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Asymptotic series solution for plane poroelastic model with non-penetrating crack driven by hydraulic fracture☆

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ABSTRACT

A new class of coupled poroelastic problems describing fluid-driven cracks (called fractures) subjected to non-penetration conditions between opposite crack faces (fracture walls) is considered in the incremental form. The nonlinear crack problem for a plane isotropic setting in a two-phase medium is expressed in polar coordinates as a variational inequality with respect to the solid phase displacement and the pore pressure. Applying nonlinear methods, the asymptotic theory and Fourier analysis, a semi-analytic solution given as the power series in the sector of angle 2π is proven using rigorous expansions with respect to the distance to the crack-tip. Here no logarithmic terms occur in the asymptotic expansion. Consequently, a square-root singularity for the poroelastic medium with a non-penetrating crack is derived, and the integral formulas for calculating the corresponding stress intensity factors are obtained.

1. Introduction

The physical model under consideration is motivated by the hydrofracking techniques used in pumping oil and natural gas from boreholes in earth reservoirs. The reservoir is modeled as a two-phase poroelastic medium comprising solid particles and fluid saturated pores. It contains a hydraulic fracture (crack) generated by pumping a fracturing fluid. The mathematical model is described by a coupled system of poroelastic equations in the incremental form with respect to the solid phase displacement and the pore pressure. The system is subjected to fluid pressure prescribed at the fracture walls (crack faces). In contrast to the classical description, we allow a compressive pressure at which the crack may close, which is physically consistent. This assumption necessitates the consideration of non-penetration conditions between the opposite crack faces.

Extending the classical theory of stress-free cracks, which allows mutual penetration between the crack faces, a variational approach to solids with non-penetrating cracks was established by Khludnev and Kovtunenko (2000) For the dynamic modeling of cracks, we cite the monograph by Bratov et al. (2009). The nonlinear concept of non-penetration was pursued accounting for the frictional contact phenomena (Itou et al., 2011), cohesion (Kovtunenko, 2011), limiting small strains (Itou et al., 2017), and nonlinear elastic moduli depending on the mean normal stress (Itou et al., 2019, 2021). Further studies of nonlinear crack problems examined effective numerical methods (see Hintermüller et al. (2009)), optimal control problems (see Lazarev et al. (2018)), etc. To investigate the singular behavior of solutions near the crack-tip, nonlinear techniques and asymptotic analysis were developed for non-penetrating cracks (Itou et al., 2012; Khludnev and

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Kozlov, 2008; Kovtunenko, 2001), and thin inclusions (Rudoy et al., 2021).

The general concept of soil and poro-mechanics was established in classic works (Barenblatt et al., 1960; Biot, 1956; Terzaghi, 1943). Further developments are related to the challenging aspects of multi-scale modeling, e.g., Meirmanov (2014). In particular, we cite (Fellner and Kovtunenko, 2016; Kovtunenko and Zubkova, 2018) for the homogenization of a two-phase medium comprising solid phase and pores, and Sazhenkov et al. (2021) for the related micro–meso–macro analysis. In our modeling, we follow the hydraulic fracturing relations for poroelastic media given in Golovin and Baykin (2018), Shelukhin et al. (2014), and the engineering description, e.g., Skopintsev et al. (2020), Valov et al. (2021). Recently, we had derived new non-penetration conditions for a fluid-driven crack in two-phase poroelastic media and had established well-posedness for the corresponding variational inequality (Kovtunenko, 2022). Furthermore, we had investigated shape perturbations of a non-penetrating crack after semi-discretization in time (Kovtunenko and Lazarev, 2022). Adopting Lagrange multiplier approach and shape sensitivity analysis methods, we derived semi-analytic formulas for calculating the strain energy release rate. In this study, we investigate asymptotic representation with the power series and obtain the so-called stress intensity factors (SIFs) for the poroelastic problem in an incremental form. SIFs are important in the Griffith–Irwin criterion of brittle fracture for crack propagation, as discussed in the concluding remarks in Section 5.

For mathematical description, let the time interval of interest be discretized by points t_k with time-steps $\Delta t_k > 0$ for $k = 0, 1, 2, \ldots$. For a fixed integer k, we consider the equilibrium equation in Cartesian coordinates for displacements $\mathbf{u}^k$ and pore pressure p^k:

$$\mu \Delta \mathbf{u}^k + (\lambda + \mu)\nabla\big(\operatorname{div}\mathbf{u}^k\big) - \alpha \nabla p^k \mathbf{I} = \mathbf{0}, \tag{1}$$

which are coupled by the k-dependent mass balance equation in the incremental form

$$S(p^k - p^{k-1}) + \alpha \operatorname{tr}\boldsymbol{\varepsilon}\big(\mathbf{u}^k - \mathbf{u}^{k-1}\big) - \varkappa \Delta t_k \Delta p^k = 0, \tag{2}$$

with Lamé parameters λ and μ, storativity S, Biot coefficient α, permeability $\varkappa$, and identity $\mathbf{I}$. After semi-discretization, the incremental problem depends on time by means of parameter k. Thereby, the output state is considered at the current $k > 0$, and the input data are prescribed apriori at the previous $(k-1)$-th state. To adjust iterations to the initial state, we assume that $\mathbf{u}$ and p at $k = 0$ satisfy the governing Eqs. (1) and (2), which should be k-independent, i.e.,

$$\mu \Delta \mathbf{u} + (\lambda + \mu)\nabla\big(\operatorname{div}\mathbf{u}\big) - \alpha \nabla p \mathbf{I} = \mathbf{0}, \quad \Delta p = 0. \tag{3}$$

These relations (3) are decoupled and correspond to a certain undrained state of the poroelastic media. Subtracting the linear governing equations, following (Atkinson and Craster, 1991), we split the solution at every k into the k-independent part solving (3) and the k-dependent remainder:

$$\mathbf{u}^k = \mathbf{u} + \tilde{\mathbf{u}}^k, \quad p^k = p + \tilde{p}^k. \tag{4}$$

The model is provided with appropriate boundary conditions, which we introduce later. It can be noted that splitting (4) is an important tool for further asymptotic analysis.

In Section 2, we give the boundary-value setting in 2d polar coordinates for the poroelastic model (1), (2) with a non-penetrating crack driven by hydraulic fracture, and establish well-posedness through its weak formulation by a variational inequality. Based on splitting (4) and developing asymptotic methods including Fourier analysis, a semi-analytic solution for the non-linear crack problem expressed as a convergent power series with respect to the distance $r > 0$ to the crack-tip is derived in Section 3. To engineers it is important to note that no log-oscillations occur in the asymptotic expansion in Appendix. In Section 4, a singular solution implying the main asymptotic term of order $\sqrt{r}$ in the series is described in details, and integral formulas for finding the respective weights called SIFs are rigorously proven.

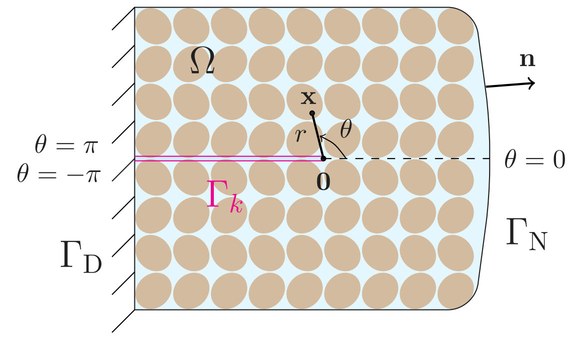

Fig. 1. Example geometry of 2d sectorial domain Ω_k with crack Γ_k.

2. Setting and well-posedness of the problem

We begin with the geometric description of a reservoir (associated with domain Ω) with a fluid-filled fracture (crack Γ_k) that typically has a planar structure. Let Ω be a 2d bounded domain with a Lipschitz continuous boundary $\partial\Omega$ consisting of mutually disjoint parts $\Gamma_{\rm N}$ and $\Gamma_{\rm D} \neq \emptyset$, and the normal vector $\mathbf{n}$ be outward from Ω. The origin $\mathbf{0}$ in the domain is associated with the tip of a semi-infinite crack, which on finite interaction with Ω builds a line segment Γ_k as illustrated in Fig. 1. Thus, the cracked domain $\Omega_k := \Omega \setminus \overline{\Gamma_k}$ presents a finite part of the sector of angle 2π bounded by $\partial\Omega$. Here, we introduce a polar coordinate system $r > 0$, $\theta \in (-\pi, \pi)$ such that the upper and lower crack faces correspond to $\theta = \pm\pi$.

In the sectorial domain Ω_k, we look for an unknown displacement vector $\mathbf{u}^k = (u^k_r, u^k_\theta)(r,\theta)$ and pore pressure $p^k(r,\theta)$, which build the linearized strain $\boldsymbol{\varepsilon}(\mathbf{u}^k)$, the Cauchy stress $\boldsymbol{\sigma}(\mathbf{u}^k)$, and the effective stress $\boldsymbol{\tau}^k(r,\theta)$, given by symmetric tensors in $\mathbb{R}^{2\times 2}_{\rm sym}$, respectively

$$\boldsymbol{\varepsilon}(\mathbf{u}^k) = \begin{pmatrix} \varepsilon_{rr}(\mathbf{u}^k) & \varepsilon_{r\theta}(\mathbf{u}^k) \\ \varepsilon_{r\theta}(\mathbf{u}^k) & \varepsilon_{\theta\theta}(\mathbf{u}^k) \end{pmatrix}, \quad \boldsymbol{\sigma}(\mathbf{u}^k) = \begin{pmatrix} \sigma_{rr}(\mathbf{u}^k) & \sigma_{r\theta}(\mathbf{u}^k) \\ \sigma_{r\theta}(\mathbf{u}^k) & \sigma_{\theta\theta}(\mathbf{u}^k) \end{pmatrix}, \quad \boldsymbol{\tau}^k = \begin{pmatrix} \tau^k_{rr} & \tau^k_{r\theta} \\ \tau^k_{r\theta} & \tau^k_{\theta\theta} \end{pmatrix}. \tag{5}$$

Relations (5) include the strain components

$$\varepsilon_{rr}(\mathbf{u}^k) = u^k_{r,r}, \quad \varepsilon_{r\theta}(\mathbf{u}^k) = \frac{1}{2}\Big(u^k_{\theta,r} + \frac{1}{r}u^k_{r,\theta} - \frac{1}{r}u^k_\theta\Big), \quad \varepsilon_{\theta\theta}(\mathbf{u}^k) = \frac{1}{r}u^k_{\theta,\theta} + \frac{1}{r}u^k_r, \tag{6}$$

where the partial derivatives $(\,\cdot\,)_{,r} = \partial(\,\cdot\,)/\partial r$ and $(\,\cdot\,)_{,\theta} = \partial(\,\cdot\,)/\partial\theta$. The stress is built according to the isotropic model in the state of plane stress as follows:

$$\sigma_{rr}(\mathbf{u}^k) = (\lambda + 2\mu)\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}^k) - 2\mu\varepsilon_{\theta\theta}(\mathbf{u}^k), \quad \sigma_{r\theta}(\mathbf{u}^k) = 2\mu\varepsilon_{r\theta}(\mathbf{u}^k),$$
$$\sigma_{\theta\theta}(\mathbf{u}^k) = (\lambda + 2\mu)\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}^k) - 2\mu\varepsilon_{rr}(\mathbf{u}^k), \tag{7}$$

where the trace (dilatation) $\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}^k) = \varepsilon_{rr}(\mathbf{u}^k) + \varepsilon_{\theta\theta}(\mathbf{u}^k)$ and the Lamé parameters

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{8}$$

with Young's modulus $E > 0$ and Poisson's ratio $\nu \in (0, 1/2)$, and

$$\boldsymbol{\tau}^k = \boldsymbol{\sigma}(\mathbf{u}^k) - \alpha p^k \mathbf{I} \tag{9}$$

with the Biot coefficient $\alpha \in (0, 1]$ and 2×2 identity tensor $\mathbf{I}$.

Excluding the dynamic terms, the equilibrium Eqs. (1) in polar coordinates are as follows:

$$\tau^k_{rr,r} + \frac{1}{r}\tau^k_{rr} + \frac{1}{r}\tau^k_{r\theta,\theta} - \frac{1}{r}\tau^k_{\theta\theta} = 0, \quad \tau^k_{r\theta,r} + \frac{2}{r}\tau^k_{r\theta} + \frac{1}{r}\tau^k_{\theta\theta,\theta} = 0. \tag{10}$$

The poroelastic mass balance law (2) can be represented by the following incremental equation

$$S(p^k - p^{k-1}) + \alpha \operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}^k - \mathbf{u}^{k-1}) - \varkappa \Delta t_k \Big(p^k_{,rr} + \frac{1}{r}p^k_{,r} + \frac{1}{r^2}p^k_{,\theta\theta}\Big) = 0, \tag{11}$$

where the storativity $S > 0$, and the permeability coefficient $\varkappa > 0$. The boundary conditions for the Eqs. (5)–(11) are set as follows.

At the outer boundary, we prescribe mixed inhomogeneous conditions using the fluid pressure f^k and force $\mathbf{g}^k = (g^k_r, g^k_\theta)$ such that

$$\mathbf{u}^k = \mathbf{0} \quad \text{on } \Gamma_{\rm D}, \quad \boldsymbol{\tau}^k\mathbf{n} = \mathbf{g}^k \quad \text{on } \Gamma_{\rm N}, \quad p^k = f^k \quad \text{on } \partial\Omega, \tag{12}$$

where $\boldsymbol{\tau}^k\mathbf{n}$ is the traction at $\partial\Omega$.

Across the crack Γ_k, the functions are discontinuous allowing non-zero jumps

$$[\![\mathbf{u}^k]\!] := \mathbf{u}^k|_{\theta=\pi} - \mathbf{u}^k|_{\theta=-\pi}, \quad [\![\boldsymbol{\tau}^k]\!] := \boldsymbol{\tau}^k|_{\theta=\pi} - \boldsymbol{\tau}^k|_{\theta=-\pi}, \quad [\![p^k]\!] := p^k|_{\theta=\pi} - p^k|_{\theta=-\pi}. \tag{13}$$

To be complementary to (13), the mean values over the crack are introduced as

$$\{\!\{\mathbf{u}^k\}\!\} := \frac{\mathbf{u}^k|_{\theta=\pi} + \mathbf{u}^k|_{\theta=-\pi}}{2}, \quad \{\!\{\boldsymbol{\tau}^k\}\!\} := \frac{\boldsymbol{\tau}^k|_{\theta=\pi} + \boldsymbol{\tau}^k|_{\theta=-\pi}}{2},$$
$$\{\!\{p^k\}\!\} := \frac{p^k|_{\theta=\pi} + p^k|_{\theta=-\pi}}{2}. \tag{14}$$

We suppose that there is no tangential stress at the crack, i.e., using notations (13) and (14),

$$[\![\tau^k_{r\theta}]\!] = 0, \quad \{\!\{\tau^k_{r\theta}\}\!\} = 0. \tag{15}$$

For the fluid pressure f^k prescribed at the opposite crack faces

$$p^k|_{\theta=\pi} = f^k|_{\theta=\pi}, \quad p^k|_{\theta=-\pi} = f^k|_{\theta=-\pi}, \tag{16}$$

where might be $f^k|_{\theta=\pi} \neq f^k|_{\theta=-\pi}$ except the crack tip $\mathbf{0}$, we suppose continuous normal stress

$$[\![\tau^k_{\theta\theta} + f^k]\!] = 0 \tag{17}$$

and non-penetration conditions set in the complementary form:

$$[\![u^k_\theta]\!] \le 0, \quad \{\!\{\tau^k_{\theta\theta} + f^k\}\!\} \le 0, \quad \{\!\{\tau^k_{\theta\theta} + f^k\}\!\}[\![u^k_\theta]\!] = 0. \tag{18}$$

The strict inequality $[\![u^k_\theta]\!] < 0$ in (18) implies an open crack (see Fig. 1) under standard condition

$$\{\!\{\tau^k_{\theta\theta} + f^k\}\!\} = 0. \tag{19}$$

Otherwise, the crack is closed when $[\![u^k_\theta]\!] = 0$ under compressive stress $\{\!\{\tau^k_{\theta\theta} + f^k\}\!\} < 0$.

We give a variational formulation to the boundary-value problem (5)–(12) and (15)–(18). In a domain Ω, which is radially convex with respect to $\mathbf{0}$, the following Green's formula

$$-\int_{\Omega_k} \Big\{ \big((r\tau^k_{rr})_{,r} + \tau^k_{r\theta,\theta} - \tau^k_{\theta\theta}\big)v_r + \big((r\tau^k_{r\theta})_{,r} + \tau^k_{r\theta} + \tau^k_{\theta\theta,\theta}\big)v_\theta \Big\}\, dr d\theta$$
$$= \int_{\Omega_k} \big(\tau^k_{rr}\varepsilon_{rr}(\mathbf{v}) + 2\tau^k_{r\theta}\varepsilon_{r\theta}(\mathbf{v}) + \tau^k_{\theta\theta}\varepsilon_{\theta\theta}(\mathbf{v})\big)\, r dr d\theta - \int_{\partial\Omega} \boldsymbol{\tau}^k\mathbf{n}\cdot\mathbf{v}\, dS$$
$$- \int_{\Gamma_k} \big([\![\tau^k_{r\theta}]\!]\{\!\{v_r\}\!\} + \{\!\{\tau^k_{r\theta}\}\!\}[\![v_r]\!] + [\![\tau^k_{\theta\theta}]\!]\{\!\{v_\theta\}\!\} + \{\!\{\tau^k_{\theta\theta}\}\!\}[\![v_\theta]\!]\big)\, dr \tag{20}$$

holds for smooth functions $\boldsymbol{\tau}^k$ and $\mathbf{v} = (v_r, v_\theta)$, where the scalar product of vectors $\boldsymbol{\tau}^k\mathbf{n}\cdot\mathbf{v}$ and identity $[\![uv]\!] = [\![u]\!]\{\!\{v\}\!\} + \{\!\{u\}\!\}[\![v]\!]$ are used. Therefore, multiplying by $v_r - u^k_r$ the first equilibrium equation in (10), and by $v_\theta - u^k_\theta$ the second one, followed by summation and application of Green's formula (20), and considering the boundary conditions (12), (15), and (17), for $\mathbf{v} = \mathbf{0}$ on $\Gamma_{\rm D}$ we obtain

$$\int_{\Omega_k} \boldsymbol{\tau}^k : \varepsilon(\mathbf{v} - \mathbf{u}^k)\, r dr d\theta$$
$$:= \int_{\Omega_k} \big(\tau^k_{rr}\varepsilon_{rr}(\mathbf{v}-\mathbf{u}^k) + 2\tau^k_{r\theta}\varepsilon_{r\theta}(\mathbf{v}-\mathbf{u}^k) + \tau^k_{\theta\theta}\varepsilon_{\theta\theta}(\mathbf{v}-\mathbf{u}^k)\big)\, r dr d\theta$$
$$= \int_{\Gamma_{\rm N}} \mathbf{g}^k\cdot(\mathbf{v}-\mathbf{u}^k) dS - \int_{\Gamma_k} \big([\![f^k]\!]\{\!\{v_\theta - u^k_\theta\}\!\} - \{\!\{\tau^k_{\theta\theta}\}\!\}[\![v_\theta - u^k_\theta]\!]\big) dr. \tag{21}$$

Consequently, using the complementarity conditions (18) rewritten in the equivalent form

$$[\![u^k_\theta]\!] \le 0, \quad \{\!\{\tau^k_{\theta\theta} + f^k\}\!\}[\![v_\theta - u^k_\theta]\!] \ge 0 \quad \text{for all } [\![v_\theta]\!] \le 0, \tag{22}$$

and a relation $[\![f^k]\!]\{\!\{v_\theta - u^k_\theta\}\!\} + \{\!\{f^k\}\!\}[\![v_\theta - u^k_\theta]\!] = [\![f^k(v_\theta - u^k_\theta)]\!]$, we infer from Eq. (21) the variational inequality

$$\int_{\Omega_k} \boldsymbol{\tau}^k : \varepsilon(\mathbf{v}-\mathbf{u}^k)\, r dr d\theta \ge \int_{\Gamma_{\rm N}} \mathbf{g}^k\cdot(\mathbf{v}-\mathbf{u}^k)\, dS - \int_{\Gamma_k} [\![f^k(v_\theta - u^k_\theta)]\!]\, r dr \tag{23}$$

for all test functions such that $[\![v_\theta]\!] \le 0$ on Γ_k. Multiplying mass balance Eq. (11) by a smooth function q such that $q = 0$ on $\partial\Omega$ after integration by parts yields the variational equation

$$\int_{\Omega_k} \Big\{ \big(S(p^k - p^{k-1}) + \alpha \mathrm{tr}\varepsilon(\mathbf{u}^k - \mathbf{u}^{k-1})\big)q + \varkappa\Delta t_k\Big(p^k_{,r}q_{,r} + \frac{1}{r^2}p^k_{,\theta}q_{,\theta}\Big) \Big\}\, r dr d\theta = 0. \tag{24}$$

Let the problem data be given in function spaces

$$\mathbf{g}^k \in L^2(\Gamma_{\rm N};\mathbb{R}^2), \quad f^k, p^{k-1} \in H^1(\Omega_k;\mathbb{R}), \quad \mathbf{u}^{k-1} \in H^1(\Omega_k;\mathbb{R}^2), \tag{25}$$

and the set of admissible displacements is defined as

$$\mathcal{K}(\Omega_k) = \{\mathbf{v} = (v_r, v_\theta) \in H^1(\Omega_k;\mathbb{R}^2)| \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{\rm D}, \quad [\![v_\theta]\!] \le 0 \text{ on } \Gamma_k\}. \tag{26}$$

Proposition 1 (*Well-posedness*). *There exists a triple $\mathbf{u}^k \in \mathcal{K}(\Omega_k)$, $p^k - f^k \in H^1_0(\Omega_k;\mathbb{R})$, and $\boldsymbol{\tau}^k \in L^2(\Omega_k;\mathbb{R}^{2\times 2}_{\rm sym})$ from (9) solving uniquely the incremental poroelastic problem with a non-penetrating crack driven by hydraulic fracture, which is stated in the weak form (23) and (24) for all test functions $\mathbf{v} \in \mathcal{K}(\Omega_k)$ and $q \in H^1_0(\Omega_k;\mathbb{R})$, respectively.*

Proof. Substituting $\boldsymbol{\tau}^k$ from (9) into (23) and summing with (24) results in inequality

$$\int_{\Omega_k} \Big\{ \boldsymbol{\sigma}(\mathbf{u}^k) : \varepsilon(\mathbf{v}-\mathbf{u}^k) - \alpha p^k \mathrm{tr}\varepsilon(\mathbf{v}-\mathbf{u}^k)$$
$$+ \big(Sp^k + \alpha\mathrm{tr}\varepsilon(\mathbf{u}^k)\big)q + \varkappa\Delta t_k\Big(p^k_{,r}q_{,r} + \frac{1}{r^2}p^k_{,\theta}q_{,\theta}\Big) \Big\}\, r dr d\theta$$
$$\ge \int_{\Gamma_{\rm N}} \mathbf{g}^k\cdot(\mathbf{v}-\mathbf{u}^k)\, dS - \int_{\Gamma_k} [\![f^k(v_\theta - u^k_\theta)]\!]\, r dr$$
$$+ \int_{\Omega_k} \big(Sp^{k-1} + \alpha\mathrm{tr}\varepsilon(\mathbf{u}^{k-1})\big)q\, r dr d\theta. \tag{27}$$

The left-hand side of (27) builds a continuous bilinear form. It is coercive when $\mathbf{v} = \mathbf{0}$, $\mathbf{v} = 2\mathbf{u}^k$, and $q = p^k$ are substituted, because terms $-\alpha p^k\mathrm{tr}\varepsilon(\mathbf{u}^k)$ and $\alpha\mathrm{tr}\varepsilon(\mathbf{u}^k)p^k$ are shortened, hence

$$\int_{\Omega_k} \Big\{ \boldsymbol{\sigma}(\mathbf{u}^k) : \varepsilon(\mathbf{u}^k) + S(p^k)^2 + \varkappa\Delta t_k\Big((p^k_{,r})^2 + \frac{1}{r^2}(p^k_{,\theta})^2\Big) \Big\}\, r dr d\theta$$
$$= \int_{\Gamma_{\rm N}} \mathbf{g}^k\cdot\mathbf{u}^k\, dS - \int_{\Gamma_k} [\![f^k u^k_\theta]\!]\, r dr + \int_{\Omega_k} \big(Sp^{k-1} + \alpha\mathrm{tr}\varepsilon(\mathbf{u}^{k-1})\big)p^k\, r dr d\theta.$$

Therefore, as per the Lions–Stampacchia theorem, there exists a unique solution.

Further, we derive the asymptotic solution as a convergent series for the boundary-value problem (5)–(11) and (15)–(18), omitting conditions (12) at the outer boundary.

3. Power series solution

To solve the inhomogeneous problem, we decompose the solution into two terms according to (4):

$$\mathbf{u}^k = \mathbf{u} + \tilde{\mathbf{u}}^k, \quad \boldsymbol{\tau}^k = \boldsymbol{\tau} + \tilde{\boldsymbol{\tau}}^k, \quad p^k = p + \tilde{p}^k, \tag{28}$$

and similarly, at $k-1$. Thereby, according to (10) and (11), the k-independent term $(\mathbf{u}, \boldsymbol{\tau}, p)$ is a solution for the homogeneous equations

$$p_{,rr} + \frac{1}{r}p_{,r} + \frac{1}{r^2}p_{,\theta\theta} = 0, \quad \tau_{rr,r} + \frac{1}{r}\tau_{rr} + \frac{1}{r}\tau_{r\theta,\theta} - \frac{1}{r}\tau_{\theta\theta} = 0, \quad \tau_{r\theta,r} + \frac{2}{r}\tau_{r\theta} + \frac{1}{r}\tau_{\theta\theta,\theta} = 0, \tag{29}$$

and k-dependent term $(\tilde{\mathbf{u}}^k, \tilde{\tau}^k, \tilde{p}^k)$ is a solution for the inhomogeneous equations

$$\tilde{\tau}^k_{rr,r} + \frac{1}{r}\tilde{\tau}^k_{rr} + \frac{1}{r}\tilde{\tau}^k_{r\theta,\theta} - \frac{1}{r}\tilde{\tau}^k_{\theta\theta} = 0, \quad \tilde{\tau}^k_{r\theta,r} + \frac{2}{r}\tilde{\tau}^k_{r\theta} + \frac{1}{r}\tilde{\tau}^k_{\theta\theta,\theta} = 0,$$
$$S(\tilde{p}^k - \tilde{p}^{k-1}) + \alpha \operatorname{tr}\varepsilon(\tilde{\mathbf{u}}^k - \tilde{\mathbf{u}}^{k-1}) - \varkappa \Delta t_k \Big(\tilde{p}^k_{,rr} + \frac{1}{r}\tilde{p}^k_{,r} + \frac{1}{r^2}\tilde{p}^k_{,\theta\theta}\Big) = 0. \tag{30}$$

Using (28), we split the linear boundary conditions at the crack (15)–(17) into homogeneous

$$p|_{\theta=\pm\pi} = 0, \quad [\![\tau_{r\theta}]\!] = 0, \quad \{\!\{\tau_{r\theta}\}\!\} = 0, \quad [\![\tau_{\theta\theta}]\!] = 0, \tag{31}$$

and inhomogeneous ones

$$\tilde{p}^k|_{\theta=\pm\pi} = f^k|_{\theta=\pm\pi}, \quad [\![\tilde{\tau}^k_{r\theta}]\!] = 0, \quad \{\!\{\tilde{\tau}^k_{r\theta}\}\!\} = 0, \quad [\![\tilde{\tau}^k_{\theta\theta} + f^k]\!] = 0, \tag{32}$$

where the complementarity conditions (18) remain coupled:

$$[\![u_\theta + \tilde{u}^k_\theta]\!] \le 0, \quad \{\!\{\tau_{\theta\theta} + \tilde{\tau}^k_{\theta\theta} + f^k\}\!\} \le 0, \quad \{\!\{\tau_{\theta\theta} + \tilde{\tau}^k_{\theta\theta} + f^k\}\!\}[\![u_\theta + \tilde{u}^k_\theta]\!] = 0. \tag{33}$$

The following results establish asymptotic solutions in the sector of angle 2π for the boundary-value problems (29), (30) and (31), (32) coupled by (33). First, we apply the power series method in the general form (see Kozlov et al. (2001, Sections 2.1 and 4.2)) to the poroelastic relations (5)–(11).

Lemma 1 (*Solution for the Poroelastic Equations*). *Excluding the term $p^k \sim \ln r$ and $\mathbf{u}^k \sim \ln r$, which do not belong to H^1, a general solution for the poroelastic relations (5)–(11) can be expressed as the functions of power $\gamma \neq 0$ for the solid displacement*

$$\mathbf{u}^k(r,\theta) = r^\gamma \Big(\sum_{i=1}^4 U_i^k \boldsymbol{\Psi}_{\gamma i}(\theta) + \frac{\beta}{4\gamma}\sum_{i=1}^2 P_i^k \boldsymbol{\Psi}_{\gamma(i+4)}(\theta)\Big), \tag{34}$$

and as the functions of power $\gamma - 1$ for the pore pressure

$$p^k(r,\theta) = r^{\gamma-1}\big(P_1^k \cos(\gamma-1)\theta + P_2^k \sin(\gamma-1)\theta\big), \tag{35}$$

with arbitrary factors $U_1^k, \dots, U_4^k, P_1^k, P_2^k$ satisfying

$$2\gamma(\kappa-1)(U_{i+2}^k - U_{i+2}^{k-1}) + (S+\beta)(P_i^k - P_i^{k-1}), \quad i = 1,2, \tag{36}$$

when expressions (34) and (35) are assumed to hold at $k-1$. In (34), the six vectors are

$$\boldsymbol{\Psi}_{\gamma 1} = \begin{pmatrix} \cos(\gamma+1)\theta \\ -\sin(\gamma+1)\theta \end{pmatrix}, \quad \boldsymbol{\Psi}_{\gamma 2} = \begin{pmatrix} \sin(\gamma+1)\theta \\ \cos(\gamma+1)\theta \end{pmatrix},$$
$$\boldsymbol{\Psi}_{\gamma 3} = \begin{pmatrix} (\gamma-\kappa)\cos(\gamma-1)\theta \\ -(\gamma+\kappa)\sin(\gamma-1)\theta \end{pmatrix},$$
$$\boldsymbol{\Psi}_{\gamma 4} = \begin{pmatrix} (\gamma-\kappa)\sin(\gamma-1)\theta \\ (\gamma+\kappa)\cos(\gamma-1)\theta \end{pmatrix}, \quad \boldsymbol{\Psi}_{\gamma 5} = \begin{pmatrix} (\gamma+1)\cos(\gamma-1)\theta \\ -(\gamma-1)\sin(\gamma-1)\theta \end{pmatrix},$$
$$\boldsymbol{\Psi}_{\gamma 6} = \begin{pmatrix} (\gamma+1)\sin(\gamma-1)\theta \\ (\gamma-1)\cos(\gamma-1)\theta \end{pmatrix} \tag{37}$$

and the parameters are defined as

$$\beta := \frac{\alpha}{\lambda+2\mu}, \quad \kappa := \frac{\lambda+3\mu}{\lambda+\mu} = 3 - 4\nu. \tag{38}$$

The corresponding strain components in (6) are

$$\varepsilon_{rr}(\mathbf{u}^k) = r^{\gamma-1}\big\{\gamma\big[U_1^k\cos(\gamma+1)\theta + U_2^k\sin(\gamma+1)\theta\big]$$
$$+ \Big(\gamma(\gamma-\kappa)U_3^k + (\gamma+1)\frac{\beta}{4}P_1^k\Big)\cos(\gamma-1)\theta$$
$$+ \Big(\gamma(\gamma-\kappa)U_4^k + (\gamma+1)\frac{\beta}{4}P_2^k\Big)\sin(\gamma-1)\theta\big\},$$
$$\varepsilon_{r\theta}(\mathbf{u}^k) = r^{\gamma-1}\big\{\gamma\big[U_2^k\cos(\gamma+1)\theta - U_1^k\sin(\gamma+1)\theta\big]$$
$$+ (\gamma-1)\Big[\Big(\gamma U_4^k + \frac{\beta}{4}P_2^k\Big)\cos(\gamma-1)\theta - \Big(\gamma U_3^k + \frac{\beta}{4}P_1^k\Big)\sin(\gamma-1)\theta\Big]\big\},$$
$$\varepsilon_{\theta\theta}(\mathbf{u}^k) = r^{\gamma-1}\big\{-\gamma\big[U_1^k\cos(\gamma+1)\theta + U_2^k\sin(\gamma+1)\theta\big]$$
$$- \Big(\gamma(\gamma-2+\kappa)U_3^k + (\gamma-3)\frac{\beta}{4}P_1^k\Big)\cos(\gamma-1)\theta$$
$$- \Big(\gamma(\gamma-2+\kappa)U_4^k + (\gamma-3)\frac{\beta}{4}P_2^k\Big)\sin(\gamma-1)\theta\big\}, \tag{39}$$

the dilatation

$$\operatorname{tr}\varepsilon(\mathbf{u}^k) = r^{\gamma-1}\big\{\big[-2\gamma(\kappa-1)U_3^k + \beta P_1^k\big]\cos(\gamma-1)\theta + \big[-2\gamma(\kappa-1)U_4^k + \beta P_2^k\big]\sin(\gamma-1)\theta\big\}, \tag{40}$$

and the stress components in (7) are

$$\tau_{rr}^k = r^{\gamma-1}\mu\big\{2\gamma\big[U_1^k\cos(\gamma+1)\theta + U_2^k\sin(\gamma+1)\theta\big]$$
$$+ (\gamma-3)\Big[\Big(2\gamma U_3^k + \frac{\beta}{2}P_1^k\Big)\cos(\gamma-1)\theta + \Big(2\gamma U_4^k + \frac{\beta}{2}P_2^k\Big)\sin(\gamma-1)\theta\Big]\big\},$$
$$\tau_{r\theta}^k = r^{\gamma-1}\mu\big\{2\gamma\big[U_2^k\cos(\gamma+1)\theta - U_1^k\sin(\gamma+1)\theta\big]$$
$$+ (\gamma-1)\Big[\Big(2\gamma U_4^k + \frac{\beta}{2}P_2^k\Big)\cos(\gamma-1)\theta - \Big(2\gamma U_3^k + \frac{\beta}{2}P_1^k\Big)\sin(\gamma-1)\theta\Big]\big\},$$
$$\tau_{\theta\theta}^k = r^{\gamma-1}\mu\big\{-2\gamma\big[U_1^k\cos(\gamma+1)\theta + U_2^k\sin(\gamma+1)\theta\big]$$
$$- (\gamma+1)\Big[\Big(2\gamma U_3^k + \frac{\beta}{2}P_1^k\Big)\cos(\gamma-1)\theta + \Big(2\gamma U_4^k + \frac{\beta}{2}P_2^k\Big)\sin(\gamma-1)\theta\Big]\big\}. \tag{41}$$

At $\gamma = 0$, the general solution takes a specific form:

$$\mathbf{u}^k(r,\theta) = \sum_{i=1}^2 U_i^k \boldsymbol{\Psi}_{0i}(\theta) + \frac{\alpha}{\lambda+3\mu}\begin{pmatrix} 0 \\ -P_2^k\cos\theta + P_1^k\sin\theta \end{pmatrix}. \tag{42}$$

As the proof of Lemma 1 is highly technical, it is presented in Appendix.

This is worth noting that the solid phase displacement in (34) and the pore pressure in (35) have different asymptotic orders of r. One of significant consequences is that the last equation in (30) is split and we have the decoupled system instead:

$$\tilde{p}^k_{,rr} + \frac{1}{r}\tilde{p}^k_{,r} + \frac{1}{r^2}\tilde{p}^k_{,\theta\theta} = 0, \quad \tilde{\tau}^k_{rr,r} + \frac{1}{r}\tilde{\tau}^k_{rr} + \frac{1}{r}\tilde{\tau}^k_{r\theta,\theta} - \frac{1}{r}\tilde{\tau}^k_{\theta\theta} = 0,$$
$$\tilde{\tau}^k_{r\theta,r} + \frac{2}{r}\tilde{\tau}^k_{r\theta} + \frac{1}{r}\tilde{\tau}^k_{\theta\theta,\theta} = 0, \quad S(\tilde{p}^k - \tilde{p}^{k-1}) + \alpha\operatorname{tr}\varepsilon(\tilde{\mathbf{u}}^k - \tilde{\mathbf{u}}^{k-1}) = 0, \tag{43}$$

where we first solve the Laplace equation in polar coordinates with respect to $\tilde{p}^k$, and then substitute the solution in the other equations for $\tilde{\tau}^k$ and $\tilde{\mathbf{u}}^k$.

As per (34) and (35), we seek for an energy solution $\tilde{\mathbf{u}}^k \in H^1(\Omega_k;\mathbb{R}^2)$ in the form of a convergent series for monotonically increasing $(\gamma_n)_{n\in\mathbb{N}}$ and $\gamma_l = \gamma_n - 1$ with integer l such that

$$\mathbf{u}^k - \mathbf{u}^k(\mathbf{0}) = \sum_{\gamma_n>0}\Big\{r^{\gamma_n}\Big(\sum_{i=1}^4 U_{ni}^k \boldsymbol{\Psi}_{\gamma_n i}(\theta) + \frac{\beta}{4\gamma_n}\sum_{i=1}^2 P_{li}^k \boldsymbol{\Psi}_{\gamma_n(i+4)}(\theta)\Big)\Big\}, \tag{44}$$

where $\boldsymbol{\Psi}_{\gamma_n 1}, \dots, \boldsymbol{\Psi}_{\gamma_n 6}$ are from (37) with $\gamma = \gamma_n$, and $\tilde{p}^k \in H^1(\Omega_k;\mathbb{R})$ yields

$$p^k - f^k(\mathbf{0}) = \sum_{\gamma_l>0}\big\{r^{\gamma_l}\big(P_{l1}^k\cos\gamma_l\theta + P_{l2}^k\sin\gamma_l\theta\big)\big\}. \tag{45}$$

These series include constant $\mathbf{u}^k(\mathbf{0})$ and $p^k(\mathbf{0})$, which are not present in the strain and stress tensors because $\varepsilon(\mathbf{u}^k) = \varepsilon(\mathbf{u}^k - \mathbf{u}^k(\mathbf{0}))$ and $\nabla p^k = \nabla(p^k - p^k(\mathbf{0}))$. Based on (44) and (45) we construct an energy solution for the k-independent boundary-value problem.

Lemma 2 (k-*independent Series*). *An energy solution for the boundary-value problem (29) and (31) is given as a convergent series with respect to integer n for the solid-phase displacement:*

$$\mathbf{u} - \mathbf{u}(\mathbf{0}) = \sum_{n=1}^\infty\Big\{r^{n/2}\Big(\sum_{i=1}^4 U_{ni}\boldsymbol{\Psi}_{\frac{n}{2}i} + \frac{\beta}{2n}\sum_{i=1}^2 P_{(n-2)i}\boldsymbol{\Psi}_{\frac{n}{2}(i+4)}\Big)\Big\}, \tag{46}$$

where vectors $\boldsymbol{\Psi}_{\frac{n}{2}1}, \dots, \boldsymbol{\Psi}_{\frac{n}{2}6}$ are from (37) with $\gamma = n/2$, and for the pore pressure:

$$p = \sum_{n=1}^\infty\Big\{r^{n/2}\Big(P_{n1}\cos\frac{n\theta}{2} + P_{n2}\sin\frac{n\theta}{2}\Big)\Big\}. \tag{47}$$

Six factors $U_{n1},\dots,U_{n4}, P_{(n-2)1}, P_{(n-2)2}$ should satisfy the conditions for even $n=2m,\ m\in\mathbb{N}$:

$$P_{(2\ m)1}=0,\quad U_{(2\ m)2}+(m-1)\Big(U_{(2\ m)4}+\frac{\beta}{4m}P_{(2m-2)2}\Big)=0, \tag{48}$$

and for odd $n=2m-1,\ m\in\mathbb{N}$:

$$P_{(2m-1)2}=0,\quad U_{(2m-1)1}+\frac{2m-3}{2}\Big(U_{(2m-1)3}+\frac{\beta}{2(2m-1)}P_{(2m-3)1}\Big)=0,$$
$$U_{(2m-1)2}+\frac{2m+1}{2}U_{(2m-1)4}=0, \tag{49}$$

where $P_{02}=P_{(-1)1}=0$ are set.

Proof. In the homogeneous boundary conditions (31), we substitute ansatz (44) for $\mathbf{u}-\mathbf{u}(\mathbf{0})$, and (45) for p with $p(\mathbf{0})=0$, and k-independent coefficients $U_{n1},\dots,U_{n4}, P_{l1}, P_{l2}$. The homogeneous Dirichlet condition for p at $\theta=\pm\pi$ is satisfied when the following holds for every $l\in\mathbb{N}$:

$$P_{l1}\cos\gamma_l\pi=P_{l2}\sin\gamma_l\pi=0.$$

Hence, $\gamma_l=l/2$ and either $\sin(l/2)\pi=0$ and $P_{l1}=0$ in (48) for even l, or $\cos(l/2)\pi=0$ and $P_{l2}=0$ in (49) for odd l.

Using expressions (41), the boundary stress at $\theta=\pm\pi$ in (31) provides three equations:

$$[\![\tau_{r\theta}]\!]=-4\mu\sum_{n=1}^{\infty}\Big\{r^{\gamma_n-1}\gamma_n\Big[U_{n1}+(\gamma_n-1)\Big(U_{n3}+\frac{\beta}{4\gamma_n}P_{l1}\Big)\Big]\sin(\gamma_n-1)\pi\Big\}=0,$$
$$\{\!\{\tau_{r\theta}\}\!\}=2\mu\sum_{n=1}^{\infty}\Big\{r^{\gamma_n-1}\gamma_n\Big[U_{n2}+(\gamma_n-1)\Big(U_{n4}+\frac{\beta}{4\gamma_n}P_{l2}\Big)\Big]\cos(\gamma_n-1)\pi\Big\}=0,$$
$$[\![\tau_{\theta\theta}]\!]=-4\mu\sum_{n=1}^{\infty}\Big\{r^{\gamma_n-1}\gamma_n\Big[U_{n2}+(\gamma_n+1)\Big(U_{n4}+\frac{\beta}{4\gamma_n}P_{l2}\Big)-\frac{1}{2\mu\gamma_n}P_{l2}\Big]\sin(\gamma_n-1)\pi\Big\}=0 \tag{50}$$

with respect to the unknown $U_{n1},\dots,U_{n4}$ for every $n\in\mathbb{N}$. As $P_{l2}=0$, the last two equations in (50) are homogeneous and solvable for non-trivial U_{n2}, U_{n4} only under zero Jacobian determinant, that is $\sin 2(\gamma_n-1)\pi=0$. Henceforth, $\gamma_n=n/2$, and $l=n-2$ since

$$\frac{l}{2}=\gamma_l=\gamma_n-1=\frac{n}{2}-1.$$

For even $n=2m$ and $\gamma_n=m$, due to $\sin(m-1)\pi=0$, the first and the third equations in (50) are satisfied identically, and the second one leads to the condition on $U_{(2\ m)2}$ and $U_{(2\ m)4}$ in (48). For odd $n=2m-1$ and $\gamma_n=m-1/2$, due to $\cos(m-3/2)\pi=0$, the second equation in (50) is satisfied, and the remaining ones result in the latter two conditions in (49). This completes the proof.

To extend Lemma 2 to the k-dependent problem (43) under inhomogeneous boundary conditions (32), we observe there that f^k should have $r^{n/2}$-asymptotic terms. Therefore, we assume that

$$f^k-f^k(\mathbf{0})=\sum_{n=1}^{\infty}\Big\{r^{n/2}\Big(F^k_{n1}\cos\frac{n\theta}{2}+F^k_{n2}\sin\frac{n\theta}{2}\Big)\Big\} \tag{51}$$

is prescribed by the given coefficients F^k_{n1}, F^k_{n2}; the same is true at $k-1$ and $f^k(\mathbf{0})=f^{k-1}(\mathbf{0})$.

Lemma 3 (k-*dependent Series*)**.** *Under assumption (51), the energy solution for the boundary-value problem (43), (32) is given as a convergent series with respect to integer n for the displacement:*

$$\tilde{\mathbf{u}}^k-\tilde{\mathbf{u}}^k(\mathbf{0})=\sum_{n=1}^{\infty}\Big\{r^{n/2}\Big(\sum_{i=1}^{4}\tilde U^k_{ni}\boldsymbol{\Psi}_{\frac{n}{2}i}+\frac{\beta}{2n}\sum_{i=1}^{2}\tilde P^k_{(n-2)i}\boldsymbol{\Psi}_{\frac{n}{2}(i+4)}\Big)\Big\}, \tag{52}$$

and for the pore pressure:

$$\tilde p^k-f^k(\mathbf{0})=\sum_{n=1}^{\infty}\Big\{r^{n/2}\Big(\tilde P^k_{n1}\cos\frac{n\theta}{2}+\tilde P^k_{n2}\sin\frac{n\theta}{2}\Big)\Big\}. \tag{53}$$

Six factors $\tilde U^k_{n1},\dots,\tilde U^k_{n4}, \tilde P^k_{(n-2)1}, \tilde P^k_{(n-2)2}$ should satisfy the conditions for even $n=2m,\ m\in\mathbb{N}$:

$$\tilde P^k_{(2\ m)1}=F^k_{(2\ m)1},\quad \tilde U^k_{(2\ m)2}+(m-1)\Big(\tilde U^k_{(2\ m)4}+\frac{\beta}{4m}\tilde P^k_{(2m-2)2}\Big)=0,$$

$$(S+\alpha\beta)\big(F^k_{(2m-2)1}-F^{k-1}_{(2m-2)1}\big)-2\ m(\kappa-1)\alpha\big(\tilde U^k_{(2\ m)3}-\tilde U^{k-1}_{(2\ m)3}\big)=0, \tag{54}$$

and for odd $n=2m-1,\ m\in\mathbb{N}$:

$$\tilde P^k_{(2m-1)2}=F^k_{(2m-1)2},$$
$$\tilde U^k_{(2m-1)1}+\frac{2m-3}{2}\Big(\tilde U^k_{(2m-1)3}+\frac{\beta}{2(2m-1)}\tilde P^k_{(2m-3)1}\Big)=0,$$
$$\tilde U^k_{(2m-1)2}+\frac{2m+1}{2}\Big(\tilde U^k_{(2m-1)4}+\frac{\beta}{2(2m-1)}F^k_{(2m-3)2}\Big)$$
$$-\frac{1}{\mu(2m-1)}F^k_{(2m-3)2}=0,$$
$$(S+\alpha\beta)\big(F^k_{(2m-3)2}-F^{k-1}_{(2m-3)2}\big)$$
$$-(2m-1)(\kappa-1)\alpha\big(\tilde U^k_{(2m-1)4}-\tilde U^{k-1}_{(2m-1)4}\big)=0, \tag{55}$$

where $\tilde P^k_{02}=\tilde P^k_{(-1)1}=F^k_{01}=F^k_{(-1)2}=0$.

Proof. We insert the ansatz (44) for $\tilde{\mathbf{u}}^k-\tilde{\mathbf{u}}^k(\mathbf{0})$ and (45) for $\tilde p^k-f^k(\mathbf{0})$ with arbitrary coefficients $\tilde U^k_{n1},\dots,\tilde U^k_{n4}, \tilde P^k_{l1}, \tilde P^k_{l2}$, in inhomogeneous boundary conditions (32) and in the last Eq. (43). From the Dirichlet condition $\tilde p^k=f^k$ at $\theta=\pm\pi$, it follows that

$$(\tilde P^k_{l1}-F^k_{l1})\cos\gamma_l\pi=(\tilde P^k_{l2}-F^k_{l2})\sin\gamma_l\pi=0$$

and $\gamma_l=l/2$, then $\tilde P^k_{l1}-F^k_{l1}=0$ in (54) for even l, and $\tilde P^k_{l2}-F^k_{l2}=0$ in (55) for odd l.

Akin to (50), the boundary stress at $\theta=\pm\pi$ in (32) implies the following relations

$$[\![\tilde\tau^k_{r\theta}]\!]=-4\mu\sum_{n=1}^{\infty}\Big\{r^{\gamma_n-1}\gamma_n\Big[\tilde U^k_{n1}+(\gamma_n-1)\Big(\tilde U^k_{n3}+\frac{\beta}{4\gamma_n}\tilde P^k_{l1}\Big)\Big]\sin(\gamma_n-1)\pi\Big\}=0,$$
$$\{\!\{\tilde\tau^k_{r\theta}\}\!\}=2\mu\sum_{n=1}^{\infty}\Big\{r^{\gamma_n-1}\gamma_n\Big[\tilde U^k_{n2}+(\gamma_n-1)\Big(\tilde U^k_{n4}+\frac{\beta}{4\gamma_n}\tilde P^k_{l2}\Big)\Big]\cos(\gamma_n-1)\pi\Big\}=0,$$
$$[\![\tilde\tau^k_{\theta\theta}+f^k]\!]=-4\mu\sum_{n=1}^{\infty}\Big\{r^{\gamma_n-1}\gamma_n\Big[\tilde U^k_{n2}+(\gamma_n+1)\Big(\tilde U^k_{n4}+\frac{\beta}{4\gamma_n}\tilde P^k_{l2}\Big)-\frac{1}{2\mu\gamma_n}F^k_{l2}\Big]$$
$$\times\sin(\gamma_n-1)\pi\Big\}=0 \tag{56}$$

and the last two equations necessitate $\sin 2(\gamma_n-1)\pi=0$, thus $\gamma_n=n/2=\gamma_l+1$ as $l=n-2$. For even n, we have $\sin(\gamma_n-1)\pi=0$ and obtain the second condition in (54); for odd n such that $\cos(\gamma_n-1)\pi=0$, we obtain the two conditions for the coefficients in (55).

Inserting (52) and (53) with $\gamma_n=n/2$ and $l=n-2$ in the dilatation (40),

$$\mathrm{tr}\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}^k)=\sum_{n=1}^{\infty}\Big\{r^{\frac{n-2}{2}}\big[\beta\tilde P^k_{(n-2)1}-n(\kappa-1)\tilde U^k_{n3}\big]\cos\frac{n-2}{2}\theta$$
$$+\big[\beta\tilde P^k_{(n-2)2}-n(\kappa-1)\tilde U^k_{n4}\big]\sin\frac{n-2}{2}\theta\Big\},$$

the same at $k-1$, and gathering the like asymptotic terms, from the last equation of (43) we have

$$\big[(S+\alpha\beta)(\tilde P^k_{(n-2)1}-\tilde P^{k-1}_{(n-2)1})-n(\kappa-1)\alpha(\tilde U^k_{n3}-\tilde U^{k-1}_{n3})\big]\cos\frac{n-2}{2}\theta$$
$$+\big[(S+\alpha\beta)(\tilde P^k_{(n-2)2}-\tilde P^{k-1}_{(n-2)2})-n(\kappa-1)\alpha(\tilde U^k_{n4}-\tilde U^{k-1}_{n4})\big]\sin\frac{n-2}{2}\theta=0,$$

and $S(f^k-f^{k-1})(\mathbf{0})=0$. This equation follows the last condition in (54) for even $n=2m$, and the last condition in (55) for odd $n=2m-1$, completing the proof.

Combining Lemmas 2 and 3, and satisfying the complementarity conditions (33), follows the main theorem.

Theorem 1 (*Power Series Solution*)**.** *Under assumption (51), the energy solution to the boundary-value problem (5)–(11) and (15)–(18) is given as a sum (28) of the convergent series (46), (52) for the solid phase displacement:*

$$\mathbf{u}^k-\mathbf{u}^k(\mathbf{0})=\sum_{n=1}^{\infty}\Big\{r^{n/2}\Big(\sum_{i=1}^{4}(U_{ni}+\tilde U_{ni}^k)\boldsymbol{\Psi}_{\frac{n}{2}i}+\frac{\beta}{2n}\sum_{i=1}^{2}(P_{(n-2)i}+\tilde P_{(n-2)i}^k)\boldsymbol{\Psi}_{\frac{n}{2}(i+4)}\Big)\Big\}, \tag{57}$$

and (47), (53) for the pore pressure:

$$p^k-f^k(\mathbf{0})=\sum_{n=1}^{\infty}\Big\{r^{n/2}\Big((P_{n1}+\tilde P_{n1}^k)\cos\frac{n\theta}{2}+(P_{n2}+\tilde P_{n2}^k)\sin\frac{n\theta}{2}\Big)\Big\}. \tag{58}$$

Twelve factors $U_{n1},\tilde U_{n1}^k,\dots,U_{n4},\tilde U_{n4}^k,P_{n1},\tilde P_{n1}^k,P_{n2},\tilde P_{n2}^k$ should satisfy the following for even $n=2m$, $m\in\mathbb{N}$:

$$\begin{aligned}
&\big[U_{(2\ m)1}+(m+1)U_{(2\ m)3}\big]\cos(m-1)\pi\ge 0,\\
&U_{(2\ m)2}+(m-1)\Big(U_{(2\ m)4}+\frac{\beta}{4m}P_{(2m-2)2}\Big)=0,\\
&\Big[\tilde U_{(2\ m)1}^k+(m+1)\Big(\tilde U_{(2\ m)3}^k+\frac{\beta}{4m}F_{(2m-2)1}^k\Big)-\frac{1}{2\mu m}F_{(2m-2)1}^k\Big]\cos(m-1)\pi\ge 0,\\
&P_{(2\ m)1}=0,\quad \tilde P_{(2\ m)1}^k=F_{(2\ m)1}^k,\\
&\tilde U_{(2\ m)2}^k+(m-1)\Big(\tilde U_{(2\ m)4}^k+\frac{\beta}{4m}\tilde P_{(2m-2)2}^k\Big)=0,\\
&(S+\alpha\beta)\big(F_{(2m-2)1}^k-F_{(2m-2)1}^{k-1}\big)-2\ m(\kappa-1)\alpha\big(\tilde U_{(2\ m)3}^k-\tilde U_{(2\ m)3}^{k-1}\big)=0,
\end{aligned} \tag{59}$$

and the following conditions for odd $n=2m-1$, $m\in\mathbb{N}$:

$$\begin{aligned}
&P_{(2m-1)2}=0,\quad U_{(2m-1)1}+\frac{2m-3}{2}\Big(U_{(2m-1)3}+\frac{\beta}{2(2m-1)}P_{(2m-3)1}\Big)=0,\\
&U_{(2m-1)2}+\frac{2m+1}{2}U_{(2m-1)4}=0,\quad U_{(2m-1)3}\sin\frac{2m-3}{2}\pi\ge 0,\\
&\tilde U_{(2m-1)3}^k\sin\frac{2m-3}{2}\pi\ge 0,\\
&\tilde P_{(2m-1)2}^k=F_{(2m-1)2}^k,\\
&\tilde U_{(2m-1)1}^k+\frac{2m-3}{2}\Big(\tilde U_{(2m-1)3}^k+\frac{\beta}{2(2m-1)}\tilde P_{(2m-3)1}^k\Big)=0,\\
&\tilde U_{(2m-1)2}^k+\frac{2m+1}{2}\Big(\tilde U_{(2m-1)4}^k+\frac{\beta}{2(2m-1)}F_{(2m-3)2}^k\Big)-\frac{1}{\mu(2m-1)}F_{(2m-3)2}^k=0,\\
&(S+\alpha\beta)\big(F_{(2m-3)2}^k-F_{(2m-3)2}^{k-1}\big)-(2m-1)(\kappa-1)\alpha\big(\tilde U_{(2m-1)4}^k-\tilde U_{(2m-1)4}^{k-1}\big)=0,
\end{aligned} \tag{60}$$

recalling that $P_{02}=P_{(-1)1}=\tilde P_{02}^k=\tilde P_{(-1)1}^k=F_{01}^k=F_{(-1)2}^k=0$.

Proof. Substituting series (57) and (58) into the jump and the stress in (41) at $\theta=\pm\pi$, the following expressions are obtained for the complementarity conditions (33):

$$[\![u_\theta^k]\!]=-2\sum_{n=1}^{\infty}\Big\{r^{n/2}\Big[(U_{n1}+\tilde U_{n1}^k)+\Big(\frac{n}{2}+\kappa\Big)(U_{n3}+\tilde U_{n3}^k)+\frac{(n-2)\beta}{4n}(P_{(n-2)1}+\tilde P_{(n-2)1}^k)\Big]\sin\frac{n-2}{2}\pi\Big\}, \tag{61}$$

$$\{\!\{\tau_{\theta\theta}^k+f^k\}\!\}=-\mu n\sum_{n=1}^{\infty}\Big\{r^{(n-2)/2}\Big[(U_{n1}+\tilde U_{n1}^k)+\frac{n+2}{2}\Big((U_{n3}+\tilde U_{n3}^k)+\frac{\beta}{2n}(P_{(n-2)1}+\tilde P_{(n-2)1}^k)\Big)-\frac{1}{\mu n}F_{(n-2)1}^k\Big]\cos\frac{n-2}{2}\pi\Big\}. \tag{62}$$

Even n implies that $\sin\pi(n-2)/2=0$ and identity $[\![u_\theta^k]\!]=0$ in (61). Then based on (62),

$$\begin{aligned}
&\Big[U_{n1}+\frac{n+2}{2}\Big(U_{n3}+\frac{\beta}{2n}P_{(n-2)1}\Big)\Big]\cos\frac{n-2}{2}\pi\ge 0,\\
&\Big[\tilde U_{n1}^k+\frac{n+2}{2}\Big(\tilde U_{n3}^k+\frac{\beta}{2n}\tilde P_{(n-2)1}^k\Big)-\frac{1}{\mu n}F_{(n-2)1}^k\Big]\cos\frac{n-2}{2}\pi\ge 0
\end{aligned}$$

leads to $\{\!\{\tau_{\theta\theta}^k+f^k\}\!\}\le 0$, which together with (48) and (54) composes conditions (59).

For odd n we have respectively $\cos\pi(n-2)/2=0$ and identity $\{\!\{\tau_{\theta\theta}^k+f^k\}\!\}=0$ in (62). Rearranging the terms in (61) as follows, the inequality $[\![u_\theta^k]\!]\le 0$ requires that

$$\begin{aligned}
&\Big[U_{n1}+\frac{n-2}{2}\Big(U_{n3}+\frac{\beta}{2n}P_{(n-2)1}\Big)+(\kappa+1)U_{n3}\Big]\sin\frac{n-2}{2}\pi\ge 0,\\
&\Big[\tilde U_{n1}^k+\frac{n-2}{2}\Big(\tilde U_{n3}^k+\frac{\beta}{2n}\tilde P_{(n-2)1}^k\Big)+(\kappa+1)\tilde U_{n3}^k\Big]\sin\frac{n-2}{2}\pi\ge 0,
\end{aligned}$$

which together with conditions (49) and (55) builds (59). The proof is completed.

It can be observed that seven relations (59) connect twelve factors with five free parameters, and nine relations (60) connect twelve factors with three free parameters. The inequalities here can be replaced with equations.

Further, we study the first asymptotic terms in (57), (58) called singular solutions, which are of primary importance in engineering practice.

4. Singular solution

Extracting the asymptotic term with $m=n=1$ in formulas (57), (58) and (60), using the Landau notation $\mathcal{O}(r)$ as $r\to 0$, we obtain the decoupled expressions for the displacement

$$\mathbf{u}^k-\mathbf{u}^k(\mathbf{0})=r^{1/2}\sum_{i=1}^{4}(U_{1i}+\tilde U_{1i}^k)\boldsymbol{\Psi}_{\frac{1}{2}i}+\mathcal{O}(r), \tag{63}$$

since $P_{(-1)i}=F_{(-1)i}^k=0$ at $i=1,2$, and the decoupled expressions for the pore pressure are

$$p^k-f^k(\mathbf{0})=r^{1/2}\Big((P_{11}+\tilde P_{11}^k)\cos\frac{\theta}{2}+(P_{12}+\tilde P_{12}^k)\sin\frac{\theta}{2}\Big)+\mathcal{O}(r), \tag{64}$$

with twelve factors $U_{11},\tilde U_{11}^k,\dots,U_{14},\tilde U_{14}^k,P_{11},\tilde P_{11}^k,P_{12},\tilde P_{12}^k$ satisfying

$$U_{11}-\frac{1}{2}U_{13}=0,\quad U_{12}+\frac{3}{2}U_{14}=0,\quad U_{13}\le 0,\quad \tilde U_{11}^k-\frac{1}{2}\tilde U_{13}^k=0,\quad \tilde U_{13}^k\le 0, \tag{65}$$

$$P_{12}=0,\quad \tilde P_{12}^k=F_{12}^k,\quad \tilde U_{12}^k+\frac{3}{2}\tilde U_{14}^k=0,\quad \tilde U_{14}^k=\tilde U_{14}^{k-1}. \tag{66}$$

The last equation $\tilde U_{14}^k=\tilde U_{14}^{k-1}$ in (66) implies k-independence, hence, k-dependent factors $\tilde U_{14}^k=\tilde U_{12}^k=0$ should be set in (66). This reduces the factors to eight unknowns $U_{11},\dots,U_{14},\tilde U_{11}^k,\tilde U_{13}^k,P_{11},\tilde P_{11}^k$ satisfying the five relations in (65).

Applying Theorem 1 and the non-energy solutions (see Maz'ya et al. (2000, Section 8.5)), we infer the following.

Theorem 2 (*Singular Solution*). *Under fluid pressure f^k prescribed such that assumption (51) holds, the energy solution to the variational problem (23) and (24) admits the following asymptotic expansion as $r\to 0$ for the solid-phase displacement:*

$$\mathbf{u}^k-\mathbf{u}^k(\mathbf{0})=\frac{1}{2\sqrt{2\pi}\mu}r^{1/2}\big(K_I^k\boldsymbol{\Psi}_I+K_{II}\boldsymbol{\Psi}_{II}\big)+\mathcal{O}(r), \tag{67}$$

and for the pore pressure:

$$p^k-f^k(\mathbf{0})=r^{1/2}\Big(P_{11}^k\cos\frac{\theta}{2}+F_{12}^k\sin\frac{\theta}{2}\Big)+\mathcal{O}(r), \tag{68}$$

with three factors $K_I^k\ge 0$, K_{II}, and P_{11}^k, where the vectors are

$$\boldsymbol{\Psi}_I(\theta)=\begin{pmatrix}-\frac{1}{2}\cos\frac{3\theta}{2}-\Big(\frac{1}{2}-\kappa\Big)\cos\frac{\theta}{2}\\ \frac{1}{2}\sin\frac{3\theta}{2}-\Big(\frac{1}{2}+\kappa\Big)\sin\frac{\theta}{2}\end{pmatrix},\quad
\boldsymbol{\Psi}_{II}(\theta)=\begin{pmatrix}-\frac{3}{2}\sin\frac{3\theta}{2}-\Big(\frac{1}{2}-\kappa\Big)\sin\frac{\theta}{2}\\ -\frac{3}{2}\cos\frac{3\theta}{2}+\Big(\frac{1}{2}+\kappa\Big)\cos\frac{\theta}{2}\end{pmatrix}. \tag{69}$$

Let $\boldsymbol{\tau}^k\mathbf{n}$ denote the normal force at the outer boundary $\partial\Omega$. For the non-energy displacement

$$\zeta(r,\theta)=r^{-1/2}Z_I\begin{pmatrix}\frac{3}{2}\cos\frac{\theta}{2}-\Big(\frac{1}{2}+\kappa\Big)\cos\frac{3\theta}{2}\\ -\frac{3}{2}\sin\frac{\theta}{2}-\Big(\frac{1}{2}-\kappa\Big)\sin\frac{3\theta}{2}\end{pmatrix}+r^{-1/2}Z_{II}\begin{pmatrix}\frac{1}{2}\sin\frac{\theta}{2}-\Big(\frac{1}{2}+\kappa\Big)\sin\frac{3\theta}{2}\\ \frac{1}{2}\cos\frac{\theta}{2}+\Big(\frac{1}{2}-\kappa\Big)\cos\frac{3\theta}{2}\end{pmatrix} \tag{70}$$

with free factors Z_I and Z_{II}, the corresponding boundary force $\sigma(\zeta)\mathbf{n}$ at $\partial\Omega$, and the dilatation

$$\mathrm{tr}\varepsilon(\zeta) = (\kappa-1)r^{-3/2}\Big(Z_I\cos\frac{3\theta}{2} + Z_{II}\sin\frac{3\theta}{2}\Big), \tag{71}$$

the stress intensity factors K_I^k and K_{II} can be calculated using the following integral formula

$$\begin{aligned}&\int_{\partial\Omega}\big[\tau^k\mathbf{n}\cdot\zeta - \sigma(\zeta)\mathbf{n}\cdot\big(\mathbf{u}^k-\mathbf{u}^k(\mathbf{0})\big)\big]\,dS \\ &\quad+\int_{\Omega_k}\alpha p^k\mathrm{tr}\varepsilon(\zeta)\,rdrd\theta + \int_{\Gamma_k}[\![\tau^k_{\theta\theta}\zeta_\theta]\!]\,rdr \\ &= \begin{cases} K_I^k \quad \text{for } Z_I = \dfrac{1}{4\sqrt{2\pi}(1-\nu)} \text{ and } Z_{II}=0, \\ K_{II} \quad \text{for } Z_I = 0 \text{ and } Z_{II} = \dfrac{1}{4\sqrt{2\pi}(1-\nu)}. \end{cases}\end{aligned} \tag{72}$$

For the normal derivative $\nabla p^k\cdot\mathbf{n}$ at $\partial\Omega$ and the non-energy pore pressure

$$\xi(r,\theta) = \frac{1}{\pi}r^{-1/2}\cos\frac{\theta}{2}, \tag{73}$$

factor P_{11}^k can be calculated using the following formula

$$P_{11}^k = \int_{\partial\Omega}\big[(\nabla p^k\cdot\mathbf{n})\xi - (\nabla\xi\cdot\mathbf{n})\big(p^k-f^k(\mathbf{0})\big)\big]\,dS - \int_{\Gamma_k}[\![\xi_{,\theta}\big(f^k-f^k(\mathbf{0})\big)]\!]\,rdr. \tag{74}$$

Proof. Setting the factors in accordance with (65),

$$\begin{aligned}&\frac{1}{2\sqrt{2\pi}\mu}K_I^k = -2(U_{11}+\tilde U_{11}^k) = -U_{13}-\tilde U_{13}^k \ge 0, \\ &\frac{1}{2\sqrt{2\pi}\mu}K_{II} = -\frac{2}{3}U_{12} = U_{14}, \quad P_{11}^k = P_{11}+\tilde P_{11}^k,\end{aligned} \tag{75}$$

and combining the respective vectors in (37),

$$\Psi_I = -\frac{1}{2}\Psi_{\frac12 1} - \Psi_{\frac12 3}, \quad \Psi_{II} = -\frac{3}{2}\Psi_{\frac12 2} + \Psi_{\frac12 4}.$$

Thus, from (63) and (64) we arrive straightforwardly at (67)–(69), and the equilibrium equations

$$\begin{aligned}&p^k_{,rr} + \frac{1}{r}p^k_{,r} + \frac{1}{r^2}p^k_{,\theta\theta} = 0, \quad \tau^k_{rr,r} + \frac{1}{r}\tau^k_{rr} + \frac{1}{r}\tau^k_{r\theta,\theta} - \frac{1}{r}\tau^k_{\theta\theta} = 0, \\ &\tau^k_{r\theta,r} + \frac{2}{r}\tau^k_{r\theta} + \frac{1}{r}\tau^k_{\theta\theta,\theta} = 0.\end{aligned} \tag{76}$$

It is well-known (e.g., Kozlov et al. (2001, Section 4.2)) that the singular solution $r^{1/2}\big(K_I^k\Psi_I + K_{II}\Psi_{II}\big)$ in (67) fulfills the stress-free conditions at the crack. Therefore, the energy solution $\mathbf{u}^k$ and p^k satisfies asymptotically the following boundary conditions at Γ_k:

$$\tau^k_{r\theta}|_{\theta=\pm\pi} = 0, \quad \tau^k_{\theta\theta}|_{\theta=\pm\pi} = \mathcal{O}(1). \tag{77}$$

As $n=-1$ in the series (46) and (47), from Lemma 2 we get a non-energy solution

$$\zeta(r,\theta) = r^{-1/2}\sum_{i=1}^{4}U_{(-1)i}\Psi_{(-\frac12)i}, \quad \xi(r,\theta) = r^{-1/2}\Big(P_{(-1)1}\cos\frac{\theta}{2} - P_{(-1)2}\sin\frac{\theta}{2}\Big) \tag{78}$$

for the Poisson and Lamé equations expressed in polar coordinates

$$\begin{aligned}&\xi_{,rr} + \frac{1}{r}\xi_{,r} + \frac{1}{r^2}\xi_{,\theta\theta} = 0, \quad \sigma_{rr,r}(\zeta) + \frac{1}{r}\sigma_{rr}(\zeta) + \frac{1}{r}\sigma_{r\theta,\theta}(\zeta) - \frac{1}{r}\sigma_{\theta\theta}(\zeta) = 0, \\ &\sigma_{r\theta,r}(\zeta) + \frac{2}{r}\sigma_{r\theta}(\zeta) + \frac{1}{r}\sigma_{\theta\theta,\theta}(\zeta) = 0,\end{aligned} \tag{79}$$

with six factors $U_{(-1)1},\dots,U_{(-1)4},P_{(-1)1},P_{(-1)2}$. To fulfill the homogeneous conditions at the crack

$$\xi|_{\theta=\pm\pi} = 0, \quad \sigma_{r\theta}(\zeta)|_{\theta=\pm\pi} = 0, \quad \sigma_{\theta\theta}(\zeta)|_{\theta=\pm\pi} = 0, \tag{80}$$

according to (49) at $m=0$, we set

$$Z_I = \frac{2}{3}U_{(-1)1} = U_{(-1)3}, \quad Z_{II} = 2U_{(-1)2} = -U_{(-1)4}, \quad P_{(-1)2} = 0, \tag{81}$$

and from (78) derive representations (70) and (73) normalized by factor $P_{(-1)1} = 1/\pi$. The formula for $\mathrm{tr}\varepsilon(\zeta)$ in (71) follows from (40) as $\gamma = -1/2$.

Let $B_\rho(\mathbf{0})$ denote a disk of radius $\rho>0$ centered at origin $\mathbf{0}$. We consider a domain $\Omega_k\setminus\overline{B_\rho(\mathbf{0})}$, bounded by the crack Γ_k, outer boundary $\partial\Omega$, and circle $\partial B_\rho(\mathbf{0})$ of radius $\rho>0$ (see Fig. 1). Excluding the neighborhood of the crack-tip and using the equilibrium Eqs. (76) in the sector of angle 2π, akin to (20), we have Green's formula

$$\int_{\Omega_k\setminus B_\rho(\mathbf{0})}\tau^k : \varepsilon(\zeta)\,rdrd\theta = \int_{\partial(\Omega_k\setminus B_\rho(\mathbf{0}))}\tau^k\mathbf{n}\cdot\zeta\,dS. \tag{82}$$

Interchanging $\mathbf{u}^k$ and ζ, by virtue of (9) and equilibrium Eqs. (79), we get

$$\begin{aligned}&\int_{\Omega_k\setminus B_\rho(\mathbf{0})}\tau^k : \varepsilon(\zeta)\,rdrd\theta = \int_{\Omega_k\setminus B_\rho(\mathbf{0})}\big(\sigma(\zeta):\varepsilon(\mathbf{u}^k) - \alpha p^k\mathrm{tr}\varepsilon(\zeta)\big)\,rdrd\theta \\ &= \int_{\partial(\Omega_k\setminus B_\rho(\mathbf{0}))}\sigma(\zeta)\mathbf{n}\cdot\big(\mathbf{u}^k-\mathbf{u}^k(\mathbf{0})\big)\,dS - \int_{\Omega_k\setminus B_\rho(\mathbf{0})}\alpha p^k\mathrm{tr}\varepsilon(\zeta)\,rdrd\theta.\end{aligned} \tag{83}$$

The respective Green's formula for the pore pressure is given by

$$\begin{aligned}&\int_{\partial(\Omega_k\setminus B_\rho(\mathbf{0}))}(\nabla p^k\cdot\mathbf{n})\xi\,dS = \int_{\Omega_k\setminus B_\rho(\mathbf{0})}\nabla p^k\cdot\nabla\xi\,rdrd\theta \\ &= \int_{\Omega_k\setminus B_\rho(\mathbf{0})}\nabla\xi\cdot\nabla p^k\,rdrd\theta = \int_{\partial(\Omega_k\setminus B_\rho(\mathbf{0}))}(\nabla\xi\cdot\mathbf{n})\big(p^k-f^k(\mathbf{0})\big)\,dS,\end{aligned} \tag{84}$$

recalling that $\varepsilon(\mathbf{u}^k) = \varepsilon(\mathbf{u}^k-\mathbf{u}^k(\mathbf{0}))$ and $\nabla p^k = \nabla(p^k-f^k(\mathbf{0}))$. From (82)–(84) it follows Green's second identities at the boundary $\partial(\Omega_k\setminus\overline{B_\rho(\mathbf{0})})$ for the displacement:

$$\begin{aligned}&\int_{\partial\Omega}\big[\tau^k\mathbf{n}\cdot\zeta - \sigma(\zeta)\mathbf{n}\cdot\big(\mathbf{u}^k-\mathbf{u}^k(\mathbf{0})\big)\big]\,dS + \int_{\Omega_k\setminus B_\rho(\mathbf{0})}\alpha p^k\mathrm{tr}\varepsilon(\zeta)\,rdrd\theta \\ &= \int_{\Gamma_k\setminus B_\rho(\mathbf{0})}[\![\tau^k\mathbf{n}\cdot\zeta - \sigma(\zeta)\mathbf{n}\cdot\big(\mathbf{u}^k-\mathbf{u}^k(\mathbf{0})\big)]\!]\,rdr \\ &-\rho\int_{-\pi}^{\pi}\big[\tau^k\mathbf{n}\cdot\zeta - \sigma(\zeta)\mathbf{n}\cdot\big(\mathbf{u}^k-\mathbf{u}^k(\mathbf{0})\big)\big]_{r=\rho}d\theta,\end{aligned} \tag{85}$$

and for the pore pressure considering $p^k = f^k$ at the crack:

$$\begin{aligned}&\int_{\partial\Omega}\big[(\nabla p^k\cdot\mathbf{n})\xi - (\nabla\xi\cdot\mathbf{n})\big(p^k-f^k(\mathbf{0})\big)\big]\,dS \\ &= \int_{\Gamma_k\setminus B_\rho(\mathbf{0})}[\![(\nabla p^k\cdot\mathbf{n})\xi - (\nabla\xi\cdot\mathbf{n})\big(p^k-f^k(\mathbf{0})\big)]\!]\,rdr \\ &-\rho\int_{-\pi}^{\pi}\big[(\nabla p^k\cdot\mathbf{n})\xi - (\nabla\xi\cdot\mathbf{n})\big(f^k-f^k(\mathbf{0})\big)\big]_{r=\rho}d\theta,\end{aligned} \tag{86}$$

where the normal force and normal derivative at the circle and the crack imply, respectively,

$$\begin{aligned}&\tau^k\mathbf{n} = -\begin{pmatrix}\tau^k_{rr}\\ \tau^k_{r\theta}\end{pmatrix}, \quad \nabla p^k\cdot\mathbf{n} = -p^k_{,r} \text{ at } \partial B_\rho(\mathbf{0}); \\ &\tau^k\mathbf{n} = -\begin{pmatrix}\tau^k_{r\theta}\\ \tau^k_{\theta\theta}\end{pmatrix}, \quad \nabla p^k\cdot\mathbf{n} = -\frac{1}{r}p^k_{,\theta} \text{ at } \Gamma_k.\end{aligned} \tag{87}$$

Inserting expressions (87) in the integrals over the crack part $\Gamma_k\setminus B_\rho(\mathbf{0})$, the terms in (85) are: $\tau^k_{r\theta}=0$, $\tau^k_{\theta\theta}\zeta_\theta = \mathcal{O}(r^{-1/2})$ in series (70), (77), and $\sigma_{r\theta}(\zeta) = \sigma_{\theta\theta}(\zeta) = 0$ due to (80), hence

$$\begin{aligned}&-\int_{\Gamma_k\setminus B_\rho(\mathbf{0})}[\![\tau^k_{r\theta}\zeta_r + \tau^k_{\theta\theta}\zeta_\theta - \sigma_{r\theta}(\zeta)\big(u^k_r-u^k_r(\mathbf{0})\big) - \sigma_{\theta\theta}(\zeta)\big(u^k_\theta-u^k_\theta(\mathbf{0})\big)]\!]\,rdr \\ &\to -\int_{\Gamma_k}[\![\tau^k_{\theta\theta}\zeta_\theta]\!]\,rdr\end{aligned} \tag{88}$$

in the limit as $\rho\to 0$. The respective terms in (86) are: $\xi_{,\theta}(f^k-f^k(\mathbf{0}))r = \mathcal{O}(r)$ in accordance with series (51), (73), and $\xi=0$ at $\theta=\pm\pi$ by (80), providing the limit

$$\begin{aligned}&-\int_{\Gamma_k\setminus B_\rho(\mathbf{0})}[\![p^k_{,\theta}\xi - \xi_{,\theta}\big(f^k-f^k(\mathbf{0})\big)]\!]\,rdr \\ &\to \int_{\Gamma_k}[\![\xi_{,\theta}\big(f^k-f^k(\mathbf{0})\big)]\!]\,rdr \quad \text{as } \rho\to 0.\end{aligned} \tag{89}$$

Due to the asymptotic $p^k \mathrm{tr}\varepsilon(\zeta) r = \mathcal{O}(r^{-1/2})$ according to (68), (71), there also exists a limit

$$\int_{\Omega_k \setminus B_\rho(\mathbf{0})} \alpha p^k \mathrm{tr}\varepsilon(\zeta)\, r dr d\theta \to \int_{\Omega_k} \alpha p^k \mathrm{tr}\varepsilon(\zeta)\, r dr d\theta \quad \text{as } \rho \to 0. \tag{90}$$

For the expressions (87) of the normal force and the normal derivative at the circle $\partial B_\rho(\mathbf{0})$, according to Eqs. (41) with $\gamma = 1/2$ and (63), we have the following asymptotic formulas

$$\begin{aligned}
\tau^k_{rr}|_{r=\rho} &= \rho^{-1/2}\mu \Big\{ U^k_{11}\cos\frac{3\theta}{2} + U^k_{12}\sin\frac{3\theta}{2} - \frac{5}{2}U^k_{13}\cos\frac{\theta}{2} \\
&\quad + \frac{5}{2}U^k_{14}\sin\frac{\theta}{2} \Big\} + \mathcal{O}(1), \\
\tau^k_{r\theta}|_{r=\rho} &= \rho^{-1/2}\mu \Big\{ -U^k_{11}\sin\frac{3\theta}{2} + U^k_{12}\cos\frac{3\theta}{2} - \frac{1}{2}U^k_{13}\sin\frac{\theta}{2} \\
&\quad - \frac{1}{2}U^k_{14}\cos\frac{\theta}{2} \Big\} + \mathcal{O}(1).
\end{aligned} \tag{91}$$

Multiplication of the normal force in (91) by $\zeta(\rho,\theta) \sim \rho^{-1/2}$ from (78) yields

$$\begin{aligned}
\rho \int_{-\pi}^{\pi} \big[\tau^k_{rr}\zeta_r + \tau^k_{r\theta}\zeta_\theta\big]_{r=\rho} d\theta &= \pi\mu \big\{ -2\kappa U^k_{11}U_{(-1)3} \\
&\quad + 2\kappa U^k_{12}U_{(-1)4} - 2U^k_{13}U_{(-1)1} + 2U^k_{14}U_{(-1)2} \big\} + \mathcal{O}(\rho^{1/2}) \\
&= \frac{\sqrt{\pi}}{2\sqrt{2}}\big\{(\kappa+3)K^k_I Z_I + (3\kappa+1)K_{II}Z_{II}\big\} + \mathcal{O}(\rho^{1/2})
\end{aligned} \tag{92}$$

according to notation (75), (81), using $U^k_{12} = U_{12}$, $U^k_{14} = U_{14}$ and the orthonormality in $L^2(-\pi,\pi)$ of a trigonometric basis:

$$\frac{1}{\sqrt{\pi}}\cos\frac{3\theta}{2}, \quad \frac{1}{\sqrt{\pi}}\sin\frac{3\theta}{2}, \quad \frac{1}{\sqrt{\pi}}\cos\frac{\theta}{2}, \quad \frac{1}{\sqrt{\pi}}\sin\frac{\theta}{2}.$$

Analogously, using (41) with $\gamma = -1/2$ and $P^k_{(-3)i} = 0$ for $i = 1,2$ we find

$$\begin{aligned}
\sigma_{rr}(\zeta)|_{r=\rho} &= \rho^{-3/2}\mu \Big\{ -U_{(-1)1}\cos\frac{3\theta}{2} - U_{(-1)2}\sin\frac{3\theta}{2} \\
&\quad + \frac{7}{2}U_{(-1)3}\cos\frac{\theta}{2} - \frac{7}{2}U_{(-1)4}\sin\frac{\theta}{2} \Big\} + \mathcal{O}(r^{-1}), \\
\sigma_{r\theta}(\zeta)|_{r=\rho} &= \rho^{-3/2}\mu \Big\{ U_{(-1)1}\sin\frac{3\theta}{2} - U_{(-1)2}\cos\frac{3\theta}{2} \\
&\quad + \frac{3}{2}U_{(-1)3}\sin\frac{\theta}{2} + \frac{3}{2}U_{(-1)4}\cos\frac{\theta}{2} \Big\} + \mathcal{O}(r^{-1}),
\end{aligned}$$

and after multiplying it by $\mathbf{u}^k - \mathbf{u}^k(\mathbf{0}) \sim \rho^{1/2}$ in (63) we calculate

$$\begin{aligned}
-\rho \int_{-\pi}^{\pi} &\big[\sigma_{rr}(\zeta)\big(u^k_r - u^k_r(\mathbf{0})\big) + \sigma_{r\theta}(\zeta)\big(u^k_\theta - u^k_\theta(\mathbf{0})\big)\big]_{r=\rho} d\theta \\
&= \pi\mu\big\{-2U^k_{11}U_{(-1)3} + 2U^k_{12}U_{(-1)4} - 2\kappa U^k_{13}U_{(-1)1} + 2\kappa U^k_{14}U_{(-1)2}\big\} + \mathcal{O}(\rho^{1/2}) \\
&= \frac{\sqrt{\pi}}{2\sqrt{2}}\big\{(1+3\kappa)K^k_I Z_I + (3+\kappa)K_{II}Z_{II}\big\} + \mathcal{O}(\rho^{1/2}).
\end{aligned} \tag{93}$$

Substituting (88),(90), (92), (93) in (85) and passing $\rho \to 0$ such that

$$\begin{aligned}
\int_{\partial\Omega} &\big[\boldsymbol{\tau}^k\mathbf{n}\cdot\zeta - \sigma(\zeta)\mathbf{n}\cdot\big(\mathbf{u}^k - \mathbf{u}^k(\mathbf{0})\big)\big]\, dS + \int_{\Omega_k} \alpha p^k \mathrm{tr}\varepsilon(\zeta)\, r dr d\theta \\
&= -\int_{\Gamma_k} [\![\tau^k_{\theta\theta}\zeta_\theta]\!]\, dr + \sqrt{2\pi}(\kappa+1)\big(K^k_I Z_I + K_{II}Z_{II}\big),
\end{aligned}$$

where $\kappa + 1 = 4(1-\nu)$ by (38), gives us the integral formula (72) for the calculation of stress intensity factors K^k_I and K_{II}.

Substituting $p^k - f^k(\mathbf{0}) \sim \rho^{1/2}$ from (68) and $\xi(\rho,\theta) \sim \rho^{-1/2}$ from (73),

$$\begin{aligned}
\rho \int_{-\pi}^{\pi} &\big[p^k_{,r}\xi_r - \xi_{,r}\big(p^k - f^k(\mathbf{0})\big)\big]_{r=\rho} d\theta \\
&= \int_{-\pi}^{\pi} \Big\{ \frac{1}{2}\Big(P^k_{11}\cos\frac{\theta}{2} + F^k_{12}\sin\frac{\theta}{2}\Big)\frac{1}{\pi}\cos\frac{\theta}{2} \\
&\quad + \frac{1}{2\pi}\cos\frac{\theta}{2}\Big(P^k_{11}\cos\frac{\theta}{2} + F^k_{12}\sin\frac{\theta}{2}\Big)\frac{1}{\pi}\Big\}\, d\theta + \mathcal{O}(\rho^{1/2}) = P^k_{11} + \mathcal{O}(\rho^{1/2}),
\end{aligned}$$

and using (89) after taking the limit as $\rho \to 0$ in (86) exactly implies integral formula (74) for finding factor P^k_{11} of the singularity of the pore pressure. This completes the proof.

We remark that the formulas proven in Theorem 2 can also be applied to determine the k-independent and k-dependent factors separately on splitting

$$K^k_I = K_I + \tilde{K}^k_I, \quad P^k_{11} = P_{11} + \tilde{P}^k_{11}$$

according to the k-independent and k-dependent solutions in (28).

5. Concluding remarks

The stress intensity factors K^k_I and K_{II} in (67) are used in the classic Griffith criterion of brittle fracture to determine whether the crack Γ_k starts to propagate when the strain energy release rate (calculated using the Griffith–Irwin formula) exceeds the prescribed threshold (the fracture toughness $G_{\mathrm{cr}} > 0$):

$$\frac{1-\nu}{2\mu}\big((K^k_I)^2 + (K_{II})^2\big) > G_{\mathrm{cr}},$$

else, Γ_k does not grow. However, the Griffith–Irwin formula is not proven for poroelastic problems.

When driven by hydraulic fracture, factor $K^k_I \geq 0$ that describes mode-I crack opening is k-dependent, whereas factor K_{II} under mode-II crack shear is k-independent.

In the integral formula (74) for factor P^k_{11} characterizing the singularity of the pore pressure, the term over the crack

$$\int_{\Gamma_k} [\![\xi_{,\theta}\big(f^k - f^k(\mathbf{0})\big)]\!]\, rdr$$

is due to the inhomogeneous Dirichlet boundary condition at $\theta = \pm\pi$. In particular, this term is not present under constant fluid pressure $f^k = f^k(\mathbf{0})$.

Under a stress-free crack causing penetration if $K^k_I < 0$, the normal stress $\tau^k_{\theta\theta} = 0$ at $\theta = \pm\pi$, hence, the integral over the crack is excluded from formula (72):

$$\int_{\Gamma_k} [\![\tau^k_{\theta\theta}\zeta_\theta]\!]\, rdr = 0.$$

The pore pressure p^k is included in the formula (72) for K^k_I and K_{II} by means of the domain integral

$$\int_{\Omega_k} \alpha p^k \mathrm{tr}\varepsilon(\zeta)\, r dr d\theta.$$

This term is caused by the fluid-driven fracture and principally distinguishes a poroelastic body with crack from the pure elastic case of $p^k = 0$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Proof of Lemma 1

We take a general power ansatz of the form

$$\mathbf{u}^k(r,\theta) = r^\gamma(\ln r)^{\gamma_0}U^k\boldsymbol{\Psi}(\theta), \quad p^k(r,\theta) = r^\beta(\ln r)^{\beta_0}P^k\Phi(\theta) \tag{A.1}$$

with scaling factors U^k, P^k, and similarly at $k-1$, then substitute it first in the mass balance Eq. (11):

$$\begin{aligned}
&r^\beta(\ln r)^{\beta_0}S(P^k - P^{k-1})\Phi + \alpha r^{\gamma-1}\big\{ (\ln r)^{\gamma_0}[\Psi'_\theta + (\gamma+1)\Psi_r] \\
&\quad + (\ln r)^{\gamma_0-1}\gamma_0\Psi_r \big\}(U^k - U^{k-1}) \\
&\quad - \varkappa\Delta t_k r^{\beta-2}\big\{ (\ln r)^{\beta_0}[\Phi'' + \beta^2\Phi] \\
&\quad + (\ln r)^{\beta_0-1}2\beta\beta_0\Phi + (\ln r)^{\beta_0-2}\beta_0(\beta_0-1)\Phi \big\} P^k = 0.
\end{aligned} \tag{A.2}$$

We can observe that $\sigma(\mathbf{u}^k) \sim r^{\gamma-1}$ and $p^k \sim r^\beta$ in the effective stress in (9) are compatible when $\beta = \gamma - 1$.

For $\beta = \gamma - 1$, gathering the terms with like powers of r in (A.2), we get two equations:

$$(\ln r)^{\beta_0} S(P^k - P^{k-1})\boldsymbol{\Phi} + \alpha \left\{ (\ln r)^{\gamma_0}[\Psi'_\theta + (\gamma+1)\Psi_r] + (\ln r)^{\gamma_0 - 1}\gamma_0 \Psi_r \right\} (U^k - U^{k-1}) = 0, \tag{A.3}$$

and

$$\left\{(\ln r)^{\beta_0}[\boldsymbol{\Phi}'' + (\gamma-1)^2\boldsymbol{\Phi}] + (\ln r)^{\beta_0 - 1} 2(\gamma-1)\beta_0 \boldsymbol{\Phi} + (\ln r)^{\beta_0-2}\beta_0(\beta_0 - 1)\boldsymbol{\Phi}\right\} P^k = 0. \tag{A.4}$$

Since the powers of $\ln r$ in (A.4) are $\beta_0 \neq \beta_0 - 1 \neq \beta_0 - 2$, then $P^k \neq 0$ is possible when

$$\boldsymbol{\Phi}'' + (\gamma - 1)^2 \boldsymbol{\Phi} = 0, \tag{A.5}$$

and either $\beta_0 = 0$; or $\gamma = 1$ and $\beta_0 = 1$, which implies that $p^k \sim \ln r$ and is not admissible in the energy H^1-space.

Inserting $\beta_0 = 0$ in (A.3) and gathering the like terms of $\ln r$, the following two cases are possible. Either the log-powers $\gamma_0 = \beta_0 = 0$, then (A.3) yields

$$S(P^k - P^{k-1})\boldsymbol{\Phi} + \alpha[\Psi'_\theta + (\gamma+1)\Psi_r](U^k - U^{k-1}) = 0; \tag{A.6}$$

or $\gamma_0 = \beta_0 + 1 = 1$ such that

$$\Psi'_\theta + (\gamma+1)\Psi_r = 0, \tag{A.7}$$

$$S(P^k - P^{k-1})\boldsymbol{\Phi} + \alpha \Psi_r (U^k - U^{k-1}) = 0. \tag{A.8}$$

Next, we substitute (A.1) in the equilibrium Eqs. (10). Because of the symmetry of the mixed derivatives, differentiating (6) yields two compatibility conditions

$$\begin{aligned}\varepsilon_{rr,\theta}(\mathbf{u}^k) &= u^k_{r,r\theta} = u^k_{r,\theta r} = \left(2r\varepsilon_{r\theta}(\mathbf{u}^k) - r u^k_{\theta,r} + u^k_\theta\right)_{,r} \\ &= 2r\varepsilon_{r\theta,r}(\mathbf{u}^k) + 2\varepsilon_{r\theta}(\mathbf{u}^k) - r u^k_{\theta,rr},\end{aligned} \tag{A.9}$$

$$\begin{aligned}r\varepsilon_{\theta\theta,r}(\mathbf{u}^k) + \varepsilon_{\theta\theta}(\mathbf{u}^k) - \varepsilon_{rr}(\mathbf{u}^k) &= \left(r\varepsilon_{\theta\theta}(\mathbf{u}^k) - u_r\right)_{,r} = u^k_{\theta,\theta r} \\ = u^k_{\theta,r\theta} = \left(2\varepsilon_{r\theta}(\mathbf{u}^k) - \frac{1}{r}u^k_{r,\theta} + \frac{1}{r}u^k_\theta\right)_{,\theta} &= 2\varepsilon_{r\theta,\theta}(\mathbf{u}^k) - \frac{1}{r}u^k_{r,\theta\theta} + \frac{1}{r}u^k_{\theta,\theta}.\end{aligned} \tag{A.10}$$

Inserting $\varepsilon_{\theta\theta,r}(\mathbf{u}^k)$ from (A.10) in the first equation and $\varepsilon_{rr,\theta}(\mathbf{u}^k)$ from (A.9) in the second equation of (10), we express them equivalently via $\operatorname{tr}\varepsilon(\mathbf{u}^k)$, $\mathbf{u}^k$ and p^k in the form

$$\begin{aligned}&(\lambda+2\mu)\left(\operatorname{tr}\varepsilon(\mathbf{u}^k)\right)_{,r} - \frac{\mu}{r}\left(u^k_{\theta,r} - \frac{1}{r}u^k_{r,\theta} + \frac{1}{r}u^k_\theta\right)_{,\theta} - \alpha p^k_{,r} = 0, \\ &\frac{\lambda+2\mu}{r}\left(\operatorname{tr}\varepsilon(\mathbf{u}^k)\right)_{,\theta} + \mu\left(u^k_{\theta,r} - \frac{1}{r}u^k_{r,\theta} + \frac{1}{r}u^k_\theta\right)_{,r} - \frac{\alpha}{r}p^k_{,\theta} = 0,\end{aligned} \tag{A.11}$$

In the case of (A.7) and (A.8), using $\beta = \gamma - 1$, the series (A.1) turns in

$$\mathbf{u}^k(r,\theta) = r^\gamma (\ln r) U^k \boldsymbol{\Psi}(\theta), \quad p^k(r,\theta) = r^{\gamma-1}P^k\boldsymbol{\Phi}(\theta), \tag{A.12}$$

where the function $\boldsymbol{\Phi}$ satisfies (A.5). In order to substitute (A.12) in (A.11), we calculate the expressions

$$\operatorname{tr}\varepsilon(\mathbf{u}^k) = r^{\gamma-1}\left(\Psi_r + \ln r[(\gamma+1)\Psi_r + \Psi'_\theta]\right)U^k = r^{\gamma-1}U^k\Psi_r$$

due to (A.7), and

$$u^k_{\theta,r} - \frac{1}{r}u^k_{r,\theta} + \frac{1}{r}u^k_\theta = r^{\gamma-1}\left(\Psi_\theta + \ln r[(\gamma+1)\Psi_\theta - \Psi'_r]\right)U^k.$$

After the substitution and division by the factor $r^{\gamma-2}$, this leads to the following equations

$$\begin{aligned}&\left\{(\lambda+2\mu)(\gamma-1)\Psi_r - \mu\left(\Psi'_\theta + \ln r[(\gamma+1)\Psi'_\theta - \Psi''_r]\right)\right\}U^k - \alpha(\gamma-1)P^k\boldsymbol{\Phi} = 0, \\ &\left\{(\lambda+2\mu)\Psi'_r + \mu\left(2\gamma\Psi_\theta - \Psi'_r + (\gamma-1)\ln r[(\gamma+1)\Psi_\theta - \Psi'_r]\right)\right\}U^k - \alpha P^k\boldsymbol{\Phi}' = 0,\end{aligned} \tag{A.13}$$

which necessitates

$$(\gamma+1)\Psi_\theta - \Psi'_r = 0. \tag{A.14}$$

On the one hand, the relations (A.7) and (A.14) together imply

$$\boldsymbol{\Psi}'' + (\gamma+1)^2\boldsymbol{\Psi} = 0. \tag{A.15}$$

On the other hand, excluding from (A.13) the derivatives Ψ'_r and Ψ'_θ with the help of (A.7) and (A.14), using identities for the parameters from (8):

$$\begin{aligned}(\lambda+2\mu)(\gamma-1) + \mu(\gamma+1) &= (\kappa\gamma - 1)(\lambda+\mu), \\ (\lambda+2\mu)(\gamma+1) + \mu(\gamma-1) &= (\kappa\gamma + 1)(\lambda+\mu),\end{aligned}$$

turns (A.13) in

$$(\lambda+\mu)(\kappa\gamma-1)\Psi_r U^k - \alpha(\gamma-1)P^k\boldsymbol{\Phi} = 0, \quad (\lambda+\mu)(\kappa\gamma+1)\Psi_\theta U^k - \alpha P^k\boldsymbol{\Phi}' = 0.$$

Due to (A.5) this follows

$$\boldsymbol{\Psi}'' + (\gamma-1)^2\boldsymbol{\Psi} = 0. \tag{A.16}$$

From (A.15) and (A.16) we conclude with only $\gamma = 0$ possible in the series (A.12). Hence $\mathbf{u}^k \sim \ln r$ in (A.12), which is not admissible in the energy H^1-space.

The Eq. (A.5) has two general solutions $\boldsymbol{\Phi} = \cos(\gamma-1)\theta$ and $\boldsymbol{\Phi} = \sin(\gamma-1)\theta$. Therefore, in the remaining case of the log-powers $\gamma_0 = \beta_0 = 0$, from (A.1) we conclude with (35) within the ansatz

$$\mathbf{u}^k(r,\theta) = r^\gamma U^k\boldsymbol{\Psi}(\theta), \quad p^k(r,\theta) = r^{\gamma-1}\left(P^k_1\cos(\gamma-1)\theta + P^k_2\sin(\gamma-1)\theta\right). \tag{A.17}$$

According to (A.17) we calculate

$$u^k_{\theta,r} - \frac{1}{r}u^k_{r,\theta} + \frac{1}{r}u^k_\theta = -r^{\gamma-1}U^k[\Psi'_r - (\gamma+1)\Psi_\theta], \quad \operatorname{tr}\varepsilon(\mathbf{u}^k) = r^{\gamma-1}U^k[\Psi'_\theta + (\gamma+1)\Psi_r]. \tag{A.18}$$

Now we introduce notation $\beta := \alpha/(\lambda+2\mu)$ from (38) and auxiliary functions $X^k(\theta)$, $Y^k(\theta)$ by

$$\begin{aligned}X^k &:= U^k[\Psi'_r - (\gamma+1)\Psi_\theta], \\ Y^k &:= U^k[\Psi'_\theta + (\gamma+1)\Psi_r] - \beta\left(P^k_1\cos(\gamma-1)\theta + P^k_2\sin(\gamma-1)\theta\right).\end{aligned} \tag{A.19}$$

Applying (A.18) and (A.19), Eqs. (A.11) divided by the factor $r^{\gamma-2}$ turn into the 1st order ODE system for X^k and Y^k:

$$(\lambda+2\mu)(\gamma-1)Y^k + \mu(X^k)' = 0, \quad (\lambda+2\mu)(Y^k)' - \mu(\gamma-1)X^k = 0. \tag{A.20}$$

After differentiation, (A.20) are decoupled within the 2nd order ODE system

$$(X^k)'' + (\gamma-1)^2X^k = 0, \quad (Y^k)'' + (\gamma-1)^2Y^k = 0,$$

which has a general solution with arbitrary coefficients c^k_1, c^k_2, $\tilde{c}^k_1$, $\tilde{c}^k_2$:

$$X^k = c^k_1\cos(\gamma-1)\theta + c^k_2\sin(\gamma-1)\theta, \quad Y^k = \tilde{c}^k_1\cos(\gamma-1)\theta + \tilde{c}^k_2\sin(\gamma-1)\theta. \tag{A.21}$$

Inserting (A.19) and (A.21) in (A.20) and using the identities (A.25), we derive the 2nd order equations for Ψ_r and Ψ_θ:

$$\begin{aligned}U^k[\Psi''_r + (\gamma+1)^2\Psi_r] &= 4\gamma\left(a^k_1\cos(\gamma-1)\theta + a^k_2\sin(\gamma-1)\theta\right), \\ U^k[\Psi''_\theta + (\gamma+1)^2\Psi_\theta] &= 4\gamma\left(\tilde{a}^k_1\cos(\gamma-1)\theta + \tilde{a}^k_2\sin(\gamma-1)\theta\right),\end{aligned} \tag{A.22}$$

where the coefficients for $\gamma \neq 0$ are given by

$$\begin{aligned}a^k_1 &= -\frac{(\gamma-\kappa)(\lambda+\mu)}{4\gamma\mu}\tilde{c}^k_1 + \frac{\beta(\gamma+1)}{4\gamma}P^k_1, \\ a^k_2 &= -\frac{(\gamma-\kappa)(\lambda+\mu)}{4\gamma\mu}\tilde{c}^k_2 + \frac{\beta(\gamma+1)}{4\gamma}P^k_2, \\ \tilde{a}^k_1 &= -\frac{(\gamma+\kappa)(\lambda+\mu)}{4\gamma(\lambda+2\mu)}c^k_1 + \frac{\beta(\gamma-1)}{4\gamma}P^k_2, \\ \tilde{a}^k_2 &= -\frac{(\gamma+\kappa)(\lambda+\mu)}{4\gamma(\lambda+2\mu)}c^k_2 - \frac{\beta(\gamma-1)}{4\gamma}P^k_1,\end{aligned} \tag{A.23}$$

and $a^k_1 = a^k_2 = \tilde{a}^k_1 = \tilde{a}^k_2 = 0$ when $\gamma = 0$.

The solution for the inhomogeneous Eqs. (A.22) can be found by summing the general and particular solutions in the following form with arbitrary factors U^k_1, U^k_2, $\tilde{U}^k_1$, $\tilde{U}^k_2$:

$$U^k\Psi_r = U^k_1\cos(\gamma+1)\theta + U^k_2\sin(\gamma+1)\theta + a^k_1\cos(\gamma-1)\theta + a^k_2\sin(\gamma-1)\theta,$$

$$U^k\Psi_\theta = \tilde U_1^k \cos(\gamma+1)\theta + \tilde U_2^k \sin(\gamma+1)\theta + \tilde a_1^k \cos(\gamma-1)\theta + \tilde a_2^k \sin(\gamma-1)\theta. \tag{A.24}$$

Using expression (A.24) we calculate X^k and Y^k in (A.19):

$$\begin{aligned} X^k &= (\gamma+1)(U_2^k - \tilde U_1^k)\cos(\gamma+1)\theta - (\gamma+1)(U_1^k + \tilde U_2^k)\sin(\gamma+1)\theta \\ &+ [(\gamma-1)a_2^k - (\gamma+1)\tilde a_1^k]\cos(\gamma-1)\theta - [(\gamma-1)a_1^k + (\gamma+1)\tilde a_2^k]\sin(\gamma-1)\theta, \\ Y^k &= (\gamma+1)(U_1^k + \tilde U_2^k)\cos(\gamma+1)\theta + (\gamma+1)(U_2^k - \tilde U_1^k)\sin(\gamma+1)\theta \\ &+ [(\gamma+1)a_1^k + (\gamma-1)\tilde a_2^k - \beta P_1^k]\cos(\gamma-1)\theta \\ &+ [(\gamma+1)a_2^k - (\gamma-1)\tilde a_1^k - \beta P_2^k]\sin(\gamma-1)\theta, \end{aligned}$$

then insert them and their derivatives in Eqs. (A.20) and use identities for the parameters from (8):

$$(\lambda+2\mu)(\gamma-1)-\mu(\gamma+1) = (\gamma-\kappa)(\lambda+\mu), \quad (\lambda+2\mu)(\gamma+1)-\mu(\gamma-1) = (\gamma+\kappa)(\lambda+\mu), \tag{A.25}$$

such that, using the notation $\tilde\beta := \alpha/(\lambda+\mu)$,

$$\begin{aligned} (\lambda+\mu)(\gamma-1)\ \{ & [(\gamma+\kappa)a_1^k + (\gamma-\kappa)\tilde a_2^k - \tilde\beta P_1^k]\cos(\gamma-1)\theta \\ &+ [(\gamma+\kappa)a_2^k - (\gamma-\kappa)\tilde a_1^k - \tilde\beta P_2^k]\sin(\gamma-1)\theta\ \} \\ &+ (\gamma+1)(\gamma-\kappa)\ \{ (U_1^k + \tilde U_2^k)\cos(\gamma+1)\theta \\ &+ (U_2^k - \tilde U_1^k)\sin(\gamma+1)\theta\ \} = 0, \\ (\lambda+\mu)(\gamma-1)\ \{ & [(\gamma+\kappa)a_2^k - (\gamma-\kappa)\tilde a_1^k - \tilde\beta P_2^k]\cos(\gamma-1)\theta \\ &- [(\gamma+\kappa)a_1^k + (\gamma-\kappa)\tilde a_2^k - \tilde\beta P_1^k]\sin(\gamma-1)\theta\ \} \\ &+ (\gamma+1)(\gamma+\kappa)\ \{ (U_2^k - \tilde U_1^k)\cos(\gamma+1)\theta \\ &- (U_1^k + \tilde U_2^k)\sin(\gamma+1)\theta\ \} = 0. \end{aligned} \tag{A.26}$$

The homogeneous system (A.26) for $\gamma \neq 0$ has a nontrivial solution

$$\tilde U_1^k = U_2^k, \quad \tilde U_2^k = -U_1^k, \quad \tilde a_1^k = \frac{\gamma+\kappa}{\gamma-\kappa}a_2^k - \frac{\tilde\beta}{\gamma-\kappa}P_2^k, \quad \tilde a_2^k = -\frac{\gamma+\kappa}{\gamma-\kappa}a_1^k + \frac{\tilde\beta}{\gamma-\kappa}P_1^k. \tag{A.27}$$

At $\gamma = 0$, the like terms in (A.26) are gathered together as follows

$$[-(\lambda+\mu)\kappa(U_1^k + \tilde U_2^k) - \alpha P_1^k]\cos\theta - [(\lambda+\mu)\kappa(U_2^k - \tilde U_1^k) + \alpha P_2^k]\sin\theta = 0,$$
$$[(\lambda+\mu)\kappa(U_2^k - \tilde U_1^k) + \alpha P_2^k]\cos\theta + [-(\lambda+\mu)\kappa(U_1^k + \tilde U_2^k) + \alpha P_1^k]\sin\theta = 0.$$

This yields

$$\tilde U_1^k = U_2^k - \frac{\alpha}{\lambda+3\mu}P_2^k, \quad \tilde U_2^k = -U_1^k + \frac{\alpha}{\lambda+3\mu}P_1^k, \quad a_1^k = a_2^k = \tilde a_1^k = \tilde a_2^k = 0. \tag{A.28}$$

Substituting into (A.24) expressions (A.23) and (A.27) for $\gamma \neq 0$, using $(\lambda+2\mu)\beta = (\lambda+\mu)\tilde\beta$, such that

$$\begin{aligned} U^k\Psi_r &= U_1^k\cos(\gamma+1)\theta + U_2^k\sin(\gamma+1)\theta \\ &- \frac{\lambda+\mu}{4\gamma\mu}(\gamma-\kappa)\big(\tilde c_1^k\cos(\gamma-1)\theta + \tilde c_2^k\sin(\gamma-1)\theta\big) \\ &+ \frac{\beta}{4\gamma}(\gamma+1)\big(P_1^k\cos(\gamma-1)\theta + P_2^k\sin(\gamma-1)\theta\big), \\ U^k\Psi_\theta &= U_2^k\cos(\gamma+1)\theta - U_1^k\sin(\gamma+1)\theta \\ &- \frac{\lambda+\mu}{4\gamma\mu}(\gamma+\kappa)\big(\tilde c_2^k\cos(\gamma-1)\theta - \tilde c_1^k\sin(\gamma-1)\theta\big) \\ &+ \frac{\beta}{4\gamma}(\gamma-1)\big(P_2^k\cos(\gamma-1)\theta - P_1^k\sin(\gamma-1)\theta\big), \end{aligned}$$

we arrive at formula (34) with the vectors from (37), where the notation is used

$$U_3^k = -\frac{\lambda+\mu}{4\gamma\mu}\tilde c_1^k, \quad U_4^k = -\frac{\lambda+\mu}{4\gamma\mu}\tilde c_2^k.$$

The respective substitution of (A.28) in (A.24) results in a specific formula (42) at $\gamma = 0$.

Applying the formulas (34) for the displacement and (35) for pore pressure, from (6) we calculate the strain components (39) and dilatation (40); from (7) we derive the stress components (41). In Eq. (A.6), inserting expressions (35) for p^k and similarly for p^{k-1}, and the expressions (40) for $\mathrm{tr}\varepsilon(\mathbf{u}^k)$ and $\mathrm{tr}\varepsilon(\mathbf{u}^{k-1})$, collecting the like terms necessitates relations (36) between the factors $U_3^k - U_3^{k-1}$ and $P_1^k - P_1^{k-1}$, and between $U_4^k - U_4^{k-1}$ and $P_2^k - P_2^{k-1}$.

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