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Three-field mixed formulation of elasticity model nonlinear in the mean normal stress for the problem of non-penetrating cracks in bodies

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A B S T R A C T

A class of models in the theory of elasticity is considered, where a material response between the linearized strain and the stress is assumed to be nonlinear with respect to the mean normal stress. The governing system is endowed with a mixed variational formulation treating the displacement, the deviatoric stress and the mean normal stress as three independent fields. The body contains an inner crack subjected to a non-penetration condition. The resulting problem is described as a pseudo-monotone variational inequality. Its well-posedness is established based on the Galerkin approximation, penalty regularization, and the existence theorem developed by Brézis.

1. Introduction

Our study is motivated by newly developed nonlinear constitutive models between the linearized strain $\boldsymbol{\varepsilon}$ and the Cauchy stress $\mathbf{T}$ useful in describing the damage of porous materials such as rocks and concrete (see Rajagopal (2021)). A subclass of models whose moduli depend on the density is represented by

$$\boldsymbol{\varepsilon} = C_1\mathbf{T} + \alpha(\mathrm{tr}\boldsymbol{\varepsilon})(\mathrm{tr}\mathbf{T})\mathbf{I}, \tag{1}$$

where $\mathbf{I}$ is the identity tensor, the constant modulus $C_1 > 0$, and α is a scalar valued function of the trace of the strain. On taking the trace of (1) we get

$$\mathrm{tr}\boldsymbol{\varepsilon} = C_1\mathrm{tr}\mathbf{T} + \alpha(\mathrm{tr}\boldsymbol{\varepsilon})3(\mathrm{tr}\mathbf{T}). \tag{2}$$

If the implicit Eq. (2) can be solved with respect to $\mathrm{tr}\boldsymbol{\varepsilon}$, then its substitution into (1) provides us with the following relation

$$\boldsymbol{\varepsilon} = C_1\mathbf{T} + \beta(\mathrm{tr}\mathbf{T})(\mathrm{tr}\mathbf{T})\mathbf{I}, \tag{3}$$

with a scalar function β depending on $\mathrm{tr}\mathbf{T}$.

Even the particular implicit constitutive Eq. (1) linear in both stress and strain when

$$\alpha(\mathrm{tr}\boldsymbol{\varepsilon}) = E_2(1 + \lambda_2\mathrm{tr}\boldsymbol{\varepsilon}), \tag{4}$$

where E_2 and λ_2 are constant, obeys neither bounded nor monotone behavior. As the consequence, no any standard existence theorem can be applied to such models. Numerical simulations investigating stress concentration at a circular hole under biaxial loading were provided in Murru and Rajagopal (2021). Using threshold of the mean normal stress, well-posedness result for an approximate model was proved in the recent paper Itou et al. (2021) based on the pseudo-monotone theory. Now we test the concept of pseudo-monotone variational inequalities for applicability to the class of nonlinear problems with non-penetrating cracks in bodies described by constitutive relations like (3).

We know that within the context of nonlinear elasticity, a material response allows the material moduli to depend on the mechanical

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pressure, that is, the mean normal stress $p = -\mathrm{tr}\mathbf{T}/3$ for a 3d body (see Itou et al. (2019b); Rajagopal and Saccomandi (2009)). Let us decompose the stress tensor as

$$\mathbf{T} = \mathbf{T}^* - p\mathbf{I}, \quad \mathrm{tr}\mathbf{T}^* = 0. \tag{5}$$

With the help of (5) and applying the volumetric-deviatoric decomposition of the strain

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^* + \frac{1}{3}(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I}, \quad \mathrm{tr}\boldsymbol{\varepsilon}^* = 0, \tag{6}$$

the response equation stemming from (3) is decoupled into deviatoric and spherical parts as

$$\boldsymbol{\varepsilon}^* = C_1\mathbf{T}^*, \quad \mathrm{tr}\boldsymbol{\varepsilon} = -C_2\mathscr{F}(p), \tag{7}$$

where the constant modulus $C_2 > 0$ and nonlinear function $\mathscr{F}(p)$ in (7) is related to specific models.

As example models, the classical linearized elasticity is recovered by

$$C_1 = \frac{1}{2\mu}, \quad C_2 = \frac{1}{3K}, \quad \mathscr{F}(p) = p, \tag{8}$$

with constant shear modulus μ and bulk modulus K. The power-law hardening was considered, e.g., in Itou et al. (2019a):

$$\mathscr{F}(p) = \frac{p}{(1+\kappa|p|^r)^{\frac{2-s}{r}}} \quad \kappa, r > 0, \quad s > 1, \tag{9}$$

and in Itou et al. (2017) the limiting small strain model as $s = 1$ in (9). Moreover, the non-constant bulk modulus $K(p)$ has been suggested to take the following form (see Itou et al. (2019b)):

$$C_2\mathscr{F}(p) = \frac{p}{3K(p)}, \quad K(p) = K_0\left(1+\kappa+\frac{2}{\pi}\arctan(rp+s)\right), \tag{10}$$

where $K_0, \kappa, r > 0$ and $s \in \mathbb{R}$. Assuming (4) leads to a discontinuous linear-fractional function (see the derivation and approximation in Itou et al. (2021)):

$$\mathscr{F}(p) = \frac{p}{1 - p/\tau_{\mathrm{cr}}}, \quad \tau_{\mathrm{cr}} \in \mathbb{R}. \tag{11}$$

We cite also (Bauer et al. (2020)) for a relevant hypoplastic model describing cohesionless granular materials and soils in geomechanics.

In the present paper, we investigate well-posedness for the problem of the equilibrium of a body described by nonlinear constitutive equations of type (7). Together with the strain $\boldsymbol{\varepsilon}$ implying the symmetric gradient of the displacement $\mathbf{u}$, the deviatoric stress $\mathbf{T}^*$ and the mean normal stress p constitute three independent fields. Therefore, we endow the governing system of equations with the corresponding mixed variational formulation. The mixed three-field formulation of the linear elastic problem is advantageous from the perspective of FEM analysis (see Anaya et al. (2019); Chiumenti et al. (2015)).

The other challenge of our modeling consists of fracture modeling of the body, which contains an inner crack (see Bratov et al. (2009)). Allowing contact between the opposite crack faces affects a Signorini-type non-penetration condition on the jump of the displacement across the crack. The variational theory describing non-penetrating cracks in elastic solids and plates was established by Khludnev and Kovtunenko (2000) and Khludnev and Sokołowski (2000). The theory was developed further for dissipative contact phenomena at the crack owing to friction (Itou et al. (2011)), cohesion (Kovtunenko (2011)), for Kirchhoff–Love plates (Lazarev and Itou (2019)), the Boussinesq indentation problem (Itou et al. (2020b)) and the limiting small strain model (Itou et al. (2017)). The extension was obtained for a class of variational problems stated in singular domains (see Fremiot et al. (2009); Itou et al. (2020a)), for notches (Kulvait et al. (2019)), anti-cracks and inclusions (Khludnev and Popova (2020); Rudoy and Shcherbakov (2020)). For appropriate numerical methods, we refer to Furtsev et al. (2020); Hintermüller et al. (2005) and Namm and Tsoy (2019).

Due to the presence of a unilateral constraint describing a non-penetrating crack in a body, the problem under our current study implies a variational inequality. Applying the general theory of pseudo-monotone variational inequalities (see Brézis (1968); Ovcharova and Gwinner (2014)), well-posedness of the problem is established based on the Galerkin approximation, penalty regularization, and existence theorem following (Troianiello, 1987, Theorem 4.16).

2. Three-field mixed formulation of the problem

We start with the geometric description of a cracked body.

Let Ω be a domain in the Euclidean space $\mathbb{R}^3$ with the boundary $\partial\Omega$ carrying the outward unit normal vector $\mathbf{n} = (n_1, n_2, n_3)$. We assume that the boundary $\partial\Omega = \overline{\Gamma_{\mathrm{N}}} \cup \overline{\Gamma_{\mathrm{D}}}$ consists of two disjointed parts: the Neumann boundary Γ_{N} and the nonempty Dirichlet boundary Γ_{D}. Let there be an interface Σ, carrying a unit normal vector $\mathbf{n} = (n_1, n_2, n_3)$, that splits Ω into two sub-domains $\Omega^{\pm}$ with Lipschitz continuous boundaries $\partial\Omega^{\pm}$, and the intersection $\partial\Omega^{\pm} \cap \overline{\Gamma_{\mathrm{D}}} \neq \varnothing$. We allow the interface part $\Gamma_{\mathrm{c}} \subsetneq \Sigma$ to be a crack inside Ω, which two opposite faces $\Gamma_{\mathrm{c}}^{\pm}$ are distinguished as the corresponding parts of $\partial\Omega^{\pm}$. The set $\Omega_{\mathrm{c}} = \Omega \setminus \overline{\Gamma_{\mathrm{c}}}$ with the boundary $\partial\Omega_{\mathrm{c}} = \partial\Omega \cup \overline{\Gamma_{\mathrm{c}}^{+}} \cup \overline{\Gamma_{\mathrm{c}}^{-}}$ is called the domain with the crack.

In the domain with crack including boundary $\overline{\Omega_{\mathrm{c}}} = \Omega_{\mathrm{c}} \cup \partial\Omega_{\mathrm{c}}$, we define a displacement vector $\mathbf{u} = (u_1, u_2, u_3)(\mathbf{x})$ over spatial points $\mathbf{x} = (x_1, x_2, x_3)$. Its jump across the crack faces is generally non-zero:

$$[\![\mathbf{u}]\!](\mathbf{x}) := \mathbf{u}|_{\mathbf{x}\in\Gamma_{\mathrm{c}}^{+}} - \mathbf{u}|_{\mathbf{x}\in\Gamma_{\mathrm{c}}^{-}}. \tag{12}$$

Following (Khludnev and Kovtunenko, 2000, Section 1.1.7), a non-penetration condition across the crack faces is imposed on a normal jump:

$$[\![\mathbf{u}\cdot\mathbf{n}]\!](\mathbf{x}) \geq 0 \quad \text{for } \mathbf{x} \in \Gamma_{\mathrm{c}}, \tag{13}$$

where $\mathbf{u}\cdot\mathbf{n} = \sum_{i=1}^{3} u_i n_i$ implies a scalar product of vectors. The crack Γ_{c} is open at point $\mathbf{x}$ where the strict inequality holds in (13); otherwise, the crack is closed if the equality takes place. We note that $[\![\mathbf{u}\cdot\mathbf{n}]\!] = [\![\mathbf{u}]\!]\cdot\mathbf{n}$ because the normal has no jump.

Now we state the elastic problem under the non-penetration condition (13).

Let the body force $\mathbf{f} = (f_1, f_2, f_3)(\mathbf{x})$ for $\mathbf{x} \in \Omega_{\mathrm{c}}$ and the boundary traction $\mathbf{g} = (g_1, g_2, g_3)(\mathbf{x})$ for $\mathbf{x} \in \Gamma_{\mathrm{N}}$ be given. For $\mathbf{x} \in \overline{\Omega_{\mathrm{c}}}$, we look for the displacement vector $\mathbf{u}(\mathbf{x})$, which determines the symmetric tensor of linearized strain $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1}^{3}(\mathbf{x})$ using the formula:

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3, \tag{14}$$

the symmetric tensor of the deviatoric stress $\mathbf{T}^* = (T_{ij}^*)_{i,j=1}^{3}(\mathbf{x})$ and the mean normal stress $p(\mathbf{x})$ according to the decomposition (5), which together satisfy the equilibrium equation

$$-\sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij}^* + \frac{\partial p}{\partial x_i} = f_i, \quad i = 1, 2, 3, \quad \text{in } \Omega_{\mathrm{c}}, \tag{15}$$

and the following constitutive equations according to the nonlinear response (7):

$$C_1\mathbf{T}^* - \boldsymbol{\varepsilon}(\mathbf{u})^* = \mathbf{0}, \quad C_2\mathscr{F}(p) + \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}) = 0. \tag{16}$$

The deviatoric strain tensor $\boldsymbol{\varepsilon}(\mathbf{u})^*$ is defined in (6), and $\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}) = \sum_{i=1}^{3}\varepsilon_{ii}(\mathbf{u}) = \mathrm{div}(\mathbf{u})$ implies divergence.

It is worth noting that, if the modulus $C_2 \to 0$, then the latter equation in (16) implies $\mathrm{div}(\mathbf{u}) = 0$, and along with (15), the limit relations describe the Stokes system for the incompressible solid:

$$-\frac{1}{2C_1}\Delta\mathbf{u}+\nabla p=\mathbf{f},\quad \mathrm{div}(\mathbf{u})=0\quad \text{in } \Omega_c.$$

The governing system (14)–(16) is augmented by the following boundary conditions: the Dirichlet condition for the clamp

$$\mathbf{u}=\mathbf{0}\quad \text{on } \Gamma_D; \tag{17}$$

the Neumann type condition for the traction

$$\mathbf{Tn}=\mathbf{g}\quad \text{on } \Gamma_N; \tag{18}$$

and the complete system of conditions due to the non-penetration (13):

$$\mathbf{Tn}-(\mathbf{Tn}\cdot\mathbf{n})\mathbf{n}=\mathbf{0}\quad \text{on } \Gamma_c^{\pm}, \tag{19}$$

$$[\![\mathbf{Tn}\cdot\mathbf{n}]\!]=0\quad \text{on } \Gamma_c, \tag{20}$$

$$[\![\mathbf{u}\cdot\mathbf{n}]\!]\ge 0,\quad \mathbf{Tn}\cdot\mathbf{n}\le 0,\quad (\mathbf{Tn}\cdot\mathbf{n})[\![\mathbf{u}\cdot\mathbf{n}]\!]=0\quad \text{on } \Gamma_c, \tag{21}$$

where the boundary stress $\mathbf{Tn}=(\sum_{j=1}^{3}T_{ij}n_j)_{i=1,2,3}$, and its jump $[\![\mathbf{Tn}]\!]$ is defined according to (12). The vector $\mathbf{Tn}$ is split into the normal $\mathbf{Tn}\cdot\mathbf{n}=\sum_{i,j=1}^{3}T_{ij}n_jn_i$ and tangential $\mathbf{Tn}-(\mathbf{Tn}\cdot\mathbf{n})\mathbf{n}$ components. Then, (19) implies zero tangential stress, (20) describes the continuity of the normal stress, and the complementarity conditions (21) imply the crack being pointwise either when open, i.e.,

$$[\![\mathbf{u}\cdot\mathbf{n}]\!]>0,\quad \mathbf{Tn}\cdot\mathbf{n}=0,$$

or closed, i.e.,

$$[\![\mathbf{u}\cdot\mathbf{n}]\!]=0,\quad \mathbf{Tn}\cdot\mathbf{n}\le 0.$$

For further use we denote the symmetric tensors by $\mathbb{R}^{3\times3}_{\mathrm{sym}}$. Let $\mathbf{f}\in L^2(\Omega_c;\mathbb{R}^3)$ and $\mathbf{g}\in L^2(\Gamma_N;\mathbb{R}^3)$. A weak solution to the nonlinear boundary value problem (14)–(21) is given by the mixed variational formulation. Find the triple of functions: the displacement $\mathbf{u}\in H^1(\Omega_c;\mathbb{R}^3)$ with $[\![\mathbf{u}\cdot\mathbf{n}]\!]\ge 0$ on Γ_c and $\mathbf{u}=\mathbf{0}$ at Γ_D, the deviatoric stress $\mathbf{T}^*\in L^2(\Omega_c;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ with $\mathrm{tr}\mathbf{T}^*=0$, and the mean normal stress $p\in L^2(\Omega_c;\mathbb{R})$, which satisfy the following system:

$$\int_{\Omega_c}(\mathbf{T}^*\cdot\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v})^*-p\,\mathrm{div}(\mathbf{u}-\mathbf{v}))\,d\mathbf{x}\ \le\ \int_{\Omega_c}\mathbf{f}\cdot(\mathbf{u}-\mathbf{v})\,d\mathbf{x}+\int_{\Gamma_N}\mathbf{g}\cdot(\mathbf{u}-\mathbf{v})\,dS_{\mathbf{x}}, \tag{22}$$

$$\int_{\Omega_c}(C_1\mathbf{T}^*-\boldsymbol{\varepsilon}(\mathbf{u})^*)\cdot\mathbf{S}^*\,d\mathbf{x}=0,\quad \int_{\Omega_c}(C_2\mathscr{T}(p)+\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}))q\,d\mathbf{x}=0, \tag{23}$$

for all admissible test functions $\mathbf{v}\in H^1(\Omega_c;\mathbb{R}^3)$ such that $[\![\mathbf{v}\cdot\mathbf{n}]\!]\ge 0$ on Γ_c and $\mathbf{v}=\mathbf{0}$ at Γ_D, for $\mathbf{S}^*\in L^2(\Omega_c;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ with $\mathrm{tr}\mathbf{S}^*=0$, and $q\in L^2(\Omega_c;\mathbb{R})$. The linearized strain tensors $\boldsymbol{\varepsilon}(\mathbf{v})$ and its deviatoric part $\boldsymbol{\varepsilon}(\mathbf{v})^*$ are defined according to formulas (14) and (6). Here, the dot stands for the scalar product of tensors $\mathbf{T}\cdot\mathbf{S}=\sum_{i,j=1}^{3}T_{ij}S_{ij}$.

The variational inequality (22) is obtained in a standard way after multiplication of the equilibrium Eq. (15) with $\mathbf{v}-\mathbf{u}$ and integration by part over Ω_c with the help of boundary conditions (17)–(21) (for details see (Khludnev and Kovtunenko, 2000, Section 1.4.4)). Whereas the variational Eq. (23) follow directly from the pointwise Eq. (16). If we test (23) with $(\mathbf{T}^*-\mathbf{S}^*, p-q)$, add (22) and cancel the same terms, then using the Frobenius norm we derive a single relation:

$$\int_{\Omega_c}\Big(C_1\|\mathbf{T}^*\|^2-\mathbf{T}^*\cdot\boldsymbol{\varepsilon}(\mathbf{v})^*-(C_1\mathbf{T}^*-\boldsymbol{\varepsilon}(\mathbf{u})^*)\cdot\mathbf{S}^*+C_2\mathscr{T}(p)(p-q)+p\,\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{v})-\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u})q\Big)\,d\mathbf{x}\le\int_{\Omega_c}\mathbf{f}\cdot(\mathbf{u}-\mathbf{v})\,d\mathbf{x}+\int_{\Gamma_N}\mathbf{g}\cdot(\mathbf{u}-\mathbf{v})\,dS_{\mathbf{x}}. \tag{24}$$

Conversely, inserting $\mathbf{v}=\mathbf{u}$, $\mathbf{S}^*=\mathbf{T}^*\pm\mathbf{R}^*$ and $q=p\pm\eta$ into (24) follows the Eq. (23)with $\mathbf{S}^*=\mathbf{R}^*$, $q=\eta$, and consequently the inequality (22).

In the next section, we establish well-posedness to the variational inequality (24) provided by the pseudo-monotone property for the nonlinearity $\mathscr{T}(p)$.

3. Well-posedness of pseudo-monotone variational inequality

Before starting well-posedness analysis, we give two preliminaries. The Korn–Poincaré inequality takes place

$$\|\mathbf{u}\|^2_{L^2(\Omega_c;\mathbb{R}^3)}\le C_{KP}\|\boldsymbol{\varepsilon}(\mathbf{u})\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\mathrm{sym}})}\quad \text{if } \mathbf{u}=\mathbf{0} \text{ on } \Gamma_D. \tag{25}$$

Together with (25), a uniform continuity of the trace operator follows the estimate

$$\|\mathbf{u}\|^2_{L^2(\Gamma_N;\mathbb{R}^3)}\le C_{tr}\|\boldsymbol{\varepsilon}(\mathbf{u})\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\mathrm{sym}})}\quad \text{if } \mathbf{u}=\mathbf{0} \text{ on } \Gamma_D. \tag{26}$$

We assume that the function $p\mapsto\mathscr{T}(p)$ is continuous and bounded, i.e.,

$$\|\mathscr{T}(p)\|_{L^2(\Omega_c;\mathbb{R})}\le\overline{b}\|p\|_{L^2(\Omega_c;\mathbb{R})}, \tag{27}$$

coercive (with constants $0<\underline{b}\le\overline{b}$), i.e.,

$$\int_{\Omega_c}\mathscr{T}(p)p\,d\mathbf{x}\ge\underline{b}\|p\|^2_{L^2(\Omega_c;\mathbb{R})}, \tag{28}$$

and pseudo-monotone (see Brézis (1968)), i.e.,

$$\begin{aligned}&\text{if } p^k\rightharpoonup p \text{ weakly in } L^2(\Omega_c;\mathbb{R}) \text{ and } \limsup_{k\to\infty}\int_{\Omega_c}\mathscr{T}(p^k)(p^k-p)\,d\mathbf{x}\le 0,\\ &\text{then } \liminf_{k\to\infty}\int_{\Omega_c}\mathscr{T}(p^k)(p^k-q)\,d\mathbf{x}\ge\int_{\Omega_c}\mathscr{T}(p)(p-q)\,d\mathbf{x}\quad \forall q\in L^2(\Omega_c;\mathbb{R}).\end{aligned} \tag{29}$$

Theorem 1. (*Well-posedness*) *Under the assumptions (27)–(29), there exists a triple $\mathbf{u}\in H^1(\Omega_c;\mathbb{R}^3)$ with $[\![\mathbf{u}\cdot\mathbf{n}]\!]\ge 0$ on Γ_c and $\mathbf{u}=\mathbf{0}$ on Γ_D, $\mathbf{T}^*\in L^2(\Omega_c;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ with $\mathrm{tr}\mathbf{T}^*=0$, and $p\in L^2(\Omega_c;\mathbb{R})$, which solves the pseudo-monotone variational inequality (24).*

The solution satisfies the a-priori estimates:

$$C_1(1-\theta C_1)\|\mathbf{T}^*\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\mathrm{sym}})}+C_2\Big(\underline{b}-\frac{1}{3}C_2\overline{b}^2\theta\Big)\|p\|^2_{L^2(\Omega_c;\mathbb{R})}\le\frac{1}{2\theta}C(f,g), \tag{30}$$

with a positive weight $\theta<\min\Big(1/C_1, 3\underline{b}/(C_2\overline{b}^2)\Big)$, and

$$\begin{aligned}\|\boldsymbol{\varepsilon}(\mathbf{u})^*\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\mathrm{sym}})}&=C_1\|\mathbf{T}^*\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\mathrm{sym}})},\ \|\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^2(\Omega_c;\mathbb{R})}\\&\le C_2\overline{b}\|p\|_{L^2(\Omega_c;\mathbb{R})},\end{aligned} \tag{31}$$

where constant $C(f,g)>0$ is related to the given forces by

$$C(f,g):=\|f\|^2_{L^2(\Omega_c;\mathbb{R}^3)}+C_{tr}\|g\|^2_{L^2(\Gamma_N;\mathbb{R}^3)}. \tag{32}$$

Proof. We regularize the pseudo-monotone variational inequality (24) by the Galerkin approximation in finite spaces of dimension $k\in\mathbb{N}$ and by the penalization with a small parameter $\delta>0$. A solution to the regularized nonlinear equation exists by the Brouwer fixed-point theorem. We subsequently pass $k\to\infty$ and $\delta\to 0$ following (Troianiello, 1987, Theorem 4.16).

Regularization

Let the sequence of subspaces $V^k\subset V^{k+1}$ and $H^k\subset H^{k+1}$ in the space of admissible functions be a conforming approximation such that $\cup_{k=1}^{\infty} V^k$ is dense in $H^1(\Omega_c;\mathbb{R}^3)$ for the functions $\mathbf{v}=\mathbf{0}$ on Γ_D, the union $\cup_{k=1}^{\infty} H^k$ is dense in $L^2(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym})$, and $\mathbf{v}\in V^k$ implies $\boldsymbol{\varepsilon}(\mathbf{v})\in H^k$. We constitute the following regularized problem: Find the triple $(\mathbf{u}^{k,\delta},(\mathbf{T}^*)^{k,\delta},p^{k,\delta})$ such that

$$\mathbf{u}^{k,\delta}\in V^k,\quad (\mathbf{T}^*)^{k,\delta}-p^{k,\delta}\mathbf{I}\in H^k,\quad \mathrm{tr}(\mathbf{T}^*)^{k,\delta}=0, \tag{33}$$

satisfying the penalized variational equation

$$\begin{aligned}&\int_{\Omega_c}\Big(C_1\parallel(\mathbf{T}^{k,\delta})^*\parallel^2-(\mathbf{T}^{k,\delta})^*\cdot\boldsymbol{\varepsilon}(\mathbf{v}^m)^*-\big(C_1(\mathbf{T}^{k,\delta})^*-\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})^*\big)\cdot(\mathbf{S}^m)^*\\&+C_2\mathscr{F}(p^{k,\delta})(p^{k,\delta}-\mathrm{tr}\mathbf{S}^m)+p^{k,\delta}\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{v}^m)-\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})\mathrm{tr}\mathbf{S}^m\Big)\,d\mathbf{x}\\&-\int_{\Gamma_c}\frac{1}{\delta}[[\mathbf{u}^{k,\delta}\cdot\mathbf{n}]]^-[[(\mathbf{u}^{k,\delta}-\mathbf{v}^m)\cdot\mathbf{n}]]\,dS_{\mathbf{x}}=\int_{\Omega_c}\mathbf{f}\cdot(\mathbf{u}^{k,\delta}-\mathbf{v}^m)\,d\mathbf{x}\\&+\int_{\Gamma_N}\mathbf{g}\cdot(\mathbf{u}^{k,\delta}-\mathbf{v}^m)\,dS_{\mathbf{x}},\end{aligned} \tag{34}$$

for all admissible test functions $\mathbf{v}^m\in V^m$ and $\mathbf{S}^m\in H^m$, where $m\le k$, and

$$[[\mathbf{u}^{k,\delta}\cdot\mathbf{n}]]^-=-\min(0,[[\mathbf{u}^{k,\delta}\cdot\mathbf{n}]])\ge0$$

implies the negative part of the jump. Upon varying the test functions in (34), it follows an equivalent system consisting of the penalty equation

$$\begin{aligned}&\int_{\Omega_c}\big((\mathbf{T}^{k,\delta})^*\cdot\boldsymbol{\varepsilon}(\mathbf{v}^m)^*-p^{k,\delta}\mathrm{div}(\mathbf{v}^m)\big)\,d\mathbf{x}-\int_{\Gamma_c}\frac{1}{\delta}[[\mathbf{u}^{k,\delta}\cdot\mathbf{n}]]^-[\![\mathbf{v}^m\cdot\mathbf{n}]\!]\,dS_{\mathbf{x}}\\&=\int_{\Omega_c}\mathbf{f}\cdot\mathbf{v}^m\,d\mathbf{x}+\int_{\Gamma_N}\mathbf{g}\cdot\mathbf{v}^m\,dS_{\mathbf{x}},\end{aligned} \tag{35}$$

and two variational equations

$$\begin{aligned}&\int_{\Omega_c}\big(C_1(\mathbf{T}^{k,\delta})^*-\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})^*\big)\cdot(\mathbf{S}^m)^*\,d\mathbf{x}=0,\\&\int_{\Omega_c}\big(C_2\mathscr{F}(p^{k,\delta})+\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})\big)\mathrm{tr}\mathbf{S}^m\,d\mathbf{x}=0,\end{aligned} \tag{36}$$

which correspond to (22) and (23).

A-priori estimates

From (34) with $\mathbf{v}^m=\mathbf{0}$ and $\mathbf{S}^m=\mathbf{0}$, using the coercivity (28) and the trace inequality (26), subsequently applying Cauchy–Schwarz and weighted Young's inequality with a weight $\theta>0$, we obtain the estimate

$$\begin{aligned}&\int_{\Omega_c}\Big(C_1\parallel(\mathbf{T}^{k,\delta})^*\parallel^2+C_2\underline{b}(p^{k,\delta})^2\Big)\,d\mathbf{x}+\int_{\Gamma_c}\frac{1}{\delta}\big([[\mathbf{u}^{k,\delta}\cdot\mathbf{n}]]^-\big)^2\,dS_{\mathbf{x}}\\&\le\Big(\|f\|_{L^2(\Omega_c;\mathbb{R}^3)}+\sqrt{C_{\rm tr}}\|g\|_{L^2(\Gamma_N;\mathbb{R}^3)}\Big)\|\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym})}\\&\le\frac{1}{2\theta}C(f,g)+\theta\|\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym})},\end{aligned} \tag{37}$$

with the constant $C(f,g)$ defined in (32).

The H^1-vector norm of the displacement is defined as

$$\|\mathbf{u}^{k,\delta}\|^2_{H^1(\Omega_c;\mathbb{R}^3)}=\|\mathbf{u}^{k,\delta}\|^2_{L^2(\Omega_c;\mathbb{R}^3)}+\|\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym})},$$

and due to the Korn–Poincaré inequality (25), it is bounded by

$$\frac{1}{1+C_{\rm KP}}\|\mathbf{u}^{k,\delta}\|^2_{H^1(\Omega_c;\mathbb{R}^3)}\le\|\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym})}. \tag{38}$$

Therefore, in the following, we estimate the strain. Inserting $\mathbf{S}^m=\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})$ into (36) and using the boundedness (27) of $\mathscr{F}$ follows the relations

$$\begin{aligned}&\|\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})^*\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym})}=C_1\|(\mathbf{T}^{k,\delta})^*\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym})},\\&\|\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})\|_{L^2(\Omega_c;\mathbb{R})}\le C_2\overline{b}\|p^{k,\delta}\|_{L^2(\Omega_c;\mathbb{R})},\end{aligned} \tag{39}$$

and leads to the upper bound

$$\begin{aligned}\|\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym})}&=\int_{\Omega_c}\Big(\|\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})^*\|^2+\frac{1}{3}\mathrm{tr}^2\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})\Big)\,d\mathbf{x}\\&\le C_1^2\|(\mathbf{T}^{k,\delta})^*\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym})}+\frac{1}{3}\big(C_2\overline{b}\big)^2\|p^{k,\delta}\|^2_{L^2(\Omega_c;\mathbb{R})}.\end{aligned}$$

Substituting this into (37) results in a uniform estimate

$$\begin{aligned}&C_1(1-\theta C_1)\|(\mathbf{T}^{k,\delta})^*\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym})}+C_2\Big(\underline{b}-\frac{1}{3}C_2\overline{b}^2\theta\Big)\|p^{k,\delta}\|^2_{L^2(\Omega_c;\mathbb{R})}\\&+\int_{\Gamma_c}\frac{1}{\delta}\big([[\mathbf{u}^{k,\delta}\cdot\mathbf{n}]]^-\big)^2\,dS_{\mathbf{x}}\le\frac{1}{2\theta}C(f,g),\end{aligned} \tag{40}$$

where the factors are positive if the weight $\theta<\min\Big(1/C_1,3\underline{b}/(C_2\overline{b}^2)\Big)$.

Owing to the uniform bounds (38)–(40) and the continuity of $\mathscr{F}$, the solution (33) to (34) is inferred from the Brouwer fixed-point theorem.

The limit passage as $k\to\infty$

From the estimates (38)–(40), we infer a weakly convergent subsequence, still denoted by k for short, such that

$$\begin{aligned}&\mathbf{u}^{k,\delta}\rightharpoonup\mathbf{u}^\delta\ \text{weakly in}\ H^1(\Omega_c;\mathbb{R}^3),\quad p^{k,\delta}\rightharpoonup p^\delta\ \text{weakly in}\ L^2(\Omega_c;\mathbb{R}),\\&(\mathbf{T}^{k,\delta})^*\rightharpoonup(\mathbf{T}^\delta)^*\ \text{weakly in}\ L^2\big(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym}\big),\end{aligned} \tag{41}$$

preserving the equality $\mathrm{tr}(\mathbf{T}^\delta)^*=0$ and the Dirichlet condition $\mathbf{u}^\delta=\mathbf{0}$ on Γ_D. Moreover, based on the compactness argument,

$$\mathbf{u}^{k,\delta}\to\mathbf{u}^\delta\ \text{strongly in}\ L^2(\Gamma_c^\pm;\mathbb{R}^3). \tag{42}$$

To pass the limit in the finite-dimensional Eq. (34), we utilize the pseudo-monotone property (29) for the non-linearity $\mathscr{F}$.

Owing to the density of the finite-dimensional approximation, there exists a sequence

$$\mathbf{v}^{k,\delta}\in V^k,\quad (\mathbf{S}^{k,\delta})^*-p^{k,\delta}\mathbf{I}\in H^k,\quad \mathrm{tr}(\mathbf{S}^{k,\delta})^*=0$$

that converges strongly as $k\to\infty$:

$$\begin{aligned}&\mathbf{v}^{k,\delta}\to\mathbf{u}^\delta\ \text{strongly in}\ H^1(\Omega_c;\mathbb{R}^3),\quad q^{k,\delta}\to p^\delta\ \text{strongly in}\ L^2(\Omega_c;\mathbb{R}),\\&(\mathbf{S}^{k,\delta})^*\to(\mathbf{T}^\delta)^*\ \text{strongly in}\ L^2\big(\Omega_c;\mathbb{R}^{3\times3}_{\rm sym}\big).\end{aligned} \tag{43}$$

Testing the penalized variational Eq. (34) with

$$\mathbf{v}^m=\mathbf{u}^\delta+\mathbf{v}^{k,\delta}-\mathbf{u}^\delta,\quad(\mathbf{S}^m)^*=(\mathbf{T}^\delta)^*+(\mathbf{S}^{k,\delta}-\mathbf{T}^\delta)^*,\quad \mathrm{tr}\mathbf{S}^m=p^\delta+q^{k,\delta}-p^\delta,$$

we rearrange the terms as follows:

$$\int_{\Omega_c}\big(C_1(\mathbf{T}^{k,\delta})^*\cdot(\mathbf{T}^{k,\delta}-\mathbf{T}^\delta)^*+C_2\mathscr{F}(p^{k,\delta})(p^{k,\delta}-p^\delta)\big)\,d\mathbf{x}=I_{k,\delta}, \tag{44}$$

where convergences (41)–(43) provide $I_{k,\delta}\to0$ for the integral term

$$\begin{aligned}I_{k,\delta}:=&\int_{\Omega_c}\mathbf{f}\cdot(\mathbf{u}^{k,\delta}-\mathbf{v}^{k,\delta})\,d\mathbf{x}+\int_{\Gamma_N}\mathbf{g}\cdot(\mathbf{u}^{k,\delta}-\mathbf{v}^{k,\delta})\,dS_{\mathbf{x}}\\&+\int_{\Omega_c}\Big((\mathbf{T}^{k,\delta})^*\cdot\boldsymbol{\varepsilon}(\mathbf{v}^{k,\delta})^*-\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})^*\cdot(\mathbf{S}^{k,\delta})^*+C_1(\mathbf{T}^{k,\delta})^*\cdot(\mathbf{S}^{k,\delta}-\mathbf{T}^\delta)^*\\&-p^{k,\delta}\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{v}^{k,\delta})+\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}^{k,\delta})q^{k,\delta}+C_2\mathscr{F}(p^{k,\delta})(q^{k,\delta}-p^\delta)\Big)\,d\mathbf{x}\\&+\int_{\Gamma_c}\frac{1}{\delta}[[\mathbf{u}^{k,\delta}\cdot\mathbf{n}]]^-[[(\mathbf{u}^{k,\delta}-\mathbf{v}^{k,\delta})\cdot\mathbf{n}]]\,dS_{\mathbf{x}}.\end{aligned}$$

Since the former quadratic term in (44) is weakly lower semi-continuous, it holds

$$\limsup_{k\to\infty}\int_{\Omega_c}C_2\mathscr{F}(p^{k,\delta})(p^{k,\delta}-p^\delta)\,d\mathbf{x}\le0,$$

and using the pseudo-monotone property (29), we conclude that

$$\liminf_{k\to\infty}\int_{\Omega_c}\mathscr{F}\left(p^{k,\delta}\right)\left(p^{k,\delta}-q\right)d\mathbf{x}\geq\int_{\Omega_c}\mathscr{F}(p^\delta)(p^\delta-q)\,d\mathbf{x}\quad\forall q\in L^2(\Omega_c;\mathbb{R}). \tag{45}$$

Using (45), the lower semi-continuity of the quadratic term

$$\liminf_{k\to\infty}\int_{\Omega_c}C_1\|\left(\mathbf{T}^{k,\delta}\right)^*\|^2\,d\mathbf{x}\geq\int_{\Omega_c}C_1\parallel(\mathbf{T}^\delta)^*\parallel^2 d\mathbf{x},$$

and convergences (41)–(43), we calculate the limit inferior in (34)

$$\int_{\Omega_c}\Big(C_1\parallel(\mathbf{T}^\delta)^*\parallel^2-(\mathbf{T}^\delta)^*\cdot\boldsymbol{\varepsilon}(\mathbf{v}^m)^*-(C_1(\mathbf{T}^\delta)^*-\boldsymbol{\varepsilon}(\mathbf{u}^\delta)^*)\cdot(\mathbf{S}^m)^*$$
$$+C_2\mathscr{F}(p^\delta)(p^\delta-\mathrm{tr}\mathbf{S}^m)+p^\delta\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{v}^m)-\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}^\delta)\mathrm{tr}\mathbf{S}^m\Big)\,d\mathbf{x}$$
$$-\int_{\Gamma_c}\frac{1}{\delta}[\![\mathbf{u}^\delta\cdot\mathbf{n}]\!]^-[\![(\mathbf{u}^\delta-\mathbf{v}^m)\cdot\mathbf{n}]\!]\,dS_\mathbf{x}\leq\int_{\Omega_c}\mathbf{f}\cdot(\mathbf{u}^\delta-\mathbf{v}^m)\,d\mathbf{x}$$
$$+\int_{\Gamma_N}\mathbf{g}\cdot(\mathbf{u}^\delta-\mathbf{v}^m)\,dS_\mathbf{x}$$

with test functions $\mathbf{v}^m\in V^m$ and $\mathbf{S}^m\in H^m$ for arbitrary $m\in\mathbb{N}$. Therefore, based on the density of the finite-dimensional approximation,

$$\int_{\Omega_c}\Big(C_1\parallel(\mathbf{T}^\delta)^*\parallel^2-(\mathbf{T}^\delta)^*\cdot\boldsymbol{\varepsilon}(\mathbf{v})^*-(C_1(\mathbf{T}^\delta)^*-\boldsymbol{\varepsilon}(\mathbf{u}^\delta)^*)\cdot\mathbf{S}^*$$
$$+C_2\mathscr{F}(p^\delta)(p^\delta-q)+p^\delta\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{v})-\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}^\delta)q\Big)\,d\mathbf{x}$$
$$-\int_{\Gamma_c}\frac{1}{\delta}[\![\mathbf{u}^\delta\cdot\mathbf{n}]\!]^-[\![(\mathbf{u}^\delta-\mathbf{v})\cdot\mathbf{n}]\!]\,dS_\mathbf{x}=\int_{\Omega_c}\mathbf{f}\cdot(\mathbf{u}^\delta-\mathbf{v})\,d\mathbf{x} \tag{46}$$
$$+\int_{\Gamma_N}\mathbf{g}\cdot(\mathbf{u}^\delta-\mathbf{v})\,dS_\mathbf{x}$$

holds for all test functions $\mathbf{v}\in H^1(\Omega_c;\mathbb{R}^3)$ such that $\mathbf{v}=\mathbf{0}$ at Γ_D, for $\mathbf{S}^*\in L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})$ with $\mathrm{tr}\mathbf{S}^*=0$, and $q\in L^2(\Omega_c;\mathbb{R})$.

Taking the limit in the uniform estimates (39) and (40) provides

$$C_1(1-\theta C_1)\|(\mathbf{T}^\delta)^*\|^2_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})}+C_2\left(\underline{b}-\frac{1}{3}C_2\overline{b}^2\theta\right)\|\,p^\delta\,\|^2_{L^2(\Omega_c;\mathbb{R})}$$
$$+\int_{\Gamma_c}\frac{1}{\delta}([\![\mathbf{u}^\delta\cdot\mathbf{n}]\!]^-)^2\,dS_\mathbf{x}\leq\frac{1}{2\theta}C(f,g), \tag{47}$$

and, respectively,

$$\|\,\boldsymbol{\varepsilon}(\mathbf{u}^\delta)^*\|_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})}=C_1\parallel(\mathbf{T}^\delta)^*\parallel_{L^2(\Omega_c;\mathbb{R}^{3\times3}_{sym})},$$
$$\|\,\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}^\delta)\|_{L^2(\Omega_c;\mathbb{R})}\leq C_2\overline{b}\|\,p^\delta\,\|_{L^2(\Omega_c;\mathbb{R})}. \tag{48}$$

The limit passage as $\delta\to0$

From the estimates (47) and (48), and recalling the Korn–Poincaré inequality (25), we infer the convergences

$$\mathbf{u}^\delta\rightharpoonup\mathbf{u}\text{ weakly in }H^1\left(\Omega_c;\mathbb{R}^3\right),\quad p^\delta\rightharpoonup p\text{ weakly in }L^2(\Omega_c;\mathbb{R}),$$
$$(\mathbf{T}^\delta)^*\rightharpoonup\mathbf{T}^*\text{ weakly in }L^2\left(\Omega_c;\mathbb{R}^{3\times3}_{sym}\right),\quad\mathrm{tr}\mathbf{T}^*=0, \tag{49}$$

where $\mathbf{u}=\mathbf{0}$ on Γ_D, and the limit relation

$$[\![\mathbf{u}\cdot\mathbf{n}]\!]^-=0\quad\text{on }\Gamma_c. \tag{50}$$

Now we pass the penalized Eq. (46) to the limit as $\delta\to0$.

Testing (46) with $\mathbf{v}=\mathbf{u}$, $\mathbf{S}^*=\mathbf{T}^*$ and $q=p$, using the well-known monotone property for the penalty operator such that

$$-\int_{\Gamma_c}\frac{1}{\delta}[\![\mathbf{u}^\delta\cdot\mathbf{n}]\!]^-[\![(\mathbf{u}^\delta-\mathbf{u})\cdot\mathbf{n}]\!]\,dS_\mathbf{x}\geq-\int_{\Gamma_c}\frac{1}{\delta}[\![\mathbf{u}\cdot\mathbf{n}]\!]^-[\![(\mathbf{u}^\delta-\mathbf{u})\cdot\mathbf{n}]\!]\,dS_\mathbf{x}=0$$

since (50) holds, and $(\mathbf{T}^\delta)^*\cdot(\mathbf{T}^\delta-\mathbf{T})^*\geq\mathbf{T}^*\cdot(\mathbf{T}^\delta-\mathbf{T})^*$, we obtain

$$\int_{\Omega_c}(C_1\mathbf{T}^*\cdot(\mathbf{T}^\delta-\mathbf{T})^*+C_2\mathscr{F}(p^\delta)(p^\delta-p))\,d\mathbf{x}\leq$$
$$-\int_{\Gamma_c}\frac{1}{\delta}[\![\mathbf{u}^\delta\cdot\mathbf{n}]\!]^-[\![(\mathbf{u}^\delta-\mathbf{u})\cdot\mathbf{n}]\!]\,dS_\mathbf{x}$$
$$+\int_{\Omega_c}(C_1(\mathbf{T}^\delta)^*\cdot(\mathbf{T}^\delta-\mathbf{T})^*+C_2\mathscr{F}(p^\delta)(p^\delta-p))\,d\mathbf{x}=\int_{\Omega_c}\mathbf{f}\cdot(\mathbf{u}^\delta-\mathbf{u})\,d\mathbf{x} \tag{51}$$
$$+\int_{\Gamma_N}\mathbf{g}\cdot(\mathbf{u}^\delta-\mathbf{u})\,dS_\mathbf{x}-\int_{\Omega_c}\Big(\boldsymbol{\varepsilon}(\mathbf{u}^\delta)^*\cdot\mathbf{T}^*-(\mathbf{T}^\delta)^*\cdot\boldsymbol{\varepsilon}(\mathbf{u})^*+p^\delta\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u})$$
$$-\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}^\delta)p\Big)\,d\mathbf{x}.$$

Taking the limit as $\delta\to0$, the convergence in (49) follows

$$\limsup_{\delta\to0}\int_{\Omega_c}C_2\mathscr{F}(p^\delta)(p^\delta-p)\,d\mathbf{x}\leq0,$$

and by the pseudo-monotone property of $\mathscr{F}$ (29),

$$\liminf_{\delta\to0}\int_{\Omega_c}\mathscr{F}(p^\delta)(p^\delta-q)\,d\mathbf{x}\geq\int_{\Omega_c}\mathscr{F}(p)(p-q)\,d\mathbf{x}\quad\forall q\in L^2(\Omega_c;\mathbb{R}). \tag{52}$$

For the test function with $[\![\mathbf{v}\cdot\mathbf{n}]\!]^-=0$ on Γ_c, using the monotonicity

$$-\int_{\Gamma_c}\frac{1}{\delta}[\![\mathbf{u}^\delta\cdot\mathbf{n}]\!]^-[\![(\mathbf{u}^\delta-\mathbf{v})\cdot\mathbf{n}]\!]\,dS_\mathbf{x}\geq0, \tag{53}$$

with the help of (52) and (53) we take the limit inferior in (46) as $\delta\to0$ and arrive at the variational inequality (24). Taking the limit in (47) and (48) leads to the a-priori estimates (30) and (31). The proof is thus completed.

□

To conclude, we remark that typical pseudomonotone operators are presented by the sum of a monotone operator and a compact operator at the boundary, see such examples in Naniewicz and Panagiotopoulos (1994).

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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