CRACK IN A SOLID UNDER COULOMB FRICTION LAW*

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Abstract. An equilibrium problem for a solid with a crack is considered. We assume that both the Coulomb friction law and a nonpenetration condition hold at the crack faces. The problem is formulated as a quasi-variational inequality. Existence of a solution is proved, and a complete system of boundary conditions fulfilled at the crack surface is obtained in suitable spaces.

Keywords: variational and quasi-variational inequalities, crack, Coulomb friction

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INTRODUCTION

Crack problems deal first of all with peculiarities concerned with the presence of tips or edges of the crack (see Cherepanov [3], Morozov [17], Telega and Lewinski [20], Duduchava and Wendland [4]). Formulation of crack problems does not usually imply any restrictions imposed a priori at the crack, for example, crack surfaces are assumed to be stress-free. Statement of the nonpenetration condition at the crack faces in Khludnev and Sokolowski [10] leads to the presence of unilateral constraints like in contact problems for systems of body-body type. The same is valid for friction conditions. Problems with friction in solid mechanics were considered by Duvaut and Lions [5], Alekhin et al. [1], Kravchuk [14] and others. For contact problems with friction, normal components of the stress vector on the contacting boundary are normally given a priori. Taking into account the Coulomb friction law leads to quasi-variational formulations of the problems. In this case, classical

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variational methods are not acceptable. To prove the existence of a solution for contact problems with Coulomb friction, fixed-point theorems are used in Nečas et al. [19], Jarušek [9], Hlaváček et al. [8], while in Eck and Jarušek [6] the penalty approximation was constructed. In both cases, additional regularity of a solution is required. We adapt the fixed point argument to establish the existence result provided the friction coefficient is small and has a compact support on the crack surface.

On the other hand, problems with cracks have nonregular character of the boundaries caused by the presence of the crack. Therefore, one needs here to apply the theory of boundary value problems in domains with nonsmooth boundaries (see Maz'ya [16], Nazarov and Plamenevskiĭ [18], Grisvard [7]). We use the spaces of traces of functions at the boundary which are adapted to the crack problems. This allows us to define the displacement and stress functions at the crack faces and to interpret the relations describing the nonpenetration and friction conditions at the crack from functional point of view.

Methods of solution for solids with cracks are proposed in Kovtunenko [11], [12], [13].

1. Domains with thin inclusions

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary Γ, $\overline{\Omega} = \Omega \cup \Gamma$. The boundary Γ belongs to the class $C^{k,1}$ if there exist two real numbers $b > 0$, $h > 0$, p coordinate systems

$$(y^j, y_3^j), \quad y^j = (y_1^j, y_2^j), \quad j = 1, \dots, p, \tag{1}$$

and p functions θ^j such that in the squares

$$\Delta^j = \{y^j \in \mathbb{R}^2 \mid |y_i^j| < b, \quad i = 1, 2\}$$

the functions θ^j belong to $C^{k,1}(\overline{\Delta}^j)$, and for

$$\begin{aligned}
\Gamma^j &= \{(y^j, y_3^j) \in \mathbb{R}^3 \mid y^j \in \Delta^j,\, y_3^j = \theta^j(y^j)\},\\
\Omega_+^j &= \{(y^j, y_3^j) \in \mathbb{R}^3 \mid y^j \in \Delta^j,\, \theta^j(y^j) < y_3^j < \theta^j(y^j) + h\},\\
\Omega_-^j &= \{(y^j, y_3^j) \in \mathbb{R}^3 \mid y^j \in \Delta^j,\, \theta^j(y^j) - h < y_3^j < \theta^j(y^j)\}
\end{aligned}$$

the following conditions hold:

$$\Gamma = \bigcup_{j=1}^{p} \Gamma^j, \quad \Omega_+^j \subset \Omega, \quad \Omega_-^j \subset \mathbb{R}^3 \setminus \overline{\Omega}, \quad j = 1, \dots, p.$$

Here $C^{k,1}(\overline{\Delta}^j)$ is the space of functions having k Lipschitz continuous derivatives in $\overline{\Delta}^j$, $k \geqslant 0$ is an integer.

Consider a domain Ω containing an open oriented surface Σ_c without self-intersections, and denote $\Omega_c = \Omega \setminus \overline{\Sigma}_c$, $\overline{\Sigma}_c = \Sigma_c \cup \partial\Sigma_c$, where $\partial\Sigma_c$ is the boundary of Σ_c. We assume that there exists a closed extension Σ of Σ_c dividing the domain Ω into two subdomains Ω_1, Ω_2 with boundaries $\partial\Omega_1$, $\partial\Omega_2$ such that $\Sigma_c \subset \Sigma$. Introduce the unit normal ν to Σ and define the opposite faces $\Sigma^{\pm}$ of the surface Σ. The signs $\pm$ fit the positive and negative directions of ν, respectively. Let $\partial\Omega_1 = \Sigma^-$, $\partial\Omega_2 = \Gamma \cup \Sigma^+$. The surfaces $\Sigma_c^{\pm}$ are the corresponding parts of $\Sigma^{\pm}$, and we denote the boundary of Ω_c by $\partial\Omega_c = \Gamma \cup \overline{\Sigma}_c^{\pm}$. We say that the boundary $\partial\Omega_c$ belongs to the class $C^{k,1}$ if $\partial\Omega_1$, $\partial\Omega_2$ belong to $C^{k,1}$.

For a domain $\Omega \subset \mathbb{R}^3$ with a boundary Γ, introduce the Sobolev space

$$H^1(\Omega) = \{u \mid u, u_{,i} \in L^2(\Omega), \quad i = 1,2,3\}, \qquad H^0(\Omega) = L^2(\Omega),$$

equipped with the norm

$$\|u\|_{1,\Omega}^2 = \|u\|_{0,\Omega}^2 + \sum_{i=1}^{3} \|u_{,i}\|_{0,\Omega}^2,$$

where $\|\cdot\|_{0,\Omega}$ is the norm in $L^2(\Omega)$. Denote by $H_0^1(\Omega)$ a completion of $C_0^\infty(\Omega)$ in the $H^1(\Omega)$-norm.

Introduce also spaces at the boundary Γ in the local coordinates (1) as follows. Let Γ belong to the class $C^{0,1}$. For a given function $s(x)$, $x \in \Gamma$, the functions

$$s^j(y^j) = s\big(y^j, \theta^j(y^j)\big), \quad y^j = (y_1^j, y_2^j) \in \Delta^j, \quad j = 1, \dots, p,$$

can be considered in the squares Δ^j. Then we define the space $H^{1/2}(\Gamma)$ equipped with the norm (Lions and Magenes [15])

$$\begin{aligned} \|s\|_{1/2,\Gamma}^2 &= \sum_{j=1}^{p} \|s^j\|_{1/2,\Delta^j}, \\ \|s^j\|_{1/2,\Delta^j}^2 &= \|s^j\|_{0,\Delta^j}^2 + \int_{-b}^{b}\int_{-b}^{b} |t-\tau|^{-2} \sum_{i=1}^{2} \|s^j(y^j\|_{y_i^j=t}) \\ &\quad - s^j(y^j|_{y_i^j=\tau})\|_{L^2(-b,b)}^2 \, dt \, d\tau. \end{aligned} \tag{2}$$

We formulate the general trace theorem (see Baiocchi and Capelo [2]).

Theorem 1. *Let the boundary Γ belong to the class $C^{0,1}$, and let a function u belong to the space $H^1(\Omega)$. Then there exists a linear continuous operator*

$\gamma\colon H^1(\Omega) \to H^{1/2}(\Gamma)$, which uniquely defines the trace $\gamma u \in H^{1/2}(\Gamma)$ of u at Γ. Conversely, there exists a linear continuous operator $H^{1/2}(\Gamma) \to H^1(\Omega)$ such that for any given $\varphi \in H^{1/2}(\Gamma)$, a function $u \in H^1(\Omega)$ can be found such that $\gamma u = \varphi$ on Γ.

In what follows, we write u on Γ meaning γu.

Consider the domain Ω_c with the boundary $\partial\Omega_c = \Gamma \cup \overline{\Sigma}_c^{\pm}$. Let a function $u \in H^1(\Omega_c)$ be given. We assume that $\partial\Omega_c$ belongs to the class $C^{0,1}$, i.e. Ω can be divided into two domains Ω_1, Ω_2 by the closed surface Σ such that $\Sigma_c \subset \Sigma$, and $\partial\Omega_1$, $\partial\Omega_2$ belong to the class $C^{0,1}$. For every Ω_k, $k = 1, 2$, we have $u \in H^1(\Omega_k)$ and, consequently, we can apply Theorem 1 and define the traces $\gamma_k u$ at $\partial\Omega_k$. The boundaries $\partial\Omega_1$, $\partial\Omega_2$ consist of Σ^-, $\Gamma \cup \Sigma^+$, respectively. Let us denote $\gamma_1 u = u^- \in H^{1/2}(\Sigma)$, $\gamma_2 u = (u|_\Gamma, u^+)$, $u|_\Gamma \in H^{1/2}(\Gamma)$, $u^+ \in H^{1/2}(\Sigma)$. The surfaces $\Sigma_c^{\pm}$ are the corresponding parts of $\Sigma^{\pm}$, therefore, $u^{\pm} \in H^{1/2}(\Sigma_c)$ are also defined.

Let us denote the jump $u^+ - u^-$ by $[u]$. Notice that, by $u \in H^1(\Omega_c)$, the uniqueness of the traces implies $u^+ = u^-$ on $\Sigma \setminus \Sigma_c$, or

$$[u] = 0 \quad \text{on} \quad \Sigma \setminus \Sigma_c. \tag{3}$$

Condition (3) gives an additional property of the traces at Σ_c which is used in studying the space $H_{00}^{1/2}(\Sigma_c)$ below.

Let Σ belong to the class $C^{k,1}$, $k \geqslant 0$ being an integer. Introduce the Hilbert space

$$H_{00}^{1/2}(\Sigma_c) = \{s \in H^{1/2}(\Sigma_c) \mid \varrho^{-1/2} s \in L^2(\Sigma_c)\}$$

equipped with the norm

$$\|s\|_{1/2,00,\Sigma_c}^2 = \|s\|_{1/2,\Sigma_c}^2 + \|\varrho^{-1/2} s\|_{0,\Sigma_c}^2,$$

where the function ϱ possesses the properties $\varrho \in C^{k,1}(\overline{\Sigma}_c)$, $\varrho > 0$ in Σ_c, $\varrho = 0$ on $\partial\Sigma_c$, $\lim\limits_{x \to x_0} \varrho(x)/\operatorname{dist}(x, \partial\Sigma_c) = d \neq 0$ for all $x_0 \in \partial\Sigma_c$. Here $\operatorname{dist}(x, \partial\Sigma_c)$ denotes the distance between the point $x \in \Sigma_c$ and the boundary $\partial\Sigma_c$.

We prove a statement characterizing the functions from $H_{00}^{1/2}(\Sigma_c)$.

Lemma 1. *The following equivalence takes place:*

$$s \in H_{00}^{1/2}(\Sigma_c) \iff \bar{s} = \begin{cases} s & \text{in } \Sigma_c \\ 0 & \text{in } \Sigma \setminus \Sigma_c \end{cases} \in H^{1/2}(\Sigma).$$

P r o o f. By utilising the local coordinate systems (1), the assertion of Lemma 1 reduces to the case

$$\Sigma = \mathbb{R}^2, \quad \Sigma_c = \Delta = \{x \in \mathbb{R}^2 \mid |x_i| < b,\ i = 1, 2\}.$$

Denote $I = (-b, b)$. By the norm definition (2), we can write

$$\|\bar{s}\|^2_{1/2,\mathbb{R}^2} = \|\bar{s}\|^2_{0,\mathbb{R}^2} + \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |t-\tau|^{-2} \sum_{i=1}^{2} \|\bar{s}(x|_{x_i=t}) - \bar{s}(x|_{x_i=\tau})\|^2_{0,\mathbb{R}} \, \mathrm{d}t \, \mathrm{d}\tau,$$

$$\|s\|^2_{1/2,\Delta} = \|s\|^2_{0,\Delta} + \int\limits_{-b}^{b}\int\limits_{-b}^{b} |t-\tau|^{-2} \sum_{i=1}^{2} \|s(x|_{x_i=t}) - s(x|_{x_i=\tau})\|^2_{0,I} \, \mathrm{d}t \, \mathrm{d}\tau.$$

Since $\bar{s}(x) = 0$ for $|x_i| \geqslant b$, $i = 1, 2$, we obtain

$$\|\bar{s}\|^2_{1/2,\mathbb{R}^2} = \|s\|^2_{0,\Delta} + \int\limits_{-b}^{b}\int\limits_{-b}^{b} |t-\tau|^{-2} \sum_{i=1}^{2} \|s(x|_{x_i=t}) - s(x|_{x_i=\tau})\|^2_{0,I} \, \mathrm{d}t \, \mathrm{d}\tau$$
$$+ 2\int\limits_{-b}^{b}\left(\int\limits_{-\infty}^{-b} + \int\limits_{b}^{\infty}\right)|t-\tau|^{-2} \sum_{i=1}^{2} \|s(x|_{x_i=\tau})\|^2_{0,I} \, \mathrm{d}t \, \mathrm{d}\tau,$$

which implies

$$\|\bar{s}\|^2_{1/2,\mathbb{R}^2} = \|s\|^2_{1/2,\Delta} + 2\int\limits_{-b}^{b} \sum_{i=1}^{2} \|s(x|_{x_i=\tau})\|^2_{0,I} \left(\int\limits_{-\infty}^{-b} + \int\limits_{b}^{\infty}\right)|t-\tau|^{-2} \, \mathrm{d}t \, \mathrm{d}\tau.$$

The integral with respect to t can be calculated for $\tau \in (-b, b)$,

$$\left(\int\limits_{-\infty}^{-b} + \int\limits_{b}^{\infty}\right)|t-\tau|^{-2} \, \mathrm{d}t = \int\limits_{-\infty}^{-b} (\tau - t)^{-2} \, \mathrm{d}t + \int\limits_{b}^{\infty} (t-\tau)^{-2} \, \mathrm{d}t = \frac{1}{b+\tau} + \frac{1}{b-\tau}.$$

Thus, we have

$$\|\bar{s}\|^2_{1/2,\mathbb{R}^2} = \|s\|^2_{1/2,\Delta} + \int\limits_{-b}^{b} \sum_{i=1}^{2} \left(\frac{(b-\tau)(b+\tau)}{4b}\right)^{-1} \|s(x|_{x_i=\tau})\|^2_{0,I} \, \mathrm{d}\tau.$$

Changing the variable τ by x_i for $i = 1, 2$ and denoting

$$\varrho^{-1}(x) = \sum_{i=1}^{2} \left(\frac{(b-x_i)(b+x_i)}{4b}\right)^{-1},$$

we obtain the equality

$$\|\bar{s}\|^2_{1/2,\mathbb{R}^2} = \|s\|^2_{1/2,\Delta} + \int\limits_{\Delta} \varrho^{-1}(x)|s(x)|^2 \, \mathrm{d}x = \|s\|^2_{1/2,00,\Delta}$$

which proves the assertion of Lemma 1. $\square$

Let $H_0^{1/2}(\Sigma_c)$ be the completion in the $H^{1/2}(\Sigma_c)$-norm of finite functions from $C^{k,1}(\Sigma_c)$ having compact supports in Σ_c. We have to note that $H_{00}^{1/2}(\Sigma_c)$ is imbedded in $H_0^{1/2}(\Sigma_c)$, $H_0^{1/2}(\Sigma_c)$ coincides with $H^{1/2}(\Sigma_c)$, and the extensions of functions from $H_0^{1/2}(\Sigma_c)$ to Σ by zero do not belong to $H^{1/2}(\Sigma)$, in general (Lions and Magenes [15]).

By Lemma 1 and property (3), Theorem 1 yields the next statement.

Theorem 2. *Let the boundary $\partial\Omega_c$ belong to the class $C^{0,1}$, and let a function u belong to the space $H^1(\Omega_c)$. Then there exists a linear continuous operator which uniquely defines at $\partial\Omega_c$ the values*

$$u\,|_\Gamma \in H^{1/2}(\Gamma), \quad u^\pm \in H^{1/2}(\Sigma_c), \quad [u] \in H_{00}^{1/2}(\Sigma_c).$$

Conversely, there exists a linear continuous operator such that for any given

$$\psi \in H^{1/2}(\Gamma), \quad \varphi^\pm \in H^{1/2}(\Sigma_c), \quad [\varphi] \in H_{00}^{1/2}(\Sigma_c),$$

a function $u \in H^1(\Omega_c)$ can be found such that

$$u = \psi \text{ on } \Gamma, \quad u^\pm = \varphi^\pm \text{ on } \Sigma_c.$$

Proof. Assume that Σ is the closed extension of Σ_c from the class $C^{0,1}$ dividing Ω into two domains Ω_1, Ω_2 as before. The boundaries $\partial\Omega_1, \partial\Omega_2$ consist of Σ^-, $\Gamma\cup\Sigma^+$, respectively. For $u \in H^1(\Omega_c)$ we have $u \in H^1(\Omega_k)$, $k = 1,2$, and, by Theorem 1, $u\,|_\Gamma \in H^{1/2}(\Gamma)$, $u^\pm \in H^{1/2}(\Sigma)$. In view of the property (3), one can write

$$[u] = 0 \text{ on } \Sigma \setminus \Sigma_c, \quad [u] \in H^{1/2}(\Sigma).$$

By Lemma 1, this means that $[u] \in H_{00}^{1/2}(\Sigma_c)$, which proves the first assertion formulated in Theorem 2.

Now we prove the converse assertion formulated in Theorem 2. Let $\psi \in H^{1/2}(\Gamma)$, $\varphi^\pm \in H^{1/2}(\Sigma_c)$ be given, $[\varphi] \in H_{00}^{1/2}(\Sigma_c)$. One can construct an arbitrary smooth extension of φ^- onto Σ^- such that

$$\tilde{\varphi}^- = \begin{cases} \varphi^- & \text{on } \Sigma_c^- \\ \xi & \text{on } \Sigma^- \setminus \Sigma_c^- \end{cases} \in H^{1/2}(\Sigma).$$

Let us define on Σ^+ the function

$$\tilde{\varphi}^+ = \begin{cases} \varphi^+ & \text{on } \Sigma_c^+ \\ \xi & \text{on } \Sigma^+ \setminus \Sigma_c^+. \end{cases}$$

Since $[\varphi] \in H_{00}^{1/2}(\Sigma_c)$ and $[\tilde{\varphi}] = 0$ at $\Sigma \setminus \Sigma_c$, then, by Lemma 1, we obtain $[\tilde{\varphi}] \in H^{1/2}(\Sigma)$. In particular, this implies that $\tilde{\varphi}^+ = [\tilde{\varphi}] + \tilde{\varphi}^- \in H^{1/2}(\Sigma)$. Hence, by Theorem 1, there exist functions $u_k \in H^1(\Omega_k)$, $k = 1, 2$, such that u_1 and u_2 coincide with $\tilde{\varphi}^-$ and ψ, $\tilde{\varphi}^+$ on Σ^- and Γ, Σ^+, respectively. In Ω_c, define the function

$$u = \begin{cases} u_1 & \text{in } \Omega_1 \\ u_2 & \text{in } \Omega_2. \end{cases}$$

By the property

$$0 = [\tilde{\varphi}] = [u] \quad \text{on } \Sigma \setminus \Sigma_c,$$

we obtain $u \in H^1(\Omega_c)$. Theorem 2 is proved. □

Let us prove two auxiliary statements.

Lemma 2. *For $s \in H_{00}^{1/2}(\Sigma_c)$, if $r \in C^{0,1}(\overline{\Sigma}_c)$, then $rs \in H_{00}^{1/2}(\Sigma_c)$.*

P r o o f. This assertion is a consequence of the norm definition (2). Indeed, utilising the local coordinate system (1) for the surface Σ_c as in Lemma 1, it is sufficient to prove the case

$$\Sigma_c = \Delta = \{x \in \mathbb{R}^2 \mid x_i \in I,\ i = 1, 2\}, \quad I = (-b, b).$$

Then we can write

$$\|rs\|^2_{1/2,00,\Delta} = \|rs\|^2_{1/2,\Delta} + \|\varrho^{-1/2} rs\|^2_{0,\Delta} = \|rs\|^2_{0,\Delta} + \|\varrho^{-1/2} rs\|^2_{0,\Delta}$$
$$+ \int_{-b}^{b}\int_{-b}^{b} |t - \tau|^{-2} \sum_{i=1}^{2} \|r(x|_{x_i=t}) s(x|_{x_i=t}) - r(x|_{x_i=\tau}) s(x|_{x_i=\tau})\|^2_{0,I} \, dt \, d\tau.$$

The following equality takes place:

$$r(x|_{x_i=t}) s(x|_{x_i=t}) - r(x|_{x_i=\tau}) s(x|_{x_i=\tau})$$
$$= r(x|_{x_i=t}) \big(s(x|_{x_i=t}) - s(x|_{x_i=\tau}) \big) + s(x|_{x_i=\tau}) \big(r(x|_{x_i=t}) - r(x|_{x_i=\tau}) \big).$$

By the Lipschitz continuity of r in $\overline{\Delta}$, we can estimate the terms

$$\|rs\|_{0,\Delta}^2 \leqslant \sup_{x\in\overline{\Delta}} |r(x)|^2 \|s\|_{0,\Delta}^2, \quad \|\varrho^{-1/2} rs\|_{0,\Delta}^2 \leqslant \sup_{x\in\overline{\Delta}} |r(x)|^2 \|\varrho^{-1/2} s\|_{0,\Delta}^2,$$

$$\int_{-b}^{b}\int_{-b}^{b} |t-\tau|^{-2} \|r(x|_{x_i=t})\big(s(x|_{x_i=t}) - s(x|_{x_i=\tau})\big)\|_{0,I}^2 \,\mathrm{d}t\,\mathrm{d}\tau$$

$$\leqslant \sup_{x\in\overline{\Delta}} |r(x)|^2 \int_{-b}^{b}\int_{-b}^{b} |t-\tau|^{-2} \|s(x|_{x_i=t}) - s(x|_{x_i=\tau})\|_{0,I}^2 \,\mathrm{d}t\,\mathrm{d}\tau,$$

$$\int_{-b}^{b}\int_{-b}^{b} \left\| \frac{s(x|_{x_i=\tau})\big(r(x|_{x_i=t}) - r(x|_{x_i=\tau})\big)}{|t-\tau|} \right\|_{0,I}^2 \mathrm{d}t\,\mathrm{d}\tau$$

$$\leqslant \sup_{x,y\in\overline{\Delta}} \left(\frac{|r(x)-r(y)|}{|x-y|} \right)^2 \cdot 2b \cdot \|s\|_{0,\Delta}^2, \quad i=1,2.$$

Hence, we obtain the estimate

$$\|rs\|_{1/2,00,\Delta} \leqslant c\|r\|_{C^{0,1}(\overline{\Delta})} \|s\|_{1/2,00,\Delta},$$

which proves the lemma. $\square$

Denote by $H_{00}^{1/2}(\Sigma_c)^*$ the space dual of $H_{00}^{1/2}(\Sigma_c)$ with the duality pairing $\langle\cdot,\cdot\rangle_{1/2,\Sigma_c}$.

Lemma 3. *For $s \in H_{00}^{1/2}(\Sigma_c)^*$, if $r \in C^{0,1}(\overline{\Sigma}_c)$, then $rs \in H_{00}^{1/2}(\Sigma_c)^*$.*

Proof. Indeed, by Lemma 2, we define rs from the formula

$$\langle rs, \varphi\rangle_{1/2,\Sigma_c} = \langle s, r\varphi\rangle_{1/2,\Sigma_c} \quad \forall \varphi \in H_{00}^{1/2}(\Sigma_c),$$

which proves Lemma 3. $\square$

2. Green's formulae

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary Γ, and let $n = (n_1, n_2, n_3)$ be a unit outward normal vector to Γ. Introduce the stress and strain tensors of linear elasticity

$$\sigma_{ij}(u) = a_{ijkl}\varepsilon_{kl}(u), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1,2,3,$$
$$a_{ijkl} = a_{jikl} = a_{klij} \in L^\infty(\Omega), \quad c_1\xi_{ij}\xi_{ij} \leqslant a_{ijkl}\xi_{kl}\xi_{ij} \leqslant c_2\xi_{ij}\xi_{ij}, \quad c_1, c_2 > 0,$$

where $u = (u_1, u_2, u_3)$ are the displacements defined in Ω.

By the symmetry $\sigma_{ij}(u) = \sigma_{ji}(u)$ we can integrate by parts,

$$\int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(v) = -\int_\Omega \sigma_{ij,j}(u)v_i + \int_\Gamma \sigma_{ij}(u)n_j v_i. \tag{4}$$

Decompose the vectors $\big(\sigma_{1j}(u)n_j, \sigma_{2j}(u)n_j, \sigma_{3j}(u)n_j\big)$, $v = (v_1, v_2, v_3)$ into normal and tangential components at the boundary as follows:

$$\sigma_{ij}(u)n_j = \sigma_n(u)n_i + \sigma_{\tau i}(u), \quad i = 1,2,3, \qquad \sigma_n(u) = \sigma_{ij}(u)n_j n_i;$$
$$v_i = v_n n_i + v_{\tau i}, \quad i = 1,2,3, \qquad v_n = v_i n_i. \tag{5}$$

Since $\sigma_{\tau i}(u)n_i = \sigma_{ij}(u)n_j n_i - \sigma_n(u) = 0$, $v_{\tau i}n_i = v_i n_i - v_n = 0$, one has

$$\sigma_{ij}(u)n_j v_i = \big(\sigma_n(u)n_i + \sigma_{\tau i}(u)\big)(v_n n_i + v_{\tau i}) = \sigma_n(u)v_n + \sigma_{\tau i}(u)v_{\tau i}.$$

Thus, for smooth functions u, v, we obtain the following Green formula instead of (4):

$$\int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(v) = -\int_\Omega \sigma_{ij,j}(u)v_i + \int_\Gamma \big(\sigma_n(u)v_n + \sigma_{\tau i}(u)v_{\tau i}\big). \tag{6}$$

Introduce the space

$$H^1_\sigma(\Omega) = \{u = (u_1, u_2, u_3) \in H^1(\Omega) \mid \sigma_{ij,j}(u) \in L^2(\Omega),\ i = 1,2,3\}$$

equipped with the norm

$$\|u\|^2_{1,\sigma,\Omega} = \sum_{i=1}^{3}\left(\|u_i\|^2_{1,\Omega} + \|\sigma_{ij,j}(u)\|^2_{0,\Omega}\right).$$

We use the second Korn inequality. There exists a constant $c > 0$ such that for all $u \in [H^1(\Omega)]^3$

$$\int_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(u) + \int_\Omega u_i u_i \geqslant c \sum_{i=1}^{3} \|u_i\|^2_{1,\Omega}, \quad u = (u_1, u_2, u_3). \tag{7}$$

Moreover, if $u = 0$ at a part of the boundary Γ, then the following estimate holds:

$$\int_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(u) \geqslant c \sum_{i=1}^{3} \|u_i\|^2_{1,\Omega}, \quad u = (u_1, u_2, u_3). \tag{8}$$

Denote by $H^{-1/2}(\Gamma)$ the space dual of $H^{1/2}(\Gamma)$ with a duality pairing $\langle\cdot,\cdot\rangle_{1/2,\Gamma}$.

The following result holds true.

Theorem 3. *Let the boundary Γ belong to the class $C^{1,1}$, and let a function u belong to the space $H^1_\sigma(\Omega)$. There exists a linear continuous operator $H^1_\sigma(\Omega) \to [H^{-1/2}(\Gamma)]^3$ which uniquely defines at the boundary Γ the values*

$$\sigma_n(u), \sigma_{\tau i}(u) \in H^{-1/2}(\Gamma), \quad i = 1,2,3, \qquad \sigma_{\tau i}(u) n_i = 0,$$

and for all $v \in [H^1(\Omega)]^3$ the generalized Green formula holds:

$$\int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(v) = -\int_\Omega \sigma_{ij,j}(u) v_i + \langle \sigma_n(u), v_n\rangle_{1/2,\Gamma} + \langle \sigma_{\tau i}(u), v_{\tau i}\rangle_{1/2,\Gamma}. \tag{9}$$

For smooth functions u defined in $\overline{\Omega}$, formula (5) is valid. Conversely, there exists a linear continuous operator $[H^{-1/2}(\Gamma)]^3 \to H^1_\sigma(\Omega)$ such that for any given $\lambda_n, \lambda_{\tau i} \in H^{-1/2}(\Gamma)$, $i = 1,2,3$, $\lambda_{\tau i} n_i = 0$, a function $u \in H^1_\sigma(\Omega)$ can be found such that

$$\sigma_n(u) = \lambda_n, \quad \sigma_{\tau i}(u) = \lambda_{\tau i}, \quad i = 1,2,3, \quad \text{on } \Gamma.$$

Proof. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in H^{1/2}(\Gamma)$ be any given function. By Theorem 1, we can find $v = (v_1, v_2, v_3) \in H^1(\Omega)$ such that

$$v = \varphi \quad \text{on } \Gamma. \tag{10}$$

For $u \in H^1_\sigma(\Omega)$ we define the linear functional

$$L_u(\varphi) = \int_\Omega \big(\sigma_{ij}(u)\varepsilon_{ij}(v) + \sigma_{ij,j}(u) v_i\big), \tag{11}$$

where the function $v \in [H^1(\Omega)]^3$ satisfies the condition (10). The functional L_u does not depend on v. Indeed, assume that $v^1, v^2 \in [H^1(\Omega)]^3$ are two functions satisfying (10). Then, for $v = v^1 - v^2$, we have $v = 0$ on Γ. Hence $v \in [H_0^1(\Omega)]^3$ and the following formula holds:

$$\int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(v) = -\int_\Omega \sigma_{ij,j}(u)v_i$$

implying that

$$\int_\Omega \left(\sigma_{ij}(u)\varepsilon_{ij}(v^1) + \sigma_{ij,j}(u)v_i^1\right) = \int_\Omega \left(\sigma_{ij}(u)\varepsilon_{ij}(v^2) + \sigma_{ij,j}(u)v_i^2\right) = L_u(\varphi).$$

By Theorem 1, using Hölder's inequality, one can obtain the estimate

$$|L_u(\varphi)| \leqslant c_1\left(\|u_i\|_{1,\Omega} + \|\sigma_{ij,j}(u)\|_{0,\Omega}\right)\|v_i\|_{1,\Omega} \leqslant c_2\|u\|_{1,\sigma,\Omega}\sum_{i=1}^{3}\|\varphi_i\|_{1/2,\Gamma}$$

which yields continuity of L_u on $[H^{1/2}(\Gamma)]^3$. Therefore, there exists a unique representation

$$L_u(\varphi) = \langle\lambda_i, \varphi_i\rangle_{1/2,\Gamma}, \quad \lambda = (\lambda_1, \lambda_2, \lambda_3) \in H^{-1/2}(\Gamma).$$

Substituting this identity into (11), by (10) one obtains

$$\int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(v) = -\int_\Omega \sigma_{ij,j}(u)v_i + \langle\lambda_i, v_i\rangle_{1/2,\Gamma} \tag{12}$$

with an arbitrary $v \in [H^1(\Omega)]^3$.

The smoothness of the boundary implies $n \in [C^{0,1}(\Gamma)]^3$. Therefore, similar to Lemma 2, for $v|_\Gamma \in [H^{1/2}(\Gamma)]^3$ we have $v_n = v_i n_i \in H^{1/2}(\Gamma)$, $v_{\tau i} = v_i - v_n n_i \in H^{1/2}(\Gamma)$, $i = 1,2,3$.

Similar to Lemma 3, introduce elements $\lambda_n, \lambda_{\tau i} \in H^{-1/2}(\Gamma)$, $i = 1,2,3$, by the formulae

$$\begin{aligned}
&\langle\lambda_n, \varphi\rangle_{1/2,\Gamma} = \langle\lambda_i, \varphi n_i\rangle_{1/2,\Gamma}, \\
&\langle\lambda_{\tau i}, \varphi\rangle_{1/2,\Gamma} = \langle\lambda_i, \varphi\rangle_{1/2,\Gamma} - \langle\lambda_n, \varphi n_i\rangle_{1/2,\Gamma}, \quad i = 1,2,3,
\end{aligned}$$

where $\varphi \in H^{1/2}(\Gamma)$, which yields the decomposition

$$\lambda_i = \lambda_n n_i + \lambda_{\tau i}, \quad i = 1,2,3. \tag{13}$$

In this notation (12) takes the form

$$(14) \qquad \int_{\Omega} \sigma_{ij}(u)\varepsilon_{ij}(v) = -\int_{\Omega} \sigma_{ij,j}(u)v_i + \langle \lambda_n, v_n\rangle_{1/2,\Gamma} + \langle \lambda_{\tau i}, v_{\tau i}\rangle_{1/2,\Gamma}.$$

If the functions u, v are sufficiently smooth, then (14) coincides with (6) and $\sigma_n(u) = \lambda_n$, $\sigma_{\tau i}(u) = \lambda_{\tau i}$, $i = 1, 2, 3$. Keeping these relations in mind, we denote

$$\lambda_n = \sigma_n(u),\ \lambda_{\tau i} = \sigma_{\tau i}(u) \in H^{-1/2}(\Gamma), \quad i = 1, 2, 3,$$

and obtain needful Green's formula stated in Theorem 3.

Conversely, let $\lambda_n, \lambda_{\tau i} \in H^{-1/2}(\Gamma)$, $i = 1, 2, 3$, $\lambda_{\tau i} n_i = 0$, be given. For a constant $\mu > 0$, consider the auxiliary problem

$$(15) \qquad \int_{\Omega} (\sigma_{ij}(u)\varepsilon_{ij}(v) + \mu u_i v_i) = \langle \lambda_n, v_n\rangle_{1/2,\Gamma} + \langle \lambda_{\tau i}, v_{\tau i}\rangle_{1/2,\Gamma} \quad \forall v \in [H^1(\Omega)]^3.$$

In view of the representation (13) and $\lambda_{\tau i} n_i = 0$, the right-hand side of (15) is equal to $\langle \lambda_i, v_i\rangle_{1/2,\Gamma}$, hence, by Theorem 1, it defines a linear continuous functional on $[H^1(\Omega)]^3$. By the Korn inequality (7), the left-hand side of (15) is a coercive continuous bilinear form on $[H^1(\Omega)]^3$. Therefore, there exists a solution $u = (u_1, u_2, u_3) \in H^1(\Omega)$ to the problem (15). On the other hand, by the Green formula (9), equation (15) is equivalent to the problem

$$-\sigma_{ij,j}(u) + \mu u_i = 0, \quad i = 1, 2, 3, \quad \text{in } \Omega,$$
$$\sigma_n(u) = \lambda_n, \quad \sigma_{\tau i}(u) = \lambda_{\tau i}, \quad i = 1, 2, 3, \quad \text{on } \Gamma.$$

This implies $u \in H^1_\sigma(\Omega)$. Theorem 3 is proved. □

Consider now the domain $\Omega_c \subset \mathbb{R}^3$ with the boundary $\partial\Omega_c = \Gamma \cup \overline{\Sigma}_c^{\pm}$, with Σ the closed extension of Σ_c as before. Let $\nu = (\nu_1, \nu_2, \nu_3)$ correspond to a unit normal vector at Σ. Introduce the space

$$H^{1,0}(\Omega_c) = \{u = (u_1, u_2, u_3) \in H^1(\Omega_c) \mid u = 0 \quad \text{on } \Gamma\}$$

equipped with the norm

$$\|u\|_1^2 = \sum_{i=1}^{3} \|u_i\|^2_{0,\Omega_c} + \sum_{i,j=1}^{3} \|u_{i,j}\|^2_{0,\Omega_c}.$$

Theorem 4. *Let the boundary $\partial\Omega_c$ belong to the class $C^{1,1}$, let a function u belong to the space $H^1_\sigma(\Omega_c)$ and $[\sigma_{ij}(u)\nu_j] = 0$, $i = 1, 2, 3$, on Σ. Then there exists* *a linear continuous operator $H^1_\sigma(\Omega_c) \to [H^{1/2}_{00}(\Sigma_c)^*]^3$ which uniquely defines at the crack Σ_c the values*

$$\sigma_\nu(u), \sigma_{\tau i}(u) \in H^{1/2}_{00}(\Sigma_c)^*, \quad i = 1,2,3, \quad \sigma_{\tau i}(u)\nu_i = 0,$$

and for all $v \in H^{1,0}(\Omega_c)$ the generalized Green formula holds:

$$(16) \quad \int\limits_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v) = -\int\limits_{\Omega_c} \sigma_{ij,j}(u)v_i - \langle \sigma_\nu(u), [v_\nu] \rangle_{1/2,\Sigma_c} - \langle \sigma_{\tau i}(u), [v_{\tau i}] \rangle_{1/2,\Sigma_c}.$$

For smooth functions u defined in $\overline{\Omega}_c = \Omega_c \cup \partial\Omega_c$, the following formula holds:

$$(17) \quad \sigma_{ij}(u)\nu_j = \sigma_\nu(u)\nu_i + \sigma_{\tau i}(u), \quad i = 1,2,3, \qquad \sigma_\nu(u) = \sigma_{ij}(u)\nu_j\nu_i \quad \text{on } \Sigma_c.$$

Conversely, there exists a linear continuous operator $[H^{1/2}_{00}(\Sigma_c)^*]^3 \to H^1_\sigma(\Omega_c)$ such that for any given $\lambda_\nu, \lambda_{\tau i} \in H^{1/2}_{00}(\Sigma_c)^*$, $i = 1,2,3$, $\lambda_{\tau i}\nu_i = 0$, a function $u \in H^1_\sigma(\Omega_c)$ can be found such that $\sigma_\nu(u), \sigma_{\tau i}(u) \in H^{1/2}_{00}(\Sigma_c)^*$, $i = 1,2,3$, are defined, and

$$\sigma_\nu(u) = \lambda_\nu, \quad \sigma_{\tau i}(u) = \lambda_{\tau i}, \quad i = 1,2,3, \quad \text{on } \Sigma_c.$$

Proof. In view of Theorem 2, for $v \in H^{1,0}(\Omega_c)$ we have $[v] = ([v_1],[v_2],[v_3])$ belongs to $H^{1/2}_{00}(\Sigma_c)$. The smoothness of the boundary implies $\nu \in [C^{0,1}(\overline{\Sigma}_c)]^3$, and consequently, by Lemma 2, we have $[v_\nu] = [v_i]\nu_i \in H^{1/2}_{00}(\Sigma_c)$, $[v_{\tau i}] = [v_i] - [v_\nu]\nu_i \in H^{1/2}_{00}(\Sigma_c)$, $i = 1,2,3$.

Let $u \in H^1_\sigma(\Omega_c)$ be a given function. By the assumption, the surface Σ divides Ω into two domains Ω_1, Ω_2 with boundaries $\partial\Omega_1 = \Sigma^-$, $\partial\Omega_2 = \Gamma \cup \Sigma^+$ of the class $C^{1,1}$. In each Ω_k, $k = 1,2$, we have $u \in H^1_\sigma(\Omega_k)$. Hence, we can apply Theorem 3 which provides existence of elements $\sigma^\pm_\nu(u), \sigma^\pm_{\tau i}(u) \in H^{-1/2}(\Sigma)$, $i = 1,2,3$, on the boundary, and obtain the Green formula

$$(18) \quad \int\limits_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v) = -\int\limits_{\Omega_c} \sigma_{ij,j}(u)v_i - \langle \sigma^+_\nu(u), v^+_\nu \rangle_{1/2,\Sigma} - \langle \sigma^+_{\tau i}(u), v^+_{\tau i} \rangle_{1/2,\Sigma} + \langle \sigma^-_\nu(u), v^-_\nu \rangle_{1/2,\Sigma} + \langle \sigma^-_{\tau i}(u), v^-_{\tau i} \rangle_{1/2,\Sigma}.$$

Here the signs $\pm$ correspond to the faces $\Sigma^\pm$ of the surface Σ; $\langle \cdot,\cdot \rangle_{1/2,\Sigma}$ means the duality pairing between $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$.

The assumption $[\sigma_{ij}(u)\nu_j] = 0$, $i = 1,2,3$, in view of the representation (5) implies

$$[\sigma_\nu(u)] = 0, \quad [\sigma_\tau(u)] = 0 \quad \text{on } \Sigma,$$

which is fulfilled in the sense

$$\left\langle \sigma_\nu^+(u) - \sigma_\nu^-(u), \psi \right\rangle_{1/2,\Sigma} = \left\langle \sigma_{\tau i}^+(u) - \sigma_{\tau i}^-(u), \psi \right\rangle_{1/2,\Sigma} = 0, \quad i = 1,2,3, \quad \forall \psi \in H^{1/2}(\Sigma). \tag{19}$$

This leads to the following identity instead of (18):

$$\int\limits_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v) = -\int\limits_{\Omega_c} \sigma_{ij,j}(u)v_i - \left\langle \sigma_\nu^\pm(u), [v_\nu] \right\rangle_{1/2,\Sigma} - \left\langle \sigma_{\tau i}^\pm(u), [v_{\tau i}] \right\rangle_{1/2,\Sigma}. \tag{20}$$

Using (19) and Lemma 1, let us define functionals $\sigma_\nu(u), \sigma_{\tau i}(u) \in H_{00}^{1/2}(\Sigma_c)^*$ by the formulae

$$\left\langle \sigma_\nu(u), \psi \right\rangle_{1/2,\Sigma_c} = \langle \sigma_\nu^\pm(u), \bar{\psi} \rangle_{1/2,\Sigma},$$
$$\left\langle \sigma_{\tau i}(u), \psi \right\rangle_{1/2,\Sigma_c} = \left\langle \sigma_{\tau i}^\pm(u), \bar{\psi} \right\rangle_{1/2,\Sigma}, \quad i = 1,2,3,$$
$$\forall \psi \in H_{00}^{1/2}(\Sigma_c), \quad \bar{\psi} = \psi \text{ in } \Sigma_c, \quad \bar{\psi} = 0 \text{ in } \Sigma \setminus \Sigma_c, \quad \bar{\psi} \in H^{1/2}(\Sigma).$$

Here $\langle \cdot, \cdot \rangle_{1/2,\Sigma_c}$ means the duality pairing between $H_{00}^{1/2}(\Sigma_c)$ and $H_{00}^{1/2}(\Sigma_c)^*$. This representation allows us to rewrite (20) in the form (16). For smooth functions u defined in $\overline{\Omega}_c$, identities (5) fulfilled at Σ imply (17).

Conversely, let $\lambda_\nu, \lambda_{\tau i} \in H_{00}^{1/2}(\Sigma_c)^*$, $i = 1,2,3$, $\lambda_{\tau i}\nu_i = 0$, be given. For a constant $\mu > 0$, $v \in H^{1,0}(\Omega_c)$, consider the auxiliary problem

$$\int\limits_{\Omega_c} \left(\sigma_{ij}(u)\varepsilon_{ij}(v) + \mu u_i v_i\right) = -\left\langle \lambda_\nu, [v_\nu] \right\rangle_{1/2,\Sigma_c} - \left\langle \lambda_{\tau i}, [v_{\tau i}] \right\rangle_{1/2,\Sigma_c}. \tag{21}$$

In view of Theorem 2, the right-hand side of (21) defines a linear continuous functional on $H^{1,0}(\Omega_c)$. By the Korn inequality (7), the left-hand side of (21) is a coercive continuous bilinear form on $[H^1(\Omega_c)]^3$. Therefore, there exists a solution $u = (u_1, u_2, u_3) \in H^1(\Omega_c)$ to the problem (21).

Substituting $v = \varphi$, $\varphi \in [C_0^\infty(\Omega_c)]^3$, in (21) as a test function, we obtain the identity

$$\int\limits_{\Omega_c} \left(\sigma_{ij}(u)\varepsilon_{ij}(\varphi) + \mu u_i \varphi_i\right) = 0 \quad \forall \varphi \in [C_0^\infty(\Omega_c)]^3.$$

This means that the equations

$$-\sigma_{ij,j}(u) + \mu u_i = 0, \quad i = 1,2,3, \tag{22}$$

hold in the sense of distributions. Consequently, $\sigma_{ij,j}(u) \in L^2(\Omega_c)$, i.e. $u \in H^1_\sigma(\Omega_c)$, and (22) is fulfilled almost everywhere in Ω_c.

On the other hand, the extension Σ divides Ω into two domains Ω_1, Ω_2 with boundaries Σ^-, $\Gamma \cup \Sigma^+$ of the class $C^{1,1}$. In each Ω_k, $k = 1, 2$, we have $u \in H^1_\sigma(\Omega_k)$. Hence, we can apply Theorem 3 which provides the existence of $\sigma^\pm_\nu(u), \sigma^\pm_{\tau i}(u) \in H^{-1/2}(\Sigma)$, $i = 1, 2, 3$, and the Green formula (9) gives

$$\int\limits_{\Omega_c} \big(\sigma_{ij}(u)\varepsilon_{ij}(v) + \sigma_{ij,j}(u)v_i\big) = -\big[\langle \sigma_\nu(u), v_\nu\rangle_{1/2,\Sigma}\big] - \big[\langle \sigma_{\tau i}(u), v_{\tau i}\rangle_{1/2,\Sigma}\big]$$

for $v \in H^{1,0}(\Omega_c)$. Utilizing (22), we deduce the formula

$$\int\limits_{\Omega_c} \big(\sigma_{ij}(u)\varepsilon_{ij}(v) + \mu u_i v_i\big) = -\big[\langle \sigma_\nu(u), v_\nu\rangle_{1/2,\Sigma}\big] - \big[\langle \sigma_{\tau i}(u), v_{\tau i}\rangle_{1/2,\Sigma}\big]. \tag{23}$$

For $\varphi \in [H^1_0(\Omega)]^3$ we have $\varphi \in H^{1,0}(\Omega_c)$ and $\varphi^\pm \in H^{1/2}(\Sigma)$, $[\varphi] = 0$ on Σ (then $[\varphi_\nu] = [\varphi_\tau] = 0$). Substituting $v = \varphi$ as a test function, from (23) we obtain

$$\int\limits_{\Omega_c} \big(\sigma_{ij}(u)\varepsilon_{ij}(\varphi) + \mu u_i \varphi_i\big) = -\big\langle [\sigma_\nu(u)], \varphi_\nu\big\rangle_{1/2,\Sigma} - \big\langle [\sigma_{\tau i}(u)], \varphi_{\tau i}\big\rangle_{1/2,\Sigma},$$

and (21) implies that

$$\int\limits_{\Omega_c} \big(\sigma_{ij}(u)\varepsilon_{ij}(\varphi) + \mu u_i \varphi_i\big) = 0.$$

Thus, one can conclude that

$$\big\langle [\sigma_\nu(u)], \varphi_\nu\big\rangle_{1/2,\Sigma} + \big\langle [\sigma_{\tau i}(u)], \varphi_{\tau i}\big\rangle_{1/2,\Sigma} = 0 \quad \forall \varphi \in [H^1_0(\Omega)]^3.$$

This equality implies $[\sigma_\nu(u)] = 0$, $[\sigma_\tau(u)] = 0$ on Σ. Therefore, we can apply the first assertion of Theorem 4 and define $\sigma_\nu(u), \sigma_{\tau i}(u) \in H^{1/2}_{00}(\Sigma_c)^*$, $i = 1, 2, 3$, $\sigma_{\tau i}(u)\nu_i = 0$. Then the Green formula (16) together with (21), (22) guarantees the fulfilment of conditions

$$\sigma_\nu(u) = \lambda_\nu, \quad \sigma_{\tau i}(u) = \lambda_{\tau i}, \quad i = 1, 2, 3, \quad \text{on } \Sigma_c.$$

Theorem 4 is proved. □

Let a solid occupy a domain $\Omega_c \subset \mathbb{R}^3$ with a crack Σ_c such that its boundary $\partial\Omega_c = \Gamma \cup \overline{\Sigma}_c^{\pm}$ belongs to the class $C^{1,1}$. We seek the displacements vector $u = (u_1, u_2, u_3)$ in the space $H^{1,0}(\Omega_c)$ which corresponds to the solid clamped at the boundary, i.e. $u = 0$ on Γ.

The nonpenetration condition of the crack surfaces has the form (Khludnev and Sokolowski [10])

$$[u_\nu] \geqslant 0 \quad \text{on } \Sigma_c.$$

Introduce the set of admissible displacements

$$K = \{u \in H^{1,0}(\Omega_c) \mid [u_\nu] \geqslant 0 \quad \text{on } \Sigma_c\}$$

which is convex and closed.

Let $F \in H_{00}^{1/2}(\Sigma_c)^*$ be a given friction force between the crack faces. Assume that $F \geqslant 0$ in the sense

$$\langle F, \varphi \rangle_{1/2,\Sigma_c} \geqslant 0 \quad \forall \varphi \in H_{00}^{1/2}(\Sigma_c),\ \varphi \geqslant 0.$$

For a given external force $f = (f_1, f_2, f_3) \in L^2(\Omega_c)$, we introduce the potential energy functional

$$P(u) = \Pi(u) + I(u),$$
$$I(u) = \langle F, |[u_\tau]| \rangle_{1/2,\Sigma_c}, \quad \Pi(u) = \frac{1}{2} \int\limits_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(u) - \int\limits_{\Omega_c} f_i u_i,$$

on the space $H^{1,0}(\Omega_c)$. By Theorem 2, for $u \in H^{1,0}(\Omega_c)$ we have $[u_\tau] = ([u_{\tau 1}], [u_{\tau 2}], [u_{\tau 3}]) \in H_{00}^{1/2}(\Sigma_c)$, hence, obviously, $|[u_\tau]| \in H_{00}^{1/2}(\Sigma_c)$, and the functional I is well-defined. Besides, I is positive since F is positive, continuous by Theorem 2, and convex.

Consider the functional Π. It is convex and continuous, consequently, weakly lower semicontinuous. Its differentiability is also obvious.

Extend Σ_c up to the boundary Γ so that Ω is divided into two domains $\mathscr{O}_1, \mathscr{O}_2$ with Lipschitz boundaries $\partial\mathscr{O}_1, \partial\mathscr{O}_2$. Assume that $\operatorname{meas}(\Gamma \cap \partial\mathscr{O}_k) > 0$, $k = 1, 2$. In each of these domains, for $u \in H^{1,0}(\Omega_c)$, the Korn inequality (8),

$$\int\limits_{\mathscr{O}_k} \varepsilon_{ij}(u)\varepsilon_{ij}(u) \geqslant c \sum_{i=1}^{3} \|u_i\|_{1,\mathscr{O}_k}^2, \quad k = 1, 2, \quad u = (u_1, u_2, u_3),$$

is fulfilled since $u = 0$ at $\Gamma \cap \partial\mathcal{O}_k$, $k = 1, 2$. Consequently, we have an estimate in Ω_c,

$$\int_{\Omega_c} \varepsilon_{ij}(u)\varepsilon_{ij}(u) \geqslant c\|u\|_1^2. \tag{24}$$

This estimate ensures the coercivity of the functional Π,

$$\Pi(u) \geqslant c\|u\|_1^2 - \|f_i\|_{0,\Omega_c}\|u_i\|_{0,\Omega_c} \to +\infty, \quad \|u\|_1 \to \infty.$$

Thus, P is a coercive, strictly convex, weakly lower semicontinuous functional on $H^{1,0}(\Omega_c)$, K is a closed convex set in $H^{1,0}(\Omega_c)$. Therefore, the equilibrium problem

$$P(u) = \inf_{v \in K} P(v), \quad P = \Pi + I,$$

is equivalent to the variational inequality

$$u \in K, \quad \Pi'_u(v-u) + I(v) - I(u) \geqslant 0 \quad \forall v \in K,$$

which has the form

$$\int_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v-u) + \big\langle F, \big|[v_\tau]\big| - \big|[u_\tau]\big|\big\rangle_{1/2,\Sigma_c} \geqslant \int_{\Omega_c} f_i(v_i - u_i) \quad \forall v \in K. \tag{25}$$

By the properties of P, there exists a unique solution $u \in K$ to the problem (25).

Theorem 5. *There exists a unique solution $u \in K$ to the problem* (25) *such that*

$$-\sigma_{ij,j}(u) = f_i, \quad i = 1,2,3, \quad \text{in } \Omega_c,$$
$$[u_\nu] \geqslant 0, \quad [\sigma_\nu(u)] = 0, \quad \sigma_\nu(u) \leqslant 0, \quad \sigma_\nu(u)[u_\nu] = 0 \quad \text{on } \Sigma_c,$$
$$[\sigma_\tau(u)] = 0, \quad |\sigma_\tau(u)| \leqslant F, \quad \sigma_{\tau i}(u)[u_{\tau i}] - F\big|[u_\tau]\big| = 0 \quad \text{on } \Sigma_c.$$

Proof. Substituting $v = u \pm \varphi$, $\varphi \in [C_0^\infty(\Omega_c)]^3$ in (25) as a test function, one obtains

$$\int_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(\varphi) = \int_{\Omega_c} f_i\varphi_i \quad \forall \varphi \in [C_0^\infty(\Omega_c)]^3. \tag{26}$$

Thus we have the equations

$$-\sigma_{ij,j}(u) = f_i, \quad i = 1,2,3, \quad \text{a.e. in } \Omega_c, \tag{27}$$

and $\sigma_{ij,j}(u) \in L^2(\Omega_c)$, $i = 1, 2, 3$. By (27) and the Green formula (9),

$$\int\limits_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v-u) = -\int\limits_{\Omega_c} \sigma_{ij,j}(u)(v_i - u_i) - \left[\langle \sigma_\nu(u), v_\nu - u_\nu \rangle_{1/2,\Sigma}\right] - \left[\langle \sigma_{\tau i}(u), v_{\tau i} - u_{\tau i} \rangle_{1/2,\Sigma}\right],$$

and from (25) one can deduce

$$\langle F, |[v_\tau]| - |[u_\tau]| \rangle_{1/2,\Sigma_c} - \left[\langle \sigma_\nu(u), v_\nu - u_\nu \rangle_{1/2,\Sigma}\right] - \left[\langle \sigma_{\tau i}(u), v_{\tau i} - u_{\tau i} \rangle_{1/2,\Sigma}\right] \geqslant 0, \tag{28}$$

where $v \in K$. For $\varphi \in [H_0^1(\Omega)]^3$ we have $[\varphi] = 0$ on Σ and, therefore, we can substitute $v = u \pm \varphi \in K$ in (28) as a test function. This gives

$$\langle [\sigma_\nu(u)], \varphi_\nu \rangle_{1/2,\Sigma} + \langle [\sigma_{\tau i}(u)], \varphi_{\tau i} \rangle_{1/2,\Sigma} = 0 \quad \forall \varphi \in [H_0^1(\Omega)]^3. \tag{29}$$

Hence $[\sigma_\nu(u)] = [\sigma_\tau(u)] = 0$ on Σ, and we can use Theorem 4 which provides the existence of $\sigma_\nu(u), \sigma_{\tau i}(u) \in H_{00}^{1/2}(\Sigma_c)^*$, $i = 1, 2, 3$, $\sigma_{\tau i}(u)\nu_i = 0$. The Green formula (16), applied to the problem (25), together with (27) yields the following inequality instead of (28):

$$\langle F, |[v_\tau]| - |[u_\tau]| \rangle_{1/2,\Sigma_c} - \langle \sigma_\nu(u), [v_\nu] - [u_\nu] \rangle_{1/2,\Sigma_c} - \langle \sigma_{\tau i}(u), [v_{\tau i}] - [u_{\tau i}] \rangle_{1/2,\Sigma_c} \geqslant 0 \quad \forall v \in K. \tag{30}$$

By the independence between normal and tangential components at the boundary, we split (30) in two inequalities

$$\langle \sigma_\nu(u), [v_\nu] \rangle_{1/2,\Sigma_c} \leqslant \langle \sigma_\nu(u), [u_\nu] \rangle_{1/2,\Sigma_c} \quad \forall v \in K, \tag{31}$$

$$\langle F, |[v_\tau]| \rangle_{1/2,\Sigma_c} - \langle \sigma_{\tau i}(u), [v_{\tau i}] \rangle_{1/2,\Sigma_c} \geqslant \langle F, |[u_\tau]| \rangle_{1/2,\Sigma_c} - \langle \sigma_{\tau i}(u), [u_{\tau i}] \rangle_{1/2,\Sigma_c} \quad \forall v \in K. \tag{32}$$

Consider the first inequality (31). Substituting here $v = 0$, $v = 2u$, one obtains

$$\langle \sigma_\nu(u), [u_\nu] \rangle_{1/2,\Sigma_c} = 0. \tag{33}$$

Consequently, $\langle \sigma_\nu(u), [v_\nu] \rangle_{1/2,\Sigma_c} \leqslant 0$ for all $v \in H^{1,0}(\Omega_c)$, $[v_\nu] \geqslant 0$. This implies the inequality

$$\langle \sigma_\nu(u), \psi \rangle_{1/2,\Sigma_c} \leqslant 0 \quad \forall \psi \in H_{00}^{1/2}(\Sigma_c),\ \psi \geqslant 0. \tag{34}$$

Relations (33), (34) imply the first line of the boundary conditions formulated in Theorem 5.

Consider now the inequality (32). We can replace v_τ by $\pm\lambda v_\tau$ in (32), $\lambda \geqslant 0$ being a constant, which gives

$$\lambda\left(\left\langle F, \left|[v_\tau]\right|\right\rangle_{1/2,\Sigma_c} \mp \left\langle \sigma_{\tau i}(u), [v_{\tau i}]\right\rangle_{1/2,\Sigma_c}\right) \geqslant \left\langle F, \left|[u_\tau]\right|\right\rangle_{1/2,\Sigma_c} - \left\langle \sigma_{\tau i}(u), [u_{\tau i}]\right\rangle_{1/2,\Sigma_c}.$$

By the arbitrariness of λ, this inequality means that

$$\begin{aligned} &\left\langle F, \left|[u_\tau]\right|\right\rangle_{1/2,\Sigma_c} - \left\langle \sigma_{\tau i}(u), [u_{\tau i}]\right\rangle_{1/2,\Sigma_c} = 0, \\ &\left\langle F, \left|[v_\tau]\right|\right\rangle_{1/2,\Sigma_c} \mp \left\langle \sigma_{\tau i}(u), [v_{\tau i}]\right\rangle_{1/2,\Sigma_c} \geqslant 0 \quad \forall v \in K. \end{aligned} \tag{35}$$

The last relation implies

$$\left|\left\langle \sigma_{\tau i}(u), \psi_i\right\rangle_{1/2,\Sigma_c}\right| \leqslant \left\langle F, |\psi|\right\rangle_{1/2,\Sigma_c} \quad \forall \psi \in [H_{00}^{1/2}(\Sigma_c)]^3,\ \psi_i\nu_i = 0. \tag{36}$$

Equations and inequalities (26), (29), (33)–(36) give the exact meaning of the relations formulated in Theorem 5. The theorem is proved. □

4. The crack under Coulomb friction

As before, we consider a solid occupying the domain Ω_c with the crack Σ_c. Let $\mathscr{F} \in C^{0,1}(\overline{\Sigma}_c)$, $\mathscr{F} \geqslant 0$, be a given friction coefficient. In accordance with the Coulomb friction law (Hlaváček et al. [8]), we assume that the friction force F between the crack surfaces $\Sigma_c^\pm$ coincides with $\mathscr{F}|\sigma_\nu(u)|$, where $|\sigma_\nu(u)|$ characterizes the contact force of these surfaces. By the nonpositiveness of $\sigma_\nu(u)$ obtained in Theorem 5 and provided the nonpenetration condition holds, we arrive at the relation

$$F = -\mathscr{F}\sigma_\nu(u) \quad \text{on } \Sigma_c. \tag{37}$$

Friction problem (25) together with condition (37) are equivalent to the problem

$$\int_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v-u) - \left\langle \mathscr{F}\sigma_\nu(u), \left|[v_\tau]\right| - \left|[u_\tau]\right|\right\rangle_{1/2,\Sigma_c} \geqslant \int_{\Omega_c} f_i(v_i - u_i) \quad \forall v \in K. \tag{38}$$

The second term on the left-hand side of (38) is well-defined thanks to Lemma 3. Notice that problem (38) is a quasi-variational inequality, and the usual variational methods are not applicable here.

Define C^- to be the dual cone of non-positive distributions in $H_{00}^{1/2}(\Sigma_c)^*$. Let $F^- \in C^-$ be given, then in view of Lemma 3, $\mathscr{F}F^- \in H_{00}^{1/2}(\Sigma_c)^*$. Consider the auxiliary problem

$$(39) \quad \int\limits_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v-u) - \langle \mathscr{F}F^-, |[v_\tau]| - |[u_\tau]| \rangle_{1/2,\Sigma_c} \geqslant \int\limits_{\Omega_c} f_i(v_i - u_i) \quad \forall v \in K.$$

Problem (39) coincides with the variational inequality (25) for $F = -\mathscr{F}F^-$. Therefore, by Theorem 5, there exists a unique solution $u \in K$ to the inequality (39) and, moreover, $\sigma_\nu(u) \in C^-$. Thus, we construct a mapping $T\colon C^- \to C^-$ given by $\sigma_\nu(u) = T(F^-)$. One can see that a solution u of the quasi-variational inequality (38) is described as a fixed point of T, i.e. $\sigma_\nu(u) = T(\sigma_\nu(u))$. Therefore, to prove solvability of (38), we look for a fixed point of T.

Take $v = \lambda u$, where $\lambda \geqslant 0$ is a constant, as a test function in the inequality (39), then

$$\int\limits_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(u) - \langle \mathscr{F}F^-, |[u_\tau]| \rangle_{1/2,\Sigma_c} = \int\limits_{\Omega_c} f_i u_i.$$

By the positiveness of the boundary term, applying the Korn and Hölder inequalities, we deduce an estimate

$$(40) \qquad \|u\|_1 \leqslant \text{ const},$$

which is uniform in F^-.

Let $F^1, F^2 \in C^-$ be arbitrary functions. Denote by u^k the solutions of the problem (39) for $F^- = F^k$, $k = 1,2$,

$$\int\limits_{\Omega_c} \sigma_{ij}(u^k)\varepsilon_{ij}(v^k - u^k) - \langle \mathscr{F}F^k, |[v_\tau^k]| - |[u_\tau^k]| \rangle_{1/2,\Sigma_c} \geqslant \int\limits_{\Omega_c} f_i(v_i^k - u_i^k) \quad \forall v^k \in K.$$

Summing up these inequalities for $v^1 = u^2$, $v^2 = u^1$, one obtains

$$\int\limits_{\Omega_c} \sigma_{ij}(u^1 - u^2)\varepsilon_{ij}(u^1 - u^2) \leqslant \langle \mathscr{F}(F^1 - F^2), |[u_\tau^1]| - |[u_\tau^2]| \rangle_{1/2,\Sigma_c}.$$

Then, applying again the Korn and Hölder inequalities, from the last relation we deduce

$$(41) \quad \|u^1 - u^2\|_1^2 \leqslant c\|\mathscr{F}(F^1 - F^2)\|_{H_{00}^{1/2}(\Sigma_c)^*} \big(\| |[u_\tau^1]| \|_{1/2,00,\Sigma_c} + \| |[u_\tau^2]| \|_{1/2,00,\Sigma_c} \big),$$

where the difference $\bigl|[u_\tau^1]\bigr|-\bigl|[u_\tau^2]\bigr|$ in $H_{00}^{1/2}(\Sigma_c)$ was estimated by the sum of the norms. The continuity of the operators described in Theorems 2, 4 implies the respective estimates,

$$
\begin{aligned}
&\bigl\|\bigl|[v_\tau]\bigr|\bigr\|_{1/2,00,\Sigma_c} \leqslant c\|v\|_1, \quad v \in H^{1,0}(\Omega_c), \qquad (42)\\
&\|\sigma_\nu(v)\|_{H_{00}^{1/2}(\Sigma_c)^*} \leqslant c\Bigl(\|v\|_1^2 + \sum_{i=1}^{3}\|\sigma_{ij,j}(v)\|_{0,\Omega_c}^2\Bigr)^{1/2}, \quad v \in H_\sigma^1(\Omega_c),
\end{aligned}
$$

or, by Theorem 5, since $\sigma_{ij,j}(u^1-u^2) = -f_i + f_i = 0$, $i = 1,2,3$, for $v = u^1 - u^2$ we have

$$
\|\sigma_\nu(u^1-u^2)\|_{H_{00}^{1/2}(\Sigma_c)^*} \leqslant c\|u^1-u^2\|_1. \tag{43}
$$

Using (40), (42), (43), from (41) we finally obtain the estimate

$$
\|\sigma_\nu(u^1-u^2)\|^2_{H_{00}^{1/2}(\Sigma_c)^*} \leqslant c\|\mathscr{F}(F^1-F^2)\|_{H_{00}^{1/2}(\Sigma_c)^*}. \tag{44}
$$

By Lemma 3, (44) implies the Hölder continuity of the mapping T, however, it is not enough for the existence of a fixed point. Using the technique of additional smoothness for solutions of contact problems with Coulomb friction developed in Nečas et al. [19], Jarušek [9], Eck and Jarušek [6], we obtain the weak continuity of T.

We need some additional assumptions on the data. First, let Σ_c be of the class $C^{2,1}$ possessing the property of local straightening, i.e. Σ_c is locally represented as the graph $y_3 = \theta(y)$, $y = (y_1, y_2)$, in local coordinates (y, y_3) with $y \in B(0)$, $B(0)$ is a ball in $\mathbb{R}^2$ centred at 0, such that $\theta \in C^{2,1}(\overline{B(0)})$, $\theta(0) = \nabla\theta(0) = 0$, and Ω_2 is locally the epigraph for this function. Perform the local coordinate transformation

$$
\Psi\colon\ y \to y,\ y_3 \to y_3 - \theta(y),
$$

which transforms a neighbourhood $\mathscr{O}(x_0)$ of any point $x_0 \in \Sigma_c$ into a cylinder $C_r(0) = B(0) \times (-r, r)$ in $\mathbb{R}^3$ such that $\mathscr{O}(x_0) \cap \Sigma_c$ is transformed onto $B(0) \times \{0\}$, the normal $\nu(x_0)$ into the third basic vector. We will denote by "hat" the result of the inverse transformation Ψ^{-1}. Second, let the friction coefficient $\mathscr{F} \in C^1(\Sigma)$ have the compact support in Σ_c,

$$
\operatorname{dist}(\operatorname{supp}\mathscr{F}, \partial\Sigma_c) = \delta_0 > 0. \tag{45}
$$

One can therefore choose an open compact set $\Sigma_{\mathscr{F}}$ in Σ_c such that $\operatorname{supp}\mathscr{F} \subset \Sigma_{\mathscr{F}}$, $\overline{\Sigma}_{\mathscr{F}} \subset \Sigma_c$. As before, Σ is a closed smooth extension of Σ_c in Ω.

We take $\varphi \in K$ with the support in $\hat{C}_r(0)$ and substitute $v = u + \varphi$ as a test function in (39), then (39) takes the form

$$\int\limits_{\Omega_c \cap \hat{C}_r(0)} \sigma_{ij}(u)\varepsilon_{ij}(\varphi) - \big\langle \mathscr{F} F^-, \big|[u_\tau] + [\varphi_\tau]\big| - \big|[u_\tau]\big|\big\rangle_{1/2,\hat{B}(0)} \geqslant \int\limits_{\Omega_c \cap \hat{C}_r(0)} f_i \varphi_i.$$

Applying here the local coordinate transformation Ψ with the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\theta_{,1} & -\theta_{,2} & 1 \end{pmatrix}$$

we arrive at the following inequality on $Q = B(0) \times (-r, 0) \cup (0, r)$:

$$\int\limits_Q \hat{\sigma}_{ij}(\hat{u})\hat{\varepsilon}_{ij}(\hat{\varphi}) + b(\hat{u}, \hat{\varphi})_Q - \big\langle \hat{\mathscr{F}} \hat{F}^- J, \big|[\hat{u}_\tau] + [\hat{\varphi}_\tau]\big| - \big|[\hat{u}_\tau]\big|\big\rangle_{1/2,B(0)} \geqslant \int\limits_Q \hat{f}_i \hat{\varphi}_i, \tag{46}$$

where

$$b(\hat{u}, \hat{\varphi})_Q = \int\limits_Q \Big(-\tfrac{1}{2}\hat{\sigma}_{ij}(\hat{u})(\hat{\varphi}_{i,3}\theta_{,j} + \hat{\varphi}_{j,3}\theta_{,i}) - \tfrac{1}{2}\hat{\sigma}_{ij}(\hat{\varphi})(\hat{u}_{i,3}\theta_{,j} + \hat{u}_{j,3}\theta_{,i})$$
$$+ \tfrac{1}{4}\hat{a}_{ijkl}(\hat{u}_{k,3}\theta_{,l} + \hat{u}_{l,3}\theta_{,k})(\hat{\varphi}_{i,3}\theta_{,j} + \hat{\varphi}_{j,3}\theta_{,i})\Big).$$

The multiplier $J = \sqrt{1 + |\nabla\theta|^2}$ is a density of the surface measure, $b(\cdot,\cdot)_Q$ is a bilinear quadratic form corresponding to this transformation. The integration over Q in (46) can be extended to the integration over $S = \mathbb{R}^2 \times (-r, 0) \cup (0, r)$ in a regular way, i.e.

$$\int\limits_S \hat{\sigma}_{ij}(\hat{u})\hat{\varepsilon}_{ij}(\hat{\varphi}) + b(\hat{u}, \hat{\varphi})_S - \big\langle \hat{\mathscr{F}} \hat{F}^- J, \big|[\hat{u}_\tau] + [\hat{\varphi}_\tau]\big| - \big|[\hat{u}_\tau]\big|\big\rangle_{1/2,\mathbb{R}^2} \geqslant \int\limits_S \hat{f}_i \hat{\varphi}_i. \tag{47}$$

Let $\chi_{C_r(0)} \in C^{2,1}(\mathbb{R}^3)$ be a cut-off function, $0 \leqslant \chi_{C_r(0)} \leqslant 1$, with the support in $C_r(0)$. Using (40), for the inequality (47) an a priori estimate was obtained in Jarušek [9]:

$$\begin{aligned} &\|\chi_{C_r(0)}\hat{\sigma}_\nu(\hat{u})J\|_{H^{-1/2+\alpha}(\mathbb{R}^2)} \\ &\quad \leqslant c\|\hat{\mathscr{F}}\|_{L^\infty(B(0))} \cdot \|\chi_{C_r(0)}\hat{F}^- J\|_{H^{-1/2+\alpha}(\mathbb{R}^2)} + \text{ const}, \quad 0 < \alpha < 1/2, \end{aligned} \tag{48}$$

where $H^{-1/2+\alpha}(\mathbb{R}^2)$ is the space dual of $H^{1/2-\alpha}(\mathbb{R}^2)$, $0 < \alpha < 1/2$, and the norm in $H^{1/2-\alpha}(\mathbb{R}^2)$ can be introduced, for example, by

$$\|s\|^2_{1/2-\alpha,\mathbb{R}^2} = \|s\|^2_{0,\mathbb{R}^2} + \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |t - \tau|^{-2+2\alpha} \sum_{i=1}^{2} \|s(x|_{x_i=t}) - s(x|_{x_i=\tau})\|^2_{0,\mathbb{R}} \, dt \, d\tau.$$

By the compactness of $\Sigma_{\mathscr{F}}$ and the assumptions on the boundary regularity, let $\{V\}$ be a finite covering of $\Sigma_{\mathscr{F}}$ such that every V is transformed into $C_r(0)$ by the local coordinate transformation Ψ, and let $\{\chi_V\}$ be a smooth partition of unity subordinate to this covering, $\sum \chi_V \equiv 1$ on $\Sigma_{\mathscr{F}}$. Choosing V small enough, in view of (45), we assume that $\bigcup \{V\colon V \cap \operatorname{supp} \mathscr{F} \neq \emptyset\}$ forms a covering of supp $\mathscr{F}$ and

$$\operatorname{closure}\Big(\bigcup\{V\} \cap \Sigma\Big) \subset \Sigma_c, \quad \operatorname{closure}\Big(\bigcup\{V\colon V \cap \operatorname{supp} \mathscr{F} \neq \emptyset\} \cap \Sigma\Big) \subset \Sigma_{\mathscr{F}}.$$

For V with $V \cap \operatorname{supp} \mathscr{F} \neq \emptyset$, it follows from (48) that

$$\|\hat{\chi}_V \hat{\sigma}_\nu(\hat{u}) J\|_{H^{-1/2+\alpha}(\mathbb{R}^2)} \leqslant c\|\hat{\mathscr{F}}\|_{L^\infty(B(0))} \cdot \|\hat{\chi}_V \hat{F}^- J\|_{H^{-1/2+\alpha}(\mathbb{R}^2)} + \text{ const},$$

for V with $V \cap \operatorname{supp} \mathscr{F} = \emptyset$, (48) implies

$$\|\hat{\chi}_V \hat{\sigma}_\nu(\hat{u}) J\|_{H^{-1/2+\alpha}(\mathbb{R}^2)} \leqslant \text{const},$$

because of $\mathscr{F} = 0$. On the space $H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})$ let us introduce the norm

$$\|s\|_{H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})} = \sum_{\{V\}} \|\hat{\chi}_V \widehat{\bar{s}} J\|_{H^{-1/2+\alpha}(\mathbb{R}^2)},$$

where "bar" denotes the zeroth extension of the function on Σ, well-defined thanks to the property $s \in H^{1/2-\alpha}(\Sigma_{\mathscr{F}}) \Leftrightarrow \bar{s} \in H^{1/2-\alpha}(\Sigma)$, $0 < \alpha < 1/2$ (Lions and Magenes [15]). Then from the last two inequalities we finally obtain an a priori estimate

$$\|\sigma_\nu(u)\|_{H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})} \leqslant c\|\mathscr{F}\|_{L^\infty(\Sigma_{\mathscr{F}})} \cdot \|F^-\|_{H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})} + \text{ const.} \tag{49}$$

Let $F^- \in C^- \cap H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})$ be given. With help of the estimate (49), for the solution u of the variational inequality (39) we then have that $\sigma_\nu(u) \in C^- \cap H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})$, i.e. we can consider the mapping $T:\ C^- \cap H^{-1/2+\alpha}(\Sigma_{\mathscr{F}}) \to C^- \cap H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})$. We have to show that T is weakly continuous. Indeed, let

$$F^n \to F \quad \text{weakly in } H^{-1/2+\alpha}(\Sigma_{\mathscr{F}}). \tag{50}$$

By (45), for a closed extension Σ of Σ_c, considerations like in Lemma 2 yield that $\mathscr{F}\xi \in H^{1/2-\alpha}(\Sigma_{\mathscr{F}})$ if $\xi \in H^{1/2-\alpha}(\Sigma)$. Then (50) yields

$$\big\langle \mathscr{F}(F^n - F), \xi \big\rangle_{1/2-\alpha,\Sigma} = \langle F^n - F, \mathscr{F}\xi \rangle_{1/2-\alpha,\Sigma_{\mathscr{F}}} \to 0$$

for any $\xi \in H^{1/2-\alpha}(\Sigma)$, i.e.

$$\mathscr{F}F^n \to \mathscr{F}F \quad \text{weakly in } H^{-1/2+\alpha}(\Sigma). \tag{51}$$

By the compact imbedding $H^{-1/2+\alpha}(\Sigma) \subset H^{-1/2}(\Sigma)$ for any $0 < \alpha < 1/2$, (51) implies

$$\mathscr{F} F^n \to \mathscr{F} F \quad \text{strongly in } H^{-1/2}(\Sigma). \tag{52}$$

In view of (45) and Lemma 1, $\mathscr{F} F^n, \mathscr{F} F \in H_{00}^{1/2}(\Sigma_c)^*$, and the definition of this space in Theorem 4 implies the estimate

$$\|\mathscr{F}(F^n - F)\|_{H_{00}^{1/2}(\Sigma_c)^*} \leqslant \|\mathscr{F}(F^n - F)\|_{H^{-1/2}(\Sigma)},$$

therefore (52) implies

$$\mathscr{F} F^n \to \mathscr{F} F \quad \text{strongly in } H_{00}^{1/2}(\Sigma_c)^*. \tag{53}$$

The estimate (44) modified by the notation $\sigma_\nu(u) = T(F)$ together with (53) gives

$$T(F^n) \to T(F) \quad \text{strongly in } H_{00}^{1/2}(\Sigma_c)^*. \tag{54}$$

On the other hand, by virtue of (49) and (50), $T(F^n)$ are bounded in $H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})$, consequently, there exist $\overline{F} \in H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})$ and a weakly convergent subsequence such that

$$T(F^{n_k}) \to \overline{F} \quad \text{weakly in } H^{-1/2+\alpha}(\Sigma_{\mathscr{F}}). \tag{55}$$

Analogously as we have obtained (53) for the sequence $\{F^n\}$ from (50) by (51), (52), for the subsequence $\{T(F^{n_k})\}$ from (55) one obtains

$$\mathscr{F} T(F^{n_k}) \to \mathscr{F} \overline{F} \quad \text{strongly in } H_{00}^{1/2}(\Sigma_c)^*. \tag{56}$$

A comparison of (54), (56) gives $\overline{F} = T(F)$, and from (55) we conclude

$$T(F^n) \to T(F) \quad \text{weakly in } H^{-1/2+\alpha}(\Sigma_{\mathscr{F}}). \tag{57}$$

Thus, (50) and (57) together mean that

$$T \colon C^- \cap H^{-1/2+\alpha}(\Sigma_{\mathscr{F}}) \to C^- \cap H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})$$

is weakly continuous.

Consider the closed sets

$$H_r = C^- \cap \{F \in H^{-1/2+\alpha}(\Sigma_{\mathscr{F}}), \quad \|F\|_{H^{-1/2+\alpha}(\Sigma_{\mathscr{F}})} \leqslant r\}.$$

For $\mathscr{F}$ small enough and such that $c\|\mathscr{F}\|_{L^\infty(\Sigma_{\mathscr{F}})} < 1$ in (49), there exists $r_0 > 0$ such that T maps H_{r_0} into itself. Then we have that $T\colon H_{r_0} \to H_{r_0}$ is weakly continuous, H_{r_0} is weakly compact, therefore, the second Schauder fixed-point theorem (see Zeidler [21]) asserts the existence of a fixed point F^* of T, $F^* = T(F^*)$. We find the solution u of the quasi-variational inequality (38) solving (39) with $F^- = F^*$, then $\sigma_\nu(u) = T(F^*) = F^* = T(\sigma_\nu(u))$.

Moreover, as was mentioned before, the quasi-variational inequality (38) is equivalent to (25), (37). Therefore, we can substitute (37) into the corresponding relations stated in Theorem 5. Like in the proof of Theorem 5, they are fulfilled in the sense that

$$\int_{\Omega_c} (-\sigma_{ij,j}(u) - f_i)\varphi = 0, \quad i = 1,2,3, \quad \forall \varphi \in L^2(\Omega_c),$$

$$\langle [\sigma_\nu(u)], \varphi\rangle_{1/2,\Sigma} = \langle [\sigma_{\tau i}(u)], \varphi\rangle_{1/2,\Sigma} = 0, \quad i = 1,2,3, \quad \forall \varphi \in H^{1/2}(\Sigma),$$

$$\langle \sigma_\nu(u), [u_\nu]\rangle_{1/2,\Sigma_c} = 0, \quad \langle \sigma_\nu(u), \varphi\rangle_{1/2,\Sigma_c} \leqslant 0 \quad \forall \varphi \in H^{1/2}_{00}(\Sigma_c), \quad \varphi \geqslant 0,$$

$$\langle \sigma_\nu(u), \mathscr{F}\big|[u_\tau]\big|\rangle_{1/2,\Sigma_c} + \langle \sigma_{\tau i}(u), [u_{\tau i}]\rangle_{1/2,\Sigma_c} = 0,$$

$$\big|\langle \sigma_{\tau i}(u), \varphi_i\rangle_{1/2,\Sigma_c}\big| \leqslant -\langle \sigma_\nu(u), \mathscr{F}|\varphi|\rangle_{1/2,\Sigma_c} \quad \forall \varphi \in [H^{1/2}_{00}(\Sigma_c)]^3, \quad \varphi_i\nu_i = 0.$$

Thus, we have proved the following theorem.

Theorem 6. *Let all the above assumptions be valid, then for the friction coefficient $\mathscr{F}$ small enough with the compact support on the crack surface, there exists a solution $u \in K$ to the problem with the Coulomb friction between the crack faces (38) such that*

$$-\sigma_{ij,j}(u) = f_i, \quad i = 1,2,3, \quad \text{in } \Omega_c,$$

$$[u_\nu] \geqslant 0, \quad [\sigma_\nu(u)] = 0, \quad \sigma_\nu(u) \leqslant 0, \quad \sigma_\nu(u)[u_\nu] = 0 \quad \text{on } \Sigma_c,$$

$$[\sigma_\tau(u)] = 0, \quad |\sigma_\tau(u)| \leqslant -\mathscr{F}\sigma_\nu(u), \quad \sigma_{\tau i}(u)[u_{\tau i}] + \mathscr{F}\sigma_\nu(u)\big|[u_\tau]\big| = 0 \quad \text{on } \Sigma_c$$

in the sense mentioned above.

References

[1] *V. V. Alekhin, B. D. Annin and S. N. Korobeinikov*: Accounting of a friction in contact elastoplastic problems. In: Fund. Prob. Mat. Mekh. 2. Novosibirsk Univ., 1996. (In Russian.)

[2] *C. Baiocchi, A. Capelo*: Variational and Quasivariational Inequalities. Applications to Free Boundary Problems. Wiley, Chichester, 1984.

[3] *G. P. Cherepanov*: On some mechanism of the development of cracks in the Earth solid shell. Izvestiya USSR Acad. Sci., Physics of the Earth *9* (1984), 3–12. (In Russian.)

[4] *R. Duduchava, W. Wendland*: The Wiener-Hopf method for system of pseudodifferential equations with applications to crack problems. Integral Equations Operator Theory *23* (1995), 294–335.

[5] *G. Duvaut, J.-L. Lions*: Les Inéquations en Mécanique et en Physique. Dunod, Paris, 1972.

[6] *C. Eck, J. Jarušek*: Existence results for the static contact problems with Coulomb friction. Math. Models Methods Appl. Sci. *8* (1998), 445–468.

[7] *P. Grisvard*: Singularities in Boundary Value Problems. Masson, Paris & Springer-Verlag, Berlin, 1992.

[8] *I. Hlaváček, J. Haslinger, J. Nečas and J. Lovíšek*: Solution of Variational Inequalities in Mechanics. Springer-Verlag, New York, 1988.

[9] *J. Jarušek*: Contact problems with bounded friction, coercive case. Czechoslovak Math. J. *33 (108)* (1983), 237–261.

[10] *A. M. Khludnev, J. Sokolowski*: Modelling and Control in Solid Mechanics. Birkhäuser, Basel-Boston-Berlin, 1997.

[11] *V. A. Kovtunenko*: Analytical solution of a variational inequality for a cutted bar. Control Cybernet. *25* (1996), 801–808.

[12] *V. A. Kovtunenko*: Iterative penalty method for plate with a crack. Adv. Math. Sci. Appl. *7* (1997), 667–674.

[13] *V. A. Kovtunenko*: A variational and a boundary problems with friction on the interior boundary. Siberian Math. J. *39* (1998), 1060–1073.

[14] *A. S. Kravchuk*: Variational and Quasivariational Inequalities in Mechanics. MGAPI, Moscow, 1997. (In Russian.)

[15] *J.-L. Lions, E. Magenes*: Problémes aux Limites non Homogénes et Applications 1. Dunod, Paris, 1968.

[16] *V. G. Maz'ya*: Spaces of S. L. Sobolev. Leningrad Univ., 1985. (In Russian.)

[17] *N. F. Morozov*: Mathematical Foundations of the Crack Theory. Nauka, Moscow, 1984. (In Russian.)

[18] *S. A. Nazarov, B. A. Plamenevskiĭ*: Elliptic Problems in Domains with Piecewise Smooth Boundaries. Nauka, Moscow, 1991. (In Russian.)

[19] *J. Nečas, J. Jarušek and J. Haslinger*: On the solution of the variational inequality to the Signorini problem with small friction. Boll. Um. Mat. Ital. *17-B* (1980), 796–811.

[20] *J. J. Telega, T. Lewinski*: Mathematical aspects of modelling the macroscopic behaviour of cross-ply laminates with intralaminar cracks. Control Cybernet. *23* (1994), 773–792.

[21] *E. Zeidler*: Nonlinear Functional Analysis and its Applications. 1. Fixed-Point Theorems. Springer-Verlag, New York, 1986.

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