EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 7, Number 3 (2016), 41 – 52

A SHAPE-TOPOLOGICAL CONTROL OF VARIATIONAL INEQUALITIES

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Communicated by P.D. Lamberti

Key words: shape-topological control, state-constrained optimization, variational inequality, singular perturbation, inhomogeneity, shape-topological derivative.

AMS Mathematics Subject Classification: 35B25, 49J40, 49Q10, 74G70.

Abstract. A shape-topological control of singularly perturbed variational inequalities is considered in the abstract framework for state-constrained optimization problems. Aiming at asymptotic analysis, singular perturbation theory is applied to the geometry-dependent objective function and results in a shape-topological derivative. This concept is illustrated analytically in a one-dimensional example problem which is controlled by an inhomogeneity posed in a domain with moving boundary.

1 Introduction

The paper aims at a shape-topological control of geometry-dependent variational inequalities. We consider a class of objective functions $J : \mathfrak{G} \mapsto \mathbb{R}$ which act on two geometric objects Γ and ω. In particular, we look how a perturbation of the topology of ω will influence the shape derivative of $J(\Gamma, \omega)$ with respect to Γ. Our example of a shape-topological control problem will refer to ω as an inhomogeneity in the given domain, and to Γ as a moving boundary of this domain.

From a mathematical viewpoint, the principal difficulty is that Γ and ω enter the objective J through a state problem which is typically expressed by partial differential equations (PDEs). Moreover, we generalize the state problem to a variational inequality subject to unilateral constraints on Γ. Therefore, to get explicit formulae, we rely on asymptotic modeling of small ω. We obtain a shape-topological derivative of the objective function, and we prove its semi-analytic expression with the help of Green type functions.

For the classical methods of the shape optimization we refer to [1, 5, 22, 24], for the topology optimization to [2, 3, 6, 16], and to [4, 7, 20] for the asymptotic theory. Our motivation comes from the crack problems in fracture mechanics, see e.g. [21], aimed either to arrest or amplify a moving crack. The control is realized by posing a trial inhomogeneity in a test medium. By this, we assume nonlinear crack models subject to contact conditions resulting in variational inequalities, see [9, 14]. The asymptotic methods of regular perturbations suitable for nonlinear crack problems are discussed in [8, 12, 15], and singular perturbations of cracks in [10, 11]. In [17] we investigated a nonlinear crack with respect to the shape-topological control by inhomogeneity in two dimensions.

In the present work, in Section 2 we state a mathematical concept of a shape-topological control for singularly perturbed variational inequalities, and we illustrate it with a one-dimensional example problem in Section 3.

2 Concept of a shape-topological control

Our construction can be outlined in the context of shape-topological differentiability, see [13, 18, 19], as follows.

For a Hilbert space H and its dual space $H^\star$, we deal with variational inequalities of the following type:

$$\text{find } u^0 - g \in K \text{ such that } \quad \langle Au^0, v - u^0 \rangle \geq 0 \quad \text{for all } v - g \in K, \tag{2.1}$$

where $g \in H$ is given, the admissible set $K \subset H$ is convex and closed, and $A : H \mapsto H^\star$ is a linear pseudo-monotone operator such that the assumption

$$u^\varepsilon \rightharpoonup u^0 \text{ weakly in } H \text{ as } \varepsilon \searrow 0^+ \text{ and } \liminf_{\varepsilon \searrow 0^+} \langle Au^\varepsilon, u^0 - u^\varepsilon \rangle \geq 0$$

implies that the following condition holds

$$\langle Au^0, v - u^0 \rangle \geq \limsup_{\varepsilon \searrow 0^+} \langle Au^\varepsilon, v - u^\varepsilon \rangle \text{ for all } v - g \in K.$$

For the theory of variational inequalities (2.1) with pseudo-monotone operators A and its solvability see [23]. In particular, if A is strongly monotone operator such that

$$\tfrac{\langle Av, v \rangle}{\|v\|^2} \geq \alpha > 0 \quad \text{for } v \in H,\ v \neq 0 \tag{2.2}$$

then the Lions–Stampacchia theorem provides the unique solution to (2.1).

We consider a singularly perturbed variational inequality: find $u^\varepsilon - g \in K$ such that

$$\langle A_\varepsilon u^\varepsilon, v - u^\varepsilon \rangle \geq 0 \quad \text{for all } v - g \in K, \tag{2.3}$$

where the perturbation $A_\varepsilon = A + \varepsilon F_\varepsilon$ of the operator A of (2.1) with a linear bounded operator $F_\varepsilon : H \mapsto H^\star$ is such that $\varepsilon \|F_\varepsilon\| = \mathrm{O}(\varepsilon)$ and A_ε is a strongly monotone operator uniformly in ε which means that

$$\tfrac{\langle A_\varepsilon v, v \rangle}{\|v\|^2} \geq \alpha > 0 \quad \text{for } v \in H,\ v \neq 0 \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{2.4}$$

Our consequent consideration aims at shape-topological control by means of the state-constrained optimization: find the geometry variables (ω, Γ) from a feasible set $\mathfrak{G}$ such that

$$\operatorname*{minimum}_{(\omega,\Gamma) \in \mathfrak{G}} J(u^{(\omega,\Gamma)}) \quad \text{subject to } \Pi(u^{(\omega,\Gamma)}) = \min_{v - g \in K} \Pi(v). \tag{2.5}$$

In (2.5) the functional $\Pi : H \mapsto \mathbb{R}$ such that

$$\Pi(v) := \langle \tfrac{1}{2} A_\varepsilon v, v \rangle$$

associates the strain energy (SE) of the state problem. Since Π is coercive by (2.4), then variational inequality (2.3) implies the first order necessary and sufficient optimality condition for the minimization of Π over $v - g \in K$. The parameter $\varepsilon \in \mathbb{R}_+$ entering (2.3) serves for variation of the geometry, we will specify this setting in examples below.

The main difficulty of the state-constrained optimization is that the geometry variables enter (2.5) in a fully implicit way. Therefore, the problem of finding its optimality condition is open. Further we rely on asymptotic models as $\varepsilon \searrow 0^+$ that needs expansion of the solution u^ε of state problem (2.3) stated below.

Theorem 2.1. *For the solutions u^0 and u^ε of variational inequalities* (2.1) *and* (2.3)*, the following properties of a corrector $\tilde{q}^\varepsilon \in H$*

$$u^0 + \epsilon\tilde{q}^\varepsilon - g \in K, \tag{2.6}$$

$$u^\varepsilon - \varepsilon\tilde{q}^\varepsilon - g \in K, \tag{2.7}$$

$$\langle A_\varepsilon\tilde{q}^\varepsilon + F_\varepsilon u^0 - R_\varepsilon, v\rangle = 0 \quad \text{for all } v \in H, \tag{2.8}$$

with a residual $R_\varepsilon \in H^\star$ such that

$$\varepsilon\|R_\varepsilon\| = \mathrm{O}(f(\varepsilon)), \tag{2.9}$$

imply the asymptotic representation in H as $\varepsilon \searrow 0^+$ of the form

$$\|u^\varepsilon - u^0 - \varepsilon\tilde{q}^\varepsilon\| = \mathrm{O}(f(\varepsilon)). \tag{2.10}$$

Proof. Indeed, plugging the test functions $v = u^\varepsilon - \varepsilon\tilde{q}^\varepsilon$ in (2.1) due to (2.7) and $v = u^0 + \varepsilon\tilde{q}^\varepsilon$ in (2.3) due to (2.6), after summation of the inequality

$$\langle A_\varepsilon u^\varepsilon - (A_\varepsilon - \varepsilon F_\varepsilon)u^0, u^\varepsilon - u^0 - \varepsilon\tilde{q}^\varepsilon\rangle \le 0$$

and equality (2.8) with $v = u^\varepsilon - u^0 - \varepsilon\tilde{q}^\varepsilon$ multiplied by $-\varepsilon$, that is

$$\langle -\varepsilon A_\varepsilon\tilde{q}^\varepsilon - \varepsilon F_\varepsilon u^0 + \varepsilon R_\varepsilon, u^\varepsilon - u^0 - \varepsilon\tilde{q}^\varepsilon\rangle = 0,$$

we get

$$\langle A_\varepsilon(u^\varepsilon - u^0 - \varepsilon\tilde{q}^\varepsilon) + \varepsilon R_\varepsilon, u^\varepsilon - u^0 - \varepsilon\tilde{q}^\varepsilon\rangle \le 0.$$

Applying here the Cauchy–Schwarz inequality together with (2.4) and (2.9) implies (2.10). □

We emphasize that $\varepsilon\tilde{q}^\varepsilon$ satisfying (2.10) is not unique but defined up to $\mathrm{o}(f(\varepsilon))$-terms. A typical example of the corrector $\tilde{q}^\varepsilon$ is $\tilde{q}\big(\frac{x}{\varepsilon}\big)$ implying a boundary layer in homogenization theory. Moreover, the asymptotic behavior $f(\varepsilon)$ of the residual in (2.10) maybe different. In the subsequent example $f(\varepsilon) = \varepsilon^{3/2}$, see Theorem 3.1.

In the following section we illustrate our construction analytically for a one-dimensional problem which obeys exact solution. In order to find a representative $\tilde{q}^\varepsilon$, in Section 3.1 we will realize sufficient conditions (2.6)–(2.9). As an alternative to the uniform asymptotic expansion (2.10), in Section 3.2, developing variational technique based on Green functions, we obtain a local asymptotic expansion in the near-field, and this expansion is unique.

3 Example problem in an inhomogeneous domain

For two variable parameters $\varepsilon, t \in \mathbb{R}_+$, we start with the description of geometry.

We define a disconnected set joining two segments $x \in (0,\varepsilon) \cup (\varepsilon, r+t)$ such that $0 < r_0 < r < r_1$, $r_0 - r < t < r_1 - r$, and $0 < \varepsilon < \varepsilon_0 < r_0$. One geometric parameter ε associates the size of inhomogeneity $\omega_\varepsilon = (0,\varepsilon)$ in the domain, and the other geometric parameter t defines the position of the moving boundary $\Gamma_t = \{x : x = r+t\}$.

The inhomogeneity is represented with the help of the characteristic function such that $\chi^\delta_{(0,\varepsilon)}(x) = \delta$ for $x < \varepsilon$, otherwise $\chi^\delta_{(0,\varepsilon)}(x) = 1$ for $x > \varepsilon$, where $\delta \in \mathbb{R}_+$ stands for a given stiffness parameter. Its two limit cases correspond to the hole as $\delta \searrow 0^+$ and to the rigid inclusion as $\delta \nearrow +\infty$.

For a fixed $g \in \mathbb{R}$, the space of functions is given by

$$H_t := \{u \in H^1(0, r+t) : \quad u(0) = 0\},$$

the admissible set is represented by the inequality constraint

$$K_t := \{u \in H_t : \quad u(r+t) + g \geq 0\},$$

hence $v - g \in K_t$ implies $v(r+t) \geq 0$ and $v(0) = g$, and variational inequality (2.3) takes the specific form:

$$\begin{aligned}&\text{find } u^{(\varepsilon,t)} - g \in K_t \text{ such that}\\ &\int_0^{r+t} \chi^\delta_{(0,\varepsilon)} (u^{(\varepsilon,t)})'(v - u^{(\varepsilon,t)})'\,dx \geq 0 \quad \text{for all } v - g \in K_t.\end{aligned} \tag{3.1}$$

Here and in what follows we mark the dependence of the solution on these two geometry variables ε and t.

Variational inequality (3.1) implies the boundary value problem:

$$-(u^{(\varepsilon,t)})''(x) = 0 \quad \text{for } x \in (0,\varepsilon) \cup (\varepsilon, r+t), \tag{3.2}$$

$$u^{(\varepsilon,t)}(0) = g, \tag{3.3}$$

$$u^{(\varepsilon,t)}(\varepsilon^+) - u^{(\varepsilon,t)}(\varepsilon^-) = 0,\ (u^{(\varepsilon,t)})'(\varepsilon^+) - \delta \cdot (u^{(\varepsilon,t)})'(\varepsilon^-) = 0, \tag{3.4}$$

$$u^{(\varepsilon,t)}(r+t) \geq 0, \quad (u^{(\varepsilon,t)})'(r+t) \geq 0, \tag{3.5}$$

$$(u^{(\varepsilon,t)})'(r+t) \cdot u^{(\varepsilon,t)}(r+t) = 0, \tag{3.6}$$

where $u^{(\varepsilon,t)}(\varepsilon^-)$ and $u^{(\varepsilon,t)}(\varepsilon^+)$ are the limit values from below and above, respectively. It is derived from (3.1) in the standard way by applying integration by parts for all $v - g \in K_t$ that

$$\begin{aligned}&-\int_0^{r+t} \chi^\delta_{(0,\varepsilon)} (u^{(\varepsilon,t)})''(v - u^{(\varepsilon,t)})\,dx + (u^{(\varepsilon,t)})'(r+t)\big(v(r+t) - u^{(\varepsilon,t)}(r+t)\big)\\ &- \big((u^{(\varepsilon,t)})'(\varepsilon^+) - \delta\,(u^{(\varepsilon,t)})'(\varepsilon^-)\big) \cdot \big(v(\varepsilon) - u^{(\varepsilon,t)}(\varepsilon)\big) \geq 0.\end{aligned}$$

We construct the solution to (3.2)–(3.6) explicitly. Indeed, for an arbitrary $c_{(\varepsilon,t)} \in \mathbb{R}$ relations (3.2)–(3.4) can be solved by

$$\begin{cases} u^{(\varepsilon,t)}(x) = g + \frac{c_{(\varepsilon,t)}}{\delta} x, & (u^{(\varepsilon,t)})' = \frac{c_{(\varepsilon,t)}}{\delta}\quad, \ x \in (0,\varepsilon) \\ u^{(\varepsilon,t)}(x) = g + c_{(\varepsilon,t)}\big(x + \varepsilon\frac{1-\delta}{\delta}\big), & (u^{(\varepsilon,t)})' = c_{(\varepsilon,t)}\quad, \ x \in (\varepsilon, r+t) \end{cases}$$

implying the piecewise-linear continuous function

$$u^{(\varepsilon,t)}(x) = g + c_{(\varepsilon,t)}\bigl(x + \tfrac{1-\delta}{\delta}\min\{\varepsilon, x\}\bigr). \tag{3.7}$$

With (3.7) complementarity condition (3.6) takes the form

$$c_{(\varepsilon,t)} \cdot \bigl(g + c_{(\varepsilon,t)}(r + t + \tfrac{1-\delta}{\delta}\varepsilon)\bigr) = 0.$$

Hence, due to (3.5), the nonnegative constant $c_{(\varepsilon,t)}$ can be found uniquely:

$$c_{(\varepsilon,t)} = \max\bigl\{0, -g\bigl(r + t + \tfrac{1-\delta}{\delta}\varepsilon\bigr)^{-1}\bigr\}. \tag{3.8}$$

As $\varepsilon \searrow 0^+$, from (3.7) and (3.8) we have the reference state

$$u^{(0,t)}(x) = g + c_{(0,t)}x, \tag{3.9}$$

$$c_{(0,t)} = \max\bigl\{0, -g(r+t)^{-1}\bigr\}, \tag{3.10}$$

which solves the reference variational inequality corresponding to (2.1):

$$\begin{aligned}&\text{find } u^{(0,t)} - g \in K_t \text{ such that}\\ &\int_0^{r+t} (u^{(0,t)})'(v - u^{(0,t)})'\,dx \geq 0 \quad \text{for all } v - g \in K_t.\end{aligned} \tag{3.11}$$

Alluding to the asymptotic expansion in Theorem 3.1 below, we need to consider a layer near the interface point $x = \varepsilon$. It is obtained after mapping $(0,\varepsilon) \mapsto (0,1)$, $x \mapsto \varepsilon y$ by solving the auxiliary transmission problem:

$$\begin{aligned}&\text{find } w \in H^1(\mathbb{R}_+) \text{ such that}\\ &\int_0^{\infty} \chi^{\delta}_{(0,1)} w'(y)v'(y)\,dy = (1-\delta)v(1) \quad \text{for all } v \in H^1(\mathbb{R}_+).\end{aligned} \tag{3.12}$$

Using integration by parts, variational equation (3.12) implies the boundary value problem:

$$\begin{gathered}-w''(y) = 0 \quad \text{for } y \in (0,1) \cup (1,\infty),\\ w(x) \to 0 \quad \text{as } x \nearrow \infty,\\ w(1^+) - w(1^-) = 0, \quad w'(1^+) - \delta \cdot w'(1^-) = -(1-\delta),\end{gathered}$$

where $w(1^-)$ and $w(1^+)$ are the limiting values from below and above, respectively. The unique solution of this problem is given by the piecewise linear continuous function

$$w(y) = \tfrac{1-\delta}{\delta}\min\{0, y-1\}. \tag{3.13}$$

After stretching the coordinates $y = \frac{x}{\varepsilon}$ in (3.13), we get the boundary layer

$$\varepsilon w(\tfrac{x}{\varepsilon}) = \tfrac{1-\delta}{\delta}\min\{0, x-\varepsilon\}, \quad \|\varepsilon w(\tfrac{x}{\varepsilon})\| = \mathrm{O}(\varepsilon^{1/2}) \quad \text{in } H^1(\mathbb{R}_+), \tag{3.14}$$

where the square root asymptotic order is due to the seminorm estimate

$$\sqrt{\int_0^{\infty} \bigl(\varepsilon w(\tfrac{x}{\varepsilon})'\bigr)^2\,dx} = \sqrt{\int_0^{\varepsilon} \bigl(\tfrac{1-\delta}{\delta}\bigr)^2\,dx} = \mathrm{O}(\sqrt{\varepsilon}).$$

In this case we justify asymptotic formula (2.10) as follows.

Theorem 3.1. *The solutions $u^{(\varepsilon,t)}$ and $u^{(0,t)}$ of variational inequalities* (3.1) *and* (3.11) *admit the following residual estimate as $\varepsilon \searrow 0^+$:*

$$u^{(\varepsilon,t)} = u^{(0,t)} + \varepsilon\tilde{q}^{(\varepsilon,t)} + \mathrm{O}(\varepsilon^{3/2}) \quad \text{in } H^1(0,r+t) \tag{3.15}$$

with the principal asymptotic term defined in H_t by

$$\varepsilon\tilde{q}^{(\varepsilon,t)}(x) := (u^{(0,t)})'(0)\cdot\big[\varepsilon w(\tfrac{x}{\varepsilon}) + \varepsilon\tfrac{1-\delta}{\delta}(1-\tfrac{x}{r+t})\big]. \tag{3.16}$$

Proof. Indeed, for sufficiently small ε we have $(r+t)(r+t+\frac{1-\delta}{\delta}\varepsilon)^{-1} > 0$, hence from (3.8) it follows that

$$c_{(\varepsilon,t)} = \big(1 + \tfrac{1-\delta}{\delta(r+t)}\varepsilon\big)^{-1}\cdot\max\big\{0, -g(r+t)^{-1}\big\},$$

and together with (3.10) this results in the expansion

$$c_{(\varepsilon,t)} = c_{(0,t)}\big(1 - \tfrac{1-\delta}{\delta(r+t)}\varepsilon + \mathrm{O}(\varepsilon)^2\big). \tag{3.17}$$

Substituting (3.17), (3.14), and (3.9) in (3.7) we get

$$u^{(\varepsilon,t)}(x) = u^{(0,t)}(x) - c_{(0,t)}x + c_{(0,t)}\big(1 - \tfrac{1-\delta}{\delta(r+t)}\varepsilon\big)\big[x + \tfrac{1-\delta}{\delta}\varepsilon + \varepsilon w(\tfrac{x}{\varepsilon})\big] + \mathrm{O}(\varepsilon^2)$$

and derive iteratively the following uniform estimates:

$$u^{(\varepsilon,t)}(x) = u^{(0,t)}(x) + \mathrm{O}(\varepsilon^{1/2}),$$
$$u^{(\varepsilon,t)}(x) = u^{(0,t)}(x) + (u^{(0,t)})'(0)\cdot\varepsilon w(\tfrac{x}{\varepsilon}) + \mathrm{O}(\varepsilon),$$
$$u^{(\varepsilon,t)}(x) = u^{(0,t)}(x) + (u^{(0,t)})'(0)\cdot\big[\varepsilon w(\tfrac{x}{\varepsilon}) + \varepsilon\tfrac{1-\delta}{\delta}(1-\tfrac{x}{r+t})\big] + \mathrm{O}(\varepsilon^{3/2}),$$

where we have used $c_{(0,t)} = (u^{(0,t)})'(0)$. The latter equality enforces (3.15) with notation (3.16), thus completing the proof. □

We remark that $\varepsilon\tilde{q}^{(\varepsilon,t)}(x)$ in Theorem 3.1 satisfies relations (2.6)–(2.9) in Theorem 2.1 with $f(\varepsilon) = \varepsilon^{3/2}$, which can be checked straightforwardly.

3.1 Uniform asymptotic expansion in the problem

We discuss examples for various objectives $J(u^{(\varepsilon,t)})$ subject to the optimal state $u^{(\varepsilon,t)}$. State-constrained optimization problem (2.5) takes the specific form:

$$\underset{(\varepsilon,t)\in(0,\varepsilon_0)\times(r_0-r,r_1-r)}{\text{minimum}} J(u^{(\varepsilon,t)}) \quad \text{subject to } \Pi(u^{(\varepsilon,t)}) = \min_{v-g\in K_t} \Pi(v), \tag{3.1}$$

and the strain energy (SE) functional $\Pi : H_t \mapsto \mathbb{R}$ is

$$\Pi(v) := \tfrac{1}{2}\int_0^{r+t} \chi^\delta_{(0,\varepsilon)}(v'(x))^2\,dx. \tag{3.2}$$

Variational inequality (3.1) implies the first order optimality condition for the constrained minimization of Π over $v - g \in K_t$.

It is important to comment that, for a fixed $\varepsilon \in (0, \varepsilon_0)$, variations of the parameter $t \in (r_0 - r, r_1 - r)$ describe regular perturbations of the moving boundary of the domain $(0, \varepsilon) \cup (\varepsilon, r+t)$, thus shape variation. In contrast, the limiting procedure $\varepsilon \searrow 0^+$ implies diminishing of the inhomogeneity $\omega_\varepsilon = (0, \varepsilon)$, and, hence, the topology change from the disconnected set to the 1-connected set $(0, r+t)$.

First, we control the optimal value function $J_{\rm SE} = \Pi$ of strain energy (3.2) with respect to the topology change as $\varepsilon \searrow 0^+$. Relying on small ε, we substitute the optimal state $u^{(\varepsilon,t)}$ with its asymptotic model (3.15) and (3.16), thus calculating the approximation of the optimal value function

$$\begin{aligned}
\Pi(u^{(\varepsilon,t)}) &= \Pi\big(u^{(0,t)} + c_{(0,t)}\big[\varepsilon w(\tfrac{x}{\varepsilon}) + \varepsilon\tfrac{1-\delta}{\delta}(1 - \tfrac{x}{r+t})\big] + \mathrm{O}(\varepsilon^{3/2})\big)\\
&= \tfrac{1}{2}\int_0^{r+t} \chi^\delta_{(0,\varepsilon)}\big((u^{(0,t)})' + c_{(0,t)}\big[\varepsilon w'(\tfrac{x}{\varepsilon}) - \tfrac{\varepsilon(1-\delta)}{\delta(r+t)}\big]\big)^2\,dx + \mathrm{O}(\varepsilon^{3/2})\\
&= \tfrac{c^2_{(0,t)}}{2}\Big[\int_0^{\varepsilon} \delta\big(1 - \tfrac{2\varepsilon(1-\delta)}{\delta(r+t)} + \tfrac{1-\delta^2}{\delta^2}\big)\,dx + \int_\varepsilon^{r+t}\big(1 - \tfrac{2\varepsilon(1-\delta)}{\delta(r+t)}\big)\,dx\Big] + \mathrm{o}(\varepsilon)\\
&= \tfrac{c^2_{(0,t)}}{2}\big(r + t - \tfrac{\varepsilon(1-\delta)}{\delta}\big) + \mathrm{o}(\varepsilon),\\
\Pi(u^{(0,t)}) &= \tfrac{c^2_{(0,t)}}{2}(r+t),
\end{aligned} \tag{3.3}$$

due to (3.9), (3.14), and (3.2). From (3.3) it follows that the function $(0, \varepsilon_0) \mapsto \mathbb{R}$, $\varepsilon \mapsto \Pi(u^{(\varepsilon,t)})$ is differentiable at $\varepsilon = 0$ with the topological derivative

$$\tfrac{d}{d\varepsilon}\Pi(u^{(\varepsilon,t)})|_{\varepsilon=0} = -c^2_{(0,t)}\tfrac{(1-\delta)}{2\delta} = -\Pi(u^{(0,t)})\tfrac{1-\delta}{\delta(r+t)}. \tag{3.4}$$

Secondly, we control the objective function $J_{\rm SERR} = -\frac{d}{dt}\Pi$ of the strain energy release rate, which implies shape variation and associates a Griffith's functional used in fracture mechanics.

To calculate $-\frac{d}{dt}\Pi$ from (3.2), we apply the constitutive formula proven in [6]. Indeed, let a cut-off function η be such that $\eta(x) = 0$ as $x < \varepsilon$ and $\eta(x) = 1$ as $x > \varepsilon + \beta$, with some β such that $\varepsilon + \beta < r_0$. For small $s \in (r_0 - r - t, r_1 - r - t)$, the translation $\Phi_s : (0, r+t) \mapsto (0, r+t+s)$, $z = x + s\eta(x)$ yields the representation of $\Pi(u^{(\varepsilon,t+s)})$ as

$$\begin{aligned}
&\tfrac{1}{2}\int_0^{r+t+s}\chi^\delta_{(0,\varepsilon)}\big((u^{(\varepsilon,t+s)})'_z\big)^2\,dz = \tfrac{1}{2}\int_0^{r+t}\chi^\delta_{(0,\varepsilon)}\big(\tfrac{(u^{(\varepsilon,t+s)}\circ\Phi_s)'_x}{1+s\eta'}\big)^2(1+s\eta')\,dx\\
&= \Pi(u^{(\varepsilon,t+s)}\circ\Phi_s) - \tfrac{s}{2}\int_0^{r+t}\chi^\delta_{(0,\varepsilon)}\big((u^{(\varepsilon,t+s)}\circ\Phi_s)'\big)^2\eta'\,dx + \mathrm{o}(s).
\end{aligned}$$

Since $u^{(\varepsilon,t+s)} \circ \Phi_s - g \in K_t$, we infer $u^{(\varepsilon,t+s)} \circ \Phi_s \to u^{(\varepsilon,t)}$ strongly in H_t as $s \to 0$, and conclude, see [6] for details, with the asymptotic expansion

$$\Pi(u^{(\varepsilon,t+s)}) = \Pi(u^{(\varepsilon,t)}) - \tfrac{s}{2}\int_0^{r+t}\chi^\delta_{(0,\varepsilon)}\big((u^{(\varepsilon,t)})'\big)^2\eta'\,dx + \mathrm{o}(s). \tag{3.5}$$

From (3.5) the explicit formula of the shape derivative follows directly:

$$J_{\rm SERR}(u^{(\varepsilon,t)}) := -\tfrac{d}{dt}\Pi(u^{(\varepsilon,t)}) = \tfrac{1}{2}\int_0^{r+t}\chi^\delta_{(0,\varepsilon)}\big((u^{(\varepsilon,t)})'\big)^2\eta'\,dx. \tag{3.6}$$

We observe that $J_{\rm SERR}$ depends on $u^{(\varepsilon,t)}$, but not on $\varepsilon\tilde{q}^{(\varepsilon,t)}$ in expansion (3.15). The latter fact is in accordance with the assertion in [18, 19].

For the shape-topological control, now we insert (3.15) in (3.6), which implies the asymptotic model

$$\begin{aligned}
&J_{\rm SERR}\big(u^{(0,t)} + c_{(0,t)}\big[\varepsilon w(\tfrac{x}{\varepsilon}) + \varepsilon\tfrac{1-\delta}{\delta}(1-\tfrac{x}{r+t})\big] + {\rm O}(\varepsilon^{3/2})\big)\\
&= \tfrac12\int_0^{r+t}\chi^\delta_{(0,\varepsilon)}\big((u^{(0,t)})' + c_{(0,t)}\big[\varepsilon w'(\tfrac{x}{\varepsilon}) - \tfrac{\varepsilon(1-\delta)}{\delta(r+t)}\big]\big)^2\eta'\,dx + {\rm O}(\varepsilon^{3/2})\\
&= \tfrac{c^2_{(0,t)}}{2}\int_\varepsilon^{\varepsilon+\beta}\big(1-\tfrac{2\varepsilon(1-\delta)}{\delta(r+t)}\big)\eta'\,dx + {\rm o}(\varepsilon) = \tfrac{c^2_{(0,t)}}{2}\big(1-\tfrac{2\varepsilon(1-\delta)}{\delta(r+t)}\big) + {\rm o}(\varepsilon)\\
&= J_{\rm SERR}(u^{(0,t)}) - \varepsilon\, c^2_{(0,t)}\tfrac{1-\delta}{\delta(r+t)} + {\rm o}(\varepsilon),\\
&J_{\rm SERR}(u^{(0,t)}) = \tfrac{c^2_{(0,t)}}{2}.
\end{aligned} \tag{3.7}$$

In particular, (3.7) follows formula for the shape-topological derivative

$$\tfrac{d}{d\varepsilon}J_{\rm SERR}(u^{(\varepsilon,t)})|_{\varepsilon=0} = -c^2_{(0,t)}\tfrac{1-\delta}{\delta(r+t)}. \tag{3.8}$$

Moreover, in view of definition (3.6), it implies the mixed second derivative $-\frac{\partial^2}{\partial\varepsilon\,\partial t}\Pi(u^{(\varepsilon,t)})|_{\varepsilon=0}$ which is symmetric: $\frac{\partial^2}{\partial\varepsilon\,\partial t}\Pi(u^{(\varepsilon,t)})|_{\varepsilon=0} = \frac{\partial^2}{\partial t\,\partial\varepsilon}\Pi(u^{(\varepsilon,t)})|_{\varepsilon=0}$. Thus, we have proved the following.

Theorem 3.2. *For the solutions $u^{(\varepsilon,t)}$ and $u^{(0,t)}$ of variational inequalities* (3.1) *and* (3.11), *there exists the shape-topological derivative*

$$\begin{aligned}
&\tfrac{d}{d\varepsilon}J_{\rm SERR}(u^{(\varepsilon,t)})|_{\varepsilon=0} = -\tfrac{\partial^2}{\partial\varepsilon\,\partial t}\Pi(u^{(\varepsilon,t)})|_{\varepsilon=0} = -\tfrac{\partial^2}{\partial t\,\partial\varepsilon}\Pi(u^{(\varepsilon,t)})|_{\varepsilon=0}\\
&= -c^2_{(0,t)}\tfrac{1-\delta}{\delta(r+t)}.
\end{aligned} \tag{3.9}$$

3.2 Local asymptotic expansion in the problem

We recall that Theorem 3.2 is derived based on the uniform asymptotic formula (3.15) which, however, is not unique. Representation (3.15) which is uniform over domain matches the near-field (the boundary layer near inhomogeneity) and the far-field (extendable to infinity) asymptotic representations, which both are unique. This is the reason of our alternative approach to the shape-topological control. Since in one dimension the far-field is trivial (zero), here we employ only the near-field.

In the near-field of the moving boundary point $x = r+t$, any solution $u^{(\varepsilon,t)}$ of homogeneous equation (3.2) can be written as a linear function

$$u^{(\varepsilon,t)}(x) = u^{(\varepsilon,t)}(r+t) + (u^{(\varepsilon,t)})'(r+t)\cdot[x-(r+t)] \text{ for } x > \varepsilon. \tag{3.1}$$

The factor in front of the principal term $x-(r+t)$ in (3.1) is called stress intensity factor (SIF) in crack mechanics. We associate it with the objective

$$J_{\rm SIF}(u^{(\varepsilon,t)}) = (u^{(\varepsilon,t)})'(r+t) =: c_{(\varepsilon,t)}, \tag{3.2}$$

and we aim at proper formula for its calculation without knowledge of the analytic solution (3.7) and (3.8) from Section 3.1.

For this reason, we construct the Green function ζ_t (called the weight function in crack mechanics) obeying the bounded singularity $\zeta_t(r+t)\neq 0$ and $\zeta_t'(r+t)\neq 0$ at the moving boundary point $x=r+t$ and solving the homogeneous problem:

$$-\zeta_t''(x)=0 \quad \text{for } x\in(0,r+t), \tag{3.3}$$
$$\zeta_t(0)=0. \tag{3.4}$$

All solutions of (3.3) and (3.4) are given by straight lines αx and defined up to arbitrary factor $\alpha\neq 0$. If we set the normalization condition

$$1=\int_0^{r+t}(\zeta_t'(x))^2\,dx=\zeta_t'(r+t)\cdot\zeta_t(r+t) \tag{3.5}$$

due to (3.3) and (3.4), then the unique αx satisfying (3.5) is

$$\zeta_t(x)=\tfrac{x}{\sqrt{r+t}}. \tag{3.6}$$

Using (3.2)–(3.4) and (3.3)-(3.4), the second Green formula yields

$$\begin{aligned}0=&\int_0^{r+t}\big[(u^{(\varepsilon,t)})''\zeta_t-u^{(\varepsilon,t)}\zeta_t''\big]\,dx=-[\![(u^{(\varepsilon,t)})'(\varepsilon)]\!]\zeta_t(\varepsilon)+g\zeta_t'(0)\\&+(u^{(\varepsilon,t)})'(r+t)\cdot\zeta_t(r+t)-u^{(\varepsilon,t)}(r+t)\cdot\zeta_t'(r+t),\end{aligned} \tag{3.7}$$

where $[\![(u^{(\varepsilon,t)})'(\varepsilon)]\!]:=(u^{(\varepsilon,t)})'(\varepsilon^+)-(u^{(\varepsilon,t)})'(\varepsilon^-)$ is the jump. Multiplying (3.7) either by $(u^{(\varepsilon,t)})'(r+t)$ or $u^{(\varepsilon,t)}(r+t)$ and using complementarity conditions (3.5), (3.6), we derive the representations

$$(u^{(\varepsilon,t)})'(r+t)=\max\big\{0,\zeta_t'(r+t)\big([\![(u^{(\varepsilon,t)})'(\varepsilon)]\!]\zeta_t(\varepsilon)-g\zeta_t'(0)\big)\big\}, \tag{3.8}$$
$$u^{(\varepsilon,t)}(r+t)=\max\big\{0,\tfrac{1}{\zeta_t'(r+t)}\big(-[\![(u^{(\varepsilon,t)})'(\varepsilon)]\!]\zeta_t(\varepsilon)+g\zeta_t'(0)\big)\big\}, \tag{3.9}$$

where we have used normalization (3.5) to get (3.8). In comparison with the explicit formula (3.8) of $c_{(\varepsilon,t)}$, expressions (3.8) and (3.9) are implicit ones. We plug in (3.8) expansion (3.15) and infer the asymptotic model

$$\begin{aligned}c_{(\varepsilon,t)}:=(u^{(\varepsilon,t)})'(r+t)=\max\big\{0,\zeta_t'(r+t)\big(-g\zeta_t'(0)\\+(u^{(0,t)})'(0)[\![w'(1)]\!]\zeta_t(\varepsilon)+\zeta_t(\varepsilon)\mathrm{O}(\varepsilon)\big)\big\}.\end{aligned} \tag{3.10}$$

Moreover, we apply to (3.10) the local representation $\zeta_t(x)=\zeta_t'(0)\,x$ following from (3.3) and (3.4), hence $\zeta_t(\varepsilon)=\zeta_t'(0)\,\varepsilon$. In this way we have proved the following.

Theorem 3.3. *For the solutions $u^{(\varepsilon,t)}$ and $u^{(0,t)}$ of variational inequalities* (3.1) *and* (3.11), *the following asymptotic representation of SIF holds:*

$$\begin{aligned}J_{\mathrm{SIF}}(u^{(\varepsilon,t)})&=c_{(\varepsilon,t)}=\max\big\{0,\zeta_t'(r+t)\zeta_t'(0)\big(-g+\varepsilon(u^{(0,t)})'(0)[\![w'(1)]\!]+\mathrm{O}(\varepsilon^2)\big)\big\},\\J_{\mathrm{SIF}}(u^{(0,t)})&=c_{(0,t)}=\max\big\{0,-g\zeta_t'(r+t)\zeta_t'(0)\big\}.\end{aligned} \tag{3.11}$$

We note that the max-function in (3.11) is, generally, nondifferentiable with respect to ε when $g = 0$. Nevertheless, further we need the square of the max-function which is differentiable with respect to its argument. Indeed, the square of (3.11) constitutes the form:

$$c^2_{(\varepsilon,t)} = c^2_{(0,t)} + 2\varepsilon c_{(0,t)}(u^{(0,t)})'(0)[\![w'(1)]\!]\zeta'_t(r+t)\zeta'_t(0) + \mathrm{O}(\varepsilon^2). \tag{3.12}$$

As the corollary of Theorem 3.3 we restate the asymptotic result on shape-topological control of J_{SERR} and J_{SE} from Section 3.1.

Inserting the exact solution (3.7) in (3.6), we get

$$J_{\mathrm{SERR}}(u^{(\varepsilon,t)}) = -\tfrac{d}{dt}\Pi(u^{(\varepsilon,t)}) = \tfrac{1}{2}c^2_{(\varepsilon,t)}. \tag{3.13}$$

With the help of (3.12), from (3.13) we immediately obtain the shape-topological derivative $-\frac{\partial^2}{\partial\varepsilon\,\partial t}\Pi(u^{(\varepsilon,t)})|_{\varepsilon=0}$ as

$$\tfrac{d}{d\varepsilon}J_{\mathrm{SERR}}(u^{(\varepsilon,t)})|_{\varepsilon=0} = c_{(0,t)}(u^{(0,t)})'(0)[\![w'(1)]\!]\zeta'_t(r+t)\zeta'_t(0). \tag{3.14}$$

In order to validate (3.14), after substitution of the exact analytic expressions (3.9), (3.14), and (3.6) of solutions $u^{(0,t)}$, w, and ζ_t, respectively, this results in $\frac{d}{d\varepsilon}J_{\mathrm{SERR}}(u^{(\varepsilon,t)})|_{\varepsilon=0} = -c^2_{(0,t)}\frac{1-\delta}{\delta(r+t)}$ thus coinciding with expression (3.9) derived in Theorem 3.2.

Similarly, substituting (3.7) in $\Pi(u^{(\varepsilon,t)})$ given in (3.2), straightforward calculation provides equivalent expression of SE-optimal value function

$$\begin{aligned}
&J_{\mathrm{SE}}(u^{(\varepsilon,t)}) = \Pi(u^{(\varepsilon,t)}) = \tfrac{1}{2}c^2_{(\varepsilon,t)}(r+t+\tfrac{1-\delta}{\delta}\varepsilon)\\
&= \big[\tfrac{c^2_{(0,t)}}{2} + \varepsilon c_{(0,t)}(u^{(0,t)})'(0)[\![w'(1)]\!]\zeta'_t(r+t)\zeta'_t(0) + \mathrm{O}(\varepsilon^2)\big](r+t+\tfrac{1-\delta}{\delta}\varepsilon)\\
&= \tfrac{c^2_{(0,t)}}{2}\big(1 - \tfrac{2(1-\delta)}{\delta(r+t)} + \mathrm{O}(\varepsilon^2)\big)(r+t+\tfrac{1-\delta}{\delta}\varepsilon) = \tfrac{c^2_{(0,t)}}{2}(r+t-\tfrac{1-\delta}{\delta}\varepsilon) + \mathrm{O}(\varepsilon^2),
\end{aligned}$$

where we have used here the expansion (3.12) of SIF $c^2_{(\varepsilon,t)}$. Thus, we arrive again at formula (3.3).

4 Discussion

In [17] this technique of a shape-topological control is extended to the nonlinear problem of crack–defect interaction in two dimensions, where no analytic solutions but only variational formulations are available. The semi-analytic expressions are proved for the shape-topological derivatives of J^2_{SIF} and J_{SERR}.

Acknowledgments

V.A. Kovtunenko is supported by the Austrian Science Fund (FWF) project P26147-N26 "Object identification problems: numerical analysis", he thanks the Austrian Academy of Sciences (OeAW) and and OeAD Scientific & Technological Cooperation WTZ CZ 01/2016. G. Leugering is supported by DFG EC 315 "Engineering of Advanced Materials".

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Received: 14.03.2016