QUASI-VARIATIONAL INEQUALITY FOR NONLINEAR INDENTATION PROBLEM: POWER-LAW HARDENING MODEL

VICTOR A. KOVTUNENKO[1,2]

ABSTRACT. The Boussinesq problem which describes quasi-static indentation of a rigid punch into a deformable body is studied within the context of nonlinear constitutive equations. By this, the material response expresses the linearized strain in terms of the stress and cannot be inverted in general. A contact area between the punch and the body is unknown a-priori, whereas the total contact force is prescribed and yields a non-local integral condition. Consequently, the unilateral indentation problem is stated as a quasi-variational inequality for unknown variables of displacement, stress, and indentation depth. The Lagrange multiplier approach is applied in order to establish well-posedness to the underlying physically and geometrically nonlinear problem based on augmented penalty regularization and applying the minimax theorem of Ekeland-Temam. A sufficient solvability condition implies response functions that are bounded, hemi-continuous, coercive and obey a convex potential. A typical example is power-law hardening models for titanium alloys, Norton-Hoff and Ramberg-Osgood materials.

1. INTRODUCTION

The indentation test is an experimental procedure of pressing a rigid punch (indenter) into a solid body (which can be small under nano-indentation) in order to determine unknown material properties of the body: elastic moduli as well as inelastic ones. Studying inelastic properties is very important, for instance, for description of modern titanium and metal alloys.

For a given shape of the punch and the amount of total contact force applied, the indentation problem consists in finding simultaneously the indentation depth, displacement and stress distributions over the body. Since the contact area is unknown a-priori, this implies a free-boundary problem, which is referred to as Boussinesq's problem. Compared to Signorini's contact problem for a prescribed obstacle, here the obstacle is unknown a-priori and has to be determined by adding to the punch shape an indentation depth after solving the problem. In the mechanical literature, the indentation problem is usually formulated in the 3d setting for a semi-infinite body (foundation) and uses analytical formulas from the classical theory of linear elasticity (see, e.g., Love [25]). A collection of analytical solutions for indentation of isotropic foundations by punches of simple axisymmetric shapes: cylinder, cone, sphere, paraboloid, etc. can be found in Argatov and Mishuris [2]. Typically, this implies an explicit relation between the indentation depth and the total contact force which is applied quasi-statically in time.

2020 *Mathematics Subject Classification.* 35Q74, 49J40, 74D10, 74M15.

Key words and phrases. indentation testing, Boussinesq problem, nonlinear equations, power-law hardening, augmented Lagrangian, quasi-variational inequality, contact, saddle point, non-local constraint, minimax, well-posedness analysis.

From the viewpoint of constrained optimization, the unilateral indentation problem is described by a quasi-variational inequality subject to a non-local constraint. Indeed, prescribing the total contact force implies an integral constraint, which itself depends on the contact stress unknown a-priori (see the relevant study of volume constraints by Aguilera et al. [1]). The concept of quasi-variational inequalities due to a solution-dependent set of admissible states was established by Bensoussan and Lions [3]. Various applications of quasi-variational inequalities in mechanics were described in the book of Kravchuk and Neittaanmäki [22]. In particular, we refer to Itou et al. [15] for frictional contact under Coulomb law, to Giuffré et al.[7] for unilateral problems with gradient constraints, and to Migórski et al. [26] for quasi-variational inequalities for p-Laplace equation.

An abstract theory of pseudo-monotone, hemi-, quasi-, and conventional variational inequalities is outlined in the recent monograph by Gwinner et al. [8] and references therein. Our particular methods of non-smooth analysis stem from the variational approach to non-penetrating cracks in solids developed by Khludnev and Kovtunenko [17] and co-authors (for example, see [16, 18, 20] and other works related to asymptotic analysis [5, 24, 27], and numerical techniques [21]).

In the previous work by Itou et al. [14], the unilateral indentation problem was studied for linear viscoelastic bodies with a non-invertible material response expressing the linearized strain in terms of the stress by the Volterra convolution operator (see Itou et al. [12]). Based on the Lagrange multiplier approach we proved well-posedness to the unilateral indentation problem. In particular, semi-explicit formulas employing Sneddon's integrals were obtained for the viscoelastic half-space intended by cone when the contact area does not increase during the loading. In Itou et al. [13], using the Fourier–Bessel transform and the Papkovich–Neuber representation, the closed-form solution for a flat-ended cylindrical punch pressed into the linear viscoelastic half-space was constructed under arbitrary loading process, since the contact area does not change.

In the current contribution, we generalize the approach of quasi-variational inequalities to nonlinear constitutive relations when the linearized strain $\boldsymbol{\varepsilon}$ is expressed with respect to the Cauchy stress $\boldsymbol{\sigma}$ in the general form

$$\boldsymbol{\varepsilon} = \mathcal{F}[\boldsymbol{\sigma}] \tag{1.1}$$

with nonlinear functions $\mathcal{F}$ of material response. It is worth noting that a nonlinear expression (1.1) cannot in general be inverted in order to express the stress in terms of the linearized strain. The linear response (1.1) implies classical Hooke's law

$$\boldsymbol{\varepsilon} = \mathbf{A}\boldsymbol{\sigma} \tag{1.2}$$

with the compliance tensor $\mathbf{A}$. The reference structure for a nonlinear response function is the power-law hardening model given by

$$\mathcal{F}[\boldsymbol{\sigma}] = \frac{1}{2\mu}\frac{\boldsymbol{\sigma}}{(1+\kappa\|\boldsymbol{\sigma}\|^r)^{\frac{2-p}{r}}}, \quad \kappa, r > 0, \quad p > 1, \tag{1.3}$$

where $\mu > 0$ is the shear modulus, $\|\boldsymbol{\sigma}\| = \sqrt{\boldsymbol{\sigma} : \boldsymbol{\sigma}}$ is the Frobenius norm, and double dot stands for the scalar product of tensors. We note that, as $\kappa \to 0$ the limit (1.3) corresponds to the isotropic linear elastic material, and as $p \to 1$ it turns into the limiting small strain model (see Itou et al. [11]).

Using the decomposition of the stress into its volumetric $\boldsymbol{\sigma}^*$ and deviatoric $\mathrm{tr}\boldsymbol{\sigma}$ parts in the 3d Euclidean space with the identity tensor $\mathbf{I}$ as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \frac{1}{3}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}, \quad \mathrm{tr}\boldsymbol{\sigma}^* = 0, \tag{1.4}$$

the expression (1.3) can be modified to the following law

$$\mathcal{F}_1[\boldsymbol{\sigma}] = \frac{1}{2\mu} \frac{\boldsymbol{\sigma}^*}{(1+\kappa_1\|\boldsymbol{\sigma}^*\|^{r_1})^{\frac{2-p_1}{r_1}}} + \frac{1}{9K} \frac{(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}}{(1+\kappa_2|\mathrm{tr}\boldsymbol{\sigma}|^{r_2})^{\frac{2-p_2}{r_2}}}, \tag{1.5}$$

where $K > 0$ is the bulk modulus, $\kappa_1, r_1, \kappa_2, r_2 > 0$ and $p_1, p_2 > 1$ are fitting parameters. A special function (1.5) with $r_1 = r_2 = 2$ was used in Kulvait et al. [23] for fitting material moduli of titanium alloys obtained in experiments. Another possible structure for the form of the response function in (1.1) combining the linear part (1.2) and the decomposition (1.4) is given by

$$\mathcal{F}_2[\boldsymbol{\sigma}] = \mathbf{A}\boldsymbol{\sigma} + \kappa\|\boldsymbol{\sigma}^*\|^{p-2}\boldsymbol{\sigma}^*, \quad \kappa > 0,\ p > 2, \tag{1.6}$$

which describes Norton–Hoff and Ramberg–Osgood materials (see Knees [19]).

In Section 2 the unilateral indentation problem governing the nonlinear constitutive equation (1.1) is stated in the quasi-variational form. In Section 3 we introduce an augmented Lagrangian (see Ito and Kunisch [10]) combined with a penalty approximation. Our penalty method is exact in the sense that a prescribed value of the total contact force is fulfilled exactly for finite penalty parameters $\delta > 0$ rather than in the limit as $\delta \to 0$. We establish well-posedness to the underlying nonlinear problem by applying the minimax theorem of Ekeland–Temam [4]. A sufficient solvability condition is derived which implies a response function $\mathcal{F}$, which is bounded, hemi-continuous, coercive and obeys a convex potential. In Section 4 we pass to the limit as $\delta \to 0$, thus proving well-posedness to the unilateral indentation problem. Finally, in Section 5 we check these properties for the reference function describing power-law hardening.

2. Unilateral indentation problem

Let the solid body occupy a domain Ω in the Euclidean space $\mathbb{R}^d$ (where $d = 2$ or $d = 3$) with the Lipschitz continuous boundary $\partial\Omega$ carrying the outward unit normal vector $\mathbf{n} = (n_1, \dots, n_d)$. We assume a disjoint union $\partial\Omega = \overline{\Sigma} \cup \overline{\Gamma_\mathrm{N}} \cup \overline{\Gamma_\mathrm{D}}$ of three non-empty boundary parts, such that the Neumann boundary Γ_N separates the contact boundary Σ from the Dirichlet boundary Γ_D. An example geometry of the reference configuration is shown in Fig. 1 in the left plot. For spacial points $\mathbf{x} = (x_1, \dots, x_d) \in \Omega$ and times $t \in (0,T)$, $T > 0$, the time-space sets will be marked by the upper index T, respectively $\Omega^T = (0,T) \times \Omega$, $\Sigma^T = (0,T) \times \Sigma$, and $\Gamma_i^T = (0,T) \times \Gamma_i$ for $i \in \{\mathrm{N}, \mathrm{D}\}$.

Let the total contact force be given by a non-negative time-dependent function $F \in L^\infty(0,T)$, $F(t) \geq 0$. We prescribe the punch shape $\psi(\mathbf{x})$ that is bounded from above by a non-negative constant $h_0 \geq 0$ such that

$$\psi(\mathbf{x}) \leq \max_{\mathbf{x}\in\overline{\Sigma}} \psi(\mathbf{x}) =: h_0 \quad \text{a.e. } \mathbf{x} \in \overline{\Sigma}. \tag{2.1}$$

In the following, the space of second order symmetric d-by-d tensors will be denoted by $\mathbb{R}^{d\times d}_{\mathrm{sym}}$. Let a response function $\mathcal{F} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \mapsto \mathbb{R}^{d\times d}_{\mathrm{sym}}$ be given. In the closed time-spacial set $(t,\mathbf{x}) \in \overline{\Omega}^T$, we look for: the *stress* tensor $\boldsymbol{\sigma}(t,\mathbf{x}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, the *displacement* vector

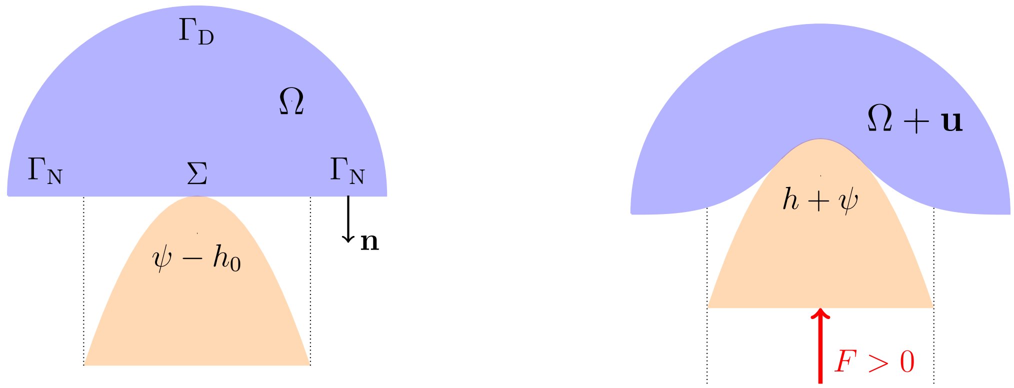

FIGURE 1. Example configuration of punch indentation: reference at $F = 0$ (left plot), and current at $F > 0$ (right plot).

$\mathbf{u}(t, \mathbf{x}) \in \mathbb{R}^d$, which determines the *linearized strain* tensor $\boldsymbol{\varepsilon}(t, \mathbf{x}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ by the symmetric part of the displacement gradient $\nabla\mathbf{u}$ and its transposed $\nabla^\top\mathbf{u}$ as

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\big(\nabla\mathbf{u} + \nabla^\top\mathbf{u}\big), \tag{2.2}$$

and the *indentation depth* $h(t)$ satisfying together the relations

$$-\operatorname{div}\boldsymbol{\sigma} = \mathbf{0} \quad \text{in } \Omega^T, \tag{2.3}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{F}[\boldsymbol{\sigma}] \quad \text{in } \Omega^T, \tag{2.4}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}^T, \tag{2.5}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{N}}^T, \tag{2.6}$$

$$\boldsymbol{\sigma}\mathbf{n} - (\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n})\mathbf{n} = \mathbf{0} \quad \text{on } \Sigma^T, \tag{2.7}$$

$$\mathbf{u}\cdot\mathbf{n} + h + \psi \le 0, \quad \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} \le 0, \quad (\mathbf{u}\cdot\mathbf{n} + h + \psi)(\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n}) = 0 \quad \text{on } \Sigma^T, \tag{2.8}$$

$$F + \int_\Sigma \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n}\, dS_{\mathbf{x}} = 0 \quad \text{for } t \in (0, T). \tag{2.9}$$

The governing system includes the homogeneous equilibrium equation (2.3) and the constitutive equation (2.4) subjected to the mixed: homogeneous Dirichlet (2.5) and homogeneous Neumann (2.6) boundary conditions. At the contact boundary Σ^T, (2.7) implies zero tangential stresses, and the unilateral contact conditions (2.8) describe non-penetration of the punch into the surface of the body in the normal direction. The integral condition (2.9) prescribes the total contact force. Here $\boldsymbol{\sigma}\mathbf{n}$ implies the matrix-vector multiplication, the dot stands for the scaler product of vectors. For a geometric example of the current configuration $\Omega + \mathbf{u}$ produced under indentation of the punch ψ with a nonzero force $F > 0$ see the right plot in Fig. 1.

For numerical computation of the unilateral problem, complementarity relations (2.8) can be expressed with the help of nonlinear complementarity problem (NLCP) functions, e.g., by the min-based function as

$$\min\big(0, \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} - c(\mathbf{u}\cdot\mathbf{n} + h + \psi)\big) - \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma^T \tag{2.10}$$

for arbitrary constant $c > 0$. Indeed, the nonlinear equation (2.10) is point-wisely equivalent to

$$\mathbf{u}\cdot\mathbf{n} + h + \psi = 0 \quad \text{on } \mathcal{A} = \{\mathbf{x}\in\Sigma|\ \big(\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} - c(\mathbf{u}\cdot\mathbf{n} + h + \psi)\big)(\mathbf{x}) < 0\} \tag{2.11}$$

over a strict coincidence set $\mathcal{A}$, where the constraint is active (the active set), and

$$\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} = 0 \quad \text{on } \mathcal{I} = \{\mathbf{x}\in\Sigma|\ \big(\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n} - c(\mathbf{u}\cdot\mathbf{n} + h + \psi)\big)(\mathbf{x}) \geq 0\} \tag{2.12}$$

over its complement (the inactive set) $\mathcal{I} := \Sigma\setminus\mathcal{A}$. Then the integral in (2.9) holds over $\mathcal{A}$ as

$$F + \int_{\mathcal{A}} \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n}\, dS_{\mathbf{x}} = 0 \quad \text{for } t\in(0,T). \tag{2.13}$$

The mixed formulation (2.11)–(2.13) is advantageous allowing to apply semi-smooth Newton methods of numerical optimization (see Hintermüller et al. [9]).

For fixed $t\in(0,T)$ we endow the boundary-value problem (2.2)–(2.9) with a weak formulation. Let the stress-stran response function be

$$\mathcal{F} : L^p(\Omega;\mathbb{R}^{d\times d}_{\text{sym}}) \mapsto L^q(\Omega;\mathbb{R}^{d\times d}_{\text{sym}}), \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q\in(1,\infty). \tag{2.14}$$

For displacements we introduce the Sobolev space due to the Dirichlet boundary condition (2.5):

$$W^{1,q}_{\Gamma_{\text{D}}}(\Omega;\mathbb{R}^d) = \{\mathbf{v}\in W^{1,q}(\Omega;\mathbb{R}^d)| \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{\text{D}}\},$$

and the function set due to the non-penetration condition in (2.8):

$$\mathcal{K}(h) = \{\mathbf{v}\in W^{1,q}_{\Gamma_{\text{D}}}(\Omega;\mathbb{R}^d)| \quad \mathbf{v}\cdot\mathbf{n} + h + \psi \leq 0 \text{ on } \Sigma\},$$

which depends on the indentation depth h.

Multiplying the equilibrium equation (2.3) by $\mathbf{v} - \mathbf{u}$ for arbitrary $\mathbf{v}\in W^{1,q}_{\Gamma_{\text{D}}}(\Omega;\mathbb{R}^d)$ and integrating it by parts over Ω, with the help of notation (2.2) for the linearized strain and boundary conditions (2.5)–(2.7) it follows

$$\begin{aligned} 0 &= -\int_\Omega \text{div}\boldsymbol{\sigma}\cdot(\mathbf{v}-\mathbf{u})\, d\mathbf{x} = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\, d\mathbf{x} - \int_{\partial\Omega} (\boldsymbol{\sigma}\mathbf{n})\cdot(\mathbf{v}-\mathbf{u})\, dS_{\mathbf{x}} \\ &= \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\, d\mathbf{x} - \int_\Sigma (\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n} + h + \psi - \mathbf{u}\cdot\mathbf{n} - h - \psi)\, dS_{\mathbf{x}}. \end{aligned} \tag{2.15}$$

Inserting into (2.15) complementarity relations (2.8) we derive a quasi-variational inequality, which together with (2.4) and (2.9) is given by the following mixed formulation: for every $t\in(0,T)$ find a triple $\mathbf{u}\in\mathcal{K}(h)$, $\boldsymbol{\sigma}\in L^p(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})$, and $h\geq -h_0$ such that

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\, d\mathbf{x} \geq 0 \quad \text{for all } \mathbf{v}\in\mathcal{K}(h), \tag{2.16}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{F}[\boldsymbol{\sigma}] \quad \text{in } \Omega, \tag{2.17}$$

$$F + \langle\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n}, \eta\rangle_{\Sigma\cup\Gamma_{\text{N}}} = 0, \quad \eta = 1 \text{ on } \Sigma, \tag{2.18}$$

$$h = \max\{\xi \in \mathbb{R} | \quad \mathbf{u} \in \mathcal{K}(\xi)\}. \tag{2.19}$$

In relation (2.18) generalizing the integral in (2.9), the brackets $\langle \cdot, \cdot \rangle_{\Sigma \cup \Gamma_{\mathrm{N}}}$ stand for the duality pairing between the Lions–Magenes space of traces $W_{00}^{1/q,p}(\Sigma \cup \Gamma_{\mathrm{N}}; \mathbb{R})$ continued by zero on Γ_{D}:

$$W_{00}^{1/q,p}(\Sigma \cup \Gamma_{\mathrm{N}}; \mathbb{R}) = \{\eta \in W^{1/q,p}(\partial\Omega; \mathbb{R}) | \quad \eta = 0 \text{ on } \Gamma_{\mathrm{D}}\}$$

and its dual space of linear continuous functionals $W_{00}^{1/q,p}(\Sigma \cup \Gamma_{\mathrm{N}}; \mathbb{R})^\star$ (see Geymonat and Suquet [6]). In (2.18) the cut-off function $\eta(\mathbf{x}) \in W_{00}^{1/q,p}(\Sigma \cup \Gamma_{\mathrm{N}}; \mathbb{R})$ can be arbitrary on Γ_{N} since $\boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n} = 0$ there.

It is worth noting that, if the non-penetration inequality $\mathbf{u} \cdot \mathbf{n} + h + \psi \leq 0$ holds, then $\mathbf{u} \cdot \mathbf{n} + \xi + \psi < 0$ and $\mathbf{u} \in \mathcal{K}(\xi)$ for all $\xi < h$. When the strict inequality takes place, the punch $\xi + \psi$ loses contact with the body, which contradicts the prescribed contact force. Taking the maximum over all possible ξ in (2.19) ensures that contact between the punch and the solid body occurs.

Moreover, (2.19) follows the lower bound $h \geq -h_0$. Indeed, if $F = 0$, then $\mathbf{u} = \mathbf{0}$, $\boldsymbol{\sigma} = \mathbf{0}$, and $h = -h_0$ solve (2.16)–(2.19), the trivial solution is feasible because $\mathbf{u} \cdot \mathbf{n} + h + \psi = -h_0 + \psi \leq 0$ according to (2.1), i.e. the punch touches the unstressed body. If $F > 0$, then contact is geometrically possible for $h > -h_0$, otherwise the active set $\mathcal{A} = \emptyset$ in (2.13) contradicts to (2.9).

In the following sections we provide sufficient conditions on response functions $\mathcal{F}$ and establish existence, uniqueness, and continuity in time for a weak solution of the variational problem (2.16)–(2.19) based on the augmented penalty method and minimax theorems.

3. Augmented penalty approximation

We start with assumptions on $\mathcal{F}$. Let there exist constant $M_0(p), M_3(p) \geq 0$ and $M_1(p), M_4(p) > 0$ such that an admissible response function in (2.14) is *bounded*:

$$\|\mathcal{F}[\boldsymbol{\sigma}]\|^q_{L^q(\Omega)} \leq M_0(p) + M_1(p)\|\boldsymbol{\sigma}\|^p_{L^p(\Omega)}, \tag{3.1}$$

hemi-continuous:

$$\int_\Omega \mathcal{F}[\boldsymbol{\sigma} + s\tilde{\boldsymbol{\sigma}}] : \overline{\boldsymbol{\sigma}} \, d\mathbf{x} \to \int_\Omega \mathcal{F}[\boldsymbol{\sigma}] : \overline{\boldsymbol{\sigma}} \, d\mathbf{x} \quad \text{as } s \to 0, \tag{3.2}$$

coercive (with respect to the Lebesgue vector norms):

$$\int_\Omega \mathcal{F}[\boldsymbol{\sigma}] : \boldsymbol{\sigma} \, d\mathbf{x} \geq M_4(p)\|\boldsymbol{\sigma}\|^p_{L^p(\Omega)} - M_3(p), \tag{3.3}$$

and obeys a *convex potential* $\mathcal{W} : L^p(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \mapsto \mathbb{R}$ such that

$$\mathcal{W}'[\boldsymbol{\sigma}] = \mathcal{F}[\boldsymbol{\sigma}], \quad \mathcal{W}[\overline{\boldsymbol{\sigma}}] - \mathcal{W}[\boldsymbol{\sigma}] \geq \mathcal{F}[\boldsymbol{\sigma}] : (\overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma}), \tag{3.4}$$

for all functions $\boldsymbol{\sigma}, \overline{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\sigma}} \in L^p(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$. From (3.4) it follows that $\mathcal{F}$ is also *monotone*:

$$\int_\Omega \big(\mathcal{F}[\boldsymbol{\sigma}] - \mathcal{F}[\overline{\boldsymbol{\sigma}}]\big) : (\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}) \, d\mathbf{x} \geq 0. \tag{3.5}$$

For a small penalty parameter $\delta > 0$, first we apply a penalization regularization to the quasi-variational inequality, and then pass $\delta \to 0$. We introduce the space product

$$V(\Omega) = W^{1,q}_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \times \mathbb{R}$$

and an augmented Lagrangian $\mathscr{L}_\delta : V(\Omega) \mapsto \mathbb{R}$ given by

$$\mathscr{L}_\delta(\mathbf{u}, \boldsymbol{\sigma}, h) := \int_\Omega \big(\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{W}[\boldsymbol{\sigma}]\big)\, d\mathbf{x} + \frac{1}{q\delta} \int_\Sigma \big([\mathbf{u}\cdot\mathbf{n} + h + \psi]^+\big)^q \, dS_{\mathbf{x}} - Fh \tag{3.6}$$

using (3.4) and decomposition into positive and negative parts

$$\mathbf{u}\cdot\mathbf{n} + h + \psi = [\mathbf{u}\cdot\mathbf{n} + h + \psi]^+ - [\mathbf{u}\cdot\mathbf{n} + h + \psi]^-.$$

In the admissible set

$$M(\Omega) = \{(\mathbf{u}, \boldsymbol{\sigma}, h) \in V(\Omega) | \quad h \geq -h_0\}$$

the following saddle-point problem is considered: find $(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, h^\delta) \in M(\Omega)$ such that

$$\mathscr{L}_\delta(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, h^\delta) = \inf_{\mathbf{u},h} \sup_{\boldsymbol{\sigma}} \mathscr{L}_\delta(\mathbf{u}, \boldsymbol{\sigma}, h) =: l_\delta \quad \text{over } (\mathbf{u}, \boldsymbol{\sigma}, h) \in M(\Omega). \tag{3.7}$$

Before investigating well-posedness of the problem (3.7) we remind some preliminaries. According to [6] there hold: the Korn–Friedrichs inequality with constant $K_{\rm KF}(q) > 0$:

$$K_{\rm KF}(q) \|\mathbf{u}\|^q_{W^{1,q}(\Omega)} \leq \|\boldsymbol{\varepsilon}(\mathbf{u})\|^q_{L^q(\Omega)} \quad \text{for } \mathbf{u} = \mathbf{0} \text{ on } \Gamma_{\rm D}, \tag{3.8}$$

the trace inequality with constant $K_{\rm tr}(q) > 0$:

$$\|\mathbf{u}\cdot\mathbf{n}\|_{L^q(\Sigma)} \leq \|\mathbf{u}\cdot\mathbf{n}\|_{W^{1/p,q}(\Sigma)} \leq K_{\rm tr}(q) \|\mathbf{u}\|_{W^{1,q}(\Omega)}, \tag{3.9}$$

and the continuous embedding with constant $K_{\rm emb}(q) > 0$:

$$\|\mathbf{u}\cdot\mathbf{n}\|_{L^1(\Sigma)} \leq K_{\rm emb}(q) \|\mathbf{u}\cdot\mathbf{n}\|_{L^q(\Sigma)} \quad \text{for } q > 1. \tag{3.10}$$

Theorem 3.1 (Well-posedness of δ-penalization)**.** *Under assumptions* (3.1)–(3.4) *on $\mathcal{F}$ there exists a saddle point $(\mathbf{u}^\delta, \boldsymbol{\sigma}^\delta, h^\delta) \in M(\Omega)$ solving the problem* (3.7)*. The solution fulfills optimality conditions:*

$$\int_\Omega \boldsymbol{\sigma}^\delta : \boldsymbol{\varepsilon}(\overline{\mathbf{u}})\, d\mathbf{x} + \frac{1}{\delta} \int_\Sigma \big([\mathbf{u}^\delta\cdot\mathbf{n} + h^\delta + \psi]^+\big)^{q-1} (\overline{\mathbf{u}}\cdot\mathbf{n})\, dS_{\mathbf{x}} = 0 \tag{3.11}$$

for all test functions $\overline{\mathbf{u}} \in W^{1,q}_{\Gamma_{\rm D}}(\Omega; \mathbb{R}^d)$, then

$$\boldsymbol{\varepsilon}(\mathbf{u}^\delta) = \mathcal{F}[\boldsymbol{\sigma}^\delta] \quad \text{in } \Omega, \tag{3.12}$$

$$F - \frac{1}{\delta} \int_\Sigma \big([\mathbf{u}^\delta\cdot\mathbf{n} + h^\delta + \psi]^+\big)^{q-1} dS_{\mathbf{x}} = 0. \tag{3.13}$$

The uniform estimates hold for $\delta \leq \delta_0$ and $\delta_0 > 0$:

$$\begin{aligned} \frac{M_4(p)}{p} \|\boldsymbol{\sigma}^\delta\|^p_{L^p(\Omega)} + \frac{1}{p\delta} \|[\mathbf{u}^\delta\cdot\mathbf{n} + h^\delta + \psi]^+\|^q_{L^q(\Sigma)} \leq M_3(p) + 2Fh_0 + \frac{F}{|\Sigma|}\|\psi\|_{L^1(\Sigma)} \\ + \frac{\delta_0}{p}\Big(\frac{F}{|\Sigma|} K_{\rm emb}(q)\Big)^p + \frac{M_1(p)}{pM_4(p)K_{\rm KF}(q)}\Big(\frac{F}{|\Sigma|} K_{\rm emb}(q) K_{\rm tr}(q)\Big)^p + \frac{M_0(p)M_4(p)}{qM_1(p)} =: C_{\boldsymbol{\sigma}}, \end{aligned} \tag{3.14}$$

$$K_{\rm KF}(q) \|\mathbf{u}^\delta\|^q_{W^{1,q}(\Omega)} \leq M_0(p) + \frac{pM_1(p)}{M_4(p)} C_{\boldsymbol{\sigma}}, \tag{3.15}$$

$$F|h^\delta| \leq C_{\boldsymbol{\sigma}} - M_3(p) - Fh_0 + \frac{p}{q} C_{\boldsymbol{\sigma}}. \tag{3.16}$$

Proof. We split the proof into three blocks: existence, optimality condition, and uniform estimate.

Existence. Due to assumption (3.4) the potential $\mathcal{W}$ is convex, continuous and Gâteaux differentiable, hence sequentially weakly lower semi-continuous (w.l.s.c.) (see [17, Theorem 1.7]). The penalty term $1/(q\delta)\int_\Sigma \big([\mathbf{u}\cdot\mathbf{n}+h+\psi]^+\big)^q\, dS_{\mathbf{x}}$ in (3.6) is also convex for $q>1$, continuous and Gâteaux differentiable. Since involving $-\mathcal{W}[\boldsymbol{\sigma}]$ the Lagrange functional $\mathscr{L}_\delta$ from (3.6) obeys the properties:

$$(\mathbf{u},h)\mapsto \mathscr{L}_\delta(\mathbf{u},\boldsymbol{\sigma},h) \text{ is convex, differentiable, w.l.s.c., and} \tag{3.17}$$

$$\boldsymbol{\sigma}\mapsto \mathscr{L}_\delta(\mathbf{u},\boldsymbol{\sigma},h) \text{ is concave, differentiable, weakly upper semi-continuous (w.u.s.c.).} \tag{3.18}$$

By the mean value theorem for integrals, due to assumptions (3.2) on continuity and (3.4) on differentiability, there exists $s\in(0,1)$ such that

$$-\int_\Omega \mathcal{W}[\boldsymbol{\sigma}]\, d\mathbf{x} = -\int_\Omega \big(\mathcal{W}[0]+\mathcal{F}[s\boldsymbol{\sigma}]:\boldsymbol{\sigma}\big)\, d\mathbf{x} \le -|\Omega|\mathcal{W}[0] - M_4(p)s^{p-1}\|\boldsymbol{\sigma}\|^p_{L^p(\Omega)} + \frac{M_3(p)}{s},$$

using the coercivity (3.3), where $s=0$ and $s=1$ are impossible because would lead to identically constant $\mathcal{F}$. Therefore, for $\mathbf{u}=\mathbf{0}$ and fixed h we have the coercivity

$$\mathscr{L}_\delta(\mathbf{0},\boldsymbol{\sigma},h)\to-\infty \quad \text{as } \|\boldsymbol{\sigma}\|_{L^p(\Omega)}\to\infty. \tag{3.19}$$

At arbitrary fixed $\boldsymbol{\sigma}$ it follows straightforwardly that

$$\mathscr{L}_\delta(\mathbf{u},\boldsymbol{\sigma},h)\to+\infty \quad \text{either as } \|\mathbf{u}\|_{W^{1,q}(\Omega)}\to\infty \text{ or } h\to+\infty. \tag{3.20}$$

Since the admissible set $M(\Omega)$ is evidently convex and closed, conditions (3.17)–(3.20) in the minimax theorem of Ekeland–Temam [4, Proposition 2.2] guarantee a saddle point to (3.7). Optimality condition. The unconstrained minimum of $\mathscr{L}_\delta$ over test functions $\overline{\mathbf{u}}\in W^{1,q}_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)$ implies the variational equation $\langle \frac{\partial\mathscr{L}_\delta}{\partial\mathbf{u}}(\mathbf{u}^\delta,\boldsymbol{\sigma}^\delta,h^\delta),\overline{\mathbf{u}}\rangle=0$, which can be easily calculated as (3.11). Similarly, the unconstrained maximum over test functions $\overline{\boldsymbol{\sigma}}\in L^p(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ implies

$$0=\langle \frac{\partial\mathscr{L}_\delta}{\partial\boldsymbol{\sigma}}(\mathbf{u}^\delta,\boldsymbol{\sigma}^\delta,h^\delta),\overline{\boldsymbol{\sigma}}\rangle = \int_\Omega \big(\boldsymbol{\varepsilon}(\mathbf{u})^\delta-\mathcal{F}[\boldsymbol{\sigma}^\delta]\big):\overline{\boldsymbol{\sigma}}\, d\mathbf{x}$$

using the Gâteaux derivative of $\mathcal{W}$ from (3.4), and leads to the constitutive equation (3.12).

If $h^\delta>-h_0$, then small $s>0$ provides admissible $\overline{h}=h^\delta\pm s\ge -h_0$ in $M(\Omega)$. Since $\mathscr{L}_\delta$ is convex with respect to h, the optimality condition over $\overline{h}$ implies a variational inequality

$$0\le \langle \frac{\partial\mathscr{L}_\delta}{\partial h}(\mathbf{u}^\delta,\boldsymbol{\sigma}^\delta,h^\delta),\overline{h}-h^\delta\rangle = \pm s\Big(\frac{1}{\delta}\int_\Sigma \big([\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+\big)^{q-1} dS_{\mathbf{x}} - F\Big),$$

thus following the total contact force (3.13). If $h^\delta=-h_0$, then the trivial solution $\mathbf{u}^\delta=\boldsymbol{\sigma}^\delta=\mathbf{0}$ and $F=0$ would satisfy the optimality system (3.11)–(3.13) due to condition (2.1). Uniform estimate. The substitution of $\overline{\mathbf{u}}=\mathbf{u}^\delta$ into (3.11) leads to the equation

$$\int_\Omega \boldsymbol{\sigma}^\delta:\boldsymbol{\varepsilon}(\mathbf{u}^\delta)\, d\mathbf{x} + \frac{1}{\delta}\int_\Sigma \big([\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+\big)^{q-1}(\mathbf{u}^\delta\cdot\mathbf{n})\, dS_{\mathbf{x}}=0. \tag{3.21}$$

After insertion in the penalty term the following identity:

$$\big([\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+\big)^{q-1}(\mathbf{u}^\delta\cdot\mathbf{n}) = \big([\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+\big)^{q} - \big([\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+\big)^{q-1}(h^\delta+\psi),$$

using the upper bound (2.1) for the punch shape ψ, and because of the total contact force (3.13):

$$\frac{1}{\delta}\int_\Sigma \big([\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+\big)^{q-1}(h^\delta+h_0)\,dS_{\mathbf{x}} = F(h^\delta+h_0),$$

from (3.21) we infer the inequality

$$\int_\Omega \boldsymbol{\sigma}^\delta : \boldsymbol{\varepsilon}(\mathbf{u}^\delta)\,d\mathbf{x} + \frac{1}{\delta}\int_\Sigma \big([\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+\big)^{q}\,dS_{\mathbf{x}} \le F(h^\delta+h_0).$$

We the help of equation $\boldsymbol{\varepsilon}(\mathbf{u}^\delta)=\mathcal{F}[\boldsymbol{\sigma}^\delta]$ and the assumption of coercivity (3.3) it follows

$$M_4(p)\|\boldsymbol{\sigma}^\delta\|^p_{L^p(\Omega)} + \frac{1}{\delta}\|[\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+\|^q_{L^q(\Sigma)} \le M_3(p)+F(|h^\delta|+h_0). \tag{3.22}$$

According to the boundedness assumption (3.1) and the Korn–Friedrichs inequality (3.8), from the constitutive equation (3.12) we estimate the displacement as

$$K_{\mathrm{KF}}(q)\|\mathbf{u}^\delta\|^q_{W^{1,q}(\Omega)} \le \|\boldsymbol{\varepsilon}(\mathbf{u}^\delta)\|^q_{L^q(\Omega)} \le M_0(p)+M_1(p)\|\boldsymbol{\sigma}^\delta\|^p_{L^p(\Omega)}. \tag{3.23}$$

To evaluate the indentation depth, avoiding the negative part $-[\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^- \le 0$ we get

$$-h_0 \le h^\delta \le [\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+ - \mathbf{u}^\delta\cdot\mathbf{n}-\psi. \tag{3.24}$$

After integration of (3.24) over Σ it follows that

$$\int_\Sigma |h^\delta|\,dS_{\mathbf{x}} \le \int_\Sigma \big(h_0 + \big|[\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+ - \mathbf{u}^\delta\cdot\mathbf{n}-\psi\big|\big)\,dS_{\mathbf{x}},$$

then with the help of trace inequality (3.9) and continuous embedding (3.10) this leads to

$$|h^\delta| \le h_0 + \frac{K_{\mathrm{emb}}(q)}{|\Sigma|}\Big(\|[\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+\|_{L^q(\Sigma)} + K_{\mathrm{tr}}(q)\|\mathbf{u}^\delta\|_{W^{1,q}(\Omega)}\Big) + \frac{1}{|\Sigma|}\|\psi\|_{L^1(\Sigma)}.$$

Applying here Young's inequality with suitable weights we can derive the upper bound:

$$\begin{aligned} F|h^\delta| \le{}& Fh_0 + \frac{\delta}{p}\Big(\frac{F}{|\Sigma|}K_{\mathrm{emb}}(q)\Big)^p + \frac{1}{q\delta}\|[\mathbf{u}^\delta\cdot\mathbf{n}+h^\delta+\psi]^+\|^q_{L^q(\Sigma)} + \frac{F}{|\Sigma|}\|\psi\|_{L^1(\Sigma)} \\ &+ \frac{M_1(p)}{pM_4(p)K_{\mathrm{KF}}(q)}\Big(\frac{F}{|\Sigma|}K_{\mathrm{emb}}(q)K_{\mathrm{tr}}(q)\Big)^p + \frac{M_4(p)}{qM_1(p)}K_{\mathrm{KF}}(q)\|\mathbf{u}^\delta\|^q_{W^{1,q}(\Omega)}. \end{aligned} \tag{3.25}$$

For $\delta \le \delta_0$, subsequently inserting the estimate (3.23) of $K_{\mathrm{KF}}(q)\|\mathbf{u}^\delta\|^q_{W^{1,q}(\Omega)}$ into (3.25), then the estimate (3.25) of $F|h^\delta|$ into (3.22) and gathering the like terms, we obtain the uniform estimate (3.14) for the stress and the penalty. When substituting (3.14) with the constant $C_{\boldsymbol{\sigma}}$ back into (3.23) gives the uniform estimate (3.15) for the displacement, and the substitution into (3.25) provides the uniform estimate (3.16) for the indentation depth. We note that $C_{\boldsymbol{\sigma}} - M_3(p) - Fh_0 > 0$ here. The proof is completed. □

We make few remarks to Theorem 3.1. After integration by parts the weak solution $(\mathbf{u}^\delta,\boldsymbol{\sigma}^\delta) \in W^{1,q}_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)\times L^p(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ to the variational equation (3.11) satisfies the boundary-value problem (compare to (2.3)–(2.8)):

$$-\operatorname{div}\boldsymbol{\sigma}^\delta = \mathbf{0} \quad \text{in } \Omega, \tag{3.26}$$

$$\mathbf{u}^\delta = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}, \qquad \boldsymbol{\sigma}^\delta\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{N}}, \tag{3.27}$$

(3.28) $$\boldsymbol{\sigma}^\delta \mathbf{n} - (\boldsymbol{\sigma}^\delta \mathbf{n}\cdot\mathbf{n})\mathbf{n} = \mathbf{0}, \quad \boldsymbol{\sigma}^\delta \mathbf{n}\cdot\mathbf{n} = -\frac{1}{\delta}\big([\mathbf{u}^\delta\cdot\mathbf{n} + h^\delta + \psi]^+\big)^{q-1} \quad \text{on } \Sigma,$$

where the penalty term $-1/\delta([\mathbf{u}^\delta\cdot\mathbf{n} + h^\delta + \psi]^+)^{q-1} \in W^{1/p,q}(\Sigma;\mathbb{R})$ according to the trace theorem (see (3.9)). Thus, the boundary condition (3.28) follows the total contact force (3.13)

(3.29) $$F + \int_\Sigma \boldsymbol{\sigma}^\delta \mathbf{n}\cdot\mathbf{n}\, dS_{\mathbf{x}} = 0$$

in the form akin to (2.9). That is: F is attained exactly by the penalty approximation at $\delta > 0$.

In the weak form, the bulk term over Ω in equation (3.11) determines a linear continuous functional in $W^{1,q}_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^d)$. Therefore, inserting there the normal stress from (3.28) and using the continuity of trace extension operator we get the upper bound with constant $K_\mathrm{ext}(q) > 0$:

$$\big|\langle \boldsymbol{\sigma}^\delta \mathbf{n}\cdot\mathbf{n}, \overline{\mathbf{u}}\cdot\mathbf{n}\rangle_{\Sigma\cup\Gamma_\mathrm{N}}\big| = \Big|\int_\Omega \boldsymbol{\sigma}^\delta : \boldsymbol{\varepsilon}(\overline{\mathbf{u}})\, d\mathbf{x}\Big| \le K_\mathrm{ext}(q)\|\boldsymbol{\sigma}^\delta\|_{L^p(\Omega)}\|\overline{\mathbf{u}}\cdot\mathbf{n}\|_{W^{1/q,p}_{00}(\Sigma\cup\Gamma_\mathrm{N})}.$$

Together with (3.14) this provides the uniform estimate for the normal stress in the dual norm:

(3.30) $$\|\boldsymbol{\sigma}^\delta \mathbf{n}\cdot\mathbf{n}\|_{W^{1/q,p}_{00}(\Sigma\cup\Gamma_\mathrm{N})^\star} \le K_\mathrm{ext}(q)\Big(\frac{p}{M_4(p)}C_{\boldsymbol{\sigma}}\Big)^{1/p},$$

which is helpful for the following use.

4. Well-posedness of the quasi-variational inequality

Based on Theorem 3.1 we pass to the limit as $\delta \to 0$.

Theorem 4.1 (Well-posedness of the quasi-variational inequality)**.** *Under assumptions* (3.1)–(3.4) *on $\mathcal{F}$ there exists a solution $\mathbf{u} \in \mathcal{K}(h)$, $\boldsymbol{\sigma} \in L^p(\Omega;\mathbb{R}^{d\times d}_\mathrm{sym})$, and $h \ge -h_0$ to the unilateral indentation problem* (2.16)–(2.19)*. It satisfies the a-priori estimates with the constant $C_{\boldsymbol{\sigma}} > 0$ from* (3.14)*:*

(4.1) $$\frac{M_4(p)}{p}\|\boldsymbol{\sigma}\|^p_{L^p(\Omega)} \le C_{\boldsymbol{\sigma}},$$

(4.2) $$K_\mathrm{KF}(q)\|\mathbf{u}\|^q_{W^{1,q}(\Omega)} \le M_0(p) + \frac{pM_1(p)}{M_4(p)}C_{\boldsymbol{\sigma}},$$

(4.3) $$F|h| \le C_{\boldsymbol{\sigma}} - M_3(p) - Fh_0 + \frac{p}{q}C_{\boldsymbol{\sigma}}.$$

If the monotone property of $\mathcal{F}$ in (3.5) *is strict, then the solution is unique.*

Let additionally the following assumptions hold: $\mathcal{F}$ is strong-to-strong continuous and strongly monotone, i.e., there exists constant $M_2(p) > 0$ such that

(4.4) $$\int_\Omega \big(\mathcal{F}[\boldsymbol{\sigma}] - \mathcal{F}[\overline{\boldsymbol{\sigma}}]\big) : (\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}})\, d\mathbf{x} \ge M_2(p)\|\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}\|^p_{L^p(\Omega)}.$$

If the total contact force $F(t)$ is continuous, then the solution $(\mathbf{u},\boldsymbol{\sigma},h)(t,\mathbf{x})$ is also continuous in time.

Proof. Again we split the proof into three blocks: limit passage, uniqueness, and time-continuity.
Limit passage. From the uniform estimates (3.14)–(3.16) and (3.30), by the compactness principle we infer a convergent sub-sequence $\delta_k \to 0$ such that

$$(\mathbf{u}^{\delta_k}, \boldsymbol{\sigma}^{\delta_k}) \rightharpoonup (\mathbf{u}, \boldsymbol{\sigma}) \quad \text{weakly in } W^{1,q}(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \quad h^{\delta_k} \to h,$$
$$[\mathbf{u}^{\delta_k} \cdot \mathbf{n} + h^{\delta_k} + \psi]^+ \to [\mathbf{u} \cdot \mathbf{n} + h + \psi]^+ = 0 \quad \text{strongly in } L^q(\Sigma; \mathbb{R}),$$
$$\boldsymbol{\sigma}^{\delta_k}\mathbf{n} \cdot \mathbf{n} \rightharpoonup \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n} \quad *\text{-weakly in } W^{1/q,p}_{00}(\Sigma \cup \Gamma_{\mathrm{N}})^\star \tag{4.5}$$

for an accumulation point $(\mathbf{u}, \boldsymbol{\sigma}, h) \in M(\Omega)$, and $\mathbf{u} \in \mathcal{K}(h)$.

We test the penalty equation (3.11) with $\overline{\mathbf{u}} = \mathbf{v} - \mathbf{u}^\delta$ for arbitrary $\mathbf{v} \in \mathcal{K}(h)$ such that

$$\int_\Omega \boldsymbol{\sigma}^\delta : \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}^\delta)\, d\mathbf{x} = \frac{1}{\delta} \int_\Sigma \big([\mathbf{u}^\delta \cdot \mathbf{n} + h^\delta + \psi]^+\big)^{q-1} (\mathbf{u}^\delta - \mathbf{v}) \cdot \mathbf{n}\, dS_{\mathbf{x}}. \tag{4.6}$$

After decomposition in the penalty term as

$$(\mathbf{u}^\delta - \mathbf{v}) \cdot \mathbf{n} = (\mathbf{u}^\delta \cdot \mathbf{n} + h^\delta + \psi) - (\mathbf{v} \cdot \mathbf{n} + h + \psi) + (h - h^\delta),$$

using well-known monotonicity of the penalty operator and $([\mathbf{v} \cdot \mathbf{n} + h + \psi]^+)^{q-1} = 0$ such that

$$\frac{1}{\delta} \int_\Sigma \big([\mathbf{u}^\delta \cdot \mathbf{n} + h^\delta + \psi]^+\big)^{q-1} \big((\mathbf{u}^\delta \cdot \mathbf{n} + h^\delta + \psi) - (\mathbf{v} \cdot \mathbf{n} + h + \psi)\big)\, dS_{\mathbf{x}} \geq 0,$$

and the following identity due to the total contact force (3.13):

$$\frac{1}{\delta} \int_\Sigma \big([\mathbf{u}^\delta \cdot \mathbf{n} + h^\delta + \psi]^+\big)^{q-1} (h - h^\delta)\, dS_{\mathbf{x}} = F(h - h^\delta),$$

from (4.6) it follows the inequality

$$\int_\Omega \big(\boldsymbol{\sigma}^\delta : \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\sigma}^\delta : \boldsymbol{\varepsilon}(\mathbf{u}^\delta)\big)\, d\mathbf{x} \geq F(h - h^\delta). \tag{4.7}$$

With the help of the constitutive equation $\boldsymbol{\varepsilon}(\mathbf{u}^\delta) = \mathcal{F}[\boldsymbol{\sigma}^\delta]$ and convexity of of the potential $\mathcal{W}$ in (3.4), the second term in the left-hand side of (4.7) allows the upper bound

$$-\mathcal{F}[\boldsymbol{\sigma}^\delta] : \boldsymbol{\sigma}^\delta \leq -\mathcal{F}[\boldsymbol{\sigma}^\delta] : \boldsymbol{\sigma} + \mathcal{W}[\boldsymbol{\sigma}] - \mathcal{W}[\boldsymbol{\sigma}^\delta] = -\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}^\delta) + \mathcal{W}[\boldsymbol{\sigma}] - \mathcal{W}[\boldsymbol{\sigma}^\delta].$$

Since $-\mathcal{W}$ is sequentially w.u.s.c. (see (3.18)), then $-\limsup \mathcal{W}[\boldsymbol{\sigma}^{\delta_k}] \leq -\mathcal{W}[\boldsymbol{\sigma}]$ as $\delta_k \to 0$ due to convergences (4.5), and in the limit of (4.7) we get the variational inequality (2.16).

In order to deduce the constitutive equation (2.17) we apply Minty's trick. Using $\boldsymbol{\varepsilon}(\mathbf{u}^\delta) = \mathcal{F}[\boldsymbol{\sigma}^\delta]$, the monotonicity of $\mathcal{F}$ in (3.5), and the inequality (4.7) with $\mathbf{v} = \mathbf{u}$ it follows

$$0 = \int_\Omega \big(\mathcal{F}[\boldsymbol{\sigma}^\delta] - \boldsymbol{\varepsilon}(\mathbf{u}^\delta)\big) : (\boldsymbol{\sigma}^\delta - \overline{\boldsymbol{\sigma}})\, d\mathbf{x} \geq \int_\Omega \big(\mathcal{F}[\overline{\boldsymbol{\sigma}}] - \boldsymbol{\varepsilon}(\mathbf{u}^\delta)\big) : (\boldsymbol{\sigma}^\delta - \overline{\boldsymbol{\sigma}})\, d\mathbf{x}$$
$$\geq \int_\Omega \big(\mathcal{F}[\overline{\boldsymbol{\sigma}}] : (\boldsymbol{\sigma}^\delta - \overline{\boldsymbol{\sigma}}) - \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\sigma}^\delta + \boldsymbol{\varepsilon}(\mathbf{u}^\delta) : \overline{\boldsymbol{\sigma}}\big)\, d\mathbf{x} + F(h - h^\delta),$$

and, after taking the limit as $\delta_k \to 0$ due to convergences (4.5) we get

$$0 \geq \int_\Omega \big(\mathcal{F}[\overline{\boldsymbol{\sigma}}] : (\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}) - \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\sigma} + \boldsymbol{\varepsilon}(\mathbf{u}) : \overline{\boldsymbol{\sigma}}\big)\, d\mathbf{x}. \tag{4.8}$$

The assignment of $\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \pm s\tilde{\boldsymbol{\sigma}}$ in (4.8) after dividing by $s > 0$ leads to

$$0 \geq \pm \int_\Omega \big(-\mathcal{F}[\boldsymbol{\sigma} \pm s\tilde{\boldsymbol{\sigma}}] + \boldsymbol{\varepsilon}(\mathbf{u})\big) : \tilde{\boldsymbol{\sigma}}\, d\mathbf{x},$$

which limit as $s \to 0$ due to the assumption of hemi-continuity (3.2) results in (2.17).

The integral in (3.29) can be expressed as the duality

$$F + \langle \boldsymbol{\sigma}^\delta \mathbf{n} \cdot \mathbf{n}, \eta \rangle_{\Sigma \cup \Gamma_\mathrm{N}} = 0$$

with the cut-off function $\eta(\mathbf{x}) \in W_{00}^{1/q,p}(\Sigma \cup \Gamma_\mathrm{N}; \mathbb{R})$ such that $\eta = 1$ on Σ. Passing here to the limit due to the last convergence in (4.5) follows directly the total contact force (2.18).

To prove the maximum indentation depth (2.19), we assume $\xi > h$ exists such that $\mathbf{u} \in \mathcal{K}(\xi)$. Because the contact condition $(\mathbf{u} \cdot \mathbf{n} + \xi + \psi)(\mathbf{x}) \leq 0$ holds for all $\mathbf{x} \in \Sigma$, then

$$\max_{\mathbf{x} \in \Sigma} (\mathbf{u} \cdot \mathbf{n} + h + \psi)(\mathbf{x}) < 0,$$

and by continuity $\max_{\mathbf{x} \in \Sigma} (\mathbf{u}^\delta \cdot \mathbf{n} + h^\delta + \psi)(\mathbf{x}) < 0$ holds for sufficiently small δ. Consequently, $[\mathbf{u}^\delta \cdot \mathbf{n} + h^\delta + \psi]^+ = 0$ follows the total contact force $F = 0$ in (3.13) implying the trivial solution $\mathbf{u}^\delta = \mathbf{0}$, $\boldsymbol{\sigma}^\delta = \mathbf{0}$, and $h^\delta = -h_0$. Thus, $\mathbf{u}^\delta \cdot \mathbf{n} + h^\delta + \psi = -h_0 + \psi < 0$ that contradicts to (2.1).

Uniqueness. Let the monotone property of $\mathcal{F}$ in (3.5) be strict. We assume there exist two different solutions $(\mathbf{u}^1, \boldsymbol{\sigma}^1, h^1) \neq (\mathbf{u}^2, \boldsymbol{\sigma}^2, h^2)$ satisfying for $i = 1, 2$ the relations:

$$\mathbf{u}^i \in \mathcal{K}(h^i), \quad \int_\Omega \boldsymbol{\sigma}^i : \boldsymbol{\varepsilon}(\mathbf{v}^i - \mathbf{u}^i)\, d\mathbf{x} \geq 0 \quad \text{for all } \mathbf{v}^i \in \mathcal{K}(h^i), \quad \boldsymbol{\varepsilon}(\mathbf{u}^i) = \mathcal{F}[\boldsymbol{\sigma}^i], \tag{4.9}$$

$$F^i + \langle \boldsymbol{\sigma}^i \mathbf{n} \cdot \mathbf{n}, \eta \rangle_{\Sigma \cup \Gamma_\mathrm{N}} = 0 \quad (\eta = 1 \text{ on } \Sigma), \quad -h_0 \leq h^i = \max\{\xi \in \mathbb{R} |\ \mathbf{u}^i \in \mathcal{K}(\xi)\} \tag{4.10}$$

for same $F^1 = F^2$. An auxiliary function $\mathbf{w} \in W_{\Gamma_\mathrm{D}}^{1,q}(\Omega; \mathbb{R}^d)$ can be constricted such that

$$\mathbf{w} \cdot \mathbf{n} = h^1 - h^2 \quad \text{on } \Sigma, \tag{4.11}$$

e.g. by solving $\int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{w})\, d\mathbf{x} \geq 0$ for all $\mathbf{v} \in W_{\Gamma_\mathrm{D}}^{1,q}(\Omega; \mathbb{R}^d)$, $\mathbf{v} \cdot \mathbf{n} = h^1 - h^2$ on Σ. Then we can test (4.9) with $\mathbf{v}^1 = \mathbf{u}^2 - \mathbf{w} \in \mathcal{K}(h^1)$ and $\mathbf{v}^2 = \mathbf{u}^1 + \mathbf{w} \in \mathcal{K}(h^2)$ because of

$$(\mathbf{u}^2 - \mathbf{w}) \cdot \mathbf{n} + h^1 + \psi = \mathbf{u}^1 \cdot \mathbf{n} + h^1 + \psi \leq 0, \quad (\mathbf{u}^1 + \mathbf{w}) \cdot \mathbf{n} + h^2 + \psi = \mathbf{u}^2 \cdot \mathbf{n} + h^2 + \psi \leq 0,$$

which after summation yields the inequality

$$\int_\Omega (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) : \boldsymbol{\varepsilon}(\mathbf{u}^1 - \mathbf{u}^2)\, d\mathbf{x} \leq \int_\Omega (\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}^1) : \boldsymbol{\varepsilon}(\mathbf{w})\, d\mathbf{x}. \tag{4.12}$$

We apply to the right-hand side of (4.12) Green's formula and the condition (4.10) on F providing

$$\begin{aligned} \int_\Omega (\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}^1) : \boldsymbol{\varepsilon}(\mathbf{w})\, d\mathbf{x} &= \langle (\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}^1)\mathbf{n} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n} \rangle_{\Sigma \cup \Gamma_\mathrm{N}} \\ &= (h^1 - h^2) \langle (\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}^1)\mathbf{n} \cdot \mathbf{n}, \eta \rangle_{\Sigma \cup \Gamma_\mathrm{N}} = (h^1 - h^2)(F^1 - F^2) = 0. \end{aligned} \tag{4.13}$$

The strict monotonicity and constitutive equations imply positive left-hand side of (4.12):

$$0 < \int_\Omega \big(\mathcal{F}[\boldsymbol{\sigma}^1] - \mathcal{F}[\boldsymbol{\sigma}^2]\big) : (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2)\, d\mathbf{x} = \int_\Omega (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) : \boldsymbol{\varepsilon}(\mathbf{u}^1 - \mathbf{u}^2)\, d\mathbf{x} \tag{4.14}$$

that contradicts to zero in (4.12). Therefore, $\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}^2$, which follows $\boldsymbol{\varepsilon}(\mathbf{u}^1) = \boldsymbol{\varepsilon}(\mathbf{u}^2)$ and $\mathbf{u}^1 = \mathbf{u}^2$. Then the maximum indentation depth guarantees $h^1 = h^2$, thus uniqueness.
Time-continuity. For the total contact force $F(t_i)$, $i = 1, 2$, at two times $t_1 \neq t_2$ let us set:

$$\mathbf{u}^i = \mathbf{u}(t_i), \quad \boldsymbol{\sigma}^i = \boldsymbol{\sigma}(t_i), \quad h^i = h(t_i), \quad F^i = F(t_i).$$

Repeating arguments (4.9)–(4.14) with $F^1 \neq F^2$, due to the strong monotonicity (4.4) we derive

$$M_2(p)\|\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2\|^p_{L^p(\Omega)} \leq \int_\Omega (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) : \boldsymbol{\varepsilon}(\mathbf{u}^1 - \mathbf{u}^2)\, d\mathbf{x} \leq (h^1 - h^2)(F^1 - F^2) \tag{4.15}$$

following $\boldsymbol{\sigma}^1 \to \boldsymbol{\sigma}^2$ if $F^1 \to F^2$. Subsequently, $\mathbf{u}^1 \to \mathbf{u}^2$ when $\mathcal{F}$ is strong-to-strong continuous, and $h^1 \to h^2$ by the uniqueness. This finishes the proof. □

As a corollary of Theorem 4.1 we consider the response function for power-law hardening.

5. Power-law hardening

We start the example with noting that the function $\mathcal{F}$ in (1.3) obeys the potential

$$\mathcal{W}[\boldsymbol{\sigma}] = \frac{1}{4\mu} \int_0^{\|\boldsymbol{\sigma}\|^2} \frac{d\xi}{(1 + \kappa \xi^{\frac{r}{2}})^{\frac{2-p}{r}}}, \tag{5.1}$$

which is convex when $\mathcal{F} = \mathcal{W}'$ in (1.3) is monotone.

Corollary 5.1 (Indentation of power-law hardening material)**.** *For a given total contact force* $0 \leq F(t) \in L^\infty(0, T)$, *there exists a unique solution*

$$\mathbf{u}(t) \in W^{1,q}_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d), \quad \boldsymbol{\sigma}(t) \in L^p(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \quad h(t) \geq -h_0$$

to the quasi-static unilateral indentation problem (2.2)–(2.9) *fulfilled in the weak form* (2.16)–(2.19) *a.e.* $t \in (0, T)$, *with the response function* $\mathcal{F}$ *from* (1.3) *describing a power-law hardening material.*

Proof. Assumptions (3.1)–(3.3) of Theorem 4.1 and the strict inequality (3.5) (hence (3.4) for $\mathcal{W}$ in (5.1)) were proved in Itou et al. [12, Appendix A] with the constants found explicitly as follows:

$$\begin{cases} M_0(p) = 0, & M_1(p) = \frac{1}{2\mu} d^{2p} \kappa^{\frac{p-2}{r}} & \text{for } 1 < p \leq 2, \\ M_0(p) = \frac{(p-q)d^2}{p(2\mu)^q} C_{\frac{r}{p-q}} |\Omega|, & M_1(p) = \frac{1}{(2\mu)^q}\big(\frac{q}{p} + d^{2p}\kappa^{\frac{p-q}{r}}\big) C_{\frac{r}{p-q}} & \text{for } p > 2, \end{cases}$$

where $C_{\frac{r}{p-q}} = 2^{\frac{p-q}{r}-1}$ for $\frac{r}{p-q} < 1$, and $C_{\frac{r}{p-q}} = 1$ for $\frac{r}{p-q} \geq 1$,

$$\begin{cases} M_3(p) = \frac{\kappa^{-\frac{2}{r}}}{2\mu p}\big((2-p)C_{\frac{r}{p}}\big)^{\frac{p-2}{p}} |\Omega|, & M_4(p) = \frac{\kappa^{\frac{p-2}{r}}}{2\mu d^2 p}\big((2-p)C_{\frac{r}{p}}\big)^{\frac{p-2}{p}} & \text{for } 1 < p < 2, \\ M_3(p) = 0, & M_4(p) = \frac{\kappa^{\frac{p-2}{r}}}{2\mu d^2} & \text{for } p \geq 2, \end{cases}$$

where $C_{\frac{r}{p}} = 2^{\frac{p}{r}-1}$ for $\frac{r}{p} < 1$, and $C_{\frac{r}{p}} = 1$ for $\frac{r}{p} \geq 1$. This proves the assertion. □

It is worth noting that the same properties as for $\mathcal{F}$ in (1.3) evidently hold for the generalized function $\mathcal{F}_1$ from (1.5). Properties of the function $\mathcal{F}_2$ in (1.6) were studied by Knees [19].

6. Concluding remarks

In conclusion, the quasi-static indentation problem consists in solving the nonlinear relations (2.2)–(2.9) for every fixed time $t \in (0, T)$. We remark that the load prescribed by a function $F(t)$ may be non-monotone, which would correspond to both increasing as well as decreasing the contact area. The obtained results are valid in the case of a non-rigid punch, when given by a time-dependent shape function $\psi(t, \mathbf{x})$.

Acknowledgments

This work was supported by the by the Austrian Science Fund (FWF) project P26147-N26: PION and the European Research Council (ERC) under the European Union's Horizon 2020 Research and Innovation Programme (advanced grant No. 668998 OCLOC).

References

[1] Aguilera N, Alt HW, Caffarelli LA. 1986 An optimization problem with volume constraint. *SIAM J. Control Optim.* **24**, 191–198. (https://doi.org/10.1137/0324011doi: 10.1137/0324011)

[2] Argatov II, Mishuris GS. 2018 *Indentation Testing of Biological Materials*. Switzerland: Springer. See https://doi.org/10.1007/978-3-319-78533-2https://doi.org/10.1007/978-3-319-78533-2.

[3] Bensoussan A, Lions J-L. 1987 *Impulse Control and Quasi Variational Inequalities*. Canada: Wiley.

[4] Ekeland I, Témam R. 1999 *Convex Analysis and Variational Problems*. Philadelphia: SIAM. See https://doi.org/10.1137/1.9781611971088https://doi.org/10.1137/1.9781611971088.

[5] Fellner K, Kovtunenko VA. 2016 A discontinuous Poisson–Boltzmann equation with interfacial transfer: homogenisation and residual error estimate. *Appl. Anal.* **95**, 2661–2682. (https://doi.org/10.1080/00036811.2015.1105962doi: 10.1080/00036811.2015.1105962)

[6] Geymonat G, Suquet P. 1986 Functional spaces for Norton–Hoff materials. *Math. Meth. Appl. Sci.* **8**, 206–222. (https://doi.org/10.1002/mma.1670080113doi: 10.1002/mma.1670080113)

[7] Giuffré S, Maugeri A, Puglisi D. 2015. Lagrange multipliers in elastic–plastic torsion problem for nonlinear monotone operators. *J. Differ. Equ.* **259**, 817–837. (https://doi.org/10.1016/j.jde.2015.02.019doi: 10.1016/j.jde.2015.02.019)

[8] Gwinner J, Jadamba B, Khan AA, Raciti F. 2021 *Uncertainty Quantification in Variational Inequalities*. CRC Press.

[9] Hintermüller M, Kovtunenko VA, Kunisch K. 2005 Generalized Newton methods for crack problems with non-penetration condition, *Numer. Meth. Part. D. E.* **21**, 586–610. (https://doi.org/10.1002/num.20053doi: 10.1002/num.20053)

[10] Ito K, Kunisch K. 2008 *Lagrange Multiplier Approach to Variational Problems and Applications*. Philadelphia: SIAM. See https://doi.org/10.1137/1.9780898718614https://doi.org/10.1137/1.9780898718614.

[11] Itou H, Kovtunenko VA, Rajagopal KR. 2017 Nonlinear elasticity with limiting small strain for cracks subject to non-penetration. *Math. Mech. Solids* **22**, 1334–1346. (https://doi.org/10.1177/1081286516632380doi: 10.1177/1081286516632380)

[12] Itou H, Kovtunenko VA, Rajagopal KR. 2019 Crack problem within the context of implicitly constituted quasi-linear viscoelasticity. *Math. Models Methods Appl. Sci.* **29**, 355–372. (https://doi.org/10.1142/S0218202519500118doi: 10.1142/S0218202519500118)

[13] Itou H, Kovtunenko VA, Rajagopal KR. 2020 The Boussinesq flat-punch indentation problem within the context of linearized viscoelasticity. *Int. J. Eng. Sci.* **151**, 103272. (https://doi.org/10.1016/j.ijengsci.2020.103272doi: 10.1016/j.ijengsci.2020.103272)

[14] Itou H, Kovtunenko VA, Rajagopal KR. 2021 Lagrange multiplier approach to unilateral indentation problems: well-posedness and application. *Math. Models Methods Appl. Sci.* **31**, 649–674. (https://doi.org/10.1142/S0218202521500159doi: 10.1142/S0218202521500159)

[15] Itou H, Kovtunenko VA, Tani A. 2011 The interface crack with Coulomb friction between two bonded dissimilar elastic media. *Appl. Math.* **56**, 69–97. (https://doi.org/10.1007/s10492-011-0010-7doi: 10.1007/s10492-011-0010-7)

[16] Khludnev AM, Fankina IV. 2021 Equilibrium problem for elastic plate with thin rigid inclusion crossing an external boundary. *Z. Angew. Math. Phys.* **72**, 121. (https://doi.org/10.1007/s00033-021-01553-3doi: 10.1007/s00033-021-01553-3)

[17] Khludnev AM, Kovtunenko VA. 2000 *Analysis of Cracks in Solids.* Southampton: WIT Press.

[18] Khludnev AM, Popova TS. 2020 The junction problem for two weakly curved inclusions in an elastic body. *Siberian Math. J.* **61**, 743–754. (https://doi.org/10.1134/S003744662004014Xdoi: 10.1134/S003744662004014X)

[19] Knees D. 2006 Griffith-formula and J-integral for a crack in a power-law hardening material, *Math. Models Methods Appl. Sci.* **16**, 1723–1749. (https://doi.org/10.1142/S0218202506001698doi: 10.1142/S0218202506001698)

[20] Kovtunenko VA. 2001 Sensitivity of cracks in 2D-Lamé problem via material derivatives. *Z. Angew. Math. Phys.* **52**, 1071–1087. (https://doi.org/10.1007/PL00001584doi: 10.1007/PL00001584)

[21] Kovtunenko VA. 2004 Numerical simulation of the non-linear crack problem with non-penetration. *Math. Meth. Appl. Sci.* **27**, 163–179. (https://doi.org/10.1002/mma.449doi: 10.1002/mma.449)

[22] Kravchuk AS, Neittaanmäki PJ. 2010 *Variational and Quasi-Variational Inequalities in Mechanics.* Netherlands: Springer. See https://doi.org/10.1007/978-1-4020-6377-0https://doi.org/10.1007/978-1-4020-6377-0.

[23] Kulvait V, Málek J, Rajagopal KR. 2017 Modeling Gum Metal and other newly developed titanium alloys within a new class of constitutive relations for elastic bodies. *Arch. Mech.* **69**, 223–241. See https://am.ippt.pan.pl/am/article/view/v69p223https://am.ippt.pan.pl/am/article/view/v69p223.

[24] Lazarev NP, Rudoy EM. 2014 Shape sensitivity analysis of Timoshenko's plate with a crack under the nonpenetration condition. *Z. Angew. Math. Mech.* **94**, 730–739. (https://doi.org/10.1002/zamm.201200229doi: 10.1002/zamm.201200229)

[25] Love AEH. 1927 *A Treatise on the Mathematical Theory of Elasticity.* Cambridge University Press.

[26] Migórski S, Khan AA, Zeng S. 2019 Inverse problems for nonlinear quasi-variational inequalities with an application to implicit obstacle problems of p-Laplacian type. *Inverse Probl.* **35**, 035004. (https://iopscience.iop.org/article/10.1088/1361-6420/aafcc9/metadoi: 10.1088/1361-6420/aafcc9/meta)

[27] Rudoy EM, Shcherbakov VV. 2020 First-order shape derivative of the energy for elastic plates with rigid inclusions and interfacial cracks. *Appl. Math. Optim.* **82**. (https://doi.org/10.1007/s00245-020-09729-5doi: 10.1007/s00245-020-09729-5)

[1]Institute for Mathematics and Scientific Computing, University of Graz, NAWI Graz, Heinrichstr.36, 8010 Graz, Austria, [2]Lavrentyev Institute of Hydrodynamics, Siberian Division of the Russian Academy of Sciences, 630090 Novosibirsk, Russia

Email address: victor.kovtunenko@uni-graz.at