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Derivatives of the Energy Functional for 2D-Problems with a Crack under Signorini and Friction Conditions

Michael Bach[1,*], Alexander M. Khludnev[2] and Victor A. Kovtunenko[1,3]

[1] *Mathematical Institute A, Stuttgart University, Pfaffenwaldring 57, Postfach 8011, 70569 Stuttgart, Deutschland, Germany*
[2] *Institute of Hydrodynamics, 630090 Novosibirsk, Russia*
[3] *Institute of Hydrodynamics, 630090 Novosibirsk, Russia*

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We consider the two-dimensional elasticity problem for an elastic body with a crack under unilateral constraints imposed at the crack. We assume that both the Signorini condition for non-penetration of the crack faces and the condition of given friction between them are fulfilled. The problem is non-linear and can be described by a variational inequality. Varying the shape of the crack by a local coordinate transformation of the domain, the first derivative of the energy functional to the problem with respect to the crack length is obtained, which gives the criterion for the crack growing. The regularity of the solution is discussed and the singular solution is performed. Copyright © 2000 John Wiley & Sons, Ltd.

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Mathematical investigations of the crack problems mainly deal with the singularities arising near the crack tips [10, 12, 14, 6]. An important question concerns the propagation of the crack [14, 1] and the closed problem of the shape sensitivity analysis [15, 9, 2]. By this, the crack faces are usually assumed to be stress-free. Taking into the consideration unilateral constraints at the crack faces [9, 8, 11], we obtain non-linear crack problems, which may not keep all the properties of the solutions for the linear ones. Therefore, it is necessary to apply general variational principles. The crack growth depends then on properties for the derivatives of the energy functional with respect to the crack length [3, 16]. Our aim is to extend the known results onto the crack problems with unilateral constraints and to generalize the methods applied.

* Correspondence to : Michael Bach, Mathematical Institute A, Stuttgart University, Pfaffenwaldring 57, Postfach 8011, 70569 Stuttgart, Deutschland, Germany

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1. Formulation of the problem

Let $\Omega \subset \mathscr{R}^2$ be a bounded domain with a boundary Γ of the class $C^{2,1}$. Define the set $\Gamma_0 = (0, l) \times \{0\}$ in $\mathscr{R}^2$ with the normal vector (0, 1) and the tangential vector (1, 0) to Γ_0. Its boundary $\partial\Gamma_0$ consists of the points (0, 0), (l, 0) and $\bar{\Gamma}_0 = \Gamma_0 \cup \partial\Gamma_0$. Assume that $[0, l + l_0] \subset \Omega$. Denote $\Omega_0 = \Omega \backslash \bar{\Gamma}_0$. We consider the two-dimensional elastic body occupying the domain Ω_0 with the crack Γ_0.

Let $f = (f_1, f_2), f_i \in C^2(\bar{\Omega})$, $i = 1, 2$ be a given load, and $g \in C^2([0, l + l_0])$, $g \geqslant 0$, be a given friction force between the crack faces $\Gamma_0^{\pm}$, where $\Gamma_0^{\pm} = \Gamma_0 \cap \{x \in \mathscr{R}^2,\ \pm x_2 \geqslant 0\}$. For example, g can be understood as an approximation of the Coulomb friction law. We consider displacements $u = (u_1, u_2)$, linear strains $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $i, j = 1, 2$, and Hooke stresses $\sigma_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u)$, $i, j = 1, 2$, in the body, $c_{ijkl} \in C^2(\bar{\Omega})$, $i, j, k, l = 1, 2$, $c_1\xi_{ij}\xi_{ij} \leqslant c_{ijkl}\xi_{kl}\xi_{ij} \leqslant c_2\xi_{ij}\xi_{ij}$. Here $u_{,j} = \partial u/\partial x_j$, $j = 1, 2$.

Let us introduce the space

$$\tilde{H}^1(\Omega_0) = \{u = (u_1, u_2) \in [H^1(\Omega_0)]^2,\ u = 0 \text{ on } \Gamma\}.$$

We assume the validity of a non-penetration condition for the crack faces, which has the following form for the jump of the displacements [9]:

$$[u_2] = u_2|_{\Gamma_0^+} - u_2|_{\Gamma_0^-} \geqslant 0,$$

and which gives us the convex closed set of admissible displacements

$$K_0 = \{u = (u_1, u_2) \in \tilde{H}^1(\Omega_0),\ [u_2] \geqslant 0 \text{ on } \Gamma_0\}.$$

In the following, we consider the functional of the potential energy for the body with given friction [4, 7]:

$$\Pi(u; \Omega_0) = \frac{1}{2}\int_{\Omega_0} \sigma_{ij}(u)\varepsilon_{ij}(u) - \int_{\Omega_0} f_i u_i + \int_{\Gamma_0} g|[u_1]|. \tag{1}$$

Here the usual summation convention over repeated indices is employed. The equilibrium problem

$$\Pi(u; \Omega_0) = \inf_{v \in K_0} \Pi(v; \Omega_0) \tag{2}$$

is equivalent to the variational inequality

$$\int_{\Omega_0} \sigma_{ij}(u)\varepsilon_{ij}(v - u) + \int_{\Gamma_0} g(|[v_1]| - |[u_1]|) \geqslant \int_{\Omega_0} f_i(v_i - u_i), \quad \forall v \in K_0. \tag{3}$$

Under the assumption of $u = 0$ on Γ, Korn's inequality

$$c\,\|u\|^2_{H^1(\Omega_0)} \leqslant \int_{\Omega_0} \sigma_{ij}(u)\varepsilon_{ij}(u), \tag{4}$$

guarantees, that problems (2) or (3) have the unique solution $u \in K_0$. From the variational inequality (3) it follows that the equilibrium equation in Ω_0,

$$-\sigma_{ij,j}(u) = f_i, \quad i = 1, 2 \tag{5}$$

and the boundary conditions on Γ_0,

$$[\sigma_{22}(u)] = 0, \quad \sigma_{22}(u) \leqslant 0, \quad [u_2] \geqslant 0, \quad \sigma_{22}(u)[u_2] = 0, \tag{6}$$

$$[\sigma_{12}(u)] = 0, \quad |\sigma_{12}(u)| \leqslant g, \quad \sigma_{12}(u)[u_1] - g|[u_1]| = 0, \tag{7}$$

hold in the weak sense (for the exact meaning see [8]). Boundary conditions (6) arise due to the Signorini condition of non-penetration, whereas conditions (7) correspond to the friction condition between the crack faces.

2. Regularity of the solution

We investigate the smoothness properties of the solution $u \in K_0$ of problem (3), following the ideas of [8].

Theorem 1. *For a smooth function ρ, $0 \leqslant \rho \leqslant 1$, with* $\operatorname{supp} \rho \cap \partial\Gamma_0 = \emptyset$, *the inclusion $\rho u \in [H^2(\Omega_0)]^2$ is valid.*

Proof. First, the needed H^2-smoothness of ρu inside Ω_0 and near Γ follows in a standard way (see, for instance [5]) from (5), $f \in [L^2(\Omega_0)]^2$, $u = 0$ on Γ and the regularity of Γ. We prove here the smoothness of ρu near Γ_0.

Let $B_\delta(x^0)$ be a circle centered in the point $x^0 \in \Gamma_0$ and with the radius $\delta < \operatorname{dist}(x^0, \partial\Gamma_0)$. We choose the smooth cut-off function ρ such that $\operatorname{supp} \rho \subset B_\delta(x^0)$. For a positive number h, define the operator $d_h = d_{+h}$ and its adjoint d_{-h} as

$$d_{\pm h}\phi := \frac{1}{h}(\phi_{\pm h} - \phi), \quad \phi_{\pm h}(x_1, x_2) = \phi(x_1 \pm h, x_2).$$

Consider the function

$$v = u - \frac{h^2}{2}\rho d_{-h} d_h(\rho u) = (1 - \rho^2)u + \frac{1}{2}\rho((\rho u)_h + (\rho u)_{-h}).$$

If $h < \operatorname{dist}(x^0, \partial\Gamma_0)$, then we have $v \in K_0$ because of

$$[v_2] = (1 - \rho^2)[u_2] + \tfrac{1}{2}\rho(\rho_h[u_2]_h + \rho_{-h}[u_2]_{-h}) \geqslant 0$$

and one can substitute v as a test function in (3). This gives

$$\int_{\Omega_0} \sigma_{ij}(d_h(\rho u))\varepsilon_{ij}(d_h(\rho u)) \leqslant I_1 + I_2 + I_3, \tag{8}$$

where

$$I_1 = \int_{\Omega_0} (\sigma_{ij}(d_h(\rho u))\varepsilon_{ij}(d_h(\rho u)) - \sigma_{ij}(u)\varepsilon_{ij}(\rho d_{-h} d_h(\rho u))),$$

$$I_2 = \int_{\Omega_0} \rho f_i d_{-h} d_h(\rho u_i), \qquad I_3 = \frac{2}{h_2}\int_{\Gamma_0} g\left(\left|\left[u_1 - \frac{h^2}{2}\rho d_{-h} d_h(\rho u_1)\right]\right| - |[u_1]|\right).$$

Let us denote $e_{ij}(u) = \frac{1}{2}(\rho_{,i}u_j + \rho_{,j}u_i)$, $i = 1, 2$, and apply the estimate

$$\|d_{\pm h}(\rho\phi)\|_{L^2(\Omega_0)} \leqslant \|(\rho\phi)_{,1}\|_{L^2(\Omega_0)} \leqslant c\,\|\rho\phi\|_{H^1(\Omega_0)}. \tag{9}$$

We obtain

$$I_1 = \int_{\Omega_0} (-\sigma_{ij}(u)e_{ij}(d_{-h}d_h(\rho u))$$

$$+\, d_h(c_{ijkl}e_{kl}(u))\varepsilon_{ij}(d_h(\rho u)) - d_h c_{ijkl}\varepsilon_{kl}(\rho u)_h\varepsilon_{ij}(d_h(\rho u)))$$

$$\leqslant c\,\|u\|_{H^1(\Omega_0)}\|d_h(\rho u)\|_{H^1(\Omega_0)},$$

$$I_2 \leqslant c\,\|\rho f\|_{L^2(\Omega_0)}\|d_h(\rho u)\|_{H^1(\Omega_0)}.$$

To evaluate I_3, we use (9), the inequalities for the modules, the positiveness of the friction force g and the continuity of the trace operator in $B_\delta^{\pm}(x^0) = B_\delta(x^0) \cap \{x \in \mathscr{R}^2,\ \pm x_2 > 0\}$ [13], and we get

$$I_3 = \frac{2}{h^2}\int_{\Gamma_0} g\left(\left|(1-\rho^2)[u_1] + \frac{1}{2}\rho[(\rho u_1)_h + (\rho u_1)_{-h}]\right| - |[u_1]|\right)$$

$$\leqslant \frac{2}{h^2}\int_{\Gamma_0} \rho g\left(\left|\frac{1}{2}[(\rho u_1)_h + (\rho u_1)_{-h}]\right| - |[\rho u_1]|\right)$$

$$\leqslant \|\rho g\|_{L^2(\Gamma_0)}\|\,|[d_{-h}d_h(\rho u_1)]|\,\|_{L^2(\Gamma_0)} \leqslant c\,\|\rho g\|_{L^2(\Gamma_0)}\|d_h(\rho u_1)\|_{H^1(\Omega_0)}.$$

Due to (4) and (8), we have then the following estimate:

$$\|d_h(\rho u)\|_{H^1(\Omega_0)} \leqslant c(\,\|u\|_{H^1(\Omega_0)} + \|f\|_{L^2(\Omega_0)} + \|g\|_{L^2(\Gamma_0)}),$$

which means that $(\rho u)_{,11}, (\rho u)_{,12} \in [L^2(\Omega_0)]^2$. Because of the ellipticity of the coefficients c_{ijkl}, the equilibrium equation (5) can be rewritten as

$$u_{,22} = \mathscr{L}(f, u, \nabla u, u_{,11}, u_{,12}).$$

In view of the shown regularity of ρu, this leads to $(\rho u)_{,22} \in [L^2(\Omega_0)]^2$. The theorem is proved.

Concerning the smoothness of the solution near $\partial\Gamma_0$, we formulate the following result.

Lemma 1. *If* $[u] = 0$ *in some neighbourhood* B *of the crack tip, then* $\rho u \in [H^2(B)]^2$ *for a cut-off function* ρ *with* $\operatorname{supp}\rho \subset B$.

Proof. Let us take $\phi \in [C_0^\infty(B)]^2$. By $[\phi] = 0$, we can substitute $v = u \pm \phi$ in (3) and obtain

$$\int_{B\cap\Omega_0} \sigma_{ij}(u)\varepsilon_{ij}(\phi) = \int_{B\cap\Omega_0} f_i\phi_i. \tag{10}$$

Applying the Green formulae in the domains $B^{\pm} = B \cap \{x \in \mathscr{R}^2,\ \pm x_2 > 0\}$, it follows from (10) that

$$
\begin{aligned}
&-\sigma_{ij,j}(u) = f_i, \quad i = 1, 2, \quad \text{in } B^{\pm}, \\
&[\sigma_{12}(u)] = [\sigma_{22}(u)] = 0 \quad \text{on } \Gamma_B = \bar{B}^{+} \cap \bar{B}^{-}.
\end{aligned}
\tag{11}
$$

For $[u] = 0$ in $B \cap \Gamma_0$, we have $u_i \in H^1(B)$, $\sigma_{ij,j}(u) \in L^2(B)$, $i = 1, 2$. Because of $f \in [L^2(B)]^2$ and (11), the following distributions are defined for $\phi \in C_0^{\infty}(B)$:

$$
\begin{aligned}
\langle -\sigma_{ij,j}(u), \phi \rangle_B &= \int_B \sigma_{ij}(u)\phi_{,j} = \int_{B^+} \sigma_{ij}(u)\phi_{,j} + \int_{B^-} \sigma_{ij}(u)\phi_{,j} \\
&= -\int_{B^+} \sigma_{ij,j}(u)\phi - \int_{B^-} \sigma_{ij,j}(u)\phi - \langle [\sigma_{i2}(u)], \phi \rangle_{\Gamma_B} \\
&= \int_B f_i \phi, \quad i = 1, 2.
\end{aligned}
$$

This means that the equilibrium equation is fulfilled in the whole domain B, and standard considerations lead to $\rho u \in [H^2(B)]^2$. The lemma is proved.

Note that, due to Lemma 1 and Theorem 1, if $[u] = 0$ near $\partial\Gamma_0$, we have $u \in [H^2(\Omega_0)]^2$.

3. Variation of the crack

We shall vary the shape of the crack by a local coordinate transformation of the domain following the ideas of [8].

For a small parameter $\varepsilon > 0$, let us consider the perturbed domain

$$
\Omega_\varepsilon = \Omega \backslash \bar{\Gamma}_\varepsilon, \qquad \Gamma_\varepsilon = (0, l + \varepsilon) \times \{0\}.
$$

In the domain Ω_ε one can consider, in analogy to K_0 and $\Pi(u; \Omega_0)$, the set of admissible displacements,

$$
K_\varepsilon = \{u = (u_1, u_2) \in \tilde{H}^1(\Omega_\varepsilon), [u_2] \geqslant 0 \text{ on } \Gamma_\varepsilon\},
$$

and the perturbed functional of the potential energy,

$$
\Pi(u; \Omega_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} \sigma_{ij}(u)\varepsilon_{ij}(u) - \int_{\Omega_\varepsilon} f_i u_i + \int_{\Gamma_\varepsilon} g|[u_1]|. \tag{12}
$$

Then the equilibrium problem

$$
\Pi(u^\varepsilon; \Omega_\varepsilon) = \inf_{v \in K_\varepsilon} \Pi(v; \Omega_\varepsilon) \tag{13}
$$

is equivalent to the variational inequality

$$\int_{\Omega_\varepsilon} \sigma_{ij}(u^\varepsilon)\varepsilon_{ij}(v-u^\varepsilon) + \int_{\Gamma_\varepsilon} g(|[v_1]| - |[u_1^\varepsilon]|) \geqslant \int_{\Omega_\varepsilon} f_i(v_i - u_i^\varepsilon) \quad \forall v \in K_\varepsilon \tag{14}$$

and has the unique solution $u^\varepsilon \in K_\varepsilon$ similar to (2) and (3), respectively.

Let B_δ be a circle of radius $\delta > 0$ centred in the crack tip $(l, 0)$, $\delta < l_0$, such that $\bar{B}_\delta \subset \Omega$. Choose the cut-off function $\chi \in C_0^\infty(\Omega)$, $0 \leqslant \chi \leqslant 1$, such that $\operatorname{supp}\chi \subset B_\delta$ and $\chi \equiv 1$ in $B_{\delta/2}$. For small $0 < \varepsilon < \delta/2$, we construct the one-to-one coordinate transformation

$$y_1 = x_1 + \varepsilon\chi(x_1, x_2), \quad y_2 = x_2, \quad (y_1, y_2) \in \Omega_\varepsilon, \quad (x_1, x_2) \in \Omega_0 \tag{15}$$

with the Jacobian $|\partial(y_1, y_2)/\partial(x_1, x_2)| = 1 + \varepsilon\chi_{,1}$, which transforms Ω_ε to Ω_0. We shall denote by $\hat{s}(x)$, $x \in \Omega_0$, the transformed function $s(y)$, $y \in \Omega_\varepsilon$, i.e.,

$$s(y) = s(x_1 + \varepsilon\chi(x), x_2) = \hat{s}(x).$$

Rewriting the derivatives of u with (15), we get

$$u_{i,j} = \hat{u}_{i,j} - \frac{\varepsilon}{1+\varepsilon\chi_{,1}}\chi_{,j}\hat{u}_{i,1}, \quad i, j = 1, 2$$

and therefore

$$\begin{aligned} &\varepsilon_{ij}(u) = \varepsilon_{ij}(\hat{u}) - \frac{\varepsilon}{1+\varepsilon\chi_{,1}} E_{ij}(\hat{u}), \qquad E_{ij}(\hat{u}) = \frac{1}{2}(\chi_{,i}\hat{u}_{j,1} + \chi_{,j}\hat{u}_{i,1}), \\ &\sigma_{ij}(u) = \hat{\sigma}_{ij}(\hat{u}) - \frac{\varepsilon}{1+\varepsilon\chi_{,1}}\hat{\Sigma}_{ij}(\hat{u}), \quad \hat{\sigma}_{ij}(\hat{u}) = \hat{c}_{ijkl}\varepsilon_{kl}(\hat{u}), \quad \hat{\Sigma}_{ij}(\hat{u}) = \hat{c}_{ijkl}E_{kl}(\hat{u}). \end{aligned} \tag{16}$$

The substitution of (16) in (12) provides

$$\begin{aligned} &\Pi(u; \Omega_\varepsilon) = \Pi_\varepsilon(\hat{u}; \Omega_0), \\ &\Pi_\varepsilon(\hat{u}; \Omega_0) = \frac{1}{2}\int_{\Omega_0} \hat{\sigma}_{ij}(\hat{u})((1+\varepsilon\chi_{,1})\varepsilon_{ij}(\hat{u}) - 2\varepsilon E_{ij}(\hat{u})) \\ &\qquad + \frac{1}{2}\int_{\Omega_0} \frac{\varepsilon^2}{1+\varepsilon\chi_{,1}}\hat{\Sigma}_{ij}(\hat{u})E_{ij}(\hat{u}) - \int_{\Omega_0}(1+\varepsilon\chi_{,1})\hat{f}_i\hat{u}_i \\ &\qquad + \int_{\Gamma_0}(1+\varepsilon\chi_{,1})\hat{g}|[\hat{u}_1]|, \end{aligned} \tag{17}$$

and, by (13),

$$\inf_{v \in K_\varepsilon} \Pi(v; \Omega_\varepsilon) = \Pi(u^\varepsilon; \Omega_\varepsilon) = \Pi_\varepsilon(\hat{u}^\varepsilon; \Omega_0) = \inf_{v \in K_0} \Pi_\varepsilon(v; \Omega_0). \tag{18}$$

Therefore, $\hat{u}^\varepsilon \in K_0$ is the unique solution of the following variational inequality:

$$\int_{\Omega_0} (\hat{\sigma}_{ij}(\hat{u}^\varepsilon)((1+\varepsilon\chi_{,1})\varepsilon_{ij}(v-\hat{u}^\varepsilon) - \varepsilon E_{ij}(v-\hat{u}^\varepsilon)) - \varepsilon\hat{\Sigma}_{ij}(\hat{u}^\varepsilon)\varepsilon_{ij}(v-\hat{u}^\varepsilon))$$

$$+ \int_{\Omega_0} \frac{\varepsilon^2}{1+\varepsilon\chi_{,1}} \hat{\Sigma}_{ij}(\hat{u}^\varepsilon)E_{ij}(v-\hat{u}^\varepsilon) + \int_{\Gamma_0} (1+\varepsilon\chi_{,1})\hat{g}(|[v_1]| - |[\hat{u}_1^\varepsilon]|)$$

$$\geqslant \int_{\Omega_0} (1+\varepsilon\chi_{,1})\hat{f}_i(v_i - \hat{u}_i^\varepsilon), \quad \forall v \in K_0. \tag{19}$$

For the connection of the solutions of problems (3) and (19), we prove the following lemma.

Lemma 2. *The solution* $\hat{u}^\varepsilon$ *of variational inequality* (19) *converges strongly to the solution* u *of variational inequality* (3) *in* $\tilde{H}^1(\Omega_0)$ *as* $\varepsilon \to 0$. *There holds the estimate*

$$\|\hat{u}^\varepsilon - u\|_{H^1(\Omega_0)} \leqslant c\varepsilon.$$

Proof. Substituting $v = u$ in (19), $v = \hat{u}^\varepsilon$ in (3) and summing, we obtain

$$\int_{\Omega_0} \sigma_{ij}(\hat{u}^\varepsilon - u)\varepsilon_{ij}(\hat{u}^\varepsilon - u) \leqslant I_1 + I_2 + I_3, \tag{20}$$

where

$$I_1 = \int_{\Omega_0} (\hat{c}_{ijkl} - c_{ijkl})\varepsilon_{kl}(\hat{u}^\varepsilon)\varepsilon_{ij}(u - \hat{u}^\varepsilon)$$

$$+ \int_{\Gamma_0} (\hat{g} - g)(|[u_1]| - |[\hat{u}_1^\varepsilon]|) + \int_{\Omega_0} (\hat{f}_i - f_i)(\hat{u}_i^\varepsilon - u_i),$$

$$I_2 = \varepsilon \int_{\Omega_0} (\hat{\sigma}_{ij}(\hat{u}^\varepsilon)(\chi_{,1}\varepsilon_{ij}(u-\hat{u}^\varepsilon) - E_{ij}(u-\hat{u}^\varepsilon)) - \hat{\Sigma}_{ij}(\hat{u}^\varepsilon)\varepsilon_{ij}(u-\hat{u}^\varepsilon))$$

$$+ \varepsilon \int_{\Gamma_0} \chi_{,1}\hat{g}(|[u_1]| - |[\hat{u}_1^\varepsilon]|) + \varepsilon \int_{\Omega_0} \chi_{,1}\hat{f}_i(\hat{u}_i^\varepsilon - u_i),$$

$$I_3 = \varepsilon^2 \int_{\Omega_0} \frac{1}{1+\varepsilon\chi_{,1}} \hat{\Sigma}_{ij}(\hat{u}^\varepsilon)E_{ij}(u-\hat{u}^\varepsilon).$$

From (3), one can derive the estimate

$$\|u\|_{H^1(\Omega_0)} \leqslant \text{const}, \tag{21}$$

from (14), analogously, $\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant \text{const}$, and

$$\|\hat{u}^\varepsilon\|_{H^1(\Omega_0)} \leqslant \text{const}, \tag{22}$$

uniformly in ε. Using (22) and the Hölder inequality, we evaluate I_3:

$$I_3 \leqslant c\varepsilon^2 \|\hat{u}^\varepsilon - u\|_{H^1(\Omega_0)}.$$

With the inequalities for the modules and the continuity of the trace operator [13] we have

$$\| |[\hat{u}_1^\varepsilon]| - |[u_1]| \|_{L^2(\Gamma_0)} \leqslant \| |[\hat{u}_1^\varepsilon - u_1]| \|_{L^2(\Gamma_0)} \leqslant c \|\hat{u}^\varepsilon - u\|_{H^1(\Omega_0)}. \tag{23}$$

Therefore, with the help of (22) and (23), the integral I_2 can be evaluated by

$$I_2 \leqslant c\varepsilon \|\hat{u}^\varepsilon - u\|_{H^1(\Omega_0)}.$$

The supposed continuous differentiability of c_{ijkl}, g, f provides the estimates

$$|\hat{c}_{ijkl} - c_{ijkl}| \leqslant c\varepsilon, \qquad |\hat{f}_i - f_i| \leqslant c\varepsilon, \qquad |\hat{g} - g| \leqslant c\varepsilon. \tag{24}$$

Altogether (21)–(24) guarantee

$$I_1 \leqslant c\varepsilon \|\hat{u}^\varepsilon - u\|_{H^1(\Omega_0)}.$$

Summing all estimates for I_1, I_2, I_3 and using Korn's inequality (4), we obtain from (20) the assertion of Lemma 2. The lemma is proved.

4. Derivative of the energy functional

Let us derive the derivative of the energy functional with respect to the crack length, i.e.

$$\Pi'(u; \Omega_0) = \lim_{\varepsilon \to 0} \frac{\Pi(u^\varepsilon; \Omega_\varepsilon) - \Pi(u; \Omega_0)}{\varepsilon}. \tag{25}$$

Our aim is to calculate the value of (25).

Lemma 3. *The following equivalence is valid*:

$$\Pi'(u; \Omega_0) = \lim_{\varepsilon \to 0} \frac{\Pi_\varepsilon(u; \Omega_0) - \Pi(u; \Omega_0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\Pi_\varepsilon(\hat{u}^\varepsilon; \Omega_0) - \Pi(\hat{u}^\varepsilon; \Omega_0)}{\varepsilon}.$$

Proof. Firstly, by (18) we have

$$\frac{\Pi(u^\varepsilon; \Omega_\varepsilon) - \Pi(u; \Omega_0)}{\varepsilon} = \frac{\Pi_\varepsilon(\hat{u}^\varepsilon; \Omega_0) - \Pi(u; \Omega_0)}{\varepsilon} \leqslant \frac{\Pi_\varepsilon(u; \Omega_0) - \Pi(u; \Omega_0)}{\varepsilon}. \tag{26}$$

On the other hand, we can deduce from (18) together with the representations (1) and (17) that

$$\frac{\Pi_\varepsilon(\hat{u}^\varepsilon; \Omega_0) - \Pi(u; \Omega_0)}{\varepsilon}$$

$$\geqslant \frac{\Pi_\varepsilon(\hat{u}^\varepsilon; \Omega_0) - \Pi(\hat{u}^\varepsilon; \Omega_0)}{\varepsilon}$$

$$= \frac{1}{2}\int_{\Omega_0} \frac{\hat{c}_{ijkl} - c_{ijkl}}{\varepsilon}\varepsilon_{kl}(\hat{u}^\varepsilon)\varepsilon_{ij}(\hat{u}^\varepsilon) - \int_{\Omega_0} \frac{\hat{f}_i - f_i}{\varepsilon}\hat{u}_i^\varepsilon + \int_{\Gamma_0} \frac{\hat{g} - g}{\varepsilon}|[\hat{u}_1^\varepsilon]|$$

$$+ \frac{1}{2}\int_{\Omega_0} \hat{\sigma}_{ij}(\hat{u}^\varepsilon)(\chi_{,1}\varepsilon_{ij}(\hat{u}^\varepsilon) - 2E_{ij}(\hat{u}^\varepsilon)) - \int_{\Omega_0} \chi_{,1}\hat{f}_i\hat{u}_i^\varepsilon + \int_{\Gamma_0} \chi_{,1}\hat{g}|[\hat{u}_1^\varepsilon]|$$

$$+ \frac{\varepsilon}{2}\int_{\Omega_0} \frac{1}{1 + \varepsilon\chi_{,1}}\hat{\Sigma}_{ij}(\hat{u}^\varepsilon)E_{ij}(\hat{u}^\varepsilon). \tag{27}$$

Because of Lemma 2 and the continuous differentiability of c_{ijkl}, f_i, g, one can take the limit infimum in (27), which coincides with the limit supermum in (26). Lemma 3 is proved.

In the proof of Lemma 3 we have calculated derivative (25) directly. Indeed, for continuously differentiable function ϕ, the following formula arises:

$$\lim_{\varepsilon \to 0} \frac{\hat{\phi} - \phi}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\phi(x_1 + \varepsilon\chi, x_2) - \phi(x_1, x_2)}{\varepsilon} = \chi\phi_{,1}.$$

Therefore, from Lemma 3 the next result follows.

Theorem 2. *The derivative of the energy functional with respect to the crack length is equal to*

$$\Pi'(u; \Omega_0) = \frac{1}{2}\int_{\Omega_0} ((\chi c_{ijkl})_{,1}\varepsilon_{kl}(u)\varepsilon_{ij}(u) - \sigma_{ij}(u)(\chi_{,i}u_{j,1} + \chi_{,j}u_{i,1})) \tag{28}$$

$$- \int_{\Omega_0} (\chi f_i)_{,1}u_i + \int_{\Gamma_0} (\chi g)_{,1}|[u_1]|.$$

We now investigate properties of the derivative $\Pi'(u; \Omega_0)$ given by (28). Let us assume that the solution u of (3) is smooth enough. Integration by parts in the domains $\Omega^\pm = \Omega_0 \cap \{x \in \mathscr{R}^2, \pm x_2 > 0\}$ with $\bar{\Omega}^+ \cap \bar{\Omega}^- = \Sigma$, $\Gamma_0 \subset \Sigma$ leads to

$$\Pi'(u; \Omega_0) = \int_{\Omega_0} \chi(\sigma_{ij,j}(u) + f_i)u_{i,1} + \int_\Sigma \chi[\sigma_{i2}(u)u_{i,1}] + \int_{\Gamma_0} (\chi g)_{,1}|[u_1]|. \tag{29}$$

For the solution u of (3), equilibrium equation (5) in Ω_0 and the boundary conditions $[\sigma_{i2}(u)] = 0, i = 1, 2$, at Σ are fulfilled. Moreover, for smooth u we also have $[u_{i,1}] = 0$ at $\Sigma \backslash \Gamma_0$, $i = 1, 2$. Therefore, (29) can be represented as

$$\Pi'(u; \Omega_0) = I_1 + I_2$$

and

$$I_1 = \int_{\Gamma_0} (\chi\sigma_{12}(u)[u_{1,1}] + (\chi g)_{,1}|[u_1]|), \qquad I_2 = \int_{\Gamma_0} \chi\sigma_{22}(u)[u_{2,1}].$$

If u is continuous, we can split Γ_0 into the sets

$$\mathscr{D}_2 = \{x \in \Gamma_0, [u_2] = 0\}, \qquad \mathscr{N}_2 = \{x \in \Gamma_0, [u_2] > 0\}.$$

On $\mathscr{D}_2$ we have $[u_{2,1}] = 0$. On $\mathscr{N}_2$, by (6), we have $\sigma_{22}(u) = 0$. Thus, $I_2 = 0$.
Analogously, Γ_0 can be splitted into the sets

$$\mathscr{D}_1 = \{x \in \Gamma_0, [u_1] = 0\}, \qquad \mathscr{N}_1^{\pm} = \{x \in \Gamma_0, \pm [u_1] > 0\}.$$

On $\mathscr{D}_1$ we have $[u_1] = 0$ and $[u_{1,1}] = 0$. On $\mathscr{N}_1^{\pm}$, the relation $|[u_1]| = \pm [u_1]$ is fulfilled, and we can integrate in I_1 by parts along x_1. This provides

$$I_1 = \int_{\mathscr{N}_1^{\pm}} \chi(\sigma_{12}(u) \mp g)[u_{1,1}].$$

By (7), we have $\sigma_{12}(u) = \pm g$ as $\pm [u_1] > 0$, and therefore $I_1 = 0$.
If u is sufficiently smooth, i.e., $u \in [H^2(\Omega_0)]^2 \subset [C(\bar{\Omega}_0)]^2$, the above considerations are correct. Thus, we have proved the following result.

Lemma 4. *If the solution u of* (3) *belong to* $[H^2(\Omega_0)]^2$, *then* $\Pi'(u; \Omega_0) = 0$.

Since, due to Theorem 1, the solution u is of the H^2-class everywhere except the neighbourhood of $\partial\Gamma_0$, we can rewrite the derivative $\Pi'(u; \Omega_0)$ in the following way.

Let $B_{\delta/2}$ be a circle centered in the crack tip $(l, 0)$ as before, with the boundary $\partial B_{\delta/2}$ and the outward normal vector $n = (n_1, n_2)$. We have $\chi \equiv 1$ in $B_{\delta/2}$ and consider now the integrals in (28) over the domains $\Omega_0 \backslash \bar{B}_{\delta/2}$ and $B_{\delta/2} \backslash \bar{\Gamma}_0$. Using Theorem 1 and (5)–(7), we apply the Green formula in $\Omega_0 \backslash \bar{B}_{\delta/2}$ and obtain, like in the proof of Lemma 4, that

$$\begin{aligned} I_1 &\equiv \frac{1}{2} \int_{\Omega_0 \backslash B_{\delta/2}} ((\chi c_{ijkl})_{,1} \varepsilon_{kl}(u) \varepsilon_{ij}(u) - \sigma_{ij}(u)(\chi_{,i} u_{j,1} + \chi_{,j} u_{i,1})) \\ &\quad - \int_{\Omega_0 \backslash B_{\delta/2}} (\chi f_i)_{,1} u_i + \int_{\Gamma_0 \backslash B_{\delta/2}} (\chi g)_{,1} |[u_1]| \\ &= \int_{\partial B_{\delta/2}} \left(-\frac{1}{2} \sigma_{ij}(u) \varepsilon_{ij}(u) n_1 + \sigma_{ij}(u) n_j u_{j,1} + f_i u_i n_1 \right) \end{aligned}$$

(the outward normal here is $-n$). By $\chi \equiv 1$ in $B_{\delta/2} \backslash \bar{\Gamma}_0$, we also have

$$\begin{aligned} I_2 &\equiv \frac{1}{2} \int_{B_{\delta/2} \backslash \Gamma_0} ((\chi c_{ijkl})_{,1} \varepsilon_{kl}(u) \varepsilon_{ij}(u) - \sigma_{ij}(u)(\chi_{,i} u_{j,1} + \chi_{,j} u_{i,1})) \\ &\quad - \int_{B_{\delta/2} \backslash \Gamma_0} (\chi f_i)_{,1} u_i + \int_{\Gamma_0 \cap B_{\delta/2}} (\chi g)_{,1} |[u_1]| \\ &= \int_{B_{\delta/2} \backslash \Gamma_0} \left(\frac{1}{2} c_{ijkl,1} \varepsilon_{kl}(u) \varepsilon_{ij}(u) + f_i u_{i,1} \right) - \int_{\partial B_{\delta/2}} f_i u_i n_1 + \int_{\Gamma_0 \cap B_{\delta/2}} g' |[u_1]|. \end{aligned}$$

The summation $I_1 + I_2 = \Pi'(u;\Omega_0)$ leads to the relation

$$\Pi'(u;\Omega_0) = \int_{B_{\delta/2}\setminus\Gamma_0} \left(\frac{1}{2} c_{ijkl,1}\,\varepsilon_{kl}(u)\varepsilon_{ij}(u) + f_i u_{i,1}\right) + \int_{\Gamma_0\cap B_{\delta/2}} g'|[u_1]|$$

$$+ \int_{\partial B_{\delta/2}} \left(-\frac{1}{2}\sigma_{ij}(u)\varepsilon_{ij}(u)n_1 + \sigma_{ij}(u)n_j u_{i,1}\right), \qquad (30)$$

where the last integral in (30),

$$C(u) = \int_{\partial B_{\delta/2}} \sigma_{ij}(u)(u_{i,1}n_j - \tfrac{1}{2}\varepsilon_{ij}(u)n_1),$$

is called the Cherepanov–Rice integral. Indeed, for $f \equiv 0$, $c_{ijkl} \equiv$ const, $g \equiv$ const in some neighbourhood of the crack tip, it follows from (30) that $\Pi'(u;\Omega_0) = C(u)$. This means that $C(u) \equiv$ const for all sufficient small δ because of the uniqueness of the derivative. Note that expression (30) does not contain χ in comparison with (28); this underlines the independence of the derivative of the energy functional on the cut-off function.

5. Rupture criterion

Knowing the value of $\Pi'(u;\Omega_0)$, we have an asymptotic representation of the energy functional as follows:

$$\Pi(u^\varepsilon,\Omega_\varepsilon) = \Pi(u;\Omega_0) + \varepsilon\Pi'(u;\Omega_0) + o(\varepsilon). \qquad (31)$$

Together with the surface energy, we can define the total potential energy for a solid with a crack [14] as

$$U(u^\varepsilon;\Omega_\varepsilon) = \Pi(u^\varepsilon;\Omega_\varepsilon) + \gamma \text{ meas } \Gamma_\varepsilon, \quad \gamma > 0.$$

Due to (31), the increment of $U(\cdot)$ is given as

$$U(u^\varepsilon;\Omega_\varepsilon) - U(u;\Omega_0) = \varepsilon(\Pi'(u;\Omega_0) + \gamma) + o(\varepsilon). \qquad (32)$$

The crack will propagate in such a way that the total potential energy turns out to be minimal, so that representation (32) leads to the following criterion for the crack growing:

$$\Pi'(u;\Omega_0) + \gamma < 0. \qquad (33)$$

Conditions like (33) are called the Griffith rupture criterion. Conversely, when the inverse condition

$$\Pi'(u;\Omega_0) + \gamma \geqslant 0 \qquad (34)$$

is fulfilled, we say that the crack is stationary. In particular, if the solution u of (3) is smooth, namely H^2-class, then Lemma 4 leads to the fulfillment of condition (34).

6. The case of the Lamé system

Let us consider the constant coefficients c_{ijkl} given by

$$\sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda\delta_{ij}\operatorname{div} u, \quad \operatorname{div} u = u_{1,1} + u_{2,2}, \tag{35}$$

where λ, μ are the Lamé constants. Then (28), (30) take the form

$$\begin{aligned}\Pi'(u;\Omega_0) &= \frac{1}{2}\int_{\Omega_0} \sigma_{ij}(u)(\chi_{,1}\varepsilon_{ij}(u) - \chi_{,i}u_{j,1} - \chi_{,j}u_{i,1}) - \int_{\Omega_0}(\chi f_i)_{,1}u_i + \int_{\Gamma_0}(\chi g)_{,1}|[u_1]| \\ &= \int_{B_{\delta/2}\setminus\Gamma_0} f_i u_{i,1} + \int_{\Gamma_0\cap B_{\delta/2}} g'|[u_1]| + \int_{\partial B_{\delta/2}} \sigma_{ij}(u)\left(u_{i,1}n_j - \frac{1}{2}\varepsilon_{ij}(u)n_1\right).\end{aligned} \tag{36}$$

We introduce the local polar coordinates in a neighbourhood of the crack tip $(l, 0)$ as follows:

$$x_1 - l = r\cos\phi, \quad x_2 = r\sin\phi, \quad r \geqslant 0, \quad |\phi| \leqslant \pi.$$

In these coordinates we have

$$B_{\delta/2} = \{0 \leqslant r < \delta/2,\ |\phi| \leqslant \pi\},\ \partial B_{\delta/2} = \{r = \delta/2,\ |\phi| \leqslant \pi\},$$

$$B_{\delta/2}\cap\Gamma_0 = \{0 < r < \delta/2,\ |\phi| = \pi\},\ n = (\cos\phi, \sin\phi).$$

The derivatives are given by

$$u_{,1} = \cos\phi\, u_{,r} - \frac{\sin\phi}{r}u_{,\phi}, \qquad u_{,2} = \sin\phi\, u_{,r} + \frac{\cos\phi}{r}u_{,\phi}.$$

Let us define the following decomposition of the displacement vector in the polar coordinates (r, ϕ),

$$\begin{pmatrix}u_1\\u_2\end{pmatrix} = \Lambda(\phi)\begin{pmatrix}u_r\\u_\phi\end{pmatrix}, \quad \Lambda(\phi) = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}, \quad \Lambda^{-1}(\phi) = \Lambda(-\phi),$$

and the corresponding decomposition of strains and stresses:

$$\begin{pmatrix}\varepsilon_{11} & \varepsilon_{12}\\ \varepsilon_{21} & \varepsilon_{22}\end{pmatrix} = \Lambda(\phi)\begin{pmatrix}\varepsilon_{rr} & \varepsilon_{r\phi}\\ \varepsilon_{\phi r} & \varepsilon_{\phi\phi}\end{pmatrix}\Lambda(-\phi),$$

$$\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix} = \Lambda(\phi)\begin{pmatrix}\sigma_{rr} & \sigma_{r\phi}\\ \sigma_{\phi r} & \sigma_{\phi\phi}\end{pmatrix}\Lambda(-\phi).$$

Because of the symmetry of $\{\varepsilon_{ij}\}$, we have

$$\varepsilon_{rr}(u) = u_{r,r}, \quad \varepsilon_{r\phi}(u) = \frac{1}{2}\left(u_{\phi,r} + \frac{1}{r}u_{r,\phi} - \frac{1}{r}u_{\phi}\right), \quad \varepsilon_{\phi\phi}(u) = \frac{1}{r}u_{\phi,\phi} + \frac{1}{r}u_r \tag{37}$$

and, due to (35),

$$\begin{aligned} &\sigma_{rr}(u) = (\lambda + 2\mu)\varepsilon_{rr}(u) + \lambda\varepsilon_{\phi\phi}(u), \quad \sigma_{r\phi}(u) = 2\mu\varepsilon_{r\phi}(u), \\ &\sigma_{\phi\phi}(u) = (\lambda + 2\mu)\varepsilon_{\phi\phi}(u) + \lambda\varepsilon_{rr}(u). \end{aligned} \tag{38}$$

In the new notation, the first derivative of the potential energy can be rewritten as

$$\begin{aligned} \Pi'(u;\Omega_0) = \int_{r=\delta/2,|\phi|\leqslant\pi} &((\sigma_{rr}(u)\varepsilon_{rr}(u) + \sigma_{r\phi}(u)u_{\phi,r})\cos\phi \\ &- \left(\sigma_{rr}(u)\left(\frac{1}{r}u_{r,\phi} - \frac{1}{r}u_{\phi}\right) + \sigma_{r\phi}(u)\varepsilon_{\phi\phi}(u)\right)\sin\phi \\ &- \frac{1}{2}(\sigma_{rr}(u)\varepsilon_{rr}(u) + 2\sigma_{r\phi}(u)\varepsilon_{r\phi}(u) + \sigma_{\phi\phi}(u)\varepsilon_{\phi\phi}(u))\cos\phi)r\,\mathrm{d}\phi \\ &+ \int_{0<r<\delta/2,|\phi|<\pi} ((f_r\varepsilon_{rr}(u) + f_{\phi}u_{\phi,r})\cos\phi \\ &- \left(f_r\left(\frac{1}{r}u_{r,\phi} - \frac{1}{r}u_{\phi}\right) + f_{\phi}\varepsilon_{\phi\phi}(u)\right)\sin\phi\right)r\,\mathrm{d}r\,\mathrm{d}\phi \\ &+ \int_{0<r<\delta/2} g'|_{x_1=r}|[u_r]|\,\mathrm{d}r. \end{aligned} \tag{39}$$

6.1. *Singular solution*

We assume that $f \equiv 0$ in the neighbourhood B_{δ_0} of the crack tip $(l, 0)$. The equilibrium equations (5) for the Lamé system in Ω_0,

$$-\mu\Delta u_i - (\lambda + \mu)(\operatorname{div} u)_{,i} = f_i, \quad i = 1, 2,$$

provide the following equations in the polar coordinates, which are fulfilled in $B_{\delta_0}\backslash\bar{\Gamma}_0$:

$$\begin{aligned} &-(\lambda + 2\mu)\left(u_{r,r} + \frac{1}{r}u_{\phi,\phi} + \frac{1}{r}u_r\right)_{,r} + \frac{\mu}{\mathrm{r}}\left(u_{\phi,r} - \frac{1}{r}u_{r,\phi} + \frac{1}{r}u_{\phi}\right)_{,\phi} = 0, \\ &-\frac{\lambda + 2\mu}{r}\left(u_{r,r} + \frac{1}{r}u_{\phi,\phi} + \frac{1}{r}u_r\right)_{,\phi} - \mu\left(u_{\phi,r} - \frac{1}{r}u_{r,\phi} + \frac{1}{r}u_{\phi}\right)_{,r} = 0. \end{aligned} \tag{40}$$

We seek the singular solution of (40), i.e. a function from the H^1-class but not from the H^2-class, in the form

$$u(r, \phi) = r^\alpha \Phi(\phi), \quad \Phi = (\Phi_r, \Phi_\phi), \quad 0 < \alpha < 1. \tag{41}$$

Substituting (41) into (40), we obtain the equation for Φ:

$$\begin{aligned} &\mu\Phi_r'' - (\lambda + 2\mu)(1 - \alpha^2)\Phi_r + (\alpha(\lambda + \mu) - \lambda - 3\mu)\Phi_\phi' = 0, \\ &(\lambda + 2\mu)\Phi_\phi'' - \mu(1 - \alpha^2)\Phi_\phi + (\alpha(\lambda + \mu) + \lambda + 3\mu)\Phi_r' = 0, \end{aligned} \tag{42}$$

where $'$ denotes the derivation to Φ. The following four vectors form the fundamental system of solutions for system (42) [14, 6]:

$$\begin{aligned} &\Phi^1 = (\cos(1 + \alpha)\phi, -\sin(1 + \alpha)\phi), \quad \Phi^2 = (\sin(1 + \alpha)\phi, \cos(1 + \alpha)\phi), \\ &\Phi^3 = ((\kappa - \alpha)\cos(1 - \alpha)\phi, \ -(\kappa + \alpha)\sin(1 - \alpha)\phi), \\ &\Phi^4 = ((\kappa - \alpha)\sin(1 - \alpha)\phi, (\kappa + \alpha)\cos(1 - \alpha)\phi), \quad \kappa = \frac{\lambda + 3\mu}{\lambda + \mu}. \end{aligned} \tag{43}$$

Let us consider the function

$$U = r^\alpha \Phi, \quad \Phi = c_1\Phi^1 + c_2\Phi^2 + c_3\Phi^3 + c_4\Phi^4 \tag{44}$$

with arbitrary constants c_i, $i = 1, 2, 3, 4$, which solves (40). Due to (37) we get

$$\begin{aligned} &\varepsilon_{rr}(U) = \alpha r^{\alpha - 1}\Phi_r, \quad \varepsilon_{\phi\phi}(U) = r^{\alpha - 1}(\Phi_\phi' + \Phi_r), \\ &\varepsilon_{r\phi}(U) = \tfrac{1}{2} r^{\alpha - 1}(\Phi_r' - (1 - \alpha)\Phi_\phi), \end{aligned} \tag{45}$$

and therefore, inserting (45) into (38),

$$\begin{aligned} &\sigma_{rr}(U) = r^{\alpha - 1}(\lambda\Phi_\phi' + (\lambda + \alpha(\lambda + 2\mu))\Phi_r), \\ &\sigma_{\phi\phi}(U) = r^{\alpha - 1}((\lambda + 2\mu)\Phi_\phi' + (\lambda + 2\mu + \alpha\lambda)\Phi_r), \\ &\sigma_{r\phi}(U) = \mu r^{\alpha - 1}(\Phi_r' - (1 - \alpha)\Phi_\phi). \end{aligned} \tag{46}$$

We assume also that $g = 0$ at $B_{\delta_0} \cap \Gamma_0$, so that the boundary conditions (6) and (7) at $B_{\delta_0} \cap \Gamma_0$ will take the form

$$\begin{aligned} &[\sigma_{\phi\phi}(u)] = 0, \qquad \sigma_{\phi\phi}(u) \leqslant 0, \qquad [u_\phi] \leqslant 0, \qquad \sigma_{\phi\phi}(u)[u_\phi] = 0, \\ &[\sigma_{r\phi}(u)] = 0, \qquad \sigma_{r\phi}(u) = 0. \end{aligned} \tag{47}$$

From (43) and (46) we deduce that

$$\begin{aligned} [\sigma_{\phi\phi}(U)] &= \sigma_{\phi\phi}(U)|_{\phi = \pi} - \sigma_{\phi\phi}(U)|_{\phi = -\pi} \\ &= 4\mu\alpha r^{\alpha - 1}(-c_2 \sin(1 + \alpha)\pi + c_4(1 + \alpha)\sin(1 - \alpha)\pi), \\ [\sigma_{r\phi}(U)] &= 4\mu\alpha r^{\alpha - 1}(-c_1 \sin(1 + \alpha)\pi + c_3(1 - \alpha)\sin(1 - \alpha)\pi). \end{aligned} \tag{48}$$

For $0 < \alpha < 1$ we have $\sin(1+\alpha)\pi < 0$, $\sin(1-\alpha)\pi > 0$, and therefore, due to (48), boundary conditions (47) give us

$$c_1 = c_3(1-\alpha)\frac{\sin(1-\alpha)\pi}{\sin(1+\alpha)\pi}, \qquad c_2 = c_4(1+\alpha)\frac{\sin(1-\alpha)\pi}{\sin(1+\alpha)\pi}. \tag{49}$$

In view of (49), the remaining boundary conditions in (47) can be calculated as

$$\begin{aligned} &\sigma_{\phi\phi}(U) = 2\mu\alpha r^{\alpha-1}\frac{\sin 2\alpha\pi}{\sin(1+\alpha)\pi}c_3, \qquad \sigma_{r\phi}(U) = -2\mu\alpha r^{\alpha-1}\frac{\sin 2\alpha\pi}{\sin(1+\alpha)\pi}c_4, \\ &[U_\phi] = -2(1+\kappa)r^\alpha c_3 \sin(1-\alpha)\pi, \quad [U_r] = 2(1+\kappa)r^\alpha c_4 \sin(1-\alpha)\pi. \end{aligned} \tag{50}$$

Substituting (50) in (47), the condition $\sigma_{r\phi}(U) = 0$ holds only when $\sin 2\alpha\pi = 0$ or $c_2 = c_4 = 0$. Similarly, the condition

$$0 = \sigma_{\phi\phi}(U)[U_\phi] = -4\mu(1+\kappa)\alpha r^{2\alpha-1}\sin 2\alpha\pi \frac{\sin(1-\alpha)\pi}{\sin(1+\alpha)\pi}c_3^2$$

holds only when $\sin 2\alpha\pi = 0$ or $c_1 = c_3 = 0$. Thus, we have obtained that the non-trivial linear combination (44) fulfills boundary conditions (47) only for $\sin 2\alpha\pi = 0$, i.e., $\alpha = \frac{1}{2}$.

For $\alpha = \frac{1}{2}$ we have $c_1 = -\frac{1}{2}c_3$, $c_2 = -\frac{3}{2}c_4$, and therefore the fundamental system (43) is reduced to the following two vectors:

$$\begin{aligned} \Psi^1 &= (2\Phi^3 - \Phi^1)|_{\alpha=1/2} \\ &= \left((2\kappa-1)\cos\frac{\phi}{2} - \cos\frac{3\phi}{2},\ -(2\kappa+1)\sin\frac{\phi}{2} + \sin\frac{3\phi}{2}\right), \end{aligned} \tag{51}$$

$$\begin{aligned} \Psi^2 &= (2\Phi^4 - 3\Phi^2)|_{\alpha=1/2} \\ &= \left((2\kappa-1)\sin\frac{\phi}{2} - 3\sin\frac{3\phi}{2},\ (2\kappa+1)\cos\frac{\phi}{2} - 3\cos\frac{3\phi}{2}\right). \end{aligned}$$

Their linear combination

$$U = r^{1/2}\Psi, \quad \Psi = K_1\Psi^1 + K_2\Psi^2, \quad K_1 > 0, \tag{52}$$

fulfill now the equilibrium equations (40) in $B_{\delta_0}\backslash\bar{\Gamma}_0$ and the following boundary conditions at $B_{\delta_0}\cap\Gamma_0$:

$$\sigma_{\phi\phi}(U) = \sigma_{r\phi}(U) = 0, \qquad [U_\phi] = -4(1+\kappa)r^{1/2}K_1 \leqslant 0.$$

Note, that (51) corresponds to the singular solution of the problem with Neumann conditions at the crack.

6.2. *The first derivative of the energy functional*

The next step is to calculate the derivative of the potential energy for the obtained singular solution U in the case of $f = 0$, $g = 0$ in $B_{\delta/2}$. Together with (37) and (38), expression (39) can be rewritten as

$$\Pi'(U;\Omega_0)=\int_{r=\delta/2,|\phi|\leqslant\pi}\Bigg(\frac{1}{2}\Bigg((\lambda+2\mu)\Bigg(U^2_{r,r}-\Bigg(\frac{1}{r}U_{\phi,\phi}+\frac{1}{r}U_r\Bigg)^2\Bigg)$$

$$+\mu\Bigg(U^2_{\phi,r}-\Bigg(\frac{1}{r}U_{r,\phi}-\frac{1}{r}U_\phi\Bigg)^2\Bigg)\cos\phi$$

$$-\Bigg((\lambda+2\mu)(U_{r,r}+\frac{1}{r}U_{\phi,\phi}+\frac{1}{r}U_r\Bigg)\Bigg(\frac{1}{r}U_{r,\phi}-\frac{1}{r}U_\phi\Bigg)$$

$$+\mu\Bigg(U_{\phi,r}-\frac{1}{r}U_{r,\phi}+\frac{1}{r}U_\phi\Bigg)\Bigg(\frac{1}{r}U_{\phi,\phi}+\frac{1}{r}U_r\Bigg)\Bigg)\sin\phi\Bigg)r\,\mathrm{d}\phi$$

or, with (52),

$$\Pi'(U;\Omega_0)$$

$$=-\int_{|\phi|\leqslant\pi}\Bigg((\lambda+2\mu)(\Psi'_\phi+\frac{3}{2}\Psi_r\Bigg)\Bigg(\frac{1}{2}\Bigg(\Psi'_\phi+\frac{1}{2}\Psi_r\Bigg)\cos\phi+(\Psi'_r-\Psi_\phi)\sin\phi\Bigg)$$

$$+\mu\Bigg(\Psi'_r-\frac{3}{2}\Psi_\phi\Bigg)\Bigg(\frac{1}{2}\Bigg(\Psi'_r-\frac{1}{2}\Psi_\phi\Bigg)\cos\phi-(\Psi'_\phi+\Psi_r)\sin\phi\Bigg)\Bigg)\mathrm{d}\phi. \tag{53}$$

Substituting (51) in (53), we have

$$\Pi'(U;\Omega_0)=-2\int_{|\phi|\leqslant\pi}\Bigg((\lambda+2\mu)(\kappa-1)\Bigg(K_1\cos\frac{\phi}{2}+K_2\sin\frac{\phi}{2}\Bigg)$$

$$\times\Bigg(K_1(\kappa+2)\sin\phi\sin\frac{\phi}{2}+K_2\Bigg(2\sin\frac{\phi}{2}-(\kappa+2)\sin\phi\sin\frac{\phi}{2}\Bigg)\Bigg)$$

$$-\mu(\kappa+1)\Bigg(K_1\sin\frac{\phi}{2}-K_2\cos\frac{\phi}{2}\Bigg)$$

$$\times\Bigg(K_1(\kappa-2)\sin\phi\cos\frac{\phi}{2}+K_2\Bigg(2\cos\frac{\phi}{2}+(\kappa-2)\sin\phi\sin\frac{\phi}{2}\Bigg)\Bigg)\Bigg)\mathrm{d}\phi$$

$$=-\frac{8\mu(\lambda+2\mu)}{\lambda+\mu}\int_{|\phi|\leqslant\pi}(K_1^2\sin^2\phi+K_2^2\cos^2\phi)$$

$$=-\frac{8\pi\mu(\lambda+2\mu)}{\lambda+\mu}(K_1^2+K_2^2).$$

This expression corresponds to the well-known Irwin formula for the linear model of a solid with a crack under the stress-free boundary conditions [3, 2].

We summarize the obtained results in the following theorem. Let $\chi(r)$ be a smooth cut-off function with the support in B_δ as before.

Theorem 3. *If* $f = 0$, $g = 0$ *in a neighbourhood of the crack tip, the singular solution* $U = \chi(r) r^{\alpha} \Phi(\phi)$, $0 < \alpha < 1$, *to problem* (3) *for the Lamé system is of the form*

$$U = \chi(r) r^{1/2} (K_1 \Psi^1(\phi) + K_2 \Psi^2(\phi)), \qquad K_1 \geqslant 0, \tag{54}$$

where Ψ^i, $i = 1, 2$, *are given by* (51). *The first derivative of the energy functional*

$$\Pi'(U; \Omega_0) = -\frac{8\pi\mu(\lambda + 2\mu)}{\lambda + \mu} (K_1^2 + K_2^2) \leqslant 0.$$

Let us make some remarks concerning the solution u of the variational inequality (3). First, if $[u] = 0$ in some neighbourhood B_δ of the crack tip, then, as it was marked before, it follows from the Lemmas 1 and 4 that $u \in [H^2(B_\delta)]^2$ and $\Pi'(u; \Omega_0) = 0$.

Second, if $[u_2] > 0$, $[u_1] > 0$ (or $[u_1] < 0$) in B_δ, then we have by (6), (7) the Neumann problem in B_δ with the boundary conditions $\sigma_{22}(u) = 0$, $\sigma_{12}(u) = g$ (or $\sigma_{12}(u) = -g$, respectively). Therefore, due to the local regularity of the solution as shown in Theorem 1 and the smoothness of g, it follows from the results of [6] that the solution u can be locally represented as $\chi u = U + w$, where $w \in H^2(B_\delta \backslash \bar{\Gamma}_0)$, U is from (54), and $\Pi'(u; \Omega_0) = \Pi'(U; \Omega_0)$. But in the general case, while the solution lose regularity, one can not provide the fulfillment of the above assumptions.

7. Second derivative of the energy functional

We look for an expression of the second derivative of the energy functional with respect to the crack length. First, by Lemma 2, the sequence $\{(\hat{u}^\varepsilon - u)/\varepsilon\}$ is bounded in $\tilde{H}^1(\Omega_0)$ uniformly in ε, so that there exists a weak limit $\dot{u} \in \tilde{H}^1(\Omega_0)$ and a subsequence ε_k such that

$$\frac{\hat{u}^{\varepsilon_k} - u}{\varepsilon_k} \to \dot{u} \text{ weakly in } \tilde{H}^1(\Omega_0) \quad \text{as } \varepsilon_k \to 0. \tag{55}$$

The function $\dot{u}$ is called the weak material derivative. The uniform estimate

$$\left\| \frac{|[\hat{u}_1^\varepsilon]| - |[u_1]|}{\varepsilon} \right\|_{L^2(\Gamma_0)} \leqslant \left\| \left[\frac{\hat{u}_1^\varepsilon - u_1}{\varepsilon} \right] \right\|_{L^2(\Gamma_0)} \leqslant \text{const}$$

provides also the following convergence for some subsequence:

$$\frac{|[\hat{u}_1^{\varepsilon_k}]| - |[u_1]|}{\varepsilon_k} \to z_g \quad \text{weakly in } L^2(\Gamma_0) \quad \text{as } \varepsilon_k \to 0. \tag{56}$$

We call z_g the boundary weak material derivative associated with the friction condition.

Due to Theorem 2, in analogy to (28), we have the first derivative of the energy functional at the displacement u^ε in the domain Ω_ε:

$$\Pi'(u^\varepsilon;\Omega_\varepsilon) = \int_{\Omega_\varepsilon} \left(\frac{1}{2} (\rho c_{ijkl})_{,1}\, \varepsilon_{kl}(u^\varepsilon)\, \varepsilon_{ij}(u^\varepsilon) - \sigma_{ij}(u^\varepsilon)\, E^\rho_{ij}(u^\varepsilon) \right) \tag{57}$$

$$- \int_{\Omega_\varepsilon} (\rho f_i)_{,1}\, u^\varepsilon_i + \int_{\Gamma_\varepsilon} (\rho g)_{,1} \,|[u^\varepsilon_1]|, \qquad E^\rho_{ij}(u^\varepsilon) = \frac{1}{2}(\rho_{,i} u^\varepsilon_{j,1} + \rho_{,j} u^\varepsilon_{i,1}),$$

where the cut-off function $\rho(y)$, $y \in \Omega_\varepsilon$, is chosen such that $\hat{\rho}(x) = \chi(x)$, $x \in \Omega_0$. Again, applying transformation (15) to (57), one obtains $\Pi'(u^\varepsilon;\Omega_\varepsilon) = \Pi'_\varepsilon(\hat{u}^\varepsilon;\Omega_0)$, where

$$\begin{aligned}\Pi'_\varepsilon(\hat{u}^\varepsilon;\Omega_0) = &\int_{\Omega_0} \left(\frac{1}{2} (\chi \hat{c}_{ijkl})_{,1}\, \varepsilon_{kl}(\hat{u}^\varepsilon)\, \varepsilon_{ij}(\hat{u}^\varepsilon) - \frac{1}{1+\varepsilon\chi_{,1}}\, \hat{\sigma}_{ij}(\hat{u}^\varepsilon)\, E_{ij}(\hat{u}^\varepsilon) \right)\\
&+ \frac{\varepsilon^2}{2} \int_{\Omega_0} \frac{(\chi \hat{c}_{ijkl})_{,1}}{(1+\varepsilon\chi_{,1})^2}\, E_{kl}(\hat{u}^\varepsilon)\, E_{ij}(\hat{u}^\varepsilon)\\
&- \int_{\Omega_0} (\chi \hat{f}_i)_{,1}\, \hat{u}^\varepsilon_i + \int_{\Gamma_0} (\chi \hat{g})_{,1} |[\hat{u}^\varepsilon_1]|\\
&+ \varepsilon \int_{\Omega_0} \left(\frac{1}{(1+\varepsilon\chi_{,1})^2}\, \hat{\Sigma}_{ij}(\hat{u}^\varepsilon) E_{ij}(\hat{u}^\varepsilon) - \frac{(\chi \hat{c}_{ijkl})_{,1}}{1+\varepsilon\chi_{,1}}\, \varepsilon_{kl}(\hat{u}^\varepsilon) E_{ij}(\hat{u}^\varepsilon) \right).\end{aligned} \tag{58}$$

We look for the expression of the second derivative:

$$\lim_{\varepsilon\to 0} \frac{\Pi'(u^\varepsilon;\Omega_\varepsilon) - \Pi'(u;\Omega_0)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\Pi'_\varepsilon(\hat{u}^\varepsilon;\Omega_0) - \Pi'(u;\Omega_0)}{\varepsilon}$$

By (28), (58), we have the following equality:

$$\begin{aligned}&\frac{\Pi'_\varepsilon(\hat{u}^\varepsilon;\Omega_0) - \Pi'(u;\Omega_0)}{\varepsilon}\\
&= \int_{\Omega_0} \left(\frac{1}{2} \left(\chi \frac{\hat{c}_{ijkl} - c_{ijkl}}{\varepsilon} \right)_{,1} \varepsilon_{kl}(\hat{u}^\varepsilon)\varepsilon_{ij}(\hat{u}^\varepsilon) - \frac{\hat{c}_{ijkl} - c_{ijkl}}{\varepsilon}\, \varepsilon_{kl}(\hat{u}^\varepsilon)\, E_{ij}(\hat{u}^\varepsilon) \right)\\
&\quad + \frac{\varepsilon}{2} \int_{\Omega_0} \frac{(\chi \hat{c}_{ijkl})_{,1}}{(1+\varepsilon\chi_{,1})^2}\, E_{kl}(\hat{u}^\varepsilon)\, E_{ij}(\hat{u}^\varepsilon) - \int_{\Omega_0} \left(\chi \frac{\hat{f}_i - f_i}{\varepsilon} \right)_{,1} \hat{u}^\varepsilon_i + \int_{\Gamma_0} \left(\chi \frac{\hat{g} - g}{\varepsilon} \right)_{,1} |[\hat{u}^\varepsilon_1]|\\
&\quad + \int_{\Omega_0} \left(\frac{1}{(1+\varepsilon\chi_{,1})^2}\, \hat{\Sigma}_{ij}(\hat{u}^\varepsilon)\, E_{ij}(\hat{u}^\varepsilon) - \frac{\chi \hat{c}_{ijkl,1}}{1+\varepsilon\chi_{,1}}\, \varepsilon_{kl}(\hat{u}^\varepsilon)\, E_{ij}(\hat{u}^\varepsilon) \right)\\
&\quad + \int_{\Omega_0} \left(\left(\frac{1}{2} (\chi c_{ijkl})_{,1}\, \varepsilon_{kl}(\hat{u}^\varepsilon + u) - \Sigma_{ij}(\hat{u}^\varepsilon) \right) \varepsilon_{ij} \left(\frac{\hat{u}^\varepsilon - u}{\varepsilon} \right) \right.\\
&\quad \left. - \sigma_{ij}(u)\, E_{ij} \left(\frac{\hat{u}^\varepsilon - u}{\varepsilon} \right) \right) - \int_{\Omega_0} (\chi f_i)_{,1} \frac{\hat{u}^\varepsilon_i - u_i}{\varepsilon} + \int_{\Gamma_0} (\chi g)_{,1} \frac{|[\hat{u}^\varepsilon_1]| - |[u_1]|}{\varepsilon}.\end{aligned}$$

In view of the convergences given in (55) and (56) and due to Lemma 2, we can pass to the limit as $\varepsilon_k \to 0$ and obtain that

$$\lim_{\varepsilon_k \to 0} \frac{\Pi'_{\varepsilon_k}(\hat{u}^{\varepsilon_k}; \Omega_0) - \Pi'(u; \Omega_0)}{\varepsilon_k} = \Pi''(u, \dot{u}, z_g; \Omega_0),$$

where

$$\begin{aligned}\Pi''(u, \dot{u}, z_g; \Omega_0) = \int_{\Omega_0} \Big(& \frac{1}{2}(\chi^2 c_{ijkl,1})_{,1}\, \varepsilon_{kl}(u)\, \varepsilon_{ij}(u) - 2\chi c_{ijkl,1}\, \varepsilon_{kl}(u)\, E_{ij}(u) \\ & + \Sigma_{ij}(u)\, E_{ij}(u) \Big) - \int_{\Omega_0} (\chi^2 f_{i,1})_{,1}\, u_i + \int_{\Gamma_0} (\chi^2 g')_{,1}\, |[u_1]| \\ & + \int_{\Omega_0} (((\chi c_{ijkl})_{,1}\, \varepsilon_{kl}(u) - \Sigma_{ij}(u))\, \varepsilon_{ij}(\dot{u}) - \sigma_{ij}(u)\, E_{ij}(\dot{u})) \\ & - \int_{\Omega_0} (\chi f_i)_{,1}\, \dot{u}_i + \int_{\Gamma_0} (\chi g)_{,1}\, z_g. \end{aligned} \tag{59}$$

Note that, dividing (20) with ε^2 and passing to the limit infimum in (20), we get in view of (55) and (56) the estimate

$$\begin{aligned}&\int_{\Omega_0} (((\chi c_{ijkl})_{,1}\, \varepsilon_{kl}(u) - \Sigma_{ij}(u))\, \varepsilon_{ij}(\dot{u}) - \sigma_{ij}(u)\, E_{ij}(\dot{u})) \\ &\quad - \int_{\Omega_0} (\chi f_i)_{,1}\, \dot{u}_i + \int_{\Gamma_0} (\chi g)_{,1}\, z_g \leqslant -\int_{\Omega_0} \sigma_{ij}(\dot{u})\, \varepsilon_{ij}(\dot{u}) \leqslant 0.\end{aligned}$$

Thus, the following result is proved.

Lemma 5. *There is a subsequence from ε such that the partial limit of the sequence* $(\Pi'(u^\varepsilon; \Omega_\varepsilon) - \Pi'(u; \Omega_0))/\varepsilon$ *exists as* $\varepsilon \to 0$ *and has the form* $\Pi''(u, \dot{u}, z_g; \Omega_0)$ *given by* (59).

Remark. While the convergences (55) and (56) do not guarantee the uniqueness of the material derivatives, (59) is not enough to provide the existence of the second derivative to the energy functional Π in the general case. When the domain is smooth, see the material derivatives of the solution with respect to the domain variation in [15].

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