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Poroelastic problem of a non-penetrating crack with cohesive contact for fluid-driven fracture

Hiromichi Itou [a], Victor A. Kovtunenko [b,c,*], Nyurgun P. Lazarev [d]

[a] *Department of Mathematics, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo, 162-8601, Japan*
[b] *Institute for Mathematics and Scientific Computing, University of Graz, NAWI Graz, Heinrichstr.36, 8010 Graz, Austria*
[c] *Lavrentyev Institute of Hydrodynamics, Siberian Division of the Russian Academy of Sciences, 630090 Novosibirsk, Russia*
[d] *North-Eastern Federal University, 677000, Kulakovsky str.48, Yakutsk, Russia*

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ABSTRACT

A new class of unilaterally constrained problems for fully coupled poroelastic models stemming from hydraulic fracturing is introduced and studied with respect to its well-posedness. The poroelastic medium contains a fluid-driven crack, which is subjected to non-penetration conditions and cohesion forces between the crack faces. Existence of solution for the governing elliptic–parabolic variational inequality under the unilateral constraint with a small cohesion is established using the incremental approximation based on Rothe's semi-discretization in time.

1. Introduction

Our physical motivation stems from hydrofracking technologies for extracting oil and natural gas from the earth by pumping a fluid through a wellbore into a fracture. The hydraulic fracture is treated mechanically as a single crack separated by two opposite faces and filled with the pumped fluid. The corresponding mathematical model is governed by a time-dependent system of coupled poroelastic equations given in the solid phase and pore space. We introduce and study well-posedness for a class of variational inequalities describing the poroelastic body with a crack, which are subjected to non-penetration conditions and cohesive forces imposed between the crack faces (fracture walls).

The classical theory of poroelasticity was developed in Biot (1956), Coussy (2004), Terzaghi (1943). The implicit model for describing the small strain response of porous elastic solids whose material moduli are dependent on the density was developed in Rajagopal (2021). Unlike Biot's model the above model stems from rigorous representation theorems. For development of a related multi-scale analysis we cite Meirmanov (2014), Sazhenkov et al. (2021). In the formulation (Baykin and Golovin, 2016; Shelukhin et al., 2014; Skopintsev et al., 2020) governing poroelastic equations are coupled with Reynolds lubrication equations for the fluid pressure to a single model. Modeling of the fluid pressure by a Darcy–Forchheimer law was suggested in Kovtunenko (2023), and by a Darcy law in Mikelić et al. (2015), where a phase-field variable was used to represent the crack. In the current modeling, the fluid pressure in the fracture is prescribed by boundary data, which can be achieved either theoretically, or from geomechanical measurement. For physical consistency we allow non-penetration conditions imposed on a crack, that admit a compressive pressure at which the crack might close. The well-posedness for non-penetrating fluid-driven cracks was studied in Kovtunenko (2022). Its shape sensitivity analysis is presented in Kovtunenko and Lazarev (2023) for the incremental formulation of the problem, and the Fourier series analysis is given in Itou et al. (2022) providing formulas for the square-root singularity and stress intensity factors. Here we continue with modeling of cohesive contact at the fracture walls.

The variational theory of solids with non-penetrating cracks and their quasi-static propagation was established in the monograph by Khludnev and Kovtunenko (2000). For a dynamic of cracks see Bratov et al. (2009). The non-penetration approach was extended to nonlinear elastic bodies in Itou et al. (2019, 2021) and viscoelastic bodies in Itou et al. (2020), for cracks subjected to contact with Coulomb friction in Itou et al. (2011), Kovtunenko (2000), cohesion in Kovtunenko (2011), Shcherbakov (2022), and other non-smooth constraints in Knees and Schröder (2012). From optimization viewpoint,

* Corresponding author at: Institute for Mathematics and Scientific Computing, University of Graz, NAWI Graz, Heinrichstr.36, 8010 Graz, Austria.
E-mail addresses: h-itou@rs.tus.ac.jp (H. Itou), victor.kovtunenko@uni-graz.at (V.A. Kovtunenko), nyurgun@ngs.ru (N.P. Lazarev).

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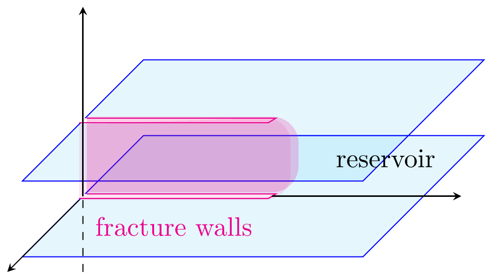

Fig. 1. Plane fracture between two fracture walls.

the non-penetrating cracks are described by variational inequalities if the mechanical energy is constituted by a smooth and convex functional, otherwise, by hemivariational inequalities for non-smooth and non-convex energies. Efficient numerical methods that are suitable for solution of the constrained crack problems can be found in Hintermüller et al. (2005, 2009), Kovtunenko (2004).

Let us consider a two-phase poroelastic material comprising solid particles and fluid-saturated pores. Within the model of linear elasticity, the solid phase is described by the Hookes's constitutive law:

$$\boldsymbol{\sigma} = \mathbf{A}\boldsymbol{\varepsilon} + \boldsymbol{\tau}^0 \tag{1.1}$$

for the linearized strain $\boldsymbol{\varepsilon}$ and Cauchy stress $\boldsymbol{\sigma}$ with a given prestress $\boldsymbol{\tau}^0$ and tensor $\mathbf{A}$ of elastic coefficients. Accounting for the pore pressure p, the effective stress is introduced as

$$\boldsymbol{\tau} = \boldsymbol{\sigma} - \alpha p \mathbf{I}, \tag{1.2}$$

where $\mathbf{I}$ is the identity transformation, and $\alpha \in (0,1]$ is the Biot coefficient. The equilibrium equation reads

$$\operatorname{div} \boldsymbol{\tau} = \mathbf{0} \tag{1.3}$$

when the inertia term and volume forces are omitted.

A Newtonian fluid in the pore space is described by the Fick's diffusion law:

$$\frac{\partial \zeta}{\partial t} = -\operatorname{div} \mathbf{q} \tag{1.4}$$

for the flow content ζ and velocity $\mathbf{q}$. The latter in turn is given by the Darcy law

$$\mathbf{q} = -\kappa \nabla p \tag{1.5}$$

recalling the pore pressure p, where the mobility $\kappa = \varkappa/\eta$ for the permeability $\varkappa > 0$ and viscosity $\eta > 0$. The governing relations for fluid are completed with the constitutive equation for ζ using dilatation $\operatorname{tr}\boldsymbol{\varepsilon}$:

$$\zeta = Sp + \alpha \operatorname{tr}\boldsymbol{\varepsilon}, \tag{1.6}$$

where $S > 0$ is the storativity. Details of the poroelastic modeling can be found in Adachi et al. (2007).

We consider a thin fracture inside the poroelastic media separated by two fracture walls and filled with the same fluid, as illustrated in 3D in Fig. 1. Let the fluid pressure f inside the fracture be prescribed. Further we introduce geometrically and physically consistent conditions suitable at the fracture walls.

In hydrofracking, the physical system is typically controlled by the rate of fluid injected through an inlet. Increasing the injection affects both the larger opening w and growth of fracture. Its decay may lead to shrinking and partial closing the fracture. Then non-penetration between the fracture walls necessitates the non-negative opening:

$$w \geq 0. \tag{1.7}$$

This description allows a compression at which the fracture can be mechanically close at $w = 0$, compared to the hydraulically open fracture when $w > w_c$ with predefined small $w_c > 0$.

From the classical brittle fracture theory, the well-known square-root singularity in the vicinity of the crack tip causes infinite stress that is physically inconsistent. A cohesive zone model suggested by Barenblatt et al. (1960) allows to avoid this theoretical drawback by closing smoothly the crack faces. In the hydraulic fracture modeling the traction–separation law is adopted with a bilinear cohesion force:

$$f_{\rm coh}(w) = f_{\rm M} \begin{cases} 0 & \text{if } w < 0, \\ w/w_{\rm M} & \text{if } 0 \leq w \leq w_{\rm M}, \\ (w_c - w)/(w_c - w_{\rm M}) & \text{if } w_{\rm M} < w \leq w_c, \\ 0 & \text{if } w > w_c, \end{cases} \tag{1.8}$$

where $0 < w_{\rm M} < w_c$ and $f_{\rm M} > 0$, which is depicted in the left plot of Fig. 2. The bilinear function in (1.8) can be derived by differentiation of a non-negative potential function drawn in the right plot:

$$\phi(w) = f_{\rm M} \begin{cases} 0 & \text{if } w < 0, \\ w^2/(2w_{\rm M}) & \text{if } 0 \leq w \leq w_{\rm M}, \\ [w(2w_c - w) - w_c w_{\rm M}]/[2(w_c - w_{\rm M})] & \text{if } w_{\rm M} < w \leq w_c, \\ w_c/2 & \text{if } w > w_c. \end{cases} \tag{1.9}$$

In general, we allow any uniformly continuous function $f_{\rm coh}$ to represent a cohesion force, if it is uniformly bounded from below and above: there exist $F_0 \geq 0$ and $\overline{F} \geq 0$ such that

$$f_{\rm coh}(w)w \geq -F_0|w|, \quad |f_{\rm coh}(w)| \leq \overline{F} \quad \text{for all } w, \tag{1.10}$$

and the following growth condition holds: there exists $\underline{F} \geq 0$ such that

$$\big(f_{\rm coh}(w_1) - f_{\rm coh}(w_2)\big)(w_1 - w_2) \geq -\underline{F}(w_1 - w_2)^2 \quad \text{for all } w_1, w_2. \tag{1.11}$$

For the cohesion in (1.8) the constants are $F_0 = 0$, $\underline{F} = f_{\rm M}/(w_c - w_{\rm M})$ and $\overline{F} = f_{\rm M}$.

Accounting for the normal stress τ_n, fluid pressure f, and cohesion $f_{\rm coh}$, the force balance holds:

$$f_c = \tau_n + f - f_{\rm coh}(w). \tag{1.12}$$

The contact force f_c and opening w should satisfy the complementarity conditions (see Khludnev and Kovtunenko, 2000, Ch.1):

$$w \geq 0, \quad f_c \leq 0, \quad f_c w = 0. \tag{1.13}$$

In particular, for the bilinear cohesion force given in (1.8), from (1.12) and (1.13) we derive three scenarios:

$$\begin{aligned} \text{(i) closed crack:} \quad & \tau_n + f \leq 0 \quad \text{if } w = 0, \\ \text{(ii) open crack with cohesion:} \quad & \tau_n + f - f_{\rm coh}(w) = 0 \quad \text{if } 0 < w \leq w_c, \\ \text{(iii) open crack without cohesion:} \quad & \tau_n + f = 0 \quad \text{if } w > w_c. \end{aligned}$$

This includes as special cases the boundary conditions known from the literature. Indeed, (ii) and (iii) coincide with the cohesive condition for the open crack from Baykin and Golovin (2016); case (iii) at $w_c = 0$ implies the standard linear condition for the open crack from Shelukhin et al. (2014); cases (i) and (iii) at $w_c = 0$ describe cohesionless non-penetration introduced earlier in Kovtunenko (2022).

For analysis of the governing system, it is worth noting that poroelastic equations (see (2.12) and (2.13)) formally coincide with thermoelastic equations when p is replaced for the temperature. From the literature we know that the system of thermoelastic equations is degenerate, due to mixed elliptic–parabolic type. Its solvability is provided in Showalter (2000) by applying the theory of accretive operators for an implicit parabolic equation. However, the parabolic problem does not conform well to unilateral conditions under consideration. Applying the pseudo-monotone theory over a compact feasible set, in Hömberg et al. (2001), Shi and Shillor (1992) the existence result for thermoelastic contact problems is proved under assumption of a

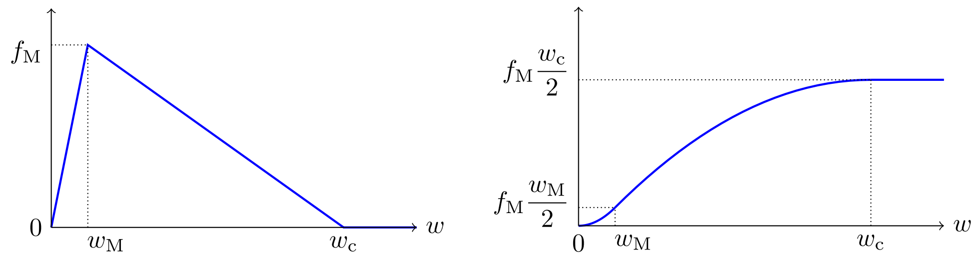

Fig. 2. The bilinear cohesion force f_{coh} (left) and its potential ϕ (right) for $w \geq 0$.

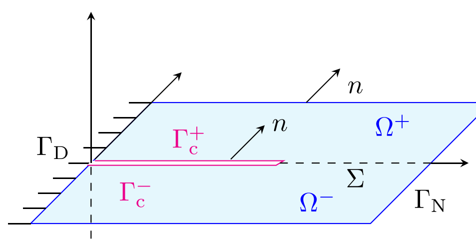

Fig. 3. Plane cross-section of the fracture.

small coupling coefficient α. For arbitrary α, we prove existence of solution for the constrained poroelastic problem using Rothe's method of temporal semi-discretization.

In Section 2 we endow with a variational formulation the poroelastic problem subjected to non-penetration conditions and cohesion forces at the crack faces. In Section 3 we rigorously prove solvability of the incremental approximation, then pass it to the limit in the virtue of uniform a-priori estimates of the incremental solution, which hold under assumption of a small coefficient $\underline{F}$ in the lower bound (1.11) for the cohesion (see the sufficient condition (3.16)).

2. Variational formulation

We start describing geometry of a poroelastic body with a fluid-driven crack in the Euclidean space of points $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, where spatial dimensions $d = 2$ and $d = 3$ are relevant physically.

Let Ω be a domain with the Lipschitz boundary $\partial\Omega$, which has normal vector $\mathbf{n} = (n_1, \dots, n_d)$ directed outward Ω, and consists of two disjoint sets $\partial\Omega = \Gamma_D \cup \Gamma_N$. We assume that a manifold Σ (that is a curve in 2D, and a surface in 3D) splits Ω into two sub-domains $\Omega^\pm$ with Lipschitz boundaries $\partial\Omega^\pm$ such that

$$\partial\Omega^+ \cap \partial\Omega^- = \Sigma, \quad \Omega = \Omega^+ \cup \Omega^- \cup \Sigma, \tag{2.1}$$

where the normal vector $\mathbf{n}$ at Σ is directed outward Ω^-, hence inward Ω^+. Let Γ_c be a part of the interface associated with the crack, and its two faces $\Gamma_c^\pm$ be such that

$$\Gamma_c \subset \Sigma, \quad \Gamma_c^\pm \subset \Sigma^\pm. \tag{2.2}$$

Then a reservoir without crack is associated by the complement set

$$\Omega_c = \Omega \setminus \overline{\Gamma_c}. \tag{2.3}$$

In time $t \in [0, T]$ with a final time $T > 0$, the time–space cylinder is determined as follows:

$$\Omega_c^T = (0,T)\times\Omega_c, \quad \partial\Omega^T = (0,T)\times\partial\Omega, \quad \Gamma_\gamma^T = (0,T)\times\Gamma_\gamma \text{ for } \gamma \in \{c, D, N\}, \tag{2.4}$$

see 2D illustration in Fig. 3.

Next we give a function setting of the problem. We assume a function prescribed in the cylinder Ω_c^T:

$$f(t, \mathbf{x}) \in H^1(0, T; H^1(\Omega_c)) \tag{2.5}$$

conforming the initial and boundary data for the fluid pressure. Let us note that such f can differ on the opposite crack faces and should coincide at the crack tip (the crack front in 3D). For the solid phase, the symmetric tensor of prestress $\boldsymbol{\tau}^0 = (\tau_{ij}^0)_{i,j=1}^d$, vectors of undrained state and boundary force are given by

$$\begin{aligned} &\boldsymbol{\tau}^0(t,\mathbf{x}) \in H^1(0,T; L^2(\Omega_c))^{d\times d}, \quad \mathbf{u}_0 = \big(u_{01}, \dots, u_{0d}\big)(\mathbf{x}) \in L^2(\Omega_c)^d, \\ &\mathbf{g} = (g_1, \dots, g_d)(t,\mathbf{x}) \in H^1(0,T; L^2(\Gamma_N))^d. \end{aligned} \tag{2.6}$$

Let the mobility for pressure allow cross-diffusion described by the tensor of inhomogeneous coefficients $\boldsymbol{\kappa} = (\kappa_{ij})_{i,j=1}^d(\mathbf{x}) \in L^\infty(\Omega_c)^{d\times d}$, which is symmetric: $\kappa_{ij} = \kappa_{ji}$, and uniformly positive definite: there exist $0 < \underline{\kappa} \leq \overline{\kappa}$ such that for all $p, q \in H^1(\Omega_c)$

$$\int_{\Omega_c} \boldsymbol{\kappa}\nabla p \cdot \nabla p \, d\mathbf{x} \geq \underline{\kappa} \|\nabla p\|^2_{L^2(\Omega_c)}, \quad \Big| \int_{\Omega_c} \boldsymbol{\kappa}\nabla p \cdot \nabla q \, d\mathbf{x} \Big| \leq \overline{\kappa} \|\nabla p\|_{L^2(\Omega_c)} \|\nabla q\|_{L^2(\Omega_c)}, \tag{2.7}$$

where "$\cdot$" stands for the scalar product, and using the multiplication of tensors $\boldsymbol{\kappa}\nabla p$. The fourth-order tensor of inhomogeneous elasticity coefficients $\mathbf{A} = (A_{ijkl})_{i,j,k,l=1}^d(\mathbf{x}) \in L^\infty(\Omega_c)^{d\times d\times d\times d}$ is assumed symmetric: $A_{ijkl} = A_{jikl} = A_{klij}$ and uniformly elliptic. Therefore, applying Korn and Poincaré inequalities to the second-order tensor of linearized strain $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1}^d$ given by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\big(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\big), \quad \boldsymbol{\varepsilon}(\mathbf{v}) = \frac{1}{2}\big(\nabla\mathbf{v} + \nabla\mathbf{v}^\top\big), \tag{2.8}$$

where $\top$ stands for transposition, there exist constants $0 < \underline{a} \leq \overline{a}$ such that for all $\mathbf{u}, \mathbf{v} \in H^1(\Omega_c)^d$:

$$\begin{aligned} &\int_{\Omega_c} \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{u}) \, d\mathbf{x} \geq \underline{a} \|\mathbf{u}\|^2_{H^1(\Omega_c)} \text{ if } \mathbf{u} = \mathbf{0} \text{ on } \Gamma_D, \\ &\Big| \int_{\Omega_c} \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, d\mathbf{x} \Big| \leq \overline{a} \|\mathbf{u}\|_{H^1(\Omega_c)} \|\mathbf{v}\|_{H^1(\Omega_c)}. \end{aligned} \tag{2.9}$$

At the boundary, the following trace inequality holds: there exists $K_{tr} > 0$ such that

$$\|\mathbf{u}\|^2_{L^2(\partial\Omega\cup\Gamma_c^+\cup\Gamma_c^-)} \leq K_{tr} \|\mathbf{u}\|^2_{H^1(\Omega_c)}. \tag{2.10}$$

In the geometry (2.1)–(2.4), we look for the unknown displacement and pore pressure in Ω_c^T:

$$\begin{aligned} &\mathbf{u} = (u_1, \dots, u_d)(t,\mathbf{x}) \in H^1(0,T; H^1(\Omega_c))^d, \\ &p(t,\mathbf{x}) \in H^1(0,T; L^2(\Omega_c)) \cap L^2(0,T; H^1(\Omega_c)). \end{aligned} \tag{2.11}$$

The symmetric second-order tensors of stress $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d$ and $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^d$ are defined according to (1.1) and (1.2). The mechanical stress $\boldsymbol{\sigma}(\mathbf{u})$ depends on $\mathbf{u}$ through the linearized strain $\boldsymbol{\varepsilon}(\mathbf{u})$ in (2.8). Avoiding redundant ζ and $\mathbf{q}$, the governing system (1.1)–(1.6) is reduced to the Stokes equation:

$$-\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) + \alpha\nabla p = 0 \quad \text{in } \Omega_c^T, \tag{2.12}$$

and the following mass balance equation:

$$\frac{\partial}{\partial t}\big(Sp + \alpha \operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u})\big) - \operatorname{div}(\boldsymbol{\kappa}\nabla p) = 0 \quad \text{in } \Omega_c^T, \tag{2.13}$$

where the trace $\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u}) = \mathrm{div}\,\mathbf{u}$. They are endowed with the initial condition:

$$\mathbf{u}(0) = \mathbf{u}_0, \quad p(0) = f(0) \quad \text{in } \Omega_c, \tag{2.14}$$

and mixed Dirichlet–Neumann conditions at the boundary of reservoir:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D^T, \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_N^T, \quad p = f \quad \text{on } \partial\Omega^T \tag{2.15}$$

for the data from (2.5) and (2.6).

In view of the presence of a crack, the fields are discontinuous allowing jumps:

$$[\![\mathbf{u}]\!] := \mathbf{u}|_{\Gamma_c^+} - \mathbf{u}|_{\Gamma_c^-}, \quad [\![\boldsymbol{\sigma}(\mathbf{u})]\!] := \boldsymbol{\sigma}(\mathbf{u})|_{\Gamma_c^+} - \boldsymbol{\sigma}(\mathbf{u})|_{\Gamma_c^-}, \quad [\![p]\!] := p|_{\Gamma_c^+} - p|_{\Gamma_c^-}. \tag{2.16}$$

We decompose vectors into the normal and tangential components at the boundary $\partial\Omega \cup \Gamma_c^+ \cup \Gamma_c^-$:

$$\mathbf{u} = u_n\mathbf{n} + \mathbf{u}_\Gamma \text{ for } u_n := \mathbf{u}\cdot\mathbf{n}, \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \sigma_n(\mathbf{u})\mathbf{n} + \boldsymbol{\sigma}_\Gamma(\mathbf{u}) \text{ for } \sigma_n(\mathbf{u}) := \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}\cdot\mathbf{n}, \tag{2.17}$$

such that $\tau_n = \sigma_n(\mathbf{u}) - \alpha p$ in (1.12). At the crack faces, there is no stress in tangential direction:

$$\boldsymbol{\sigma}_\Gamma(\mathbf{u}) = \mathbf{0} \quad \text{on } \Gamma_c^+ \cup \Gamma_c^- \text{ for } t \in (0,T), \tag{2.18}$$

and the pore pressure p is continuous

$$p = f \quad \text{on } \Gamma_c^+ \cup \Gamma_c^- \text{ for } t \in (0,T), \tag{2.19}$$

with the fluid pressure f prescribed in (2.5). In the normal direction, the crack opening $w = [\![u_n]\!]$ and $p = f$ according to (2.19), then from (1.12) and (1.13) we infer the following complementarity conditions:

$$\begin{aligned} &[\![\sigma_n(\mathbf{u}) + (1-\alpha)f]\!] = 0, \quad \sigma_n(\mathbf{u}) + (1-\alpha)f - f_{\mathrm{coh}}\big([\![u_n]\!]\big) \le 0, \\ &[\![u_n]\!] \ge 0, \quad \big[\sigma_n(\mathbf{u}) + (1-\alpha)f - f_{\mathrm{coh}}\big([\![u_n]\!]\big)\big][\![u_n]\!] = 0 \quad \text{on } \Gamma_c^T. \end{aligned} \tag{2.20}$$

For H^1-functions in (2.11), and similarly in (2.5), the jump and the normal stress are defined as

$$[\![u_n]\!], [\![p]\!], [\![f]\!] \in H_{00}^{1/2}(\Gamma_c), \quad \sigma_n(\mathbf{u}) \in H_{00}^{1/2}(\Gamma_c)^\star \tag{2.21}$$

in the Lions–Magenes space $H_{00}^{1/2}(\Gamma_c)$ of functions, which continuation by zero belongs to $H^{1/2}(\Sigma)$, and its adjoint space of linear continuous functionals $H_{00}^{1/2}(\Gamma_c)^\star$. Then relations in (2.20) are well defined in the sense of distributions in the dual spaces, see Khludnev and Kovtunenko (2000, Ch.1) for details.

Now we give a variational formulation to the problem.

Proposition 2.1 (*Variational Problem*). *The poroelastic problem with fluid-driven crack subjected to non-penetration and cohesion conditions consists in finding* $\mathbf{u}$ *and* p *in* (2.11) *which satisfy the initial condition* (2.14)*, equality and inequality constraints from* (2.15)*,* (2.19) *and* (2.20)*:*

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma_D^T, \quad p = f \text{ on } \partial\Omega \cup \Gamma_c^+ \cup \Gamma_c^- \text{ for } t \in (0,T), \quad [\![u_n]\!] \ge 0 \text{ on } \Gamma_c^T, \tag{2.22}$$

at $t \in (0,T)$*. They solve the following variational inequality:*

$$\begin{aligned} &\int_{\Omega_c} \big(\boldsymbol{\sigma}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}) - \alpha p\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\big)\,d\mathbf{x} + \int_{\Gamma_c} f_{\mathrm{coh}}\big([\![u_n]\!]\big)[\![v_n - u_n]\!]\,dS_{\mathbf{x}} \\ &\ge \int_{\Gamma_N} \mathbf{g}\cdot(\mathbf{v}-\mathbf{u})\,dS_{\mathbf{x}} + \int_{\Gamma_c} [\![f(v_n - u_n)]\!]\,dS_{\mathbf{x}} \end{aligned} \tag{2.23}$$

for all test functions $\mathbf{v} \in H^1(\Omega_c)^d$ *with* $\mathbf{v} = \mathbf{0}$ *at* Γ_D *and* $[\![v_n]\!] \ge 0$ *at* Γ_c*, and variational equation:*

$$\int_{\Omega_c} \Big[\frac{\partial}{\partial t}\big(Sp + \alpha\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u})\big)q + \boldsymbol{\kappa}\nabla p\cdot\nabla q\Big]\,d\mathbf{x} = 0 \tag{2.24}$$

for all test functions $q \in H_0^1(\Omega_c)$ *such that* $q = 0$ *at* $\partial\Omega \cup \Gamma_c^+ \cup \Gamma_c^-$.

Proof. For smooth functions $\mathbf{u}$, $\mathbf{v}$ and p, the following Green formula associated to Stokes equation holds:

$$\begin{aligned} &-\int_{\Omega_c} \big(\mathrm{div}\,\boldsymbol{\sigma}(\mathbf{u}) - \alpha\nabla p\big)\cdot\mathbf{v}\,d\mathbf{x} = \int_{\Omega_c} \big(\boldsymbol{\sigma}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{v}) - \alpha p\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{v})\big)\,d\mathbf{x} \\ &-\int_{\Gamma_N} \big(\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n}\big)\cdot\mathbf{v}\,dS_{\mathbf{x}} + \int_{\Gamma_c} [\![\big(\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n}\big)\cdot\mathbf{v}]\!]\,dS_{\mathbf{x}}, \end{aligned} \tag{2.25}$$

if $\mathbf{v} = \mathbf{0}$ on Γ_D. For the transport, Green's formula takes place for all smooth functions p and q:

$$-\int_{\Omega_c} \mathrm{div}(\boldsymbol{\kappa}\nabla p)q\,d\mathbf{x} = \int_{\Omega_c} \boldsymbol{\kappa}\nabla p\cdot\nabla q\,d\mathbf{x}, \quad \text{if } q = 0 \text{ on } \partial\Omega \cup \Gamma_c^+ \cup \Gamma_c^-. \tag{2.26}$$

Since pointwise conditions in (2.20) can be expressed equivalently in the variational form:

$$[\![\sigma_n(\mathbf{u}) + (1-\alpha)f]\!] = 0, \quad [\![u_n]\!] \ge 0, \quad \big[\sigma_n(\mathbf{u}) + (1-\alpha)f - f_{\mathrm{coh}}\big([\![u_n]\!]\big)\big][\![v_n - u_n]\!] \le 0 \tag{2.27}$$

for all $[\![v_n]\!] \ge 0$. Substitution of the equilibrium equation (2.12) and boundary conditions from (2.15), (2.18), (2.19) and (2.27) into Green's formula (2.25) tested with $\mathbf{v}-\mathbf{u}$ yields the variational inequality (2.23). Inserting the mass balance equation (2.13) into (2.26) leads straightforwardly to (2.24).

Conversely, H^2-smooth solution of (2.22)–(2.24) after integration by parts justifies equations and inequalities (2.12)–(2.15) and (2.18)–(2.20). This completes the proof. □

Within the variational theory, it is worth noting the following issue. If there exists a potential ϕ constituting the cohesion force $f_{\mathrm{coh}} = \phi'$, then an energy functional can be introduced by

$$\begin{aligned} \mathcal{E}(\mathbf{u}, p) = &\int_{\Omega_c} \Big(\frac{1}{2}\boldsymbol{\sigma}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{u}) - \alpha p\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u})\Big)\,d\mathbf{x} + \int_{\Gamma_c} \phi\big([\![u_n]\!]\big)\,dS_{\mathbf{x}} \\ &- \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{u}\,dS_{\mathbf{x}} - \int_{\Gamma_c} [\![f u_n]\!]\,dS_{\mathbf{x}} \end{aligned}$$

such that constrained minimization

$$\min_{\mathbf{v}} \mathcal{E}(\mathbf{v}, p) \quad \text{subject to } \mathbf{v} \in H^1(\Omega_c)^d \text{ with } \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \text{ and } [\![v_n]\!] \ge 0 \text{ on } \Gamma_c \tag{2.28}$$

yields the first-order necessary optimality condition (2.23). However, a solution of the variational inequality (2.23) does not provide the minimum in (2.28) because of the lack of convexity of ϕ, e.g. given by (1.9).

3. Existence theory

In order to prove a variational solution for the poroelastic problem with fluid-driven crack subjected to non-penetration and cohesion conditions, we approximate (2.12)–(2.15) by applying Rothe's method of temporal semi-discretization, where the time derivative is approximated by a difference quotient.

For a fixed final time T and integer $N > 0$, let the uniform mesh of size $\delta = T/N > 0$ be given by points:

$$t_0^\delta = 0, \quad t_1^\delta = \delta, \ \ldots, \ t_k^\delta = k\delta, \ \ldots, \ t_N^\delta = N\delta = T. \tag{3.1}$$

The time-continuous functions from (2.5) and (2.6) constitute the sequence of data:

$$\begin{aligned} &f_k^\delta := f(t_k^\delta) \in H^1(\Omega_c) \cap L^2(\Gamma_N \cup \Gamma_c^+ \cup \Gamma_c^-), \quad \boldsymbol{\tau}_k^{0\delta} := \boldsymbol{\tau}^0(t_k^\delta) \in L^2(\Omega_c)^{d\times d}, \\ &\mathbf{g}_k^\delta := \mathbf{g}(t_k^\delta) \in L^2(\Gamma_N)^d \end{aligned} \tag{3.2}$$

for $k = 1, \ldots, N$. Initializing with the initial conditions (2.14), we look for the unknown pore pressure $p_k^\delta(\mathbf{x}) - f_k^\delta(\mathbf{x}) \in H_0^1(\Omega_c)$ and displacement

$\mathbf{u}_k^\delta(\mathbf{x}) \in \mathcal{K}$ from the feasible set

$$\mathcal{K} = \{\mathbf{v} \in H^1(\Omega_c)^d \mid \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, \quad [\![v_n]\!] \geq 0 \text{ on } \Gamma_c\}, \tag{3.3}$$

which solve subsequently the recursive relations for $k = 1, \dots, N$:

$$\int_{\Omega_c} \big(\mathbf{A}\varepsilon(\mathbf{u}_k^\delta) \cdot \varepsilon(\mathbf{v} - \mathbf{u}_k^\delta) - \alpha p_k^\delta \mathrm{tr}\varepsilon(\mathbf{v} - \mathbf{u}_k^\delta)\big)\, d\mathbf{x} + \int_{\Gamma_c} f_{\mathrm{coh}}\big([\![(u_k^\delta)_n]\!]\big)[\![v_n - (u_k^\delta)_n]\!]\, dS_{\mathbf{x}} \geq \int_{\Omega_c} \boldsymbol{\tau}_k^{0\delta} \cdot \varepsilon(\mathbf{v} - \mathbf{u}_k^\delta)\, d\mathbf{x} + \int_{\Gamma_N} \mathbf{g}_k^\delta \cdot (\mathbf{v} - \mathbf{u}_k^\delta)\, dS_{\mathbf{x}} + \int_{\Gamma_c} [\![f_k^\delta\big(v_n - (u_k^\delta)_n\big)]\!]\, dS_{\mathbf{x}}, \tag{3.4}$$

$$\int_{\Omega_c} \big[\big(Sp_k^\delta + \alpha \mathrm{tr}\varepsilon(\mathbf{u}_k^\delta)\big)q + \delta\boldsymbol{\kappa}\nabla p_k^\delta \cdot \nabla q\big]\, d\mathbf{x} = \int_{\Omega_c} \big(Sp_{k-1}^\delta + \alpha \mathrm{tr}\varepsilon(\mathbf{u}_{k-1}^\delta)\big)q\, d\mathbf{x} \tag{3.5}$$

for all test functions $\mathbf{v} \in \mathcal{K}$ and $q \in H_0^1(\Omega_c)$.

Theorem 3.1 (*Solvability of the Incremental Problem*). *Let the uniform conditions* (2.7) and (2.9) *on the coefficients* $\boldsymbol{\kappa}$ *and* $\mathbf{A}$*, and* (1.10) *on the cohesion force* f_{coh} *hold. For every* $k = 1, \dots, N$ *there exists a solution* $(\mathbf{u}_k^\delta, p_k^\delta - f_k^\delta) \in \mathcal{K} \times H_0^1(\Omega_c)$ *to the incremental poroelastic problem* (3.4) and (3.5) *for the fluid-driven crack subject to non-penetration and cohesion. If* f_{coh} *monotonically increases, then it is unique.*

Proof. Summation of (3.4) and (3.5) builds a single variational inequality:

$$a^\delta(\mathbf{u}_k^\delta, p_k^\delta, \mathbf{v} - \mathbf{u}_k^\delta, q) + b(\mathbf{u}_k^\delta, \mathbf{v} - \mathbf{u}_k^\delta) \geq l_k^\delta(\mathbf{v} - \mathbf{u}_k^\delta, q) \quad \text{for all } (\mathbf{v}, q) \in \mathcal{K} \times H_0^1(\Omega_c), \tag{3.6}$$

with a bilinear function:

$$a^\delta(\mathbf{u}, p, \mathbf{v}, q) := \int_{\Omega_c} \big[\mathbf{A}\varepsilon(\mathbf{u}) \cdot \varepsilon(\mathbf{v}) + Spq + \alpha\big(\mathrm{tr}\varepsilon(\mathbf{u})q - p\,\mathrm{tr}\varepsilon(\mathbf{v})\big) + \delta\boldsymbol{\kappa}\nabla p \cdot \nabla q\big]\, d\mathbf{x} \tag{3.7}$$

and a nonlinear bifunction in the left-hand side:

$$b(\mathbf{u}, \mathbf{v}) := \int_{\Gamma_c} f_{\mathrm{coh}}\big([\![u_n]\!]\big)[\![v_n]\!]\, dS_{\mathbf{x}}, \tag{3.8}$$

and a linear function in the right-hand side:

$$l_k^\delta(\mathbf{v}, q) := \int_{\Omega_c} \big[\boldsymbol{\tau}_k^{0\delta} \cdot \varepsilon(\mathbf{v}) + \big(Sp_{k-1}^\delta + \alpha \mathrm{tr}\varepsilon(\mathbf{u}_{k-1}^\delta)\big)q\big]\, d\mathbf{x} + \int_{\Gamma_N} \mathbf{g}_k^\delta \cdot \mathbf{v}\, dS_{\mathbf{x}} + \int_{\Gamma_c} [\![f_k^\delta v_n]\!]\, dS_{\mathbf{x}}. \tag{3.9}$$

For a penalty parameter $\epsilon > 0$ and the bifunction associated with the non-penetration in $\mathcal{K}$:

$$\beta_\epsilon(\mathbf{u}, \mathbf{v}) := \frac{1}{\epsilon} \int_{\Gamma_c} \min\big(0, [\![u_n]\!]\big)[\![v_n]\!]\, dS_{\mathbf{x}} \tag{3.10}$$

such that $\beta_\epsilon(\mathbf{u}, \mathbf{u}) \geq 0$, we introduce a standard penalization of the variational inequality (3.6):

$$a^\delta(\mathbf{u}_k^{\delta\epsilon}, p_k^{\delta\epsilon}, \mathbf{v}, q) + b(\mathbf{u}_k^{\delta\epsilon}, \mathbf{v}) + \beta_\epsilon(\mathbf{u}_k^{\delta\epsilon}, \mathbf{v}) = l_k^\delta(\mathbf{v}, q) \tag{3.11}$$

for all test functions $(\mathbf{v}, q) \in H^1(\Omega_c)^d \times H_0^1(\Omega_c)$ such that $\mathbf{v} = \mathbf{0}$ on Γ_D. The uniform conditions (2.7) and (2.9) on $\boldsymbol{\kappa}$ and $\mathbf{A}$, and (1.10) on f_{coh} guarantee boundedness and coercivity of $[a^\delta + b + \beta_\epsilon]$ in the left-hand side of Eq. (3.11), noting that the term $\alpha(\mathrm{tr}\varepsilon(\mathbf{u})q - p\,\mathrm{tr}\varepsilon(\mathbf{v})) = 0$ in (3.7) for $(\mathbf{u}, p) = (\mathbf{v}, q)$. Moreover, the nonlinear bifunction $[b + \beta_\epsilon]$ is weakly continuous in the following sense: if $\mathbf{u}^m \rightharpoonup \mathbf{u}$ weakly in $H^1(\Omega_c)^d$ as $m \to \infty$, then $[\![u_n^m]\!] \to [\![u_n]\!]$ strongly in $L^2(\Gamma_c)$ by the compact embedding, and uniformly continuous functions preserve the strong convergence:

$$[b + \beta_\epsilon](\mathbf{u}^m, \mathbf{v}) \to [b + \beta_\epsilon](\mathbf{u}, \mathbf{v}) \quad \text{as } m \to \infty. \tag{3.12}$$

Therefore, Galerkin's approximation and Brouwer's fixed point theorem proves existence of a solution $(\mathbf{u}_k^{\delta\epsilon}, p_k^{\delta\epsilon})$ to the penalty equation (3.11). Taking the limit as $\epsilon \to 0$ in the standard way, and using the weak continuity

$$b(\mathbf{u}_k^{\delta\epsilon}, \mathbf{u}_k^{\delta\epsilon}) \to b(\mathbf{u}_k^\delta, \mathbf{u}_k^\delta) \quad \text{as } \mathbf{u}_k^{\delta\epsilon} \rightharpoonup \mathbf{u}_k^\delta \text{ weakly in } H^1(\Omega_c)^d, \tag{3.13}$$

it justifies that $(\mathbf{u}_k^\delta, p_k^\delta - f_k^\delta) \in \mathcal{K} \times H_0^1(\Omega_c)$ is a solution to the variational inequality (3.6).

Testing (3.6) with $q = 0$ implies the variational inequality (3.4), next choosing $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = 2\mathbf{u}_k^\delta$ yields the variational equation (3.5).

The uniqueness result is well-known under the monotony assumption. The proof is completed. □

Next we derive uniform estimates for the solution of the incremental problem, which are independent of the time step δ. For this task we introduce a piecewise-affine interpolant for the semi-discrete functions:

$$\mathbf{u}^\delta(t) = \frac{t - t_{k-1}^\delta}{\delta}\mathbf{u}_k^\delta + \frac{t_k^\delta - t}{\delta}\mathbf{u}_{k-1}^\delta, \quad p^\delta(t) = \frac{t - t_{k-1}^\delta}{\delta}p_k^\delta + \frac{t_k^\delta - t}{\delta}p_{k-1}^\delta \tag{3.14}$$

at $t \in (t_{k-1}^\delta, t_k^\delta] =: I_k^\delta$ for $k = 1, \dots, N$, which have piecewise-constant time derivatives:

$$\frac{\partial \mathbf{u}^\delta}{\partial t}(t) = \frac{\mathbf{u}_k^\delta - \mathbf{u}_{k-1}^\delta}{\delta}, \quad \frac{\partial p^\delta}{\partial t}(t) = \frac{p_k^\delta - p_{k-1}^\delta}{\delta} \quad \text{at } t \in I_k^\delta. \tag{3.15}$$

The interpolants $f^\delta(t)$, $\boldsymbol{\tau}^{0\delta}(t)$, $\mathbf{g}^\delta(t)$ are similarly defined from (3.2).

Theorem 3.2 (*Uniform Estimate of the Incremental Solution*). *Let the growth condition* (1.11) *on the cohesion force* f_{coh} *hold with a sufficiently small lower bound* $\underline{F}$ *such that*

$$K := \frac{\underline{a}}{5} - \underline{F}K_{\mathrm{tr}} > 0. \tag{3.16}$$

The solution $(\mathbf{u}^\delta, p^\delta - f^\delta)$ *of the incremental poroelastic problem for the fluid-driven crack* (3.4) and (3.5)*, after interpolation* (3.14) and (3.15)*, possesses the following estimates:*

$$\begin{aligned} &\underline{a}\|\mathbf{u}^\delta\|^2_{L^\infty(0,T;H^1(\Omega_c))} + S\|p^\delta\|^2_{L^\infty(0,T;L^2(\Omega_c))} + \underline{\kappa}\|\nabla p^\delta\|^2_{L^2(\Omega_c^T)} \\ &\leq \overline{a}\|\mathbf{u}_0\|^2_{H^1(\Omega_c)} + S\|f(0)\|^2_{L^2(\Omega_c)} + \frac{\overline{\kappa}\delta}{2}\|\nabla f(0)\|^2_{L^2(\Omega_c)} \\ &+ T\overline{F} + 3\|P^\delta\|_{L^2(0,T)} - \delta P^\delta(0) + \|Q^\delta\|_{L^2(0,T)}, \\ &K\left\|\frac{\partial \mathbf{u}^\delta}{\partial t}\right\|^2_{L^2(0,T;H^1(\Omega_c))} + S\left\|\frac{\partial p^\delta}{\partial t}\right\|^2_{L^2(\Omega_c^T)} \leq \frac{\overline{\kappa}}{2}\|\nabla f(0)\|^2_{L^2(\Omega_c)} + \|R^\delta\|_{L^2(0,T)}, \end{aligned} \tag{3.17}$$

where the constants are taken from conditions (1.10) and (1.11) *on* f_{coh}, (2.7) *on* $\boldsymbol{\kappa}$, (2.9) *on* $\mathbf{A}$*, and* (2.10) *on the boundary trace, and the data are gathered within*

$$\begin{aligned} P^\delta &:= \|\boldsymbol{\tau}^{0\delta}\|^2_{L^2(\Omega_c)} + (\alpha + S)\|f^\delta\|^2_{L^2(\Omega_c)} + (\overline{\kappa} + K_{\mathrm{tr}})\|f^\delta\|^2_{H^1(\Omega_c)} + \|\mathbf{g}^\delta\|^2_{L^2(\Gamma_N)}, \\ Q^\delta &:= \big(1 + \alpha d + (2 + \overline{F})K_{\mathrm{tr}}\big)\left\|\frac{\partial \mathbf{u}^\delta}{\partial t}\right\|^2_{H^1(\Omega_c)} + S\left\|\frac{\partial p^\delta}{\partial t}\right\|^2_{L^2(\Omega_c)}, \\ R^\delta &:= \frac{5}{4\underline{a}}\left\{\left\|\frac{\partial \boldsymbol{\tau}^{0\delta}}{\partial t}\right\|^2_{L^2(\Omega_c)} + K_{\mathrm{tr}}\left\|\frac{\partial \mathbf{g}^\delta}{\partial t}\right\|^2_{L^2(\Gamma_N)} + K_{\mathrm{tr}}\left\|\frac{\partial f^\delta}{\partial t}\right\|^2_{L^2(\Gamma_c^+\cup\Gamma_c^-)} + \alpha^2 d\left\|\frac{\partial f^\delta}{\partial t}\right\|^2_{L^2(\Omega_c)}\right\} \\ &+ \frac{\overline{\kappa}T}{2}\left\|\frac{\partial f^\delta}{\partial t}\right\|^2_{H^1(\Omega_c)}. \end{aligned} \tag{3.18}$$

Proof. *(i) Uniform estimate of interpolates in* (3.14). Let us test (3.5) with $q = p_k^\delta - f_k^\delta \in H_0^1(\Omega_c)$:

$$\int_{\Omega_c} \big[\big(Sp_k^\delta + \alpha \mathrm{tr}\varepsilon(\mathbf{u}_k^\delta)\big)(p_k^\delta - f_k^\delta) + \delta\boldsymbol{\kappa}\nabla p_k^\delta \cdot \nabla(p_k^\delta - f_k^\delta)\big]\, d\mathbf{x} = \int_{\Omega_c} \big(Sp_{k-1}^\delta + \alpha \mathrm{tr}\varepsilon(\mathbf{u}_{k-1}^\delta)\big)(p_k^\delta - f_k^\delta)\, d\mathbf{x}$$

and subtract (3.4) tested with $\mathbf{v} = \mathbf{u}_{k-1}^{\delta} \in \mathcal{K}$:

$$\int_{\Omega_c} \left(\mathbf{A}\varepsilon(\mathbf{u}_k^{\delta}) \cdot \varepsilon(\mathbf{u}_{k-1}^{\delta} - \mathbf{u}_k^{\delta}) - \alpha p_k^{\delta} \mathrm{tr}\varepsilon(\mathbf{u}_{k-1}^{\delta} - \mathbf{u}_k^{\delta})\right) d\mathbf{x}$$
$$+ \int_{\Gamma_c} f_{\mathrm{coh}}\left([\![(u_k^{\delta})_n]\!]\right)[\![(u_{k-1}^{\delta})_n - (u_k^{\delta})_n]\!]\, dS_{\mathbf{x}}$$
$$\geq \int_{\Omega_c} \boldsymbol{\tau}_k^{0\delta} \cdot \varepsilon(\mathbf{u}_{k-1}^{\delta} - \mathbf{u}_k^{\delta})\, d\mathbf{x}$$
$$+ \int_{\Gamma_N} \mathbf{g}_k^{\delta} \cdot (\mathbf{u}_{k-1}^{\delta} - \mathbf{u}_k^{\delta})\, dS_{\mathbf{x}} + \int_{\Gamma_c} [\![f_k^{\delta}\left((u_{k-1}^{\delta})_n - (u_k^{\delta})_n\right)]\!]\, dS_{\mathbf{x}}, \tag{3.19}$$

such that the term $\alpha p_k^{\delta} \mathrm{tr}\varepsilon(\mathbf{u}_{k-1}^{\delta} - \mathbf{u}_k^{\delta})$ is canceled, then we have

$$\int_{\Omega_c} \left(\mathbf{A}\varepsilon(\mathbf{u}_k^{\delta}) \cdot \varepsilon(\mathbf{u}_k^{\delta}) + S(p_k^{\delta})^2 + \delta\boldsymbol{\kappa}\nabla p_k^{\delta} \cdot \nabla p_k^{\delta}\right) d\mathbf{x} \leq \sum_{l=1}^{5} I_l \tag{3.20}$$

at $t \in I_k^{\delta}$ for $k = 1, \dots, N$, where the integrals are given by

$$I_1 := \int_{\Omega_c} \left(\mathbf{A}\varepsilon(\mathbf{u}_k^{\delta}) \cdot \varepsilon(\mathbf{u}_{k-1}^{\delta}) + \boldsymbol{\tau}_k^{0\delta} \cdot \varepsilon(\mathbf{u}_k^{\delta} - \mathbf{u}_{k-1}^{\delta}) + \alpha \mathrm{tr}\varepsilon(\mathbf{u}_k^{\delta} - \mathbf{u}_{k-1}^{\delta}) f_k^{\delta}\right) d\mathbf{x},$$
$$I_2 := \int_{\Omega_c} \left(S p_k^{\delta} p_{k-1}^{\delta} + S(p_k^{\delta} - p_{k-1}^{\delta}) f_k^{\delta} + \delta\boldsymbol{\kappa}\nabla p_k^{\delta} \cdot \nabla f_k^{\delta}\right) d\mathbf{x},$$
$$I_3 := \int_{\Gamma_N} \mathbf{g}_k^{\delta} \cdot (\mathbf{u}_k^{\delta} - \mathbf{u}_{k-1}^{\delta})\, dS_{\mathbf{x}},$$
$$I_4 := \int_{\Gamma_c} [\![f_k^{\delta}\left((u_k^{\delta})_n - (u_{k-1}^{\delta})_n\right)]\!]\, dS_{\mathbf{x}},$$
$$I_5 := \int_{\Gamma_c} f_{\mathrm{coh}}\left([\![(u_k^{\delta})_n]\!]\right)[\![(u_{k-1}^{\delta})_n - (u_k^{\delta})_n]\!]\, dS_{\mathbf{x}}.$$

Using the symmetry of $\mathbf{A}$ and $\boldsymbol{\kappa}$, notation (3.15), $\mathrm{tr}^2\varepsilon(\mathbf{u}) \leq d\|\varepsilon(\mathbf{u})\|^2$, Cauchy–Schwarz, weighted Young and trace (2.10) inequalities, the upper bounds in (1.10), (2.7) and (2.9) provide estimates of I_1–I_5:

$$|I_1| \leq \frac{1}{2}\int_{\Omega_c} \left(\mathbf{A}\varepsilon(\mathbf{u}_k^{\delta}) \cdot \varepsilon(\mathbf{u}_k^{\delta}) + \mathbf{A}\varepsilon(\mathbf{u}_{k-1}^{\delta}) \cdot \varepsilon(\mathbf{u}_{k-1}^{\delta})\right) d\mathbf{x}$$
$$+ \frac{\delta}{2}\|\boldsymbol{\tau}_k^{0\delta}\|_{L^2(\Omega_c)}^2 + \frac{\delta}{2}(1+\alpha d)\left\|\frac{\partial \mathbf{u}^{\delta}}{\partial t}\Big|_{I_k^{\delta}}\right\|_{H^1(\Omega_c)}^2 + \frac{\alpha\delta}{2}\|f_k^{\delta}\|_{L^2(\Omega_c)}^2,$$

$$|I_2| \leq \frac{S}{2}\left(\|p_k^{\delta}\|_{L^2(\Omega_c)}^2 + \|p_{k-1}^{\delta}\|_{L^2(\Omega_c)}^2\right) + \frac{S\delta}{2}\left(\left\|\frac{\partial p^{\delta}}{\partial t}\Big|_{I_k^{\delta}}\right\|_{L^2(\Omega_c)}^2 + \|f_k^{\delta}\|_{L^2(\Omega_c)}^2\right)$$
$$+ \frac{\delta}{2}\int_{\Omega_c} \left(\boldsymbol{\kappa}\nabla p_k^{\delta} \cdot \nabla p_k^{\delta} + \boldsymbol{\kappa}\nabla f_k^{\delta} \cdot \nabla f_k^{\delta}\right) d\mathbf{x},$$

$$|I_3| \leq \|\mathbf{g}_k^{\delta}\|_{L^2(\Gamma_N)}\left\|\delta\frac{\partial \mathbf{u}^{\delta}}{\partial t}\Big|_{I_k^{\delta}}\right\|_{L^2(\Gamma_N)} \leq \frac{\delta}{2}\|\mathbf{g}_k^{\delta}\|_{L^2(\Gamma_N)}^2 + \frac{\delta K_{\mathrm{tr}}}{2}\left\|\frac{\partial \mathbf{u}^{\delta}}{\partial t}\Big|_{I_k^{\delta}}\right\|_{H^1(\Omega_c)}^2,$$
$$|I_4| \leq \|f_k^{\delta}\|_{L^2(\Gamma_c^+\cup\Gamma_c^-)}\left\|\delta\frac{\partial \mathbf{u}^{\delta}}{\partial t}\Big|_{I_k^{\delta}}\right\|_{L^2(\Gamma_c^+\cup\Gamma_c^-)}$$
$$\leq \frac{\delta K_{\mathrm{tr}}}{2}\left(\|f_k^{\delta}\|_{H^1(\Omega_c)}^2 + \left\|\frac{\partial \mathbf{u}^{\delta}}{\partial t}\Big|_{I_k^{\delta}}\right\|_{H^1(\Omega_c)}^2\right),$$
$$|I_5| \leq \overline{F}\left\|\left[\!\!\left[\delta\left(\frac{\partial \mathbf{u}^{\delta}}{\partial t}\right)_n\Big|_{I_k^{\delta}}\right]\!\!\right]\right\|_{L^2(\Gamma_c)} \leq \frac{\delta\overline{F}}{2}\left(1 + K_{\mathrm{tr}}\left\|\frac{\partial \mathbf{u}^{\delta}}{\partial t}\Big|_{I_k^{\delta}}\right\|_{H^1(\Omega_c)}^2\right).$$

Inserting these estimates into (3.20), using the upper bound for $\boldsymbol{\kappa}$ in (2.7) and gathering the same terms, the result multiplied by 2 yields

$$\int_{\Omega_c} \mathbf{A}\varepsilon(\mathbf{u}_k^{\delta}) \cdot \varepsilon(\mathbf{u}_k^{\delta})\, d\mathbf{x} + S\|p_k^{\delta}\|_{L^2(\Omega_c)}^2 + \delta\int_{\Omega_c} \boldsymbol{\kappa}\nabla p_k^{\delta} \cdot \nabla p_k^{\delta}\, d\mathbf{x}$$
$$\leq \int_{\Omega_c} \mathbf{A}\varepsilon(\mathbf{u}_{k-1}^{\delta}) \cdot \varepsilon(\mathbf{u}_{k-1}^{\delta})\, d\mathbf{x} + S\|p_{k-1}^{\delta}\|_{L^2(\Omega_c)}^2 + \delta(\overline{F} + P_k^{\delta} + Q^{\delta}|_{I_k^{\delta}}), \tag{3.21}$$

where the notation Q^{δ} in the right-hand side is given in (3.18), and

$$P_k^{\delta} := \|\boldsymbol{\tau}_k^{0\delta}\|_{L^2(\Omega_c)}^2 + (\alpha + S)\|f_k^{\delta}\|_{L^2(\Omega_c)}^2 + (\overline{\kappa} + K_{\mathrm{tr}})\|f_k^{\delta}\|_{H^1(\Omega_c)}^2 + \|\mathbf{g}_k^{\delta}\|_{L^2(\Gamma_N)}^2.$$

Summing up (3.21) over $k = 1, \dots, m$ and using the telescope rule, we have

$$\int_{\Omega_c} \mathbf{A}\varepsilon(\mathbf{u}_m^{\delta}) \cdot \varepsilon(\mathbf{u}_m^{\delta})\, d\mathbf{x} + S\|p_m^{\delta}\|_{L^2(\Omega_c)}^2 + \delta\sum_{k=1}^{m}\int_{\Omega_c} \boldsymbol{\kappa}\nabla p_k^{\delta} \cdot \nabla p_k^{\delta}\, d\mathbf{x}$$
$$\leq \int_{\Omega_c} \mathbf{A}\varepsilon(\mathbf{u}_0^{\delta}) \cdot \varepsilon(\mathbf{u}_0^{\delta})\, d\mathbf{x} + S\|p_0^{\delta}\|_{L^2(\Omega_c)}^2 + \delta\sum_{k=1}^{m}(\overline{F} + P_k^{\delta} + Q^{\delta}|_{I_k^{\delta}}). \tag{3.22}$$

The left-hand side of (3.22) is estimated due to the lower bounds in (2.7) for $\boldsymbol{\kappa}$ and (2.9) for $\mathbf{A}$. Taking maximum over $m \in [1, N]$ in (3.22), recalling the initial condition (2.14) and the upper bound for $\mathbf{A}$ in (2.9), it follows the estimate:

$$\underline{a}\max_{k\in[1,N]}\|\mathbf{u}_k^{\delta}\|_{H^1(\Omega_c)}^2 + S\max_{k\in[1,N]}\|p_k^{\delta}\|_{L^2(\Omega_c)}^2 + \underline{\kappa}\delta\sum_{k=1}^{N}\|\nabla p_k^{\delta}\|_{L^2(\Omega_c)}^2$$
$$\leq \overline{a}\|\mathbf{u}_0\|_{H^1(\Omega_c)}^2 + S\|f(0)\|_{L^2(\Omega_c)}^2 + \delta N\overline{F} + \delta\sum_{k=1}^{N}(P_k^{\delta} + Q^{\delta}|_{I_k^{\delta}}). \tag{3.23}$$

Since Kepler's integration rule applied to the piecewise-affine interpolant from (3.14) gives the following form

$$\int_0^T \|\mathbf{u}^{\delta}\|_{H^1(\Omega_c)}^2\, dt = \sum_{k=1}^{N}\int_{t_{k-1}^{\delta}}^{t_k^{\delta}} \|\mathbf{u}^{\delta}\|_{H^1(\Omega_c)}^2\, dt$$
$$= \frac{\delta}{3}\sum_{k=1}^{N}\left(\|\mathbf{u}_k^{\delta}\|_{H^1(\Omega_c)}^2 + \|\mathbf{u}_{k-1}^{\delta}\|_{H^1(\Omega_c)}^2 + \|\mathbf{u}_k^{\delta}\|_{H^1(\Omega_c)}\|\mathbf{u}_{k-1}^{\delta}\|_{H^1(\Omega_c)}\right),$$

by use of algebraic inequality $2xy \leq x^2 + y^2$ we obtain

$$\frac{\delta}{3}\left(\|\mathbf{u}_0^{\delta}\|_{H^1(\Omega_c)}^2 + 2\sum_{k=1}^{N-1}\|\mathbf{u}_k^{\delta}\|_{H^1(\Omega_c)}^2 + \|\mathbf{u}_N^{\delta}\|_{H^1(\Omega_c)}^2\right) \leq \int_0^T \|\mathbf{u}^{\delta}\|_{H^1(\Omega_c)}^2\, dt$$
$$\leq \frac{\delta}{2}\left(\|\mathbf{u}_0^{\delta}\|_{H^1(\Omega_c)}^2 + 2\sum_{k=1}^{N-1}\|\mathbf{u}_k^{\delta}\|_{H^1(\Omega_c)}^2 + \|\mathbf{u}_N^{\delta}\|_{H^1(\Omega_c)}^2\right).$$

This provides the lower and upper bounds for the sum:

$$\int_0^T \|\mathbf{u}^{\delta}\|_{H^1(\Omega_c)}^2\, dt - \frac{\delta}{2}\|\mathbf{u}_0^{\delta}\|_{H^1(\Omega_c)}^2 \leq \delta\sum_{k=1}^{N}\|\mathbf{u}_k^{\delta}\|_{H^1(\Omega_c)}^2$$
$$\leq 3\int_0^T \|\mathbf{u}^{\delta}\|_{H^1(\Omega_c)}^2\, dt - \delta\|\mathbf{u}_0^{\delta}\|_{H^1(\Omega_c)}^2.$$

With its help, the estimate (3.23) can be rewritten for the time-dependent interpolant functions:

$$\underline{a}\max_{t\in[0,T]}\|\mathbf{u}^{\delta}\|_{H^1(\Omega_c)}^2 + S\max_{t\in[0,T]}\|p^{\delta}\|_{L^2(\Omega_c)}^2 + \underline{\kappa}\int_0^T \|\nabla p^{\delta}\|_{L^2(\Omega_c)}^2\, dt$$
$$\leq \overline{a}\|\mathbf{u}_0\|_{H^1(\Omega_c)}^2 + S\|f(0)\|_{L^2(\Omega_c)}^2 + \frac{\overline{\kappa}\delta}{2}\|\nabla f_0^{\delta}\|_{L^2(\Omega_c)}^2$$
$$+ T\overline{F} + 3\int_0^T P^{\delta}\, dt - \delta P^{\delta}(0) + \int_0^T Q^{\delta}\, dt, \tag{3.24}$$

where $\delta N = T$, piecewise-affine P^{δ} and piecewise-constant Q^{δ} are defined in (3.18).

(ii) Uniform estimate of interpolates in (3.15). Summation of the variational inequality (3.4) at $t = t_{k-1}^{\delta}$ gives

$$\int_{\Omega_c} \left(\mathbf{A}\varepsilon(\mathbf{u}_{k-1}^{\delta}) \cdot \varepsilon(\mathbf{v} - \mathbf{u}_{k-1}^{\delta}) - \alpha p_{k-1}^{\delta}\mathrm{tr}\varepsilon(\mathbf{v} - \mathbf{u}_{k-1}^{\delta})\right) d\mathbf{x}$$
$$+ \int_{\Gamma_c} f_{\mathrm{coh}}\left([\![(u_{k-1}^{\delta})_n]\!]\right)[\![v_n - (u_{k-1}^{\delta})_n]\!]\, dS_{\mathbf{x}}$$
$$\geq \int_{\Omega_c} \boldsymbol{\tau}_{k-1}^{0\delta} \cdot \varepsilon(\mathbf{v} - \mathbf{u}_{k-1}^{\delta})\, d\mathbf{x} + \int_{\Gamma_N} \mathbf{g}_{k-1}^{\delta} \cdot (\mathbf{v} - \mathbf{u}_{k-1}^{\delta})\, dS_{\mathbf{x}}$$
$$+ \int_{\Gamma_c} [\![f_{k-1}^{\delta}\left(v_n - (u_{k-1}^{\delta})_n\right)]\!]\, dS_{\mathbf{x}}. \tag{3.25}$$

Taking $\mathbf{v} = \mathbf{u}_k^{\delta} \in \mathcal{K}$ in (3.25) and using (3.19), after division by δ^2 it yields the inequality:

$$\int_{\Omega_c} \left\{\mathbf{A}\varepsilon\left(\frac{\partial \mathbf{u}^{\delta}}{\partial t}\right) \cdot \varepsilon\left(\frac{\partial \mathbf{u}^{\delta}}{\partial t}\right) - \alpha\frac{\partial p^{\delta}}{\partial t}\mathrm{tr}\varepsilon\left(\frac{\partial \mathbf{u}^{\delta}}{\partial t}\right)\right\}\Big|_{I_k^{\delta}} d\mathbf{x}$$
$$+ \int_{\Gamma_c} \frac{f_{\mathrm{coh}}\left([\![(u_k^{\delta})_n]\!]\right) - f_{\mathrm{coh}}\left([\![(u_{k-1}^{\delta})_n]\!]\right)}{\delta}\left[\!\!\left[\left(\frac{\partial \mathbf{u}^{\delta}}{\partial t}\right)_n\Big|_{I_k^{\delta}}\right]\!\!\right] dS_{\mathbf{x}} \leq \sum_{l=6}^{8} I_l|_{I_k^{\delta}} \tag{3.26}$$

at $t \in I_k^\delta$ for $k = 1, \dots, N$, where the integrals I_6–I_8 are determined as

$$I_6 := \int_{\Omega_c} \frac{\partial \boldsymbol{\tau}^{0\delta}}{\partial t} \cdot \boldsymbol{\varepsilon}\Big(\frac{\partial \mathbf{u}^\delta}{\partial t}\Big)\, d\mathbf{x}, \quad I_7 := \int_{\Gamma_N} \frac{\partial \mathbf{g}^\delta}{\partial t} \cdot \frac{\partial \mathbf{u}^\delta}{\partial t}\, dS_{\mathbf{x}},$$
$$I_8 := \int_{\Gamma_c} \Big[\!\Big[\frac{\partial f^\delta}{\partial t} \frac{\partial \mathbf{u}^\delta}{\partial t} \Big]\!\Big] \cdot \mathbf{n}\, dS_{\mathbf{x}}.$$

Testing the variational equation (3.5) with $q = p_k^\delta - p_{k-1}^\delta - f_k^\delta + f_{k-1}^\delta \in H_0^1(\Omega_c)$ and dividing by δ^2 we get

$$\int_{\Omega_c} \Big\{ \Big[S\frac{\partial p^\delta}{\partial t} + \alpha \frac{\partial p^\delta}{\partial t} \mathrm{tr}\boldsymbol{\varepsilon}\Big(\frac{\partial \mathbf{u}^\delta}{\partial t}\Big)\Big] \frac{\partial (p^\delta - f^\delta)}{\partial t} + \boldsymbol{\kappa} \nabla p_k^\delta \cdot \nabla \frac{\partial (p^\delta - f^\delta)}{\partial t} \Big\}\Big|_{I_k^\delta} d\mathbf{x} = 0 \tag{3.27}$$

at $t \in I_k^\delta$ for $k = 1, \dots, N$. In the sum of (3.26) and (3.27) the term $\alpha \partial p^\delta / \partial t\, \mathrm{tr}\boldsymbol{\varepsilon}(\partial \mathbf{u}^\delta / \partial t)$ is canceled, then lower bounds in (1.11) for f_{coh} and (2.9) for $\mathbf{A}$ provide

$$\begin{aligned} &\underline{a} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t}\Big|_{I_k^\delta} \Big\|_{H^1(\Omega_c)}^2 + S \Big\| \frac{\partial p^\delta}{\partial t}\Big|_{I_k^\delta} \Big\|_{L^2(\Omega_c)}^2 + \frac{1}{\delta} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_k^\delta \cdot \nabla p_k^\delta\, d\mathbf{x} \\ &- \underline{F} \Big\| \frac{\partial [\![u_n^\delta]\!]}{\partial t}\Big|_{I_k^\delta} \Big\|_{L^2(\Gamma_c)}^2 \le \sum_{l=6}^{9} I_l|_{I_k^\delta} + I_{10}, \end{aligned} \tag{3.28}$$

where the integrals I_9 and I_{10} are given by

$$I_9 := \int_{\Omega_c} \alpha \mathrm{tr}\boldsymbol{\varepsilon}\Big(\frac{\partial \mathbf{u}^\delta}{\partial t}\Big) \frac{\partial f^\delta}{\partial t}\, d\mathbf{x},$$
$$I_{10} := \int_{\Omega_c} \Big(\frac{1}{\delta} \boldsymbol{\kappa} \nabla p_k^\delta \cdot \nabla p_{k-1}^\delta + \boldsymbol{\kappa} \nabla p_k^\delta \cdot \nabla \frac{\partial f^\delta}{\partial t}\Big|_{I_k^\delta} \Big) d\mathbf{x}.$$

Applying Cauchy–Schwarz and weighted Young inequalities, using $\mathrm{tr}^2 \boldsymbol{\varepsilon}(\mathbf{u}) \le d \|\boldsymbol{\varepsilon}(\mathbf{u})\|^2$, it follows from the upper bounds in (2.7), (2.9), and (2.10) that

$$|I_6| \le \Big\| \frac{\partial \boldsymbol{\tau}^{0\delta}}{\partial t} \Big\|_{L^2(\Omega_c)} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{H^1(\Omega_c)} \le \frac{\underline{a}}{5} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{H^1(\Omega_c)}^2 + \frac{5}{4\underline{a}} \Big\| \frac{\partial \boldsymbol{\tau}^{0\delta}}{\partial t} \Big\|_{L^2(\Omega_c)}^2,$$

$$|I_7| \le \Big\| \frac{\partial \mathbf{g}^\delta}{\partial t} \Big\|_{L^2(\Gamma_N)} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{L^2(\Gamma_N)} \le \frac{\underline{a}}{5} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{H^1(\Omega_c)}^2 + \frac{5K_{\mathrm{tr}}}{4\underline{a}} \Big\| \frac{\partial \mathbf{g}^\delta}{\partial t} \Big\|_{L^2(\Gamma_N)}^2,$$

$$\begin{aligned} |I_8| &\le \Big\| \frac{\partial f^\delta}{\partial t} \Big\|_{L^2(\Gamma_c^+ \cup \Gamma_c^-)} \Big\| \frac{\partial u_n^\delta}{\partial t} \Big\|_{L^2(\Gamma_c^+ \cup \Gamma_c^-)} \\ &\le \frac{\underline{a}}{5} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{H^1(\Omega_c)}^2 + \frac{5K_{\mathrm{tr}}}{4\underline{a}} \Big\| \frac{\partial f^\delta}{\partial t} \Big\|_{L^2(\Gamma_c^+ \cup \Gamma_c^-)}^2, \end{aligned}$$

$$|I_9| \le \alpha \sqrt{d} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{H^1(\Omega_c)} \Big\| \frac{\partial f^\delta}{\partial t} \Big\|_{L^2(\Omega_c)} \le \frac{\underline{a}}{5} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{H^1(\Omega_c)}^2 + \frac{5\alpha^2 d}{4\underline{a}} \Big\| \frac{\partial f^\delta}{\partial t} \Big\|_{L^2(\Omega_c)}^2,$$

$$\begin{aligned} |I_{10}| &\le \frac{1}{2\delta} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_k^\delta \cdot \nabla p_k^\delta\, d\mathbf{x} + \frac{1}{2\delta} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_{k-1}^\delta \cdot \nabla p_{k-1}^\delta\, d\mathbf{x} \\ &+ \frac{1}{2T} \max_{k \in [1,N]} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_k^\delta \cdot \nabla p_k^\delta\, d\mathbf{x} + \frac{T}{2} \int_{\Omega_c} \Big(\boldsymbol{\kappa} \nabla \frac{\partial f^\delta}{\partial t} \cdot \nabla \frac{\partial f^\delta}{\partial t} \Big)\Big|_{I_k^\delta} d\mathbf{x}. \end{aligned}$$

Inserting these estimates into (3.28) and using the trace inequality, after gathering the same terms give

$$\begin{aligned} &\Big(\frac{\underline{a}}{5} - \underline{F} K_{\mathrm{tr}} \Big) \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t}\Big|_{I_k^\delta} \Big\|_{H^1(\Omega_c)}^2 + S \Big\| \frac{\partial p^\delta}{\partial t}\Big|_{I_k^\delta} \Big\|_{L^2(\Omega_c)}^2 + \frac{1}{2\delta} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_k^\delta \cdot \nabla p_k^\delta\, d\mathbf{x} \\ &\le \frac{1}{2\delta} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_{k-1}^\delta \cdot \nabla p_{k-1}^\delta\, d\mathbf{x} + \frac{1}{2T} \max_{k \in [1,N]} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_k^\delta \cdot \nabla p_k^\delta\, d\mathbf{x} + R^\delta|_{I_k^\delta}, \end{aligned} \tag{3.29}$$

where the term R^δ is determined in (3.18).

Summing (3.29) over $k = 1, \dots, m$ for integer m and using the telescope rule and lower bound in (2.7), we obtain

$$\begin{aligned} &\Big(\frac{\underline{a}}{5} - \underline{F} K_{\mathrm{tr}} \Big) \sum_{k=1}^{m} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t}\Big|_{I_k^\delta} \Big\|_{H^1(\Omega_c)}^2 + S \sum_{k=1}^{m} \Big\| \frac{\partial p^\delta}{\partial t}\Big|_{I_k^\delta} \Big\|_{L^2(\Omega_c)}^2 \\ &+ \frac{1}{2\delta} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_m^\delta \cdot \nabla p_m^\delta\, d\mathbf{x} \\ &\le \frac{1}{2\delta} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_0^\delta \cdot \nabla p_0^\delta\, d\mathbf{x} + \frac{m}{2T} \max_{k \in [1,N]} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_k^\delta \cdot \nabla p_k^\delta\, d\mathbf{x} + \sum_{k=1}^{m} R^\delta|_{I_k^\delta}. \end{aligned}$$

Taking maximum over $m \in [1, N]$ we have

$$\begin{aligned} &\Big(\frac{\underline{a}}{5} - \underline{F} K_{\mathrm{tr}} \Big) \sum_{k=1}^{N} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t}\Big|_{I_k^\delta} \Big\|_{H^1(\Omega_c)}^2 + S \sum_{k=1}^{N} \Big\| \frac{\partial p^\delta}{\partial t}\Big|_{I_k^\delta} \Big\|_{L^2(\Omega_c)}^2 \\ &+ \frac{1}{2\delta} \max_{m \in [1,N]} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_m^\delta \cdot \nabla p_m^\delta\, d\mathbf{x} \\ &\le \frac{1}{2\delta} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_0^\delta \cdot \nabla p_0^\delta\, d\mathbf{x} + \frac{N}{2T} \max_{k \in [1,N]} \int_{\Omega_c} \boldsymbol{\kappa} \nabla p_k^\delta \cdot \nabla p_k^\delta\, d\mathbf{x} + \sum_{k=1}^{N} R^\delta|_{I_k^\delta}. \end{aligned}$$

Since the max-norm of ∇p^δ is canceled due to $T = \delta N$, after multiplication by δ, by use of the upper bound in (2.7) and the interpolant from (3.15) with the norm

$$\int_0^T \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{H^1(\Omega_c)}^2 dt = \sum_{k=1}^{N} \int_{t_{k-1}^\delta}^{t_k^\delta} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{H^1(\Omega_c)}^2 dt = \delta \sum_{k=1}^{N} \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t}\Big|_{I_k^\delta} \Big\|_{H^1(\Omega_c)}^2,$$

the following inequality holds:

$$\begin{aligned} &\Big(\frac{\underline{a}}{5} - \underline{F} K_{\mathrm{tr}} \Big) \int_0^T \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{H^1(\Omega_c)}^2 dt + S \int_0^T \Big\| \frac{\partial p^\delta}{\partial t} \Big\|_{L^2(\Omega_c)}^2 dt \\ &\le \frac{\overline{\kappa}}{2} \|\nabla p_0\|_{L^2(\Omega_c)}^2 + \int_0^T R^\delta\, dt. \end{aligned} \tag{3.30}$$

From (3.24) and (3.30) we infer assertions (3.16)–(3.18) of the theorem. The proof is completed. □

Finally, on the basis of uniform estimates we prove the main existence theorem.

Theorem 3.3 (*Solvability of the Temporal Problem*)**.** *Under assumptions of Theorems 3.1 and 3.2, there exists a variational solution* $(\mathbf{u}, p)$ *to the poroelastic problem for the fluid-driven crack subject to non-penetration and cohesion, which is defined in (2.11) and satisfies the initial condition (2.14), variational inequality (2.23) and variational equality (2.24).*

Proof. Let Theorems 3.1 and 3.2 hold true. In the virtue of convergence of the interpolants as $\delta \to 0$:

$$\begin{aligned} &\boldsymbol{\tau}^{0\delta} \to \boldsymbol{\tau}^0, \ \frac{\partial \boldsymbol{\tau}^{0\delta}}{\partial t} \to \frac{\partial \boldsymbol{\tau}^0}{\partial t}, \quad f^\delta \to f, \ \frac{\partial f^\delta}{\partial t} \to \frac{\partial f}{\partial t}, \\ &\mathbf{g}^\delta \to \mathbf{g}, \ \frac{\partial \mathbf{g}^\delta}{\partial t} \to \frac{\partial \mathbf{g}}{\partial t} \quad \text{strongly in } L^2(0,T), \end{aligned} \tag{3.31}$$

from (3.17) and (3.18), under condition (3.16) we infer the uniform estimates:

$$\underline{a} \|\mathbf{u}^\delta\|_{L^\infty(0,T;H^1(\Omega_c))}^2 + S \|p^\delta\|_{L^\infty(0,T;L^2(\Omega_c))}^2 + \underline{\kappa} \|\nabla p^\delta\|_{L^2(\Omega_c^T)}^2 \le \text{const},$$
$$K \Big\| \frac{\partial \mathbf{u}^\delta}{\partial t} \Big\|_{L^2(0,T;H^1(\Omega_c))}^2 + S \Big\| \frac{\partial p^\delta}{\partial t} \Big\|_{L^2(\Omega_c^T)}^2 \le \text{const}.$$

Therefore, there exists a convergent subsequence δ_m, and an accumulation point $(\mathbf{u}, p)$ from the function space described in (2.11), such that as $\delta_m \to 0$:

$$\begin{aligned} &\mathbf{u}^{\delta_m} \to \mathbf{u} \quad \text{weakly in } H^1(0,T;H^1(\Omega_c)), \\ &\text{strongly in } L^2(0,T;L^2(\Gamma_N \cup \Gamma_c^+ \cup \Gamma_c^-)), \end{aligned} \tag{3.32}$$

where the strong convergence is provided by the compact embedding, and

$$p^{\delta_m} \to p \quad \text{weakly in } H^1(0,T;L^2(\Omega_c)) \cap L^2(0,T;H^1(\Omega_c)), \quad \text{strongly in } L^2(\Omega_c^T), \tag{3.33}$$

where the strong convergence is according to Aubin–Lions lemma (cf. Simon, 1986, Theorem 5 on p.84).

Taking the limit in the incremental problem (3.4) and (3.5) as $\delta_m \to 0$ on the basis of convergences (3.31)–(3.33), we conclude that $(\mathbf{u}, p)$ solves the variational inequality (2.23) and the variational equation (2.24). This finishes the proof. □

This is worth noting that passing in (3.17) and (3.18) to the limit as $\delta_m \to 0$ according to the convergences (3.31)–(3.33) will justify corresponding a-priori estimates for the solution from Theorem 3.3.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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