Derivatives of the Energy Functional for 2D-Problems with a Crack under Signorini and Friction Conditions

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We consider the two-dimensional elasticity problem for an elastic body with a crack under unilateral constraints imposed at the crack. We assume that both the Signorini condition for non-penetration of the crack faces and the condition of given friction between them are fulfilled. The problem is non-linear and can be described by a variational inequality. Varying the shape of the crack by a local coordinate transformation of the domain, the first derivative of the energy functional to the problem with respect to the crack length is obtained, which gives the criterion for the crack growing. The regularity of the solution is discussed and the singular solution is performed. Copyright © 2000 John Wiley & Sons, Ltd.

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Mathematical investigations of the crack problems mainly deal with the singularities arising near the crack tips [10, 12, 14, 6]. An important question concerns the propagation of the crack [14, 1] and the closed problem of the shape sensitivity analysis [15, 9, 2]. By this, the crack faces are usually assumed to be stress-free. Taking into the consideration unilateral constraints at the crack faces [9, 8, 11], we obtain non-linear crack problems, which may not keep all the properties of the solutions for the linear ones. Therefore, it is necessary to apply general variational principles. The crack growth depends then on properties for the derivatives of the energy functional with respect to the crack length [3, 16]. Our aim is to extend the known results onto the crack problems with unilateral constraints and to generalize the methods applied.

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1. Formulation of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a boundary $\Gamma$ of the class $C^{2,1}$. Define the set $\Gamma_0 = (0, 1) \times \{0\}$ in $\mathbb{R}^2$ with the normal vector $(0, 1)$ and the tangential vector $(1, 0)$ to $\Gamma_0$. Its boundary $\partial \Gamma_0$ consists of the points $(0, 0), (l, 0)$ and $\Gamma_0 = \Gamma_0 \cup \partial \Gamma_0$. Assume that $[0, l + l_0] \subset \Omega$. Denote $\Omega_0 = \Omega \setminus \Gamma_0$. We consider the two-dimensional elastic body occupying the domain $\Omega_0$ with the crack $\Gamma_0$.

Let $f = (f_1, f_2), f_i \in C^2(\Omega)$, $i = 1, 2$ be a given load, and $g \in C^2([0, l + l_0])$, $g \geq 0$, be a given friction force between the crack faces $\Gamma_0^\pm$, where $\Gamma_0^\pm = \Gamma_0 \cap \{x \in \mathbb{R}^2, \pm x_2 \geq 0\}$. For example, $g$ can be understood as an approximation of the Coulomb friction law.

In the following, we consider the functional of the potential energy for the body with given friction $[4, 7]$:

$$\Pi(u; \Omega_0) = \inf_{v \in K_0} \Pi(v; \Omega_0)$$

is equivalent to the variational inequality

$$\int_{\Omega_0} \sigma_{ij}(u) \varepsilon_{ij}(v - u) - \int_{\Gamma_0} f_i(v) \varepsilon_i(u) - \int_{\partial \Gamma_0} g(\{v_1\} - \{u_1\}) \geq \int_{\Omega_0} f_i(v_i - u_i), \quad \forall v \in K_0.$$ (3)

Under the assumption of $u = 0$ on $\Gamma$, Korn’s inequality

$$c \|u\|_{H^1(\Omega_0)} \leq \int_{\Omega_0} \sigma_{ij}(u) \varepsilon_{ij}(u),$$ (4)

guarantees, that problems (2) or (3) have the unique solution $u \in K_0$. From the variational inequality (3) it follows that the equilibrium equation in $\Omega_0$,

$$- \sigma_{ij,j}(u) = f_i, \quad i = 1, 2.$$ (5)
and the boundary conditions on $\Gamma_0$,

\[
\begin{align*}
[\sigma_{22}(u)] &= 0, \quad \sigma_{22}(u) \leq 0, \quad [u_2] \geq 0, \quad \sigma_{22}(u)[u_2] = 0, \\
[\sigma_{12}(u)] &= 0, \quad |\sigma_{12}(u)| \leq g, \quad \sigma_{12}(u)[u_1] - g[|u_1|] = 0,
\end{align*}
\]

(6) (7)

hold in the weak sense (for the exact meaning see [8]). Boundary conditions (6) arise due to the Signorini condition of non-penetration, whereas conditions (7) correspond to the friction condition between the crack faces.

2. Regularity of the solution

We investigate the smoothness properties of the solution $u \in K_0$ of problem (3), following the ideas of [8].

**Theorem 1.** For a smooth function $\rho$, $0 \leq \rho \leq 1$, with $\text{supp} \rho \cap \partial \Gamma_0 = \emptyset$, the inclusion \( \rho u \in H^2(\Omega_0)^2 \) is valid.

**Proof.** First, the needed $H^2$-smoothness of $\rho u$ inside $\Omega_0$ and near $\Gamma$ follows in a standard way (see, for instance [5]) from (5), $f \in [L^2(\Omega_0)]^2$, $u=0$ on $\Gamma$ and the regularity of $\Gamma$. We prove here the smoothness of $\rho u$ near $\Gamma_0$.

Let $B_\delta(x^0)$ be a circle centered in the point $x^0 \in \Gamma_0$ and with the radius $\delta < \text{dist}(x^0, \partial \Gamma_0)$. We choose the smooth cut-off function $\rho$ such that $\text{supp} \rho \subset B_\delta(x^0)$. For a positive number $h$, define the operator $d_h = d_+ h$ and its adjoint $d_- h$ as

\[
d_\pm h \phi := \frac{1}{h}(\phi \pm h) - \phi), \quad \phi \pm h(x_1, x_2) = \phi(x_1 \pm h, x_2).
\]

Consider the function

\[
\rho u - \frac{h^2}{2} \rho d_- h d_h(\rho u) = (1 - \rho^2)u + \frac{1}{2} \rho ((\rho u)_h + (\rho u)_{-h})
\]

If $h < \text{dist}(x^0, \partial \Gamma_0)$, then we have $v \in K_0$ because of

\[
[\sigma_{ij}(d_h(\rho u))e_{ij}(d_h(\rho u)) \leq I_1 + I_2 + I_3,
\]

(8)

where

\[
I_1 = \int_{\Omega_0} (\sigma_{ij}(d_h(\rho u))e_{ij}(d_h(\rho u)) - \sigma_{ij}(u)e_{ij}(\rho d_- h d_h(\rho u))),
\]

\[
I_2 = \int_{\Omega_0} \rho f_i d_- h d_h(\rho u_i), \quad I_3 = \frac{2}{h^2} \int_{\Gamma_0} g\left(\left[\frac{1}{h} - \frac{h^2}{2} \rho d_- h d_h(\rho u_i)\right] - [u_i]\right).
\]
Let us denote $e_{ij}(u) = \frac{1}{2}(\rho_j u_j + \rho_i u_i)$, $i = 1, 2$, and apply the estimate
\[
\|d_{\pm h}(\rho \phi)\|_{L^2(\Omega_0)} \leqslant \|(\rho \phi),(1)\|_{L^2(\Omega_0)} \leqslant c \|\rho \phi\|_{H^1(\Omega_0)}.
\]
We obtain
\[
I_1 = \int_{\Omega_0} (-\sigma_{ij}(u)e_{ij}(d_{-h}d_h(\rho u))
+ d_h(c_{ijkl}e_{kl}(u))e_{ij}(d_h(\rho u)) - d_h(c_{ijkl}e_{kl}(\rho u))e_{ij}(d_h(\rho u)))
\leqslant c \|u\|_{H^1(\Omega_0)} \|d_h(\rho u)\|_{H^1(\Omega_0)},
\]
\[
I_2 \leqslant c \|\rho f\|_{L^2(\Omega_0)} \|d_h(\rho u)\|_{H^1(\Omega_0)}.
\]
To evaluate $I_3$, we use (9), the inequalities for the modules, the positiveness of the friction force $g$ and the continuity of the trace operator in $B_\delta^\pm (x^0) = B_\delta(x^0) \cap \{x \in R^2, \quad \pm x_2 > 0\}$ [13], and we get
\[
I_3 = \frac{2}{h^2} \int_{\Gamma_0} g \left( \left| (1 - \rho^2)[u_1] + \frac{1}{2} \rho [(\rho u_1)_h + (\rho u_1)_{-h}] \right| - |[u_1]| \right)
\leqslant \frac{2}{h^2} \int_{\Gamma_0} \rho g \left( \frac{1}{2} [(\rho u_1)_h + (\rho u_1)_{-h}] - |[\rho u_1]| \right)
\leqslant \|\rho g\|_{L^2(\Gamma_0)} \|[d_{-h}d_h(\rho u_1)]\|_{L^2(\Gamma_0)} \leqslant c \|\rho g\|_{L^2(\Gamma_0)} \|d_h(\rho u_1)\|_{H^1(\Omega_0)}.
\]
Due to (4) and (8), we have then the following estimate:
\[
\|d_h(\rho u)\|_{H^1(\Omega_0)} \leqslant c (\|u\|_{H^1(\Omega_0)} + \|f\|_{L^2(\Omega_0)} + \|g\|_{L^2(\Gamma_0)}),
\]
which means that $(\rho u),_{11}, (\rho u),_{12} \in [L^2(\Omega_0)]^2$. Because of the ellipticity of the coefficients $c_{ijkl}$, the equilibrium equation (5) can be rewritten as
\[
u,_{22} = L(f, u, \nabla u, u,_{11}, u,_{12}).
\]
In view of the shown regularity of $\rho u$, this leads to $(\rho u),_{22} \in [L^2(\Omega_0)]^2$. The theorem is proved.

Concerning the smoothness of the solution near $\partial \Omega_0$, we formulate the following result.

**Lemma 1.** If $[u] = 0$ in some neighbourhood $B$ of the crack tip, then $\rho u \in [H^2(B)]^2$ for a cut-off function $\rho$ with supp $\rho \subset B$.

**Proof.** Let us take $\phi \in [C_0^\infty(B)]^2$. By $[\phi] = 0$, we can substitute $v = u \pm \phi$ in (3) and obtain
\[
\int_{B \cap \Omega_0} \sigma_{ij}(u)e_{ij}(\phi) = \int_{B \cap \Omega_0} f_i \phi_i.
\]
Applying the Green formulae in the domains \( B^\pm = B \cap \{ x \in \mathbb{R}^2, \pm x_2 > 0 \} \), it follows from (10) that

\[
- \sigma_{ij,j}(u) = f_i, \quad i = 1, 2, \quad \text{in } B^\pm,
\]

\[
[\sigma_{12}(u)] = [\sigma_{22}(u)] = 0 \quad \text{on } \Gamma_B = \bar{B}^+ \cap \bar{B}^-.
\]

For \([u] = 0\) in \( B \cap \Gamma_o\), we have \( u_i \in H^1(B)\), \( \sigma_{ij,j}(u) \in L^2(B)\), \( i = 1, 2\). Because of \( f \in [L^2(B)]^2\) and (11), the following distributions are defined for \( \phi \in C^0_0(B)\):

\[
\langle -\sigma_{ij,j}(u), \phi \rangle_B = \int_B \sigma_{ij}(u) \phi_{,j} = \int_{B^+} \sigma_{ij}(u) \phi_{,j} + \int_{B^-} \sigma_{ij}(u) \phi_{,j}
\]

\[
= -\int_{B^+} \sigma_{ij,j}(u) \phi - \int_{B^-} \sigma_{ij,j}(u) \phi - \langle [\sigma_{12}(u)], \phi \rangle_{\Gamma_B}
\]

\[
= \int_B f_i \phi, \quad i = 1, 2.
\]

This means that the equilibrium equation is fulfilled in the whole domain \( B\), and standard considerations lead to \( \rho u \in [H^2(B)]^2\). The lemma is proved.

Note that, due to Lemma 1 and Theorem 1, if \([u] = 0\) near \( \partial \Gamma_o\), we have \( u \in [H^2(\Omega_0)]^2\).

3. Variation of the crack

We shall vary the shape of the crack by a local coordinate transformation of the domain following the ideas of [8].

For a small parameter \( \varepsilon > 0\), let us consider the perturbed domain

\[
\Omega_\varepsilon = \Omega \setminus \bar{\Gamma}_\varepsilon, \quad \Gamma_\varepsilon = (0, l + \varepsilon) \times \{ 0 \}.
\]

In the domain \( \Omega_\varepsilon\) one can consider, in analogy to \( K_0 \) and \( \Pi(u; \Omega_0)\), the set of admissible displacements,

\[
K_\varepsilon = \{ u = (u_1, u_2) \in \bar{H}^1(\Omega_\varepsilon), [u_2] \geq 0 \text{ on } \Gamma_\varepsilon \},
\]

and the perturbed functional of the potential energy,

\[
\Pi(u; \Omega_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} \sigma_{ij}(u) e_{ij}(u) - \int_{\Omega_\varepsilon} f_i u_i + \int_{\Gamma_\varepsilon} g[u_1].
\] (12)

Then the equilibrium problem

\[
\Pi(u^*; \Omega_\varepsilon) = \inf_{v \in K_\varepsilon} \Pi(v; \Omega_\varepsilon)
\] (13)
is equivalent to the variational inequality

\[ \int_{\Omega_{\varepsilon}} \sigma_{ij}(u) \varepsilon_{ij}(v - u) + \int_{\Gamma_{\varepsilon}} g(|[v_{1}]| - |[u_{1}]|) \geq \int_{\Omega_{\varepsilon}} f_{i}(v_{i} - u_{i}) \quad \forall v \in K_{\varepsilon} \]  

(14)

and has the unique solution \( u^{\varepsilon} \in K_{\varepsilon} \) similar to (2) and (3), respectively.

Let \( B_{\delta} \) be a circle of radius \( \delta > 0 \) centred in the crack tip \( (l, 0) \), \( \delta < l_{0} \), such that \( \overline{B}_{\delta} \subset \Omega \). Choose the cut-off function \( \chi \in C_{0}^{0}(\Omega), \; 0 \leq \chi \leq 1 \), such that \( \text{supp} \chi \subset B_{\delta} \) and \( \chi \equiv 1 \) in \( B_{0/2} \). For small \( 0 < \varepsilon < \delta/2 \), we construct the one-to-one coordinate transformation

\[ y_{1} = x_{1} + \varepsilon \chi(x_{1}, x_{2}), \quad y_{2} = x_{2}, \quad (y_{1}, y_{2}) \in \Omega_{\varepsilon}, \quad (x_{1}, x_{2}) \in \Omega_{0} \]

(15)

with the Jacobian \( |\partial y_{1}/\partial x_{1}(y_{1}, y_{2})/\partial x_{1}(x_{1}, x_{2})| = 1 + \varepsilon \chi_{,1} \), which transforms \( \Omega_{\varepsilon} \) to \( \Omega_{0} \). We shall denote by \( \hat{\chi}(x), \; x \in \Omega_{0} \), the transformed function \( s(y), \; y \in \Omega_{\varepsilon} \), i.e.,

\[ s(y) = s(x_{1} + \varepsilon \chi(x_{1}, x_{2})) = \hat{\chi}(x). \]

Rewriting the derivatives of \( u \) with (15), we get

\[ u_{i,j} = \hat{u}_{i,j} - \frac{\varepsilon}{1 + \varepsilon \chi_{,1}} \chi_{,j} \hat{u}_{i,1}, \quad i, j = 1, 2 \]

and therefore

\[ \varepsilon_{ij}(u) = \varepsilon_{ij}(\hat{u}) - \frac{\varepsilon}{1 + \varepsilon \chi_{,1}} E_{ij}(\hat{u}), \quad E_{ij}(\hat{u}) = \frac{1}{2} (\chi_{,i} \hat{u}_{j,1} + \chi_{,j} \hat{u}_{i,1}), \]

(16)

\[ \sigma_{ij}(u) = \hat{\sigma}_{ij}(\hat{u}) - \frac{\varepsilon}{1 + \varepsilon \chi_{,1}} \hat{\Sigma}_{ij}(\hat{u}), \quad \hat{\sigma}_{ij}(\hat{u}) = \hat{\chi}_{ijkl} \hat{e}_{kl}(\hat{u}), \quad \hat{\Sigma}_{ij}(\hat{u}) = \hat{\chi}_{ijkl} E_{kl}(\hat{u}). \]

The substitution of (16) in (12) provides

\[ \Pi(u; \Omega_{\varepsilon}) = \Pi_{\varepsilon}(\hat{u}; \Omega_{0}), \]

\[ \Pi_{\varepsilon}(\hat{u}; \Omega_{0}) = \frac{1}{2} \int_{\Omega_{0}} \hat{\sigma}_{ij}(\hat{u}) ((1 + \varepsilon \chi_{,1}) \varepsilon_{ij}(\hat{u}) - 2\varepsilon E_{ij}(\hat{u})) \]

\[ + \frac{1}{2} \int_{\Omega_{0}} \frac{\varepsilon^{2}}{1 + \varepsilon \chi_{,1}} \hat{\Sigma}_{ij}(\hat{u}) E_{ij}(\hat{u}) \]

\[ + \int_{\Gamma_{0}} (1 + \varepsilon \chi_{,1}) \hat{g} |[\hat{u}_{1}]|, \]

(17)

and, by (13),

\[ \inf_{v \in K_{\varepsilon}} \Pi(v; \Omega_{\varepsilon}) = \Pi(u^{\varepsilon}; \Omega_{\varepsilon}) = \Pi_{\varepsilon}(\hat{u}^{\varepsilon}; \Omega_{0}) = \inf_{v \in K_{0}} \Pi_{\varepsilon}(v; \Omega_{0}). \]

(18)
Therefore, \( \hat{u} \in K_0 \) is the unique solution of the following variational inequality:

\[
\int_{\Omega_0} (\hat{\sigma}_{ij}(\hat{u}')(1 + e\chi_{\Omega})e_{ij}(v - \hat{u}') - eE_{ij}(v - \hat{u}') - e\hat{\Sigma}_{ij}(\hat{u}')e_{ij}(v - \hat{u}'))
+ \int_{\Omega_0} \frac{e^2}{1 + e\chi_{\Omega}} \hat{\Sigma}_{ij}(\hat{u}')E_{ij}(v - \hat{u}') + \int_{\Gamma_0} (1 + e\chi_{\Omega})\hat{g}([v_1] - [\hat{u}_1])
\geq \int_{\Omega_0} (1 + e\chi_{\Omega})\hat{f}_i(v_i - \hat{u}_i), \quad \forall v \in K_0.
\tag{19}
\]

For the connection of the solutions of problems (3) and (19), we prove the following lemma.

**Lemma 2.** The solution \( \hat{u} \) of variational inequality (19) converges strongly to the solution \( u \) of variational inequality (3) in \( \tilde{H}^1(\Omega_0) \) as \( e \to 0 \). There holds the estimate

\[
\| \hat{u} - u \|_{H^1(\Omega_0)} \leq ce.
\]

**Proof.** Substituting \( v = u \) in (19), \( v = \hat{u} \) in (3) and summing, we obtain

\[
\int_{\Omega_0} \sigma_{ij}(\hat{u} - u)e_{ij}(\hat{u} - u) \leq I_1 + I_2 + I_3,
\tag{20}
\]

where

\[
I_1 = \int_{\Omega_0} (\hat{\epsilon}_{ijkl} - c_{ijkl})e_{kl}(\hat{u}' - u')
+ \int_{\Gamma_0} (\hat{g} - g)([u_1] - [\hat{u}_1]) + \int_{\Omega_0} (\hat{f}_i - f_i)(\hat{u}_i - u_i),
\]

\[
I_2 = e \int_{\Omega_0} (\hat{\sigma}_{ij}(\hat{u}')(\chi_{\Omega}e_{ij}(u - \hat{u}')) - E_{ij}(u - \hat{u}')) - \hat{\Sigma}_{ij}(\hat{u}')e_{ij}(u - \hat{u}'))
+ e \int_{\Gamma_0} \chi_{\Omega}\hat{g}([u_1] - [\hat{u}_1]) + e \int_{\Omega_0} \chi_{\Omega}\hat{f}_i(\hat{u}_i - u_i),
\]

\[
I_3 = e^2 \int_{\Omega_0} \frac{1}{1 + e\chi_{\Omega}} \hat{\Sigma}_{ij}(\hat{u}')E_{ij}(u - \hat{u}').
\]

From (3), one can derive the estimate

\[
\| u \|_{H^1(\Omega_0)} \leq \text{const},
\tag{21}
\]

from (14), analogously, \( \| \hat{u} \|_{H^1(\Omega_0)} \leq \text{const} \), and

\[
\| \hat{u} \|_{H^1(\Omega_0)} \leq \text{const},
\tag{22}
\]
uniformly in $\varepsilon$. Using (22) and the Hölder inequality, we evaluate $I_3$:

$$I_3 \leqslant c\varepsilon^2 \|\hat{u} - u\|_{H^1(\Omega_0)}.$$

With the inequalities for the modules and the continuity of the trace operator [13] we have

$$\|[[\hat{u}]] - [[u_1]]\|_{L^2(\Gamma_0)} \leqslant \|[[\hat{u} - u_1]]\|_{L^2(\Gamma_0)} \leqslant c \|\hat{u} - u\|_{H^1(\Omega_0)}.$$  (23)

Therefore, with the help of (22) and (23), the integral $I_2$ can be evaluated by

$$I_2 \leqslant c\varepsilon \|\hat{u} - u\|_{H^1(\Omega_0)}.$$

The supposed continuous differentiability of $c_{ijkl}, g, f$ provides the estimates

$$|\hat{c}_{ijkl} - c_{ijkl}| \leqslant c\varepsilon, \quad |\hat{f}_i - f_i| \leqslant c\varepsilon, \quad |\hat{g} - g| \leqslant c\varepsilon.$$  (24)

Altogether (21)–(24) guarantee

$$I_1 \leqslant c\varepsilon \|\hat{u} - u\|_{H^1(\Omega_0)}.$$

Summing all estimates for $I_1, I_2, I_3$ and using Korn’s inequality (4), we obtain from (20) the assertion of Lemma 2. The lemma is proved.

4. Derivative of the energy functional

Let us derive the derivative of the energy functional with respect to the crack length, i.e.

$$\Pi'(u; \Omega_0) = \lim_{\varepsilon \to 0} \frac{\Pi(u'\varepsilon; \Omega_0) - \Pi(u; \Omega_0)}{\varepsilon}.$$  (25)

Our aim is to calculate the value of (25).

**Lemma 3.** The following equivalence is valid:

$$\Pi'(u; \Omega_0) = \lim_{\varepsilon \to 0} \frac{\Pi_\varepsilon(u; \Omega_0) - \Pi(u; \Omega_0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\Pi_\varepsilon(\hat{u}; \Omega_0) - \Pi(\hat{u}; \Omega_0)}{\varepsilon}.$$  (26)

**Proof.** Firstly, by (18) we have

$$\frac{\Pi(u'; \Omega_0) - \Pi(u; \Omega_0)}{\varepsilon} = \frac{\Pi_\varepsilon(u'; \Omega_0) - \Pi(u; \Omega_0)}{\varepsilon} \leqslant \frac{\Pi_\varepsilon(\hat{u}; \Omega_0) - \Pi(\hat{u}; \Omega_0)}{\varepsilon}.$$  (26)

On the other hand, we can deduce from (18) together with the representations (1) and (17) that

$$\frac{\Pi_\varepsilon(\hat{u}; \Omega_0) - \Pi(u; \Omega_0)}{\varepsilon} \geqslant \frac{\Pi_\varepsilon(\hat{u}; \Omega_0) - \Pi(\hat{u}; \Omega_0)}{\varepsilon}.$$
\[ \frac{1}{2} \int_{\Omega_0} \left[ \frac{\epsilon_{ijkl}}{\epsilon} e_{ikl}(\bar{u}) \epsilon_{ij}(\bar{u}) - \int_{\Omega_0} \frac{f_i}{\epsilon} \bar{u}_i^c + \int_{\Gamma_0} \frac{\hat{g} - g}{\epsilon} \left[ \bar{u}_i^c \right] \right] \]

\[ + \frac{1}{2} \int_{\Omega_0} \left[ \delta_{ij}(\bar{u}) (\chi, \epsilon_{ij}(\bar{u}) - 2E_{ij}(\bar{u})) - \int_{\Omega_0} \chi, f_i \bar{u}_i^c + \int_{\Gamma_0} \chi, \hat{g} \left[ \bar{u}_i^c \right] \right] \]

\[ + \frac{\epsilon}{2} \int_{\Omega_0} \frac{1}{1 + \epsilon \chi, 1} \sum_{ij} (\epsilon_{ij}(\bar{u})) E_{ij}(\bar{u}). \]  

(27)

Because of Lemma 2 and the continuous differentiability of \( c_{ijkl}, f, g, \) one can take the limit infimum in (27), which coincides with the limit supremum in (26). Lemma 3 is proved.

In the proof of Lemma 3 we have calculated derivative (25) directly. Indeed, for continuously differentiable function \( \phi \), the following formula arises:

\[ \lim_{\epsilon \to 0} \frac{\phi - \phi}{\epsilon} = \lim_{\epsilon \to 0} \frac{\phi(x_1 + \epsilon \chi, x_2) - \phi(x_1, x_2)}{\epsilon} = \chi \phi, 1. \]

Therefore, from Lemma 3 the next result follows.

**Theorem 2.** The derivative of the energy functional with respect to the crack length is equal to

\[ \Pi(u; \Omega_0) = \frac{1}{2} \int_{\Omega_0} \left[ \sigma_{ij}(\epsilon_{ij}(u)) - \sigma_{ij}(u) \right] \epsilon_{ij}(u) \left[ \sigma_{ij}(u) \right] \epsilon_{ij}(u) - \int_{\Omega_0} \chi \sigma_{ij} f_i u_i + \int_{\Gamma_0} \chi \sigma_{ij} g \left[ u_i \right]. \]  

(28)

We now investigate properties of the derivative \( \Pi(u; \Omega_0) \) given by (28). Let us assume that the solution \( u \) of (3) is smooth enough. Integration by parts in the domains \( \Omega^\pm = \Omega_0 \cap \{ x \in \mathbb{R}^2, \pm x_2 > 0 \} \) with \( \Omega^+ \cap \Omega^- = \Sigma, \Gamma_0 \subset \Sigma \) leads to

\[ \Pi(u; \Omega_0) = \int_{\Omega_0} \chi (\sigma_{ij}(u) + f_i) u_i + \int_{\Sigma} \chi \sigma_{ij}(u) u_i + \int_{\Gamma_0} \chi \sigma_{ij} g \left[ u_i \right]. \]  

(29)

For the solution \( u \) of (3), equilibrium equation (5) in \( \Omega_0 \) and the boundary conditions \( [\sigma_{ij}(u)] = 0, i = 1, 2, \) at \( \Sigma \) are fulfilled. Moreover, for smooth \( u \) we also have \( [u_i] = 0 \) at \( \Sigma \setminus \Gamma_0, i = 1, 2 \). Therefore, (29) can be represented as

\[ \Pi(u; \Omega_0) = I_1 + I_2 \]

and

\[ I_1 = \int_{\Gamma_0} (\chi \sigma_{ij}(u) [u_i] + \chi \sigma_{ij} g \left[ u_i \right]), \quad I_2 = \int_{\Gamma_0} \chi \sigma_{ij} g \left[ u_i \right]. \]
If \( u \) is continuous, we can split \( \Gamma_0 \) into the sets
\[
\mathcal{D}_2 = \{ x \in \Gamma_0, [u_2] = 0 \}, \quad \mathcal{N}_2 = \{ x \in \Gamma_0, [u_2] > 0 \}.
\]
On \( \mathcal{D}_2 \) we have \([u_2,1] = 0\). On \( \mathcal{N}_2 \), by (6), we have \( \sigma_{22}(u) = 0 \). Thus, \( I_2 = 0 \).

Analogously, \( \Gamma_0 \) can be split into the sets
\[
\mathcal{D}_1 = \{ x \in \Gamma_0, [u_1] = 0 \}, \quad \mathcal{N}_1^\pm = \{ x \in \Gamma_0, \pm [u_1] > 0 \}.
\]
On \( \mathcal{D}_1 \) we have \([u_1] = 0 \) and \([u_1,1] = 0 \). On \( \mathcal{N}_1^\pm \), the relation \([u_1] = \pm [u_1] \) is fulfilled, and we can integrate in \( \mathcal{I}_1 \) by parts along \( x_1 \). This provides
\[
I_1 = \int_{\mathcal{N}_1^\pm} \chi(\sigma_{12}(u) + g)[u_{1,1}].
\]

By (7), we have \( \sigma_{12}(u) = \pm g \) as \( \pm [u_1] > 0 \), and therefore \( I_1 = 0 \).

If \( u \) is sufficiently smooth, i.e., \( u \in [H^2(\Omega_0)]^2 \subset \{ C(\bar{\Omega}_0) \}^2 \), the above considerations are correct. Thus, we have proved the following result.

**Lemma 4.** If the solution \( u \) of (3) belong to \([H^2(\Omega_0)]^2\), then \( \Pi(u; \Omega_0) = 0 \).

Since, due to Theorem 1, the solution \( u \) is of the \( H^2 \)-class everywhere except the neighbourhood of \( \Gamma_0 \), we can rewrite the derivative \( \Pi(u; \Omega_0) \) in the following way.

Let \( B_{\delta/2} \) be a circle centered in the crack tip \((l, 0)\) as before, with the boundary \( \partial B_{\delta/2} \) and the outward normal vector \( n = (n_1, n_2) \). We have \( \chi \equiv 1 \) in \( B_{\delta/2} \) and consider now the integrals in (28) over the domains \( \Omega_0 \setminus \bar{B}_{\delta/2} \) and \( B_{\delta/2} \setminus \bar{\Gamma}_0 \). Using Theorem 1 and (5)–(7), we apply the Green formula in \( \Omega_0 \setminus \bar{B}_{\delta/2} \) and obtain, like in the proof of Lemma 4, that
\[
I_1 = \frac{1}{2} \int_{B_{\delta/2} \setminus \bar{\Gamma}_0} (\chi c_{ijkl}, 1 \varepsilon_{kl}(u) \varepsilon_{ij}(u) - \sigma_{ij}(u)(\chi, iu_{j,1} + \chi, j u_{i,1}))
\]
\[- \int_{B_{\delta/2} \setminus \bar{\Gamma}_0} (\chi f_i), 1 u_i + \int_{B_{\delta/2} \setminus \bar{\Gamma}_0} (\chi g), 1 [u_1] \]
\[= \int_{\partial B_{\delta/2} \setminus \Gamma_0} \left( -\frac{1}{2} \sigma_{ij}(u) \varepsilon_{ij}(u) n_1 + \sigma_{ij}(u) n_j u_{i,1} + f_i u_i n_1 \right) \]
(the outward normal here is \( -n \)). By \( \chi \equiv 1 \) in \( B_{\delta/2} \setminus \bar{\Gamma}_0 \), we also have
\[
I_2 = \frac{1}{2} \int_{B_{\delta/2} \setminus \bar{\Gamma}_0} (\chi c_{ijkl}, 1 \varepsilon_{kl}(u) \varepsilon_{ij}(u) - \sigma_{ij}(u)(\chi, iu_{j,1} + \chi, j u_{i,1}))
\[- \int_{B_{\delta/2} \setminus \bar{\Gamma}_0} (\chi f_i), 1 u_i + \int_{B_{\delta/2} \setminus \bar{\Gamma}_0} (\chi g), 1 [u_1] \]
\[= \int_{B_{\delta/2} \setminus \bar{\Gamma}_0} \left( \frac{1}{2} c_{ijkl} \varepsilon_{kl}(u) \varepsilon_{ij}(u) + f_i u_i n_1 \right) - \int_{\partial B_{\delta/2} \setminus \Gamma_0} f_i u_i n_1 + \int_{\partial B_{\delta/2} \setminus \Gamma_0} g', [u_1]. \]
The summation $I_1 + I_2 = \Pi'(u; \Omega_0)$ leads to the relation

\[ \Pi'(u; \Omega_0) = \int_{B_{u/2}} \left( \frac{1}{2} c_{ijkl} \varepsilon_{kl}(u) \varepsilon_{ij}(u) + f_i u_{i,1} \right) + \int_{\Gamma_0 \cap B_{u/2}} g'([u_1]) \]

\[ + \int_{\partial B_{u/2}} \left( -\frac{1}{2} \sigma_{ij}(u) \varepsilon_{ij}(u) n_1 + \sigma_{ij}(u) n_j u_{i,1} \right), \tag{30} \]

where the last integral in (30),

\[ C(u) = \int_{\partial B_{u/2}} \sigma_{ij}(u)(u_{i,1} n_j - \frac{1}{2} \varepsilon_{ij}(u) n_1), \]

is called the Cherepanov–Rice integral. Indeed, for $f \equiv 0$, $c_{ijkl} \equiv \text{const}$, $g \equiv \text{const}$ in some neighbourhood of the crack tip, it follows from (30) that $\Pi'(u; \Omega_0) = C(u)$. This means that $C(u) \equiv \text{const}$ for all sufficient small $\delta$ because of the uniqueness of the derivative. Note that expression (30) does not contain $\gamma$ in comparison with (28); this underlines the independence of the derivative of the energy functional on the cut-off function.

### 5. Rupture criterion

Knowing the value of $\Pi'(u; \Omega_0)$, we have an asymptotic representation of the energy functional as follows:

\[ \Pi(u'; \Omega_\varepsilon) = \Pi(u; \Omega_0) + \varepsilon \Pi'(u; \Omega_0) + o(\varepsilon). \tag{31} \]

Together with the surface energy, we can define the total potential energy for a solid with a crack [14] as

\[ U(u'; \Omega_\varepsilon) = \Pi(u'; \Omega_\varepsilon) + \gamma \text{ meas } \Gamma_\varepsilon, \quad \gamma > 0. \]

Due to (31), the increment of $U(\cdot)$ is given as

\[ U(u'; \Omega_\varepsilon) - U(u; \Omega_0) = \varepsilon(\Pi'(u; \Omega_0) + \gamma) + o(\varepsilon). \tag{32} \]

The crack will propagate in such a way that the total potential energy turns out to be minimal, so that representation (32) leads to the following criterion for the crack growing:

\[ \Pi'(u; \Omega_0) + \gamma < 0. \tag{33} \]

Conditions like (33) are called the Griffith rupture criterion. Conversely, when the inverse condition

\[ \Pi'(u; \Omega_0) + \gamma \geq 0 \tag{34} \]
is fulfilled, we say that the crack is stationary. In particular, if the solution $u$ of (3) is smooth, namely $H^2$-class, then Lemma 4 leads to the fulfillment of condition (34).

6. The case of the Lamé system

Let us consider the constant coefficients $c_{ijkl}$ given by

$$\sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda\delta_{ij}\text{div} \, u, \quad \text{div} \, u = u_{1,1} + u_{2,2},$$

where $\lambda, \mu$ are the Lamé constants. Then (28), (30) take the form

$$\Pi'(u; \Omega_0) = \frac{1}{2} \int_{\Omega_0} \sigma_{ij}(u)(\chi_{,1}\varepsilon_{ij}(u) - \chi_{,i}u_{j,1} - \chi_{,j}u_{i,1}) - \int_{\Omega_0} (\chi f_i)_{,1}u_i + \int_{\Gamma_0} (\chi g)_{,1}[u_1]$$

$$= \int_{B_{\delta/2}\cap \Gamma_0} f_iu_{i,1} + \int_{\Gamma_0 - B_{\delta/2}} g'[u_1] + \int_{\partial B_{\delta/2}} \sigma_{ij}(u)(u_{i,1} - 1/2 \varepsilon_{ij}(u)n_1).$$

We introduce the local polar coordinates in a neighbourhood of the crack tip $(l, 0)$ as follows:

$$x_1 - l = r \cos \phi, \quad x_2 = r \sin \phi, \quad r \geq 0, \quad |\phi| \leq \pi.$$

In these coordinates we have

$$B_{\delta/2} = \{0 \leq r < \delta/2, |\phi| \leq \pi\}, \quad \partial B_{\delta/2} = \{r = \delta/2, |\phi| \leq \pi\},$$

$$B_{\delta/2} \cap \Gamma_0 = \{0 < r < \delta/2, |\phi| = \pi\}, \quad n = (\cos \phi, \sin \phi).$$

The derivatives are given by

$$u_{,1} = \cos \phi u_r - \frac{\sin \phi}{r} u_\phi, \quad u_{,2} = \sin \phi u_r + \frac{\cos \phi}{r} u_\phi.$$

Let us define the following decomposition of the displacement vector in the polar coordinates $(r, \phi)$,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \Lambda(\phi) \begin{pmatrix} u_r \\ u_\phi \end{pmatrix}, \quad \Lambda(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}, \quad \Lambda^{-1}(\phi) = \Lambda(-\phi),$$

and the corresponding decomposition of strains and stresses:

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \Lambda(\phi) \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\phi} \end{pmatrix} \Lambda(-\phi),$$

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \Lambda(\phi) \begin{pmatrix} \sigma_{rr} & \sigma_{r\phi} \\ \sigma_{\phi r} & \sigma_{\phi\phi} \end{pmatrix} \Lambda(-\phi).$$
Because of the symmetry of $\{e_{ij}\}$, we have

$$
e_{rr}(u) = u_{r,r}, \quad e_{r\phi}(u) = \frac{1}{2} \left( u_{\phi,r} + \frac{1}{r} u_{r,\phi} - \frac{1}{r} u_{r} \right), \quad e_{\phi\phi}(u) = \frac{1}{r} u_{\phi,\phi} + \frac{1}{r} u_{\phi} \tag{37}$$

and, due to (35),

$$
\sigma_{rr}(u) = (\lambda + 2\mu) e_{rr}(u) + \lambda e_{\phi\phi}(u), \quad \sigma_{r\phi}(u) = 2\mu e_{r\phi}(u), \\
\sigma_{\phi\phi}(u) = (\lambda + 2\mu) e_{\phi\phi}(u) + \lambda e_{rr}(u). \tag{38}
$$

In the new notation, the first derivative of the potential energy can be rewritten as

$$
\Pi'(u, \Omega_0) = \int_{r = \delta/2, |\phi| \leq \pi} ((\sigma_{rr}(u) e_{rr}(u) + \sigma_{r\phi}(u) u_{\phi,\phi}) \cos \phi \\
- \left( \sigma_{rr}(u) \left( \frac{1}{r} u_{r,\phi} - \frac{1}{r} u_{\phi} \right) + \sigma_{r\phi}(u) e_{\phi\phi}(u) \right) \sin \phi \\
- \frac{1}{2} (\sigma_{rr}(u) e_{rr}(u) + 2\sigma_{r\phi}(u) e_{r\phi}(u) + \sigma_{\phi\phi}(u) e_{\phi\phi}(u)) \cos \phi r \, d\phi \\
+ \int_{0 < r < \delta/2, |\phi| < \pi} ((f_r e_{rr}(u) + f_\phi u_{\phi,r}) \cos \phi \\
- \left( f_r \left( \frac{1}{r} u_{r,\phi} - \frac{1}{r} u_{\phi} \right) + f_\phi e_{\phi\phi}(u) \right) \sin \phi ) r \, dr \, d\phi \\
+ \int_{0 < r < \delta/2} g' |u_r| r |u_r| \, dr.
\tag{39}
$$

6.1. Singular solution

We assume that $f \equiv 0$ in the neighbourhood $B_{\delta_0}$ of the crack tip $(l, 0)$. The equilibrium equations (5) for the Lamé system in $\Omega_0$,

$$
- \mu \Delta u_i - (\lambda + \mu) (\text{div } u)_i = f_i, \quad i = 1, 2,
$$

provide the following equations in the polar coordinates, which are fulfilled in $B_{\delta_0} \setminus \Gamma_0$:

$$
- (\lambda + 2\mu) \left( u_{r,r} + \frac{1}{r} u_{\phi,\phi} + \frac{1}{r} u_{r} \right) + \frac{\mu}{r} \left( u_{\phi,r} - \frac{1}{r} u_{r,\phi} + \frac{1}{r} u_{\phi} \right) = 0, \\
- \frac{\mu}{r} \left( u_{r,r} + \frac{1}{r} u_{\phi,\phi} + \frac{1}{r} u_{r} \right) - \mu \left( u_{\phi,r} - \frac{1}{r} u_{r,\phi} + \frac{1}{r} u_{\phi} \right) = 0. \tag{40}
$$
We seek the singular solution of (40), i.e. a function from the $H^1$-class but not from the $H^2$-class, in the form
\[ u(r, \phi) = r^2 \Phi(\phi), \quad \Phi = (\Phi_r, \Phi_\phi), \quad 0 < \alpha < 1. \] (41)

Substituting (41) into (40), we obtain the equation for $\Phi$:
\[
\begin{align*}
\mu \Phi'' - (\lambda + 2\mu)(1 - \alpha^2)\Phi_r + (\alpha(\lambda + \mu) - \lambda - 3\mu)\Phi_\phi &= 0, \\
(\lambda + 2\mu)\Phi_\phi'' - \mu(1 - \alpha^2)\Phi_\phi + (\alpha(\lambda + \mu) + \lambda + 3\mu)\Phi_r' &= 0,
\end{align*}
\] (42)

where $'$ denotes the derivation to $\Phi$. The following four vectors form the fundamental system of solutions for system (42) [14, 6]:
\[
\begin{align*}
\Phi^1 &= (\cos(1 + \alpha)\phi, -\sin(1 + \alpha)\phi), \\
\Phi^2 &= (\sin(1 + \alpha)\phi, \cos(1 + \alpha)\phi), \\
\Phi^3 &= ((\kappa - \alpha)\cos(1 - \alpha)\phi, (\kappa + \alpha)\sin(1 - \alpha)\phi), \\
\Phi^4 &= ((\kappa - \alpha)\sin(1 - \alpha)\phi, (\kappa + \alpha)\cos(1 - \alpha)\phi), \quad \kappa = \frac{\lambda + 3\mu}{\lambda + \mu}.
\end{align*}
\] (43)

Let us consider the function
\[ U = r^2 \Phi, \quad \Phi = c_1 \Phi^1 + c_2 \Phi^2 + c_3 \Phi^3 + c_4 \Phi^4 \] (44)

with arbitrary constants $c_i$, $i = 1, 2, 3, 4$, which solves (40). Due to (37) we get
\[
\begin{align*}
\varepsilon_{rr}(U) &= \alpha r^{\alpha - 1} \Phi_r, \\
\varepsilon_{\phi\phi}(U) &= r^\alpha (\Phi_\phi' + \Phi_r), \\
\varepsilon_{r\phi}(U) &= \frac{1}{2} r^{\alpha - 1} (\Phi_r' - (1 - \alpha)\Phi_\phi),
\end{align*}
\] (45)

and therefore, inserting (45) into (38),
\[
\begin{align*}
\sigma_{rr}(U) &= r^{\alpha - 1} (\lambda \Phi_\phi' + (\lambda + \alpha(\lambda + 2\mu))\Phi_r), \\
\sigma_{\phi\phi}(U) &= r^{\alpha - 1} ((\lambda + 2\mu)\Phi_\phi' + (\lambda + 2\mu + \alpha\lambda)\Phi_r), \\
\sigma_{r\phi}(U) &= \mu r^{\alpha - 1} (\Phi_r' - (1 - \alpha)\Phi_\phi).
\end{align*}
\] (46)

We assume also that $g = 0$ at $B_{\phi_0} \cap \Gamma_0$, so that the boundary conditions (6) and (7) at $B_{\phi_0} \cap \Gamma_0$ will take the form
\[
\begin{align*}
[\sigma_{\phi\phi}(u)] = 0, & \quad \sigma_{\phi\phi}(u) \leq 0, & \quad [u_\phi] \leq 0, & \quad \sigma_{\phi\phi}(u)[u_\phi] = 0, \\
[\sigma_{r\phi}(u)] = 0, & \quad \sigma_{r\phi}(u) = 0.
\end{align*}
\] (47)

From (43) and (46) we deduce that
\[
\begin{align*}
[\sigma_{\phi\phi}(U)] &= \sigma_{\phi\phi}(U)|_{\phi = \pi} - \sigma_{\phi\phi}(U)|_{\phi = -\pi} \\
&= 4\mu \alpha r^{\alpha - 1} (-c_2 \sin(1 + \alpha)\pi + c_4 (1 + \alpha)\sin(1 - \alpha)\pi), \\
[\sigma_{r\phi}(U)] &= 4\mu \alpha r^{\alpha - 1} (-c_1 \sin(1 + \alpha)\pi + c_3 (1 - \alpha)\sin(1 - \alpha)\pi).
\end{align*}
\] (48)
For $0 < \alpha < 1$ we have $\sin(1 + \alpha)\pi < 0$, $\sin(1 - \alpha)\pi > 0$, and therefore, due to (48), boundary conditions (47) give us

$$c_1 = c_3(1 - \alpha)\frac{\sin(1 - \alpha)\pi}{\sin(1 + \alpha)\pi}, \quad c_2 = c_4(1 + \alpha)\frac{\sin(1 - \alpha)\pi}{\sin(1 + \alpha)\pi}. \quad (49)$$

In view of (49), the remaining boundary conditions in (47) can be calculated as

$$\sigma_{\phi\phi}(U) = 2\mu x r^{x-1} \frac{\sin 2\alpha \pi}{\sin(1 + \alpha)\pi} c_3, \quad \sigma_{r\phi}(U) = -2\mu x r^{x-1} \frac{\sin 2\alpha \pi}{\sin(1 + \alpha)\pi} c_4,$$

$$[U_\phi] = -2(1 + \kappa)r^2c_3\sin(1 - \alpha)\pi, \quad [U_r] = 2(1 + \kappa)r^2c_4\sin(1 - \alpha)\pi. \quad (50)$$

Substituting (50) in (47), the condition $\sigma_{r\phi}(U) = 0$ holds only when $\sin 2\alpha \pi = 0$ or $c_2 = c_4 = 0$. Similarly, the condition

$$0 = \sigma_{\phi\phi}(U)[U_\phi] = -4\mu(1 + \kappa)x r^{2x-1} \sin 2\alpha \pi \frac{\sin(1 - \alpha)\pi}{\sin(1 + \alpha)\pi} c_3^2$$

holds only when $\sin 2\alpha \pi = 0$ or $c_1 = c_3 = 0$. Thus, we have obtained that the non-trivial linear combination (44) fulfills boundary conditions (47) only for $\sin 2\alpha \pi = 0$, i.e., $\alpha = \frac{1}{2}$.

For $\alpha = \frac{1}{2}$ we have $c_1 = -\frac{1}{2}c_3$, $c_2 = -\frac{3}{2}c_4$, and therefore the fundamental system (43) is reduced to the following two vectors:

$$\Psi^1 = (2\Phi^3 - \Phi^1)|_{x=1/2}$$

$$= \left( (2\kappa - 1)\cos \phi \frac{3}{2} - \cos \frac{3\phi}{2}, -(2\kappa + 1)\sin \phi \frac{3}{2} + \sin \frac{3\phi}{2} \right). \quad (51)$$

$$\Psi^2 = (2\Phi^4 - 3\Phi^2)|_{x=1/2}$$

$$= \left( (2\kappa - 1)\sin \phi \frac{3}{2} - 3\sin \frac{3\phi}{2}, (2\kappa + 1)\cos \phi \frac{3}{2} - 3\cos \frac{3\phi}{2} \right).$$

Their linear combination

$$U = r^{1/2}\Psi, \quad \Psi = K_1\Psi^1 + K_2\Psi^2, \quad K_1 > 0, \quad (52)$$

fulfills now the equilibrium equations (40) in $B_{\alpha_e}\setminus \Gamma_0$ and the following boundary conditions at $B_{\alpha_e} \cap \Gamma_0$:

$$\sigma_{\phi\phi}(U) = \sigma_{r\phi}(U) = 0, \quad [U_\phi] = -4(1 + \kappa)r^{1/2}K_1 \leq 0.$$

Note, that (51) corresponds to the singular solution of the problem with Neumann conditions at the crack.

6.2. The first derivative of the energy functional

The next step is to calculate the derivative of the potential energy for the obtained singular solution $U$ in the case of $f = 0$, $g = 0$ in $B_{\alpha_e^{1/2}}$. Together with (37) and (38),
expression (39) can be rewritten as

\[
\Pi'(U; \Omega_0) = \int_{r = \delta/2, |\phi| < \pi} \left( \frac{1}{2} \left( \lambda + 2\mu \right) \left( U_{rr}^2 - \left( \frac{1}{r} U_{\phi,\phi} + \frac{1}{r} U_{r} \right)^2 \right) + \mu \left( U_{\phi,r}^2 - \left( \frac{1}{r} U_{r,\phi} - \frac{1}{r} U_{\phi} \right)^2 \right) \cos \phi - \left( \lambda + 2\mu \right) (U_{rr} + \frac{1}{r} U_{\phi,r} + \frac{1}{r} U_{r}) \left( \frac{1}{r} U_{r,\phi} - \frac{1}{r} U_{\phi} \right) + \mu \left( U_{\phi,r}^2 - \left( \frac{1}{r} U_{r,\phi} + \frac{1}{r} U_{\phi} \right) \left( \frac{1}{r} U_{\phi,\phi} + \frac{1}{r} U_{r} \right) \right) \sin \phi \right) r \, d\phi
\]

or, with (52),

\[
\Pi'(U; \Omega_0) = -\int_{|\phi| < \pi} \left( \lambda + 2\mu \right) (\Psi'_{\phi} + \frac{3}{2} \Psi_r) \left( \frac{1}{2} \left( \Psi_{\phi} + \frac{1}{2} \Psi_r \right) \cos \phi + (\Psi'_r - \Psi'_{\phi}) \sin \phi \right) + \mu \left( \Psi'_{r} - \frac{3}{2} \Psi_{\phi} \right) \left( \frac{1}{2} \left( \Psi'_r - \frac{1}{2} \Psi_{\phi} \right) \cos \phi - (\Psi'_{\phi} + \Psi'_r) \sin \phi \right) \right) d\phi. \tag{53}
\]

Substituting (51) in (53), we have

\[
\Pi'(U; \Omega_0) = -2 \int_{|\phi| < \pi} \left( \lambda + 2\mu \right) (\kappa - 1) \left( K_1 \cos \frac{\phi}{2} + K_2 \sin \frac{\phi}{2} \right) \times \left( K_1 (\kappa + 2) \sin \phi \sin \frac{\phi}{2} + K_2 2 \sin \frac{\phi}{2} - (\kappa + 2) \sin \phi \sin \frac{\phi}{2} \right) - \mu(\kappa + 1) \left( K_1 \sin \frac{\phi}{2} - K_2 \cos \frac{\phi}{2} \right) \times \left( K_1 (\kappa - 2) \sin \phi \cos \frac{\phi}{2} + K_2 \left( 2 \cos \frac{\phi}{2} + (\kappa - 2) \sin \phi \sin \frac{\phi}{2} \right) \right) \right) d\phi
\]

\[
= -\frac{8\mu(\lambda + 2\mu)}{\lambda + \mu} \int_{|\phi| < \pi} \left( K_1^2 \sin^2 \phi + K_2^2 \cos^2 \phi \right) \right) d\phi
\]

\[
= -\frac{8\pi \mu(\lambda + 2\mu)}{\lambda + \mu} \left( K_1^2 + K_2^2 \right).
\]

This expression corresponds to the well-known Irwin formula for the linear model of a solid with a crack under the stress-free boundary conditions [3, 2].

We summarize the obtained results in the following theorem. Let \( \chi(r) \) be a smooth cut-off function with the support in \( B_\delta \) as before.
Theorem 3. If \( f = 0, \ g = 0 \) in a neighbourhood of the crack tip, the singular solution \( U = \chi(r)r^\alpha \Phi(\phi), \ 0 < \alpha < 1, \) to problem (3) for the Lamé system is of the form

\[
U = \chi(r)r^{1/2} \left( K_1 \Psi_1(\phi) + K_2 \Psi_2(\phi) \right), \quad K_1 \geq 0,
\]

where \( \Psi^i, \ i = 1, 2, \) are given by (51). The first derivative of the energy functional

\[
\Pi'(U; \Omega_0) = -\frac{8\pi \mu(\lambda + 2\mu)}{\lambda + \mu} (K_1^2 + K_2^2) \leq 0.
\]

Let us make some remarks concerning the solution \( u \) of the variational inequality (3). First, if \( [u] = 0 \) in some neighbourhood \( B_\delta \) of the crack tip, then, as it was marked before, it follows from the Lemmas 1 and 4 that \( u \in \mathcal{H}^2(B_\delta) \) and \( \Pi'(u; \Omega_0) = 0. \)

Second, if \( [u_2] > 0, \ [u_1] > 0 \) (or \( [u_1] < 0 \)) in \( B_\delta \), then we have by (6), (7) the Neumann problem in \( B_\delta \) with the boundary conditions \( \sigma_{22}(u) = 0, \ \sigma_{12}(u) = g \) (or \( \sigma_{12}(u) = -g \), respectively). Therefore, due to the local regularity of the solution as shown in Theorem 1 and the smoothness of \( g \), it follows from the results of [6] that the solution \( u \) can be locally represented as \( \chi u = U + w \), where \( w \in \mathcal{H}^2(B_\delta \setminus \Gamma_0), \ U \) is from (54), and \( \Pi'(u; \Omega_0) = \Pi'(U; \Omega_0) \). But in the general case, while the solution lose regularity, one can not provide the fulfillment of the above assumptions.

7. Second derivative of the energy functional

We look for an expression of the second derivative of the energy functional with respect to the crack length. First, by Lemma 2, the sequence \( \{[\tilde{u}^\varepsilon - u]/\varepsilon\} \) is bounded in \( \mathcal{H}^1(\Omega_0) \) uniformly in \( \varepsilon \), so that there exists a weak limit \( \hat{u} \in \mathcal{H}^1(\Omega_0) \) and a subsequence \( \varepsilon_k \) such that

\[
\frac{\tilde{u}^\varepsilon - u}{\varepsilon_k} \rightharpoonup \hat{u} \text{ weakly in } \mathcal{H}^1(\Omega_0) \quad \text{as } \varepsilon_k \to 0.
\]

The function \( \hat{u} \) is called the weak material derivative. The uniform estimate

\[
\left\| \frac{[\tilde{u}^\varepsilon_1] - [u_1]}{\varepsilon} \right\|_{L^2(\Gamma_0)} \leq \left\| \frac{\tilde{u}^\varepsilon - u}{\varepsilon} \right\|_{L^2(\Gamma_0)} \leq \text{const}
\]

provides also the following convergence for some subsequence:

\[
\frac{[\tilde{u}^\varepsilon_1] - [u_1]}{\varepsilon_k} \rightharpoonup z_g \quad \text{weakly in } L^2(\Gamma_0) \quad \text{as } \varepsilon_k \to 0.
\]

We call \( z_g \) the boundary weak material derivative associated with the friction condition.
Due to Theorem 2, in analogy to (28), we have the first derivative of the energy functional at the displacement \( u' \) in the domain \( \Omega_\varepsilon \):

\[
\Pi'(u'; \Omega_\varepsilon) = \int_{\Omega_\varepsilon} \left( \frac{1}{2} \left( \rho c_{ijkl},_1 \varepsilon_{kl}(u') \varepsilon_{ij}(u') - \sigma_{ij}(u') E_{ij}''(u') \right) \right. \\
- \left. \int_{\Gamma_\varepsilon} (\rho f_i),_1 u'_i \right) \\
- \left( \rho g \right),_1 \left[ [u'_1] \right], \quad E_{ij}''(u') = \frac{1}{2} \left( \rho_i u'_j, + \rho_j u'_i, \right),
\]

where the cut-off function \( \rho(y), y \in \Omega_\varepsilon \) is chosen such that \( \hat{\rho}(x) = \chi(x), x \in \Omega_0 \). Again, applying transformation (15) to (57), one obtains \( \Pi'(u'; \Omega_\varepsilon) = \Pi'_\varepsilon(\hat{u}; \Omega_0) \), where

\[
\Pi'_\varepsilon(\hat{u}; \Omega_0) = \int_{\Omega_0} \left( \frac{1}{2} (\chi \hat{c}_{ijkl},_1 \varepsilon_{kl}(\hat{u}) \varepsilon_{ij}(\hat{u}) - \frac{1}{1 + \varepsilon \chi_1} \hat{\sigma}_{ij}(\hat{u}) E_{ij}(\hat{u})) \right) \\
+ \frac{\varepsilon^2}{2} \int_{\Omega_0} \frac{(\chi \hat{c}_{ijkl},_1)}{(1 + \varepsilon \chi_1)^2} E_{kl}(\hat{u}) E_{ij}(\hat{u}) \\
- \int_{\Omega_0} (\chi \hat{f},_1 \hat{u}_i \right) + \int_{\Gamma_0} (\chi \hat{g},_1 [\hat{u}_1]) \\
+ \varepsilon \int_{\Omega_0} \left( \frac{1}{(1 + \varepsilon \chi_1)^2} \hat{S}_{ij}(\hat{u}) E_{ij}(\hat{u}) - \frac{(\chi \hat{c}_{ijkl},_1)}{1 + \varepsilon \chi_1} \varepsilon_{kl}(\hat{u}) E_{ij}(\hat{u}) \right)
\]

We look for the expression of the second derivative:

\[
\lim_{\varepsilon \to 0} \frac{\Pi'(u'; \Omega_\varepsilon) - \Pi'(u; \Omega_0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\Pi'_\varepsilon(\hat{u}; \Omega_0) - \Pi'(u; \Omega_0)}{\varepsilon}
\]

By (28), (58), we have the following equality:

\[
\frac{\Pi'_\varepsilon(\hat{u}; \Omega_0) - \Pi'(u; \Omega_0)}{\varepsilon} = \int_{\Omega_0} \left( \frac{1}{2} \left( \chi \hat{c}_{ijkl} - c_{ijkl},_1 \right) \varepsilon_{kl}(\hat{u}) \varepsilon_{ij}(\hat{u}) - \frac{\chi \hat{c}_{ijkl} - c_{ijkl},_1}{\varepsilon} \varepsilon_{kl}(\hat{u}) E_{ij}(\hat{u}) \right) \\
+ \frac{\varepsilon^2}{2} \int_{\Omega_0} \frac{(\chi \hat{c}_{ijkl},_1)}{(1 + \varepsilon \chi_1)^2} E_{kl}(\hat{u}) E_{ij}(\hat{u}) - \int_{\Omega_0} \left( \chi \hat{f}_i - f_i \right),_1 \hat{u}_i + \int_{\Gamma_0} \left( \chi \hat{g} - g \right),_1 [\hat{u}_1] \\
+ \int_{\Omega_0} \left( \frac{1}{(1 + \varepsilon \chi_1)^2} \hat{S}_{ij}(\hat{u}) E_{ij}(\hat{u}) - \frac{\chi \hat{c}_{ijkl},_1}{1 + \varepsilon \chi_1} \varepsilon_{kl}(\hat{u}) E_{ij}(\hat{u}) \right) \\
+ \int_{\Omega_0} \left( \frac{1}{2} (\chi c_{ijkl},_1) \varepsilon_{kl}(\hat{u}^e + u) - \hat{S}_{ij}(\hat{u}) \right) \varepsilon_{ij} \left( \frac{\hat{u}^e - u}{\varepsilon} \right) \\
- \sigma_{ij}(u) E_{ij} \left( \frac{\hat{u}^e - u}{\varepsilon} \right) - \int_{\Omega_0} (\chi \hat{f}_i),_1 \frac{\hat{u}_i - u_i}{\varepsilon} + \int_{\Gamma_0} (\chi \hat{g},_1 \left[ [\hat{u}_1] \right] - [u_1]) \right.
\]
In view of the convergences given in (55) and (56) and due to Lemma 2, we can pass to the limit as \( \varepsilon_k \to 0 \) and obtain that

\[
\lim_{\varepsilon_k \to 0} \frac{\Pi'_{\varepsilon_k}(\hat{u}; \Omega_0) - \Pi'(u; \Omega_0)}{\varepsilon_k} = \Pi''(u, \hat{u}, z_g; \Omega_0),
\]

where

\[
\Pi''(u, \hat{u}, z_g; \Omega_0) = \int_{\Omega_0} \left( \frac{1}{2} \left( \chi^2 c_{ijkl,1},1 \varepsilon_{kl}(u) \varepsilon_{ij}(u) - 2 \chi c_{ijkl,1} \varepsilon_{kl}(u) E_{ij}(u) \right) + \Sigma_{ij}(u) E_{ij}(u) - \int_{\Omega_0} (\chi^2 f_{i,1},1 u_i + \int_{\Gamma_0} (\chi^2 g',1 [u_1]) \\
+ \int_{\Omega_0} (\chi f_{i,1},1 \hat{u}_i + \int_{\Gamma_0} (\chi g,1 z_g) \right) + \int_{\Omega_0} (\chi f_{i,1},1 \hat{u}_i + \int_{\Gamma_0} (\chi g,1 z_g).
\]

(59)

Note that, dividing (20) with \( \varepsilon^2 \) and passing to the limit infimum in (20), we get in view of (55) and (56) the estimate

\[
\int_{\Omega_0} ((\chi c_{ijkl,1} \varepsilon_{kl}(u) - \Sigma_{ij}(u) \varepsilon_{ij}(\hat{u}) - \sigma_{ij}(u) E_{ij}(\hat{u})) \\
- \int_{\Omega_0} (\chi f_{i,1},1 \hat{u}_i + \int_{\Gamma_0} (\chi g,1 z_g \leq \int_{\Omega_0} \sigma_{ij}(\hat{u}) \varepsilon_{ij}(\hat{u}) \leq 0.
\]

Thus, the following result is proved.

**Lemma 5.** There is a subsequence from \( \varepsilon \) such that the partial limit of the sequence \((\Pi'(u'; \Omega_{\varepsilon}) - \Pi'(u; \Omega_0))/\varepsilon \) exists as \( \varepsilon \to 0 \) and has the form \( \Pi''(u, \hat{u}, z_g; \Omega_0) \) given by (59).

**Remark.** While the convergences (55) and (56) do not guarantee the uniqueness of the material derivatives, (59) is not enough to provide the existence of the second derivative to the energy functional \( \Pi \) in the general case. When the domain is smooth, see the material derivatives of the solution with respect to the domain variation in [15].

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