\int_{\Gamma_N}\mathbf{g}\cdot(\mathbf{v}-\mathbf{u}^\delta)\,dS_{\mathbf{x}}$$

based on the properties of monotony (A.9) and continuity (A.10) of $\mathcal{F}$ from Lemma 2 and non-negativeness of Π^δ, thus deriving the reference relations (27), (31), (30), (36).

On taking the limit of (44) and (46) we get the a-priori estimates (38) and (39), respectively. The uniqueness of the solution follows from the strong monotonous property (A.9) of the response function $\mathcal{F}$. The proof is completed. □

4. Discussion

We introduced a class of nonlinear elastic models in which the linearized strain is expressed as a function of the stress with the material moduli depending on the mean normal stress. Such models can describe the behavior of several metallic alloys as well as materials like concrete. Our model stems as an approximation from implicit models developed to describe the response of elastic bodies. We established the well-posedness of the problem of bodies that contain non-penetrating cracks.

Admissible functions for the material moduli μ and K are uniformly positive, bounded, and should satisfy the growth condition (37). The example (23) is based on the inverse tangent, the other admissible function is

$$K(\zeta)=K_0\Big(1+\varkappa+\frac{\alpha_1\zeta+\alpha_2}{\sqrt{1+(\alpha_1\zeta+\alpha_2)^2}}\Big). \tag{47}$$

Although our study is carried out for the fully three dimensional spatial formulation, it can be easily adapted to the two dimensional anti-plane and in-plane setting when the material moduli depend on the mean normal stress satisfying the conditions imposed on them in this study.

Acknowledgments

H. Itou is partially supported by Grant-in-Aid for Scientific Research (C)(No. 18K03380) and (B)(No. 17H02857) of Japan Society for the Promotion of Science.

V.A. Kovtunenko is supported by the Austrian Science Fund (FWF) project P26147-N26: PION and the Austrian Academy of Sciences (OeAW).

K.R. Rajagopal thanks the Office of Naval Research and the National Science Foundation for their support.

Appendix A. Auxiliary lemmata

Lemma 1. *The material moduli K and μ defined (23) and (24) as*

$$K(\zeta)=K_0\Big(1+\varkappa+\frac{2}{\pi}\arctan(\alpha_1\zeta+\alpha_2)\Big) \tag{A.1}$$

with positive parameters K_0, α_1, $\varkappa$, and

$$\mu(\zeta)=\mu_0\Big(1+\gamma+\frac{2}{\pi}\arctan(\beta_1\zeta+\beta_2)\Big) \tag{A.2}$$

with positive parameters μ_0, β_1, γ, for

$$\alpha_2>-\frac{\pi}{2}(1+\varkappa),\quad \beta_2>-\frac{\pi}{2}(1+\gamma) \tag{A.3}$$

satisfy the uniform conditions (37). That is, there exist constant $0<c_\mu\leq C_\mu<\infty$ and $0<c_K\leq C_K<\infty$ such that

$$c_\mu\leq\frac{d}{d\zeta}\Big(\frac{\zeta}{\mu(\zeta)}\Big)\leq C_\mu,\quad c_K\leq\frac{d}{d\zeta}\Big(\frac{\zeta}{K(\zeta)}\Big)\leq C_K\quad\text{for }\zeta\in\mathbb{R}. \tag{A.4}$$

Proof. It suffices to prove the property (A.4) for the function K in (A.1), since μ in (A.2) is of the same structure.

We easy calculate the derivative $\frac{d}{d\zeta}\big(\frac{\zeta}{K(\zeta)}\big)=\frac{K_0}{K(\zeta)^2}f(\alpha_1\zeta+\alpha_2)$, where the auxiliary f is defined with respect $\eta:=\alpha_1\zeta+\alpha_2$ as follows

$$f(\eta):=1+\varkappa+\frac{2}{\pi}\Big(\arctan(\eta)-\frac{\eta-\alpha_2}{1+\eta^2}\Big).$$

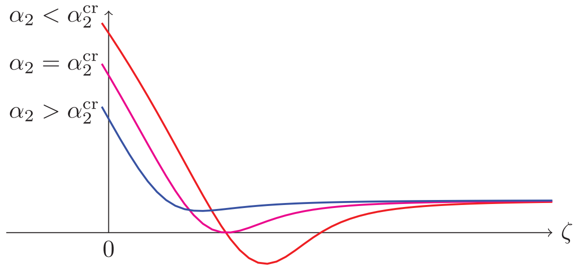

Fig. A.2. The curves $\frac{d}{d\zeta}\left(\frac{\zeta}{K(\zeta)}\right)$ in (A.4) for the example function $K(\zeta)$ from (A.1) with $K_0 = 1, \varkappa = 0.1, \alpha_1 = 1$ and $\alpha_2 \in \{-2.5, \alpha_2^{\rm cr}, -1\}$, where $\alpha_2^{\rm cr} = -\frac{\pi}{2}(1+\varkappa) \approx -1.728$.

The function $\frac{d}{d\zeta}\left(\frac{\zeta}{K(\zeta)}\right)$ is continuous, and both its limits $\frac{d}{d\zeta}\left(\frac{\zeta}{K(\zeta)}\right) = \frac{1}{K(\zeta)} - \frac{\zeta K'(\zeta)}{K(\zeta)^2} \to \frac{1}{K_0\varkappa}$ as $\zeta \to -\infty$ and $\frac{d}{d\zeta}\left(\frac{\zeta}{K(\zeta)}\right) \to \frac{1}{K_0(2+\varkappa)}$ as $\zeta \to \infty$ are positive. For illustration, three curves of $\frac{d}{d\zeta}\left(\frac{\zeta}{K(\zeta)}\right)$ from (A.4) for the example function $K(\zeta)$ from (A.1) are sketched in Fig. A.2, when varying α_2 with respect to the critical value $\alpha_2^{\rm cr} = -\frac{\pi}{2}(1+\varkappa)$ due to (A.3).

To estimate minimum of this function we study the following cases:

- if $\eta = 0$, then $f(0) = 1 + \varkappa + \frac{2}{\pi}\alpha_2$;
- if $\eta > 0$, since $\arctan(\eta) > \frac{\eta}{\sqrt{1+\eta^2}}$, then

$$f(\eta) > \frac{(1+\varkappa)(1+\eta^2)^{3/2} + \frac{2}{\pi}\eta(\sqrt{1+\eta^2}-1) + \frac{2}{\pi}\alpha_2}{(1+\eta^2)^{3/2}} > \frac{1+\varkappa+\frac{2}{\pi}\alpha_2}{(1+\eta^2)^{3/2}};$$

- if $-\frac{\pi}{2}(1+\varkappa) < \eta < 0$, since $\arctan(\eta) > \eta$, then

$$f(\eta) > \frac{1+\varkappa+(1+\varkappa+\frac{2}{\pi}\eta)\eta^2 + \frac{2}{\pi}\alpha_2}{1+\eta^2} > \frac{1+\varkappa+\frac{2}{\pi}\alpha_2}{1+\eta^2};$$

- if $\eta \le -\frac{\pi}{2}(1+\varkappa)$, since $\arctan(\eta) > -\frac{\pi}{2}$, then

$$f(\eta) > \varkappa + \frac{-\frac{2}{\pi}\eta + \frac{2}{\pi}\alpha_2}{1+\eta^2} \ge \varkappa + \frac{1+\varkappa+\frac{2}{\pi}\alpha_2}{1+\eta^2}.$$

In all cases, $f(\eta) > 0$ for $\alpha_2 > -\frac{\pi}{2}(1+\varkappa)$. On the contrary, for $\alpha_2 \le -\frac{\pi}{2}(1+\varkappa)$ we have $f(0) = 1+\varkappa+\frac{2}{\pi}\alpha_2 \le 0$, that finishes the proof. □

Lemma 2. *For material moduli $\mu : \mathbb{R} \mapsto (a_\mu, b_\mu)$ with $0 < a_\mu \le b_\mu < \infty$ and $K : \mathbb{R} \mapsto (a_K, b_K)$ with $0 < a_K \le b_K < \infty$, the response function $\mathcal{F} : \mathbb{R}^{3\times3}_{\rm sym} \mapsto \mathbb{R}^{3\times3}_{\rm sym}$ defined from (25) as*

$$\mathcal{F}(\mathbf{T}) := \frac{\mathbf{T}^*}{2\mu(\|\mathbf{T}^*\|)} + \frac{\mathrm{tr}\mathbf{T}}{9K(\mathrm{tr}\mathbf{T})}\mathbf{I} \tag{A.5}$$

is coercive:

$$\mathcal{F}(\mathbf{T}) \cdot \mathbf{T} \ge \min\left\{\frac{1}{2b_\mu}, \frac{1}{9b_K}\right\}\|\mathbf{T}\|^2 \tag{A.6}$$

and bounded:

$$\|\mathcal{F}(\mathbf{T})\| \le \max\left\{\frac{1}{2a_\mu}, \frac{1}{9a_K}\right\}\sqrt{2}\|\mathbf{T}\|. \tag{A.7}$$

In particular, for the functions K and μ from (23) and (24) we have the bounds:

$$a_\mu = \mu_0\gamma, \quad b_\mu = \mu_0(2+\gamma), \quad a_K = K_0\varkappa, \quad b_K = K_0(2+\varkappa). \tag{A.8}$$

Under the conditions (37) (equivalently (A.4)) $\mathcal{F}$ is strongly monotone:

$$\big(\mathcal{F}(\mathbf{T}) - \mathcal{F}(\overline{\mathbf{T}})\big) \cdot (\mathbf{T} - \overline{\mathbf{T}}) \ge \min\left\{\frac{1}{2}c_\mu, \frac{1}{9}c_K\right\}\|\mathbf{T} - \overline{\mathbf{T}}\|^2 \tag{A.9}$$

and continuous:

$$\|\mathcal{F}(\mathbf{T}) - \mathcal{F}(\overline{\mathbf{T}})\| \le \max\left\{\frac{1}{2}C_\mu, \frac{1}{9}C_K\right\}\sqrt{2}\|\mathbf{T} - \overline{\mathbf{T}}\|. \tag{A.10}$$

Proof. The properties (A.6) and (A.7) follow in a straightforward manner from the uniform bounds $a_K < K(\zeta) < b_K$ and $a_\mu < \mu(\zeta) < b_\mu$ for all $\zeta \in \mathbb{R}$.

To prove (A.9) and (A.10) we express in the integral form the difference

$$\mathcal{F}(\mathbf{T}) - \mathcal{F}(\overline{\mathbf{T}}) = \int_0^1 \frac{d}{d\zeta}\mathcal{F}(\boldsymbol{\xi})\, d\zeta, \quad \boldsymbol{\xi} := \zeta\mathbf{T} + (1-\zeta)\overline{\mathbf{T}}, \tag{A.11}$$

then apply $\frac{d}{d\zeta}\boldsymbol{\xi} = \mathbf{T} - \overline{\mathbf{T}}$ and decomposition $\boldsymbol{\xi} = \boldsymbol{\xi}^* + \frac{\mathrm{tr}\boldsymbol{\xi}}{3}\mathbf{I}$ from (13) to calculate the derivative of the nonlinear function (A.5) as follows

$$\frac{d}{d\zeta}\mathcal{F}(\boldsymbol{\xi}) = \frac{1}{2}\frac{d}{d\|\boldsymbol{\xi}^*\|}\left(\frac{\|\boldsymbol{\xi}^*\|}{\mu(\|\boldsymbol{\xi}^*\|)}\right)\frac{\boldsymbol{\xi}^*}{\|\boldsymbol{\xi}^*\|}(\mathbf{T}^* - \overline{\mathbf{T}}^*) + \frac{1}{9}\frac{d}{d(\mathrm{tr}\boldsymbol{\xi})}\left(\frac{\mathrm{tr}\boldsymbol{\xi}}{K(\mathrm{tr}\boldsymbol{\xi})}\right)\mathrm{tr}(\mathbf{T} - \overline{\mathbf{T}})\mathbf{I}. \tag{A.12}$$

On the one side, estimating from above the norm of (A.12) with the help of the upper bounds in (A.4), using $\|\mathbf{T}^* - \overline{\mathbf{T}}^*\| + |\mathrm{tr}(\mathbf{T} - \overline{\mathbf{T}})| \le \sqrt{2}\|\mathbf{T} - \overline{\mathbf{T}}\|$, and inserting the result in (A.11) we get the continuity (A.10). On the other side, multiplying (A.12) element-wisely with $\mathbf{T} - \overline{\mathbf{T}}$ and inserting it in (A.11), with the help of the lower bounds in (A.4) we derive directly the monotony inequality (A.9). The proof is completed. □

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