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ITERATIVE PENALTY METHOD FOR PLATE WITH A CRACK[1]

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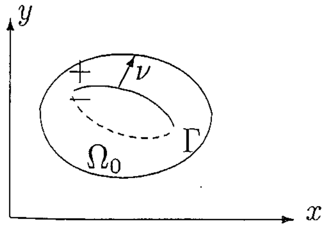
Abstract. Problems with cracks often arise in applications (solid mechanics, geophysics) and need to essentially describe mathematical conditions fulfilled on the crack. We can quote the works [1,2,10,11] and others. In this paper we deal with the nonpenetration condition on the crack faces stated by A.M. Khludnev in [4,5] as the inequality. The obtained problem with the unilateral constraint is described by the variational inequality (see [3,6,9]). Here we construct approximate solutions of this variational inequality using penalty and iterative methods. Convergence of the solutions is proved and it's application at the one-dimensional problem is discussed. Similar approaches for elastic and plastic plates contacted with an obstacle were considered by the author in [7,8].

1. Introduction. A thin isotropic homogeneous plate is assumed to occupy a bounded domain $\Omega_0 \subset R^2$ with a smooth boundary $\partial\Omega_0$. A crack Γ inside Ω_0 is described by a sufficiently smooth function. Choosen direction of the normal $\nu = (\nu^1, \nu^2)$ to Γ defines positive Γ^+ and negative Γ^- crack faces (see the figure).



Let us denote $\Omega = \Omega_0 \setminus \Gamma$. Vector $u = (u^1, u^2)$ of the plate horizontal displacements must satisfy the following boundary conditions. Firstly, the jam condition $u = 0$ must hold on $\partial\Omega_0$. Secondly, the nonpenetration condition of the crack faces without friction condition is imposed on the internal boundary [4,5]:

$$[u]\nu \equiv [u^1]\nu^1 + [u^2]\nu^2 \geq 0,$$

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$[u]$ is the jump of u on Γ, i.e. $[u] = u|_{\Gamma^+} - u|_{\Gamma^-}$. Here we consider $u|_{\Gamma^+}$ and $u|_{\Gamma^-}$ as the traces of the function u from $(H^1(\Omega^+))^2$ and $(H^1(\Omega^-))^2$, respectively, for the domain Ω_0 divided on Ω^+ and Ω^- by means of a smooth continuation of Γ to some closed curve inside Ω_0 (broken line on the figure).

Let us define the basic Hilbert space

$$X = \{u \in \left(H^1(\Omega)\right)^2, \quad u = 0 \quad \text{on } \partial\Omega_0\}$$

and the close and convex set

$$K = \{u \in X, \quad [u]\nu \geq 0\}.$$

Using the Poincare inequality for X

$$\|u^i\|_0^2 \leq c_1 \|\nabla u^i\|_0^2 \quad i = 1, 2,$$

where $\|\cdot\|_0$ is the norm in $L^2(\Omega)$, we define the inner product in X by

$$(u, v) = \langle Du, Dv\rangle + \int_\Gamma [u]\nu \cdot [v]\nu d\Gamma$$

and the norm in X by

$$\|u\|^2 = (u, u).$$

Here $Du = (u_x^1, u_y^1, u_x^2, u_y^2)$ and brackets $\langle\cdot,\cdot\rangle$ denote integration over Ω. We introduce the following bilinear form known in the elasticity by

$$a(u, v) = \int_\Omega \left(u_x^1 v_x^1 + u_y^2 v_y^2 + \kappa(u_x^1 v_y^2 + u_y^2 v_x^1) + \frac{1-\kappa}{2}(u_y^1 + u_x^2)(v_y^1 + v_x^2)\right) d\Omega.$$

Constant $0 < \kappa < 0.5$ is given. The following first Korn inequality will be valid:

$$a(u, u) \geq M\|u\|^2, \quad M > 0. \tag{1}$$

Let $f = (f^1, f^2) \in (L_2(\Omega))^2$ be the given vector of external forces. The equilibrium problem for the thin elastic plate with the crack is formulated as follows (f is reduced by a factor $E(1-\kappa^2)^{-1}$) [4,5]:

$$u \in K, \quad a(u, v - u) \geq \langle f, v - u\rangle, \quad \forall v \in K. \tag{2}$$

The unique solution of the variational inequality (2) exists by virtue of the coercivity (1), boundedness of bilinear form $a(\cdot,\cdot)$ [9].

2. Approximate models. Formally integrating by the parts, we can obtain that the following relation hold

$$a(u, v) = \langle Au, v\rangle - \int_\Gamma [\sigma(u)\nu \cdot v\nu + \sigma(u)\tau \cdot v\tau] d\Gamma,$$

where

$$A(u) = \Big(-u^1_{xx} - \frac{1-\kappa}{2}u^1_{yy} - \frac{1+\kappa}{2}u^2_{xy}, \quad -u^2_{yy} - \frac{1-\kappa}{2}u^2_{xx} - \frac{1+\kappa}{2}u^1_{xy}\Big),$$

$$\sigma(u) = \Big((u^1_x + \kappa u^2_y)\nu^1 + \frac{1-\kappa}{2}(u^1_y + u^2_x)\nu^2, \quad (u^2_y + \kappa u^1_x)\nu^2 + \frac{1-\kappa}{2}(u^1_y + u^2_x)\nu^1\Big).$$

Here $\tau = (-\nu^2, \nu^1)$ is the tangent vector on Γ. Let the solution u be smooth enough. Then we can rewrite (2) as follows

$$\langle Au - f, v-u\rangle - \int_\Gamma\Big([\sigma(u)\nu\cdot(v-u)\nu] + [\sigma(u)\tau\cdot(v-u)\tau]\Big)d\Gamma \geq 0.$$

By varying the test function $v \in K$, it can be deduced (see [3]) that the variational inequality (2) with a smooth enough solution u is equivalent to the following boundary problem

$$Au = f \qquad \text{in } \Omega,$$

$$[\sigma(u)] = 0, \quad \sigma(U)\tau = 0, \qquad \text{on } \Gamma$$

$$[u]\nu \geq 0, \quad \sigma(u)\nu \leq 0, \quad [u]\nu\cdot\sigma(u)\nu = 0.$$

The exact meaning of boundary relations on Γ is studied in [4,11].

To construct a penalty problem, we introduce a penalty operator $\beta : X \to X^*$ by the relation

$$\langle\beta(u), v\rangle = -\int_\Gamma([u]\nu)^-([v]\nu)d\Gamma.$$

Here $\langle\cdot,\cdot\rangle$ means the duality between X and it's dual space X^*. By the upper minus sign we have denoted the negative part of a function, i.e. $s = s^+ - s^-, \quad s^+, s^- \geq 0, \quad s^+s^- = 0$. It is easy to see that β is the monotonous operator. By $u^\varepsilon \in X$ we shall denote the unique solution of the following penalty equation depending on a small parameter $\varepsilon > 0$:

(3) $$a(u^\varepsilon, v) + \varepsilon^{-1}\langle\beta(u^\varepsilon), v\rangle = \langle f, v\rangle, \quad \forall v \in X.$$

The last is interpretated in the above sense as follows

$$Au^\varepsilon = f \qquad \text{in } \Omega,$$

$$[\sigma(u^\varepsilon)] = 0, \quad \sigma(u^\varepsilon)\tau = 0, \qquad \text{on } \Gamma$$

$$\sigma(u^\varepsilon)\nu = -\varepsilon^{-1}([u^\varepsilon]\nu)^-.$$

Let us fix ε. To linearize the left side of (3), we construct the following iterations for an arbitrary $u^{\varepsilon,0} \in X, \quad n = 0, 1, 2, ...$

(4) $$a(u^{\varepsilon,n+1}, v) + \varepsilon^{-1}(u^{\varepsilon,n+1}, v) = \langle f, v\rangle + \varepsilon^{-1}(u^{\varepsilon,n}, v) - \varepsilon^{-1}\langle\beta(u^{\varepsilon,n}), v\rangle.$$

It is obvious that $u^{\varepsilon,n+1} \in X$ exists for the operator's properties marked. The appropriate boundary problem is of the form

$$Au^{\varepsilon,n+1} - \varepsilon^{-1}\triangle(u^{\varepsilon,n+1} - u^{\varepsilon,n}) = f \qquad \text{in } \Omega,$$

$$\left[\sigma(u^{\varepsilon,n+1}) + \varepsilon^{-1}\partial(u^{\varepsilon,n+1} - u^{\varepsilon,n})/\partial\nu\right] = 0,$$

$$\left(\sigma(u^{\varepsilon,n+1}) + \varepsilon^{-1}\partial(u^{\varepsilon,n+1} - u^{\varepsilon,n})/\partial\nu\right)\tau = 0, \qquad \text{on } \Gamma$$

$$\left(\sigma(u^{\varepsilon,n+1}) + \varepsilon^{-1}\partial(u^{\varepsilon,n+1} - u^{\varepsilon,n})/\partial\nu - \varepsilon^{-1}[u^{\varepsilon,n+1} - u^{\varepsilon,n}]\right)\nu = -\varepsilon^{-1}([u^{\varepsilon,n}]\nu)^{-}.$$

Here used notations mean

$$\triangle u = (\triangle u^1, \triangle u^2), \quad \partial u/\partial\nu = (\partial u^1/\partial\nu, \partial u^2/\partial\nu).$$

Theorem 1 *$u^{\varepsilon,n+1} \to u^\varepsilon$ strongly in X as $n \to \infty$ and*

$$(5) \qquad \|u^{\varepsilon,n+1} - u^\varepsilon\|^2 \le (1 + 2M\varepsilon)^{-(n+1)}\|u^{\varepsilon,0} - u^\varepsilon\|^2,$$

$$u^\varepsilon \to u \quad \textit{strongly in } X \textit{ as } \varepsilon \to 0,$$

where $u^{\varepsilon,n+1}, u^\varepsilon, u$ are the solutions of (4),(3),(2), respectively .

Proof. By subtracting (3) from (4) and adding $-\varepsilon^{-1}(u^\varepsilon, v)$ to the both parts, we get

$$a(u^{\varepsilon,n+1} - u^\varepsilon, v) + \varepsilon^{-1}(u^{\varepsilon,n+1} - u^\varepsilon, v) = \varepsilon^{-1}(u^{\varepsilon,n} - u^\varepsilon, v) - \varepsilon^{-1}\langle\beta(u^{\varepsilon,n}) - \beta(u^\varepsilon), v\rangle.$$

Let us consider this equation with the test function $v = u^{\varepsilon,n+1} - u^\varepsilon$ and express it's right side as integrals:

$$(6) \quad a(u^{\varepsilon,n+1} - u^\varepsilon, u^{\varepsilon,n+1} - u^\varepsilon) + \varepsilon^{-1}\|u^{\varepsilon,n+1} - u^\varepsilon\|^2 = \varepsilon^{-1}\langle D(u^{\varepsilon,n} - u^\varepsilon), D(u^{\varepsilon,n+1} - u^\varepsilon)\rangle +$$

$$+\varepsilon^{-1}\int_\Gamma \left([u^{\varepsilon,n} - u^\varepsilon]\nu + ([u^{\varepsilon,n}]\nu)^- - ([u^\varepsilon]\nu)^-\right)\cdot [u^{\varepsilon,n+1} - u^\varepsilon]\nu d\Gamma.$$

Since $s_1 - s_2 + s_1^- - s_2^- = s_1^+ - s_2^+ \le |s_1 - s_2|$, the right side of (6), thanks to the Holder inequality, is no greater than

$$(2\varepsilon)^{-1}\left(\|D(u^{\varepsilon,n} - u^\varepsilon)\|_0^2 + \|D(u^{\varepsilon,n+1} - u^\varepsilon)\|_0^2 + \int_\Gamma \left(([u^{\varepsilon,n} - u^\varepsilon]\nu)^2 + ([u^{\varepsilon,n+1} - u^\varepsilon]\nu)^2\right)d\Gamma\right) =$$

$$= (2\varepsilon)^{-1}\left(\|u^{\varepsilon,n} - u^\varepsilon\|^2 + \|u^{\varepsilon,n+1} - u^\varepsilon\|^2\right).$$

On the other hand, the left side of (6) is no less than

$$(M + \varepsilon^{-1})\|u^{\varepsilon,n+1} - u^\varepsilon\|^2.$$

Therefore

$$\|u^{\varepsilon,n+1} - u^\varepsilon\|^2 \le (1 + 2M\varepsilon)^{-1}\|u^{\varepsilon,n} - u^\varepsilon\|^2.$$

By repeating the last estimate as $n, n-1, ..., 0$, we get that (5) holds and, therefore, the first convergence result is also true.

The weak convergence

$$(7) \qquad u^\varepsilon \to u \quad \text{weakly in } X \text{ as } \varepsilon \to 0$$

is proved by familiar methods in [9] using the properties of operators $a(\cdot,\cdot), \beta(\cdot)$. Indeed, equation (3) with $v = u^\varepsilon - \xi, \quad \xi \in K$ (i.e. $\beta(\xi) = 0$) gives

$$a(u^\varepsilon, u^\varepsilon - \xi) \leq a(u^\varepsilon, u^\varepsilon - \xi) + \varepsilon^{-1}\langle \beta(u^\varepsilon) - \beta(\xi), u^\varepsilon - \xi\rangle = \langle f, u^\varepsilon - \xi\rangle.$$

Hence, $\|u^\varepsilon\| \leq c(M, c_1, f, \xi) = const$ and some subsequence exists such that

$$u^\varepsilon \to u_0 \quad \text{weakly in } X \text{ as } \varepsilon \to 0.$$

Then

$$a(u^\varepsilon, \xi) \to a(u_0, \xi), \quad lim\,inf\; a(u^\varepsilon, u^\varepsilon) \geq a(u_0, u_0),$$
$$\langle \beta(u^\varepsilon), \xi\rangle = \varepsilon\left(\langle f, \xi\rangle - a(u^\varepsilon, \xi)\right) \to 0 \text{ as } \varepsilon \to 0.$$

Therefore, we can obtain that $\beta(u_0) = 0$, i.e. $u_0 \in K$ and pass to a limit infinum in the following inequality:

$$a(u^\varepsilon, v - u^\varepsilon) - \langle f, v - u^\varepsilon\rangle = \varepsilon^{-1}\langle \beta(v) - \beta(u^\varepsilon), v - u^\varepsilon\rangle \geq 0 \quad \forall v \in K.$$

This gives

$$a(u_0, v - u_0) \geq \langle f, v - u_0\rangle \quad \forall v \in K$$

and $u_0 = u$ owing to the uniqness property of the solution.

Subtracting $a(u, v)$ from (3) and considering this equation with the test element $v = u^\varepsilon - u$, one obtaines

$$a(u^\varepsilon - u, u^\varepsilon - u) - \varepsilon^{-1}\int_\Gamma \left([u^\varepsilon]\nu\right)^- [u^\varepsilon - u]\nu d\Gamma = \langle f, u^\varepsilon - u\rangle - a(u, u^\varepsilon - u).$$

Owing to

$$-\left([u^\varepsilon]\nu\right)^- [u^\varepsilon - u]\nu = \left(\left([u^\varepsilon]\nu\right)^-\right)^2 + \left([u^\varepsilon]\nu\right)^- [u]\nu, \quad [u]\nu \geq 0$$

and (1), we have

$$M\|u^\varepsilon - u\|^2 + \varepsilon^{-1}\int_\Gamma \left(\left([u^\varepsilon]\nu\right)^-\right)^2 d\Gamma \leq \langle f, u^\varepsilon - u\rangle - a(u, u^\varepsilon - u).$$

Therefore, (7) leads to the second strong convergence to be proved. The proof is completed.
Remark. Obviously, we may use another inner product in X, for instance

$$(u, v) = a(u, v) + \int_\Gamma [u]\nu \cdot [v]\nu d\Gamma.$$

Then (4) takes the form

$$(1 + \varepsilon^{-1})a(u^{\varepsilon,n+1} - u^{\varepsilon,n}, v) + \varepsilon^{-1}\int_\Gamma [u^{\varepsilon,n+1} - u^{\varepsilon,n}]\nu \cdot [v]\nu d\Gamma =$$
$$= \langle f, v\rangle - a(u^{\varepsilon,n}, v) - \varepsilon^{-1}\langle \beta(u^{\varepsilon,n}), v\rangle$$

and Theorem 1 is also valid.

3. Application. We will consider the one-dimensional crack problem, i.e. a thin bar $\Omega_0 = (a,b)$ with a cut $\Gamma = \{y\}, a < y < b$. Hence, $\Omega = (a, y-0) \cup (y+0, b)$, $X = \{u \in H^1(\Omega), u(a) = u(b) = 0\}$, $K = \{u \in X, [u] \equiv u(y+0) - u(y-0) \geq 0\}$ and the equilibrium problem (2) takes the form

$$u \in K, \quad \langle u_x, v_x - u_x \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K$$

for the load $f \in L^2(\Omega)$. Here $\langle f, g \rangle = \int_a^y fg\,dx + \int_y^b fg\,dx$. The corresponding boundary problem is as follows

$$-u_{xx} = f \qquad \text{in } \Omega, \tag{8}$$

$$[u_x] = 0, \quad [u] \geq 0, \quad u_x(y) \leq 0, \quad [u]\, u_x(y) = 0.$$

The penalty equation (3) is transformed in

$$\langle u_x^\varepsilon, v_x \rangle - \varepsilon^{-1}[u^\varepsilon]^-[v] = \langle f, v \rangle \quad \forall v \in X$$

or

$$-u_{xx}^\varepsilon = f \qquad \text{in } \Omega, \tag{9}$$

$$[u_x^\varepsilon] = 0, \quad u_x^\varepsilon(y) + \varepsilon^{-1}[u^\varepsilon]^- = 0$$

and the iterations (4) are

$$(1+\varepsilon^{-1})\langle u_x^{\varepsilon,n+1}, v_x \rangle + \varepsilon^{-1}[u^{\varepsilon,n+1}][v] = \langle f, v \rangle + \varepsilon^{-1}\langle u_x^{\varepsilon,n}, v_x \rangle + \varepsilon^{-1}[u^{\varepsilon,n}]^+[v].$$

We can also write the iterative boundary problem

$$-(1+\varepsilon^{-1})u_{xx}^{\varepsilon,n+1} = f - \varepsilon^{-1}u_{xx}^{\varepsilon,n} \qquad \text{in } \Omega, \tag{10}$$

$$[u_x^{\varepsilon,n+1}] = 0, \quad (1+\varepsilon^{-1})u_x^{\varepsilon,n+1}(y) - \varepsilon^{-1}[u^{\varepsilon,n+1}] = \varepsilon^{-1}u_x^{\varepsilon,n}(y) - \varepsilon^{-1}[u^{\varepsilon,n}]^+.$$

Lemma 1 *Boundary problem*

$$-s_{xx} = f \qquad \text{in } \Omega,$$

$$[s_x] = 0, \quad c_1 s_x(y) - c_2[s] = g$$

has the solution

$$s = w + \frac{g + c_2[w]}{c_1 + c_2(b-a)}\alpha, \tag{11}$$

where $w \in H^2(\Omega) \cap X$ *is the solution of*

$$-w_{xx} = f \qquad \text{in } \Omega,$$

$$[w_x] = 0, \quad w_x(y) = 0$$

and $\alpha(x) = \begin{cases} x - a & , x \in (a, y-0), \\ x - b & , x \in (y+0, b), \end{cases} \qquad \alpha \in C^\infty(\Omega) \cap X.$

This Lemma can be easily proved in view of the following properties of the function α:

$$[\alpha] = -(b-a), \quad \alpha_x \equiv 1, \quad \alpha_{xx} \equiv 0.$$

It seems to be natural that we will find the solution of (10) as $u^{\varepsilon,n+1} = w + c^{n+1}(\varepsilon)\alpha, n = 0, 1, 2, ..., c^{n+1}(\varepsilon) \in R$. Indeed, then the equation (10) is fulfilled in the domain Ω for any $c^{n+1}(\varepsilon)$:

$$-(1+\varepsilon^{-1})u_{xx}^{\varepsilon,n+1} = (1+\varepsilon^{-1})(-w_{xx} - c^{n+1}(\varepsilon)\alpha_{xx}) = (1+\varepsilon^{-1})f =$$
$$= f - \varepsilon^{-1}(w_{xx} + c^n(\varepsilon)\alpha_{xx}) = f - \varepsilon^{-1}u_{xx}^{\varepsilon,n}$$

and it needs to fulfil the corresponding boundary conditions on Γ by choosing $c^{n+1}(\varepsilon)$.

Theorem 2 *Solutions of (8),(9) and (10) have the following presentations:*

$$u = w - \frac{[w]^-}{b-a}\alpha, \quad u^\varepsilon = w - \frac{[w]^-}{\varepsilon + b - a}\alpha,$$

$$u^{\varepsilon,n+1} = w - \frac{(1-\rho^{n+1})[w]^-}{\varepsilon + b - a}\alpha \quad \left(\rho = \frac{1}{1+\varepsilon+b-a}\right).$$

Proof. Let us choose $u^{\varepsilon,0} = w$ for simplicity. By substituting $u_x^{\varepsilon,0}(y) - [u^{\varepsilon,0}]^+ = -[w]^+$ in (10) and by virtue of the Lemma 1 results, one obtain

$$u^{\varepsilon,1} = w + \frac{-[w]^+ + [w]}{1+\varepsilon+b-a}\alpha = w - \rho[w]^-\alpha.$$

Futher, by calculating $u_x^{\varepsilon,1}(y) - [u^{\varepsilon,1}]^+ = -\rho[w]^- - ([w] + (b-a)\rho[w]^-)^+ = -\rho[w]^- - ([w]^+ - (1+\varepsilon)\rho[w]^-)^+ = -\rho[w]^- - [w]^+$, the equations (10) and (11) give

$$u^{\varepsilon,2} = w + \frac{-\rho[w]^- - [w]^+ + [w]}{1+\varepsilon+b-a}\alpha = w - (\rho+\rho^2)[w]^-\alpha.$$

By iterating as n increase we get by the similar way that

$$u^{\varepsilon,n} = w - (\rho + \rho^2 + ... + \rho^n)[w]^-\alpha = w - \frac{\rho(1-\rho^n)}{1-\rho}\alpha = w - \frac{1-\rho^n}{\varepsilon+b-a}[w]^-\alpha.$$

Then we passage to a limit in the last relation as $n \to \infty$ and $\varepsilon \to 0$ thanks to the Theorem 1. The proof is completed.

Theorem 2 can be proved by direct substitution of the obtained solutions in (8),(9) and (10), respectively.

Example. Let $f(x) = \begin{cases} c & , x \in (a, y-0), \\ -c & , x \in (y+0, b), \end{cases}$ that corresponds to uniform compression for $c > 0$ or stretch for $c < 0$. Then

$$w(x) = c\begin{cases} -\frac{(x-a)^2}{2} + (x-a)(y-a) & , x \in (a, y-0), \\ \frac{(x-b)^2}{2} - (x-b)(y-b) & , x \in (y+0, b), \end{cases}$$

$[w] = -\frac{c}{2}((y-a)^2 + (y-b)^2)$. If $c \leq 0$, then $[w] \geq 0$ (i.e. $[w]^- = 0$) and $u = w$. If $c > 0$, then $u = w - \frac{[w]^-}{b-a}\alpha$, i.e.

$$u(x) = \frac{c}{2}\begin{cases} -(x-a)^2 - (x-a)\left(((y-a)^2 + (y-b)^2)/(b-a) - 2(y-a)\right) & , x \in (a, y-0), \\ (x-b)^2 - (x-b)\left(((y-a)^2 + (y-b)^2)/(b-a) + 2(y-b)\right) & , x \in (y+0, b) \end{cases}$$

and $[u] = 0$.

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