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PROPAGATION OF A MODE-I CRACK UNDER THE IRWIN AND KHRISTIANOVICH–BARENBLATT CRITERIA

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We study the problem of propagation of a crack admitting the closure of its surfaces near the crack front. The variational inequality for variations of the crack front is deduced by using the Nazarov asymptotic formula for the stress intensity factor. An asymptotic solution is obtained in a special case.

1. Introduction

We consider an elastic isotropic medium $\mathbf{R}^3$ containing a plane crack G. It is assumed that a symmetric normal load P applied to the crack surfaces is such that the mode-I stress-strain state is realized at the crack front Γ. We study the case of quasistatic crack propagation.

According to the well-known Irwin criterion for the stress intensity factor (SIF) K_1, the crack-front growth is described by the relations:

$$h(t;s) = 0 \Rightarrow 0 \le K_1(t;s) \le K_{1c}, \tag{1}$$

$$h(t;s) > 0 \Rightarrow K_1(t;s) = K_{1c}, \tag{2}$$

where h is a function describing the increment of the crack front $\Gamma(t)$ at a point s, $t \ge 0$ is a timelike parameter of loading, and K_{1c} is the critical SIF (a material parameter). In addition we assume that the following condition is satisfied:

$$h(t;s) < 0 \Rightarrow K_1(t;s) = 0. \tag{3}$$

Relations (1)–(3) mean that, for some t, the crack $\Gamma(t)$ turns out to be in an equilibrium, i.e., the two-sided inequality for the SIF

$$0 \le K_1(t;s) \le K_{1c}, \tag{4}$$

holds at all points of the crack front $\Gamma(t)$. Condition (2) characterizes the growth of the crack into the medium (new crack surfaces appear). Condition (3) allows the closure of crack surfaces in the vicinity of the crack front.

According to the Papkovich–Neuber representation, the equilibrium problem for an elastic medium containing a crack Γ (with $h \equiv 0$ for $t = 0$) is reduced to finding a harmonic function f in $\mathbf{R}^3_+ = \{(y;s): z > 0\}$ which vanishes at infinity and satisfies the following boundary conditions:

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$$f(\mathbf{y}, 0) < 0, \qquad \mathbf{y} = (y_1; y_2) \in \mathbf{R}^2 \setminus G, \tag{5}$$

$$-\frac{\partial f}{\partial z}(\mathbf{y}, 0) = p^0(\mathbf{y}) \equiv \alpha^{-1} = P^0(\mathbf{y}), \qquad \mathbf{y} \in G, \tag{6}$$

where $\alpha = (1-\nu)^{-1}\mu$ with μ and ν being the shear modulus and Poisson's ratio, respectively.

In the neighborhood of the contour Γ, the function f has the following asymptotic representation:

$$f(\mathbf{y}, z) = \frac{1}{\alpha}\sqrt{\frac{2}{\pi}}\left\{K_1^0(s) r^{1/2} \sin\left(\frac{\varphi}{2}\right) + \frac{1}{3} k_1^0(s) r^{3/2} \sin\left(\frac{3\varphi}{2}\right)\right.$$
$$\left. + \kappa(s) K_1^0(s) r^{3/2}\left[\frac{1}{4}\sin\left(\frac{\varphi}{2}\right) - \frac{1}{12}\sin\left(\frac{3\varphi}{2}\right)\right]\right\} - p_\Gamma^0(s)\sin\varphi + O(r^2), \tag{7}$$

where s is the arc length along Γ, (r, φ) are polar coordinates in the plane normal to Γ, $r = \operatorname{dist}\{(\mathbf{y}, z), \Gamma\}$, $\varphi \in (-\pi, \pi)$, $\kappa(s)$ is the curvature of Γ at the point s, $p_\Gamma^0(s)$ is the value of $p^0(\mathbf{y})$ for $\mathbf{y} \in \Gamma$, and $k_1^0(s)$ denotes the "junior" SIF.

We emphasise that, in order to consider the problem of the crack propagation, it is necessary to assume that K_1 is a function of h in relations (1)–(3).

The problem of propagation of a mode-I crack was studied under the Irwin criterion in [2, 7, 10] and under the Griffith criterion in [11]. Hypothesis (3) was used in [9] (see the supplementary chapter by G. M. Barenblatt). The propagation of a crack by the mechanism of fatigue under the Paris law was numerically studied in [3].

2. Variational Inequality

By h_+ we denote the positive part of the function h, i.e., $h_+(s) = 0$ if $h(s) \le 0$ and $h_+(s) = h(s)$ if $h(s) > 0$ for all $s \in \Gamma$.

It is easy to see that, relations (1)–(3) imply the equality

$$K_1(t; s) h(t; s) = K_{1c} h_+(t; s), \qquad s \in \Gamma. \tag{8}$$

In view of (4), for any smooth function χ, we have

$$K_1(t; s)\chi(t; s) \le K_{1c}\chi_+(t; s), \qquad s \in \Gamma. \tag{9}$$

If we now find the sum of relations (8) and (9) and integrate over Γ, then we get

$$\langle K_1, \chi - h\rangle \le \langle K_{1c}, \chi_+ - h_+\rangle, \qquad s \in \Gamma. \tag{10}$$

Conversely, for sufficiently smooth K_1 and h, in a standard way [4], inequality (10) leads to relations (1)–(3).

In view of the dependence of K_1 on h, relation (10) is interpreted as a variational inequality [4] for finding the function h.

3. Asymptotic Relation for the SIF

The behavior of the SIF for the perturbed crack contour was investigated in [3], [5], and [10]. In accordance with [10], the following formula holds for small h:

$$K_1(t;s) \cong K_1^0(s) + tK_1'(s) + B(K_1^0 h; s) + h(t;s)\left[b(s)K_1^0(s) + \frac{1}{2}k_1^0(s) - \frac{3}{8}\kappa(s)K_1^0(s)\right], \tag{11}$$

where $K_1^0(s)$ and $k_1^0(s)$ are defined in (7) and K' is the SIF corresponding to an increment of the load P' (for the sake of simplicity, we assume that $P(t;\mathbf{y}) = P^0(\mathbf{y}) + tP'(\mathbf{y})$ for small t). The integral operator B has the form

$$B(H;s) = \text{v.p.}\int_\Gamma (H(\tilde{s}) - H(s))Z(\tilde{s}, s)d\tilde{s} \tag{12}$$

and its kernel

$$Z(\tilde{s}, s) = \frac{1}{2\pi}\frac{1}{|\tilde{s} - s|^2} + O(|\ln||\tilde{s} - s|) \tag{13}$$

is symmetric and positive. The function b is given in [10, 7].

For a circular crack of radius a, we have [6] $b(s) = 0$, $s = a\sigma$, and

$$Z(a\tilde{\sigma}, \sigma) = \frac{1}{2\pi}\frac{1}{2a\left|\sin\left[\frac{1}{2}(\tilde{\sigma} - \sigma)\right]\right|^2}. \tag{14}$$

Note that the explicit expressions for the kernel of operator (12) are known for an external circular crack [5] [this formula coincides with (14)], for a half-plane crack [10] and for a tunnel crack [8].

4. Variational Inequality for the Crack Front

Substituting the asymptotic representation (11) of the SIF depending on h in inequality (10), we obtain

$$\langle \beta^0 h - B^0(h), \chi - h\rangle + j_c(\chi) - j_c(h) \geq \langle f, \chi - h\rangle \quad \forall\chi, \tag{15}$$

where we have used the following notation:

$$\beta^0(s) = \frac{3}{8}(\kappa(s)K_1^0(s)) - \frac{1}{2}k_1^0(s) - b(s)K_1^0(s), \quad B^0(h;s) = B(K_1^0 h; s),$$
$$f(s) = K_1^0(s) + tK_1'(s), \quad j_c(h;s) = \langle K_{1c}, h_+\rangle. \tag{16}$$

Relation (15) is a standard variational inequality containing a nondifferentiable functional j_c. As proposed in [10], we seek the solution h of inequality (15) in the space $V = W_2^{1/2}(\Gamma)$. The linear operator $-B$ (and, therefore, $-B^0$) is positive (i.e., $-\langle B(H), H\rangle \geq 0$ for any $H \in V$). Also note that K_{1c} is a positive constant. The unique solvability of the variational inequality (15) follows from the classical results [4] provided that

$$\beta^0(s) \geq \text{const} > 0, \qquad s \in \Gamma. \tag{17}$$

5. Statement of the Problem for the Closure of Crack Surfaces

The function K_1^0 is a principal characteristic of the initial $(t = 0)$ position of the crack. If the strict inequality $0 < K_1^0(s) < K_{1c}$ holds everywhere in Γ (the crack is open and does not reach the extreme state), then, for any sufficiently small perturbation $t > 0$, the solution of inequality (15) is trivial, i.e., $h \equiv 0$.

Further, there are three possible cases:

(i) at some point $s_1 \in \Gamma$, the initial SIF takes the critical value $K_1^0(s_1) = K_{1c}$ and the inequality $0 < K_1^0(s) < K_{1c}$ holds at all other points $s \neq s_1$;

(ii) the equality $K_1^0(s_2) = 0$ holds at some point $s_2 \in \Gamma$, and the inequality $0 < K_1^0(s) < K_{1c}$ is true for all $s \neq s_2$;

(iii) there are two points $s_1, s_2 \in \Gamma$ such that $K_1^0(s_1) = K_{1c}$, $K_1^0(s_2) = 0$, and $0 < K_1^0(s) < K_{1c}$ for all $s \neq s_1$, $s \neq s_2$.

Finally, one can assume that the conditions presented above for K_1^0 are realized at several points or in parts of Γ.

The first case (more useful in applications) was studied in [7], where the asymptotics of the solution of the variational inequality was constructed.

We consider the second case and use the following assumptions:

It is assumed, first, that the following expansion is true in the neighborhood of the point $s_2 \equiv s_0$:

$$K_1^0(s) = a_0(s - s_0)^2 + O(|s - s_0|^3), \qquad a_0 > 0, \tag{18}$$

and, second, that the condition

$$K_1'(s_0) < 0 \tag{19}$$

guarantees the decrease in the SIF in the neighborhood of the point s_0 as the load varies.

In the next section, we construct the asymptotic solution of inequality (15) as $t \to 0$ under assumptions (18) and (19).

6. Asymptotic Solution

For small t, the contour $\Gamma(t)$ coincides with Γ outside a certain neighborhood of the point s_0. In this region, in view of relations (1), (11), and (12), we have

$$h(t;s) = 0 \Rightarrow = \text{v.p.}\int_{\Gamma} K_1^0(\tilde{s})h(t;\tilde{s})Z(\tilde{s},s)d\tilde{s} \le K_1^0(s) + tK_1'(s). \tag{20}$$

In the perturbed part of $\Gamma(t)$ [for all s such that $h(t;s) < 0$], by using relations (3), (11) and (16), we obtain

$$\beta^0(s)h(t;s) - B^0(h;s) = K_1^0(s) + tK_1'(s). \tag{21}$$

Note that if the right-hand side of relation (21) is positive, then the inequality in (20) is true. This means that the necessary condition for $h(t;s) = 0$ is satisfied. On the other hand, in view of relations (18) and (19), the right-hand side of (21) is negative in the neighborhood $O(t^{1/2})$ of the point s_0.

To describe perturbations of the initial contour Γ localized near s_0, we introduce a "stretched" coordinate

$$\sigma = t^{-1/2}(s - s_0). \tag{22}$$

The asymptotic solution is sought in the form (cf. [7]):

$$h(t;s) \cong tY(t^{-1/2}(s - s_0)). \tag{23}$$

Substituting relation (23) in Eq. (21) and separating the principal parts of the asymptotic expansions, we obtain

$$\beta^0(s_0)\,t\,Y(\sigma) = t\,a_0\sigma^2 - tA_0,$$

where $A_0 = -K_1'(s_0)$. Thus, we have

$$Y(\sigma) = -\frac{(A_0 - a_0\sigma^2)_+}{\beta^0(s_0)}. \tag{24}$$

Recall that the subscript "+" denotes the positive part of a function.

The asymptotic solution (23), (24) leaves the residual $O(t^{3/2})$ in Eq. (21). Moreover, inequality (20) is satisfied outside the interval

$$(s_0 - l_0,\ s_0 + l_0), \qquad l_0 = t^{1/2}\sqrt{\frac{A_0}{a_0}}, \tag{25}$$

with the exception of zones $O(t^{3/2})$ near the ends of interval (25). Relation (25) gives the first approximation to the perturbed part of the contour $\Gamma(t)$.

CONCLUSIONS

By using the results obtained in the present work and in [7], one can also study the third case (see Sec. 5). It is easy to see that the principal asymptotic terms are constructed independently of each other. In order to investigate the interaction between the points of the first and the second types, it is necessary to construct the next terms of the asymptotic expansion (23). In this way (see [1]), one can obtain the correction of the first approximation for the *a priori* unknown location of the perturbed part of $\Gamma(t)$.

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