ITERATIVE PENALTY METHOD FOR PLATE WITH A CRACK\(^1\)

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Abstract. Problems with cracks often arise in applications (solid mechanics, geophysics) and need to essentially describe mathematical conditions fulfilled on the crack. We can quote the works [1,2,10,11] and others. In this paper we deal with the nonpenetration condition on the crack faces stated by A.M. Khludnev in [4,5] as the inequality. The obtained problem with the unilateral constraint is described by the variational inequality (see [3,6,9]). Here we construct approximate solutions of this variational inequality using penalty and iterative methods. Convergence of the solutions is proved and it's application at the one-dimensional problem is discussed. Similar approaches for elastic and plastic plates contacted with an obstacle were considered by the author in [7,8].

1. Introduction. A thin isotropic homogeneous plate is assumed to occupy a bounded domain \( \Omega_0 \subset R^2 \) with a smooth boundary \( \partial \Omega_0 \). A crack \( \Gamma \) inside \( \Omega_0 \) is described by a sufficiently smooth function. Choose direction of the normal \( \nu = (\nu^1, \nu^2) \) to \( \Gamma \) defines positive \( \Gamma^+ \) and negative \( \Gamma^- \) crack faces (see the figure).

Let us denote \( \Omega = \Omega_0 \setminus \Gamma \). Vector \( u = (u^1, u^2) \) of the plate horizontal displacements must satisfy the following boundary conditions. Firstly, the jam condition \( u = 0 \) must hold on \( \partial \Omega_0 \). Secondly, the nonpenetration condition of the crack faces without friction condition is imposed on the internal boundary [4,5]:

\[
[u]\nu \equiv [u^1]\nu^1 + [u^2]\nu^2 \geq 0,
\]

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\([u]\) is the jump of \(u\) on \(\Gamma\), i.e. \([u] = u|_{\Gamma^+} - u|_{\Gamma^-}\). Here we consider \(u|_{\Gamma^+}\) and \(u|_{\Gamma^-}\) as the traces of the function \(u\) from \((H^1(\Omega^+))^2\) and \((H^1(\Omega^-))^2\), respectively, for the domain \(\Omega_0\) divided on \(\Omega^+\) and \(\Omega^-\) by means of a smooth continuation of \(\Gamma\) to some closed curve inside \(\Omega_0\) (broken line on the figure).

Let us define the basic Hilbert space
\[
X = \{u \in (H^1(\Omega))^2, \quad u = 0 \quad \text{on } \partial \Omega_0\}
\]
and the close and convex set
\[
K = \{u \in X, \quad [u]u \geq 0\}.
\]
Using the Poincare inequality for \(X\)
\[
\|u_i\|_0^2 \leq c_1 \|\nabla u_i\|_0^2 \quad i = 1, 2,
\]
where \(\|\cdot\|_0\) is the norm in \(L^2(\Omega)\), we define the inner product in \(X\) by
\[
(u, v) = \langle Du, Dv \rangle + \int_{\Gamma} [u]u \cdot [v]v d\Gamma
\]
and the norm in \(X\) by
\[
\|u\|^2 = (u, u).
\]
Here \(Du = (u^1_x, u^1_y, u^2_x, u^2_y)\) and brackets \(\langle \cdot, \cdot \rangle\) denote integration over \(\Omega\). We introduce the following bilinear form known in the elasticity by
\[
a(u, v) = \int_{\Omega} \left( u^1_x v^1_x + u^2_y v^2_y + \kappa(u^1_x v^2_y + u^2_y v^1_x) + \frac{1-\kappa}{2}(u^1_y + u^2_x)(v^1_y + v^2_x) \right) d\Omega.
\]
Constant \(0 < \kappa < 0.5\) is given. The following first Korn inequality will be valid:
\[
(1) \quad a(u, u) \geq M\|u\|^2, \quad M > 0.
\]
Let \(f = (f^1, f^2) \in (L^2(\Omega))^2\) be the given vector of external forces. The equilibrium problem for the thin elastic plate with the crack is formulated as follows (\(f\) is reduced by a factor \(E(1-\kappa^2)^{-1}\) [4,5]):
\[
(2) \quad u \in K, \quad a(u, v - u) \geq \langle f, v - u \rangle, \quad \forall v \in K.
\]
The unique solution of the variational inequality (2) exists by virtue of the coercivity (1), boundedness of bilinear form \(a(\cdot, \cdot)\) [9].

2. **Approximate models.** Formally integrating by the parts, we can obtain that the following relation hold
\[
a(u, v) = \langle Au, v \rangle + \int_{\Gamma}[\sigma(u)\nu \cdot \nu v + \sigma(u)\tau \cdot \nu \tau]d\Gamma,
\]
where
\[
A(u) = \left(-u^1_{xx} - \frac{1 - \kappa}{2} u^1_{yy}, \quad \frac{1 + \kappa}{2} u^2_{xx} - u^2_{yy} - \frac{1 - \kappa}{2} u^2_{xy}, \quad -u^2_{xy} - \frac{1 - \kappa}{2} u^1_{xx} - \frac{1 + \kappa}{2} u^1_{xy}\right),
\]
\[
\sigma(u) = \left(\left(u^1_x + \kappa u^2_y\right)\nu^1 + \frac{1 - \kappa}{2} \left(u^1_y + u^2_x\right)\nu^2, \quad \left(u^2_y + \kappa u^1_x\right)\nu^2 + \frac{1 - \kappa}{2} \left(u^1_y + u^2_x\right)\nu^1\right).
\]
Here \(\tau = (-\nu^2, \nu^1)\) is the tangent vector on \(\Gamma\). Let the solution \(u\) be smooth enough. Then we can rewrite (2) as follows
\[
\langle Au - f, v - u \rangle - \int_{\Gamma} \left(\left[\sigma (u) \nu \cdot (v - u) \nu\right] + \left[\sigma (u) \tau \cdot (v - u) \tau\right] \right) d\Gamma \geq 0.
\]
By varying the test function \(v \in K\), it can be deduced (see [3]) that the variational inequality (2) with a smooth enough solution \(u\) is equivalent to the following boundary problem
\[
Au = f \quad \text{in} \Omega,
\]
\[
[\sigma(u)] = 0, \quad \sigma(U)\tau = 0, \quad \text{on} \Gamma
\]
\[
[u]\nu \geq 0, \quad [u]\nu \cdot \sigma(u)\nu = 0, \quad [u]\nu \cdot \sigma(u)\nu = 0.
\]
The exact meaning of boundary relations on \(\Gamma\) is studied in [4,11].

To construct a penalty problem, we introduce a penalty operator \(\beta : X \to X^*\) by the relation
\[
\langle \beta(u), v \rangle = -\int_{\Gamma} ([u]\nu - ([u]\nu)^{-}) ([v]\nu) d\Gamma.
\]
Here \(\langle \cdot, \cdot \rangle\) means the duality between \(X\) and it's dual space \(X^*\). By the upper minus sign we have denoted the negative part of a function, i.e. \(s = s^+ - s^-\), \(s^+, s^- \geq 0, \quad s^+ s^- = 0\). It is easy to see that \(\beta\) is the monotonous operator. By \(u^\varepsilon \in X\) we shall denote the unique solution of the following penalty equation depending on a small parameter \(\varepsilon > 0:\)
\[
a(u^\varepsilon, v) + \varepsilon^{-1} \langle \beta(u^\varepsilon), v \rangle = \langle f, v \rangle, \quad \forall v \in X.
\]
The last is interpreted in the above sense as follows
\[
Au^\varepsilon = f \quad \text{in} \Omega,
\]
\[
[\sigma(u^\varepsilon)] = 0, \quad \sigma(u^\varepsilon)\tau = 0, \quad \text{on} \Gamma
\]
\[
\sigma(u^\varepsilon)\nu = -\varepsilon^{-1} ([u^\varepsilon]\nu)^-.
\]
Let us fix \(\varepsilon\). To linearize the left side of (3), we construct the following iterations for an arbitrary \(u^{\varepsilon,0} \in X\), \(n = 0, 1, 2, \ldots\)
\[
a(u^{\varepsilon,n+1}, v) + \varepsilon^{-1} (u^{\varepsilon,n+1}, v) = \langle f, v \rangle + \varepsilon^{-1} (u^{\varepsilon,n}, v) - \varepsilon^{-1} \langle \beta(u^{\varepsilon,n}), v \rangle.
\]
It is obvious that \(u^{\varepsilon,n+1} \in X\) exists for the operator's properties marked. The appropriate boundary problem is of the form
\[
Au^{\varepsilon,n+1} - \varepsilon^{-1} \Delta (u^{\varepsilon,n+1} - u^{\varepsilon,n}) = f \quad \text{in} \Omega,
\]
\[
\left[ \sigma(u^{\varepsilon,n+1}) + \varepsilon^{-1} \partial(u^{\varepsilon,n+1} - u^{\varepsilon,n})/\partial\nu \right] = 0,
\]
\[
\left( \sigma(u^{\varepsilon,n+1}) + \varepsilon^{-1} \partial(u^{\varepsilon,n+1} - u^{\varepsilon,n})/\partial\nu \right) \tau = 0, \quad \text{on } \Gamma
\]
\[
\left( \sigma(u^{\varepsilon,n+1}) + \varepsilon^{-1} \partial(u^{\varepsilon,n+1} - u^{\varepsilon,n})/\partial\nu - \varepsilon^{-1}[u^{\varepsilon,n+1} - u^{\varepsilon,n}] \right) \nu = -\varepsilon^{-1}([u^{\varepsilon,n}] \nu)^{-}.
\]

Here used notations mean
\[
\triangle u = (\triangle u^{1}, \triangle u^{2}), \quad \partial u/\partial\nu = (\partial u^{1}/\partial\nu, \partial u^{2}/\partial\nu).
\]

**Theorem 1** \(u^{\varepsilon,n+1} \to u^{\varepsilon}\) strongly in \(X\) as \(n \to \infty\) and

\[
\|u^{\varepsilon,n+1} - u^{\varepsilon}\| \leq (1 + 2M\varepsilon)^{-(n+1)}\|u^{\varepsilon,0} - u^{\varepsilon}\|^{2},
\]

\(u^{\varepsilon} \to u\) strongly in \(X\) as \(\varepsilon \to 0\),

where \(u^{\varepsilon,n+1}, u^{\varepsilon}, u\) are the solutions of (4), (3), (2), respectively.

**Proof.** By subtracting (3) from (4) and adding \(-\varepsilon^{-1}(u^{\varepsilon}, v)\) to the both parts, we get

\[
a(u^{\varepsilon,n+1} - u^{\varepsilon}, u^{\varepsilon,n+1} - u^{\varepsilon}, v) + \varepsilon^{-1}(u^{\varepsilon,n+1} - u^{\varepsilon}, v) = \varepsilon^{-1}(u^{\varepsilon,n} - u^{\varepsilon}, v) - \varepsilon^{-1}(\beta(u^{\varepsilon,n}) - \beta(u^{\varepsilon}), v).
\]

Let us consider this equation with the test function \(v = u^{\varepsilon,n+1} - u^{\varepsilon}\) and express it's right side as integrals:

\[
a(u^{\varepsilon,n+1} - u^{\varepsilon}, u^{\varepsilon,n+1} - u^{\varepsilon}) + \varepsilon^{-1}\|u^{\varepsilon,n+1} - u^{\varepsilon}\|^{2} = \varepsilon^{-1}(D(u^{\varepsilon,n} - u^{\varepsilon}), D(u^{\varepsilon,n+1} - u^{\varepsilon})) + \\
+\varepsilon^{-1}\int_{\Gamma} \left( [u^{\varepsilon,n} - u^{\varepsilon}] \nu + ([u^{\varepsilon,n}] \nu)^{-} - ([u^{\varepsilon}] \nu)^{-} \right) \cdot [u^{\varepsilon,n+1} - u^{\varepsilon}] \nu d\Gamma.
\]

Since \(s_{1} - s_{2} + s_{1}^{-} - s_{2}^{-} = s_{1}^{+} - s_{2}^{+} \leq |s_{1} - s_{2}|\), the right side of (6), thanks to the Holder inequality, is no greater than

\[
(2\varepsilon)^{-1} \left( \|D(u^{\varepsilon,n} - u^{\varepsilon})\|^{2} + \|D(u^{\varepsilon,n+1} - u^{\varepsilon})\|^{2} \right) - \int_{\Gamma} \left( ([u^{\varepsilon,n} - u^{\varepsilon}] \nu)^{2} + ([u^{\varepsilon,n+1} - u^{\varepsilon}] \nu)^{2} \right) d\Gamma = \\
= (2\varepsilon)^{-1} \left( \|u^{\varepsilon,n} - u^{\varepsilon}\|^{2} + \|u^{\varepsilon,n+1} - u^{\varepsilon}\|^{2} \right).
\]

On the other hand, the left side of (6) is no less than

\[
(M + \varepsilon^{-1})\|u^{\varepsilon,n+1} - u^{\varepsilon}\|^{2}.
\]

Therefore

\[
\|u^{\varepsilon,n+1} - u^{\varepsilon}\|^{2} \leq (1 + 2M\varepsilon)^{-1}\|u^{\varepsilon,n} - u^{\varepsilon}\|^{2}.
\]

By repeating the last estimate as \(n, n-1, ..., 0\), we get that (5) holds and, therefore, the first convergence result is also true.

The weak convergence

\[
u^{\varepsilon} \to u\quad \text{weakly in } X \text{ as } \varepsilon \to 0
\]
is proved by familiar methods in [9] using the properties of operators \(a(\cdot, \cdot), \beta(\cdot)\). Indeed, equation (3) with \(v = u^\varepsilon - \xi, \quad \xi \in K\) (i.e. \(\beta(\xi) = 0\)) gives

\[
a(u^\varepsilon, u^\varepsilon - \xi) \leq a(u^\varepsilon, u^\varepsilon - \xi) + \varepsilon^{-1}\langle \beta(u^\varepsilon) - \beta(\xi), u^\varepsilon - \xi \rangle = \langle f, u^\varepsilon - \xi \rangle.
\]

Hence, \(\|u^\varepsilon\| \leq c(M, c_1, f, \xi) = \text{const}\) and some subsequence exists such that

\[u^\varepsilon \to u_0 \quad \text{weakly in } X \quad \text{as } \varepsilon \to 0.\]

Then

\[
a(u^\varepsilon, \xi) \to a(u_0, \xi), \quad \liminf a(u^\varepsilon, u^\varepsilon) \geq a(u_0, u_0),
\]

\[
\langle \beta(u^\varepsilon), \xi \rangle = \varepsilon \langle (f, \xi) - a(u^\varepsilon, \xi) \rangle \to 0 \quad \text{as } \varepsilon \to 0.
\]

Therefore, we can obtain that \(\beta(u_0) = 0\), i.e. \(u_0 \in K\) and pass to a limit infimum in the following inequality:

\[
a(u^\varepsilon, v - u^\varepsilon) - \langle f, v - u^\varepsilon \rangle = \varepsilon^{-1}\langle \beta(v) - \beta(u^\varepsilon), v - u^\varepsilon \rangle \geq 0 \quad \forall v \in K.
\]

This gives

\[
a(u_0, v - u_0) \geq \langle f, v - u_0 \rangle \quad \forall v \in K
\]

and \(u_0 = u\) owing to the uniqueness property of the solution.

Subtracting \(a(u, v)\) from (3) and considering this equation with the test element \(v = u^\varepsilon - u\), one obtains

\[
a(u^\varepsilon - u, u^\varepsilon - u) - \varepsilon^{-1} \int_\Gamma [u^\varepsilon] \nu [-u^\varepsilon - v] \nu d\Gamma = \langle f, u^\varepsilon - u \rangle - a(u, u^\varepsilon - u).
\]

Owing to

\[-([u^\varepsilon] \nu) - [u^\varepsilon - u] \nu = \left(\left([u^\varepsilon] \nu\right)^{-} \right)^2 + \left([u^\varepsilon] \nu\right)^{-} [u] \nu, \quad [u] \nu \geq 0\]

and (1), we have

\[
M\|u^\varepsilon - u\|^2 + \varepsilon^{-1} \int_\Gamma \left(\left([u^\varepsilon] \nu\right)^{-}\right)^2 d\Gamma \leq \langle f, u^\varepsilon - u \rangle - a(u, u^\varepsilon - u).
\]

Therefore, (7) leads to the second strong convergence to be proved. The proof is completed.

**Remark.** Obviously, we may use another inner product in \(X\), for instance

\[
(u, v) = a(u, v) + \int_\Gamma [u] \nu \cdot [v] \nu d\Gamma.
\]

Then (4) takes the form

\[
(1 + \varepsilon^{-1}) a(u^{\varepsilon,n+1} - u^{\varepsilon,n}, v) + \varepsilon^{-1} \int_\Gamma [u^{\varepsilon,n+1} - u^{\varepsilon,n}] \nu \cdot [v] \nu d\Gamma =
\]

\[
= \langle f, v \rangle - a(u^{\varepsilon,n}, v) - \varepsilon^{-1} \langle \beta(u^{\varepsilon,n}), v \rangle
\]

and Theorem 1 is also valid.
3. Application. We will consider the one-dimensional crack problem, i.e. a thin bar
\( \Omega_0 = (a, b) \) with a cut \( \Gamma = \{ y \}, a < y < b \). Hence, \( \Omega = (a, y - 0) \cup (y + 0, b) \), \( X = \{ u \in H^1(\Omega), u(a) = u(b) = 0 \}, \quad K = \{ u \in X, [u] \equiv u(y + 0) - u(y - 0) \geq 0 \} \) and the equilibrium problem (2) takes the form

\[
 u \in K, \quad (u_x, v_x - u_x) \geq (f, v - u), \quad \forall v \in K
\]

for the load \( f \in L^2(\Omega) \). Here \( \langle f, g \rangle = \int_a^b fg \, dx + \int_y^b f g \, dx \). The corresponding boundary problem is as follows

\[
 -u_{xx} = f \quad \text{in } \Omega, \\
 [u_x] = 0, \quad [u] \geq 0, \quad u_x(y) \leq 0, \quad [u] \, u_x(y) = 0.
\]

The penalty equation (3) is transformed in

\[
 \langle u^\varepsilon_x, v_x \rangle - \varepsilon^{-1}[u_x^-][v] = \langle f, v \rangle \quad \forall v \in X
\]
or

\[
 -u_{xx}^\varepsilon = f \quad \text{in } \Omega, \\
 [u_x^\varepsilon] = 0, \quad u_x^\varepsilon(y) - \varepsilon^{-1}[u_x^-] = 0
\]

and the iterations (4) are

\[
 (1 + \varepsilon^{-1})u_{xx}^{\varepsilon,n+1}, u_x \rangle + \varepsilon^{-1}[u_x^{\varepsilon,n+1}][v] = \langle f, v \rangle + \varepsilon^{-1}\langle u_x^{\varepsilon,n}, v_x \rangle + \varepsilon^{-1}[u_x^{\varepsilon,n}][v].
\]

We can also write the iterative boundary problem

\[
 -(1 + \varepsilon^{-1})u_{xx}^{\varepsilon,n+1} = f - \varepsilon^{-1}u_{xx}^{\varepsilon,n} \quad \text{in } \Omega, \\
 [u_x^{\varepsilon,n+1}] = 0, \quad (1 + \varepsilon^{-1})u_x^{\varepsilon,n+1}(y) - \varepsilon^{-1}[u_x^{\varepsilon,n+1}] = \varepsilon^{-1}u_x^{\varepsilon,n}(y) - \varepsilon^{-1}[u_x^{\varepsilon,n}]^+.
\]

**Lemma 1** Boundary problem

\[
 -s_{xx} = f \quad \text{in } \Omega, \\
 [s_x] = 0, \quad c_1 s_x(y) - c_2 [s] = g
\]

has the solution

\[
 s = w + \frac{g + c_2 [w]}{c_1 + c_2 (b - a)} \alpha,
\]

where \( w \in H^2(\Omega) \cap X \) is the solution of

\[
 -w_{xx} = f \quad \text{in } \Omega, \\
 [w_x] = 0, \quad w_x(y) = 0
\]

and \( \alpha(x) = \begin{cases} x - a, & x \in (a, y - 0), \\ x - b, & x \in (y + 0, b), \end{cases} \quad \alpha \in C^\infty(\Omega) \cap X. \)
This Lemma can be easily proved in view of the following properties of the function $\alpha$:

$$[\alpha] = -(b - a), \quad \alpha_x = 1, \quad \alpha_{xx} = 0.$$ 

It seems to be natural that we will find the solution of (10) as $u^{\varepsilon,n+1} = w + c^{n+1}(\varepsilon)\alpha, n = 0, 1, 2, \ldots, c^{n+1}(\varepsilon) \in R$. Indeed, then the equation (10) is fulfilled in the domain $\Omega$ for any $c^{n+1}(\varepsilon)$:

$$-(1 + \varepsilon^{-1})u_{xx}^{\varepsilon,n+1} = (1 + \varepsilon^{-1})(-w_{xx} - c^{n+1}(\varepsilon)\alpha_{xx}) = (1 + \varepsilon^{-1})f =$$

$$= f - \varepsilon^{-1}(w_{xx} + c^n(\varepsilon)\alpha_{xx}) = f - \varepsilon^{-1}u_{xx}^{\varepsilon,n}$$

and it needs to fulfill the corresponding boundary conditions on $\Gamma$ by choosing $c^{n+1}(\varepsilon)$.

**Theorem 2** Solutions of (8), (9) and (10) have the following presentations:

$$u = w - \frac{[w]}{b - a} \alpha, \quad u^\varepsilon = w - \frac{[w]}{\varepsilon + b - a} \alpha,$$

$$u^{\varepsilon,n+1} = w - \frac{(1 - \rho^{n+1})[w]}{\varepsilon + b - a} \alpha \quad (\rho = \frac{1}{1 + \varepsilon + b - a}).$$

**Proof.** Let us choose $u^{\varepsilon,0} = w$ for simplicity. By substituting $u^{\varepsilon,0}(y) = [u^{\varepsilon,0}]^+ = -[w]^+$ in (10) and by virtue of the Lemma 1 results, one obtain

$$u^{\varepsilon,1} = w + \frac{-[w]^+ + [w]}{1 + \varepsilon + b - a} \alpha = w - \rho [w]^+ \alpha.$$ 

Futhermore, by calculating $u^{\varepsilon,1}(y) - [u^{\varepsilon,1}]^+ = -\rho [w]^+ - ([w] + (b - a)\rho[w]^+)^+ = -[w]^+ - (1 + \varepsilon)\rho[w]^-)^+ = -[w]^+ - [w]^+$, the equations (10) and (11) give

$$u^{\varepsilon,2} = w + \frac{-\rho[w]^+ - [w]^+ + [w]}{1 + \varepsilon + b - a} \alpha = w - (\rho + \rho^2) [w]^+ \alpha.$$ 

By iterating as $n$ increase we get by the similar way that

$$u^{\varepsilon,n} = w - (\rho + \rho^2 + \ldots + \rho^n) [w]^+ \alpha = w - \frac{\rho(1 - \rho^n)}{1 - \rho} \alpha = w - \frac{1 - \rho^n}{\varepsilon + b - a} [w]^+ \alpha.$$ 

Then we pass to a limit in the last relation as $n \to \infty$ and $\varepsilon \to 0$ thanks to the Theorem 1. The proof is completed.

Theorem 2 can be proved by direct substitution of the obtained solutions in (8), (9) and (10), respectively.

**Example.** Let $f(x) = \begin{cases} c, & x \in (a, y - 0), \\ -c, & x \in (y + 0, b), \end{cases}$ that corresponds to uniform compression for $c > 0$ or stretch for $c < 0$. Then

$$w(x) = c \begin{cases} \frac{(x-a)^2}{2} + (x-a)(y-a) & x \in (a, y - 0), \\ \frac{(x-b)^2}{2} - (x-b)(y-b) & x \in (y + 0, b), \end{cases}$$
\[ [w] = -\frac{c}{2}((y-a)^2 + (y-b)^2). \] If \( c \leq 0 \), then \([w] \geq 0\) (i.e. \([w] = 0\)) and \( u = w \). If \( c > 0 \), then \( u = w - \frac{[w]_{-\overline{a}}}{b-a} \), i.e.

\[
u(x) = \frac{c}{2} \left\{ \begin{array}{ll}
-(x-a)^2 - (x-a) \left( \frac{(y-a)^2 + (y-b)^2}{b-a} - 2(y-a) \right) & , x \in (a, y-0), \\
(x-b)^2 - (x-b) \left( \frac{(y-a)^2 + (y-b)^2}{b-a} + 2(y-b) \right) & , x \in (y+0, b)
\end{array} \right.
\]

and \( [u] = 0 \).

References.


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