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On the topological derivative due to kink of a crack with non-penetration. Anti-plane model

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Abstract

A topological derivative is defined, which is caused by kinking of a crack, thus, representing the topological change. Using variational methods, the anti-plane model of a solid subject to a non-penetration condition imposed at the kinked crack is considered. The objective function of the potential energy is expanded with respect to the diminishing branch of the incipient crack. The respective sensitivity analysis is provided by a Saint-Venant principle and a local decomposition of the solution of the variational problem in the Fourier series.

Résumé

On définit une dérivée topologique qui provient du branchement d'une fissure constituant le changement topologique. En utilisant des méthodes variationnelles, on considère le modèle anti-plan d'un solide soumis à une condition de non pénétration imposée sur la fissure ou s'opère le branchement. La fonction coût de l'énergie potentielle est développée par rapport à la branche de petite taille de la fissure naissante. L'analyse de sensibilité associée est fournie pas un principe de Saint-Venant et une décomposition locale de la solution du problème variationnel en série de Fourier.

MSC: 49J40; 49K10; 49Q12; 74R10

Keywords: Kink of crack; Non-penetration condition; Variational inequality; Shape sensitivity analysis; Topological change; Structure optimization

0. Introduction

In fracture mechanics, the principal question is to determine an a priori unknown propagation of an incipient crack in a solid. For the classic concepts of cracks adopted in mechanics we refer to [12,39]. Within a 3-dimensional elasticity, there is common agreement to separate the crack deformation into three fracture modes corresponding to

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opening (mode-1), in-plane shear (mode-2), and anti-plane shear (mode-3). Experimental results show, however, that real cracks are mixed of all of these modes. As the consequence, the crack propagation is influenced generally by all of the fracture modes; see the related investigation in [34,35]. Moreover, interaction phenomena between the crack faces are involved certainly in the processing of cracks.

To get a proper geometric description of admissible cracks is rather complicated task in the 3-d setting. Indeed, cracks are generally non-smooth structures due to various micro- and macro-phenomena involved. Along a 2-d cross-section, we consider the admissible geometry of cracks which allows kink of its path. In the planar cross-section, under suitable assumptions, the 3-d elasticity system splits into in-plane and anti-plane relations. The fracture modes remain, however, mixed due to twisting around the direction of propagation; see [34]. Therefore, separating the fracture modes, the in-plane model is commonly used to describe the phenomenon of kinking of cracks of mode-1 and mode-2. The anti-plane model of a crack of mode-3 with kink is reasonable for multi-material junctions, slanting cores, delamination of wedges, and alike; see the related topic in [17,38].

Relying on a planar geometric setting, we focus on the problem of kinking of a crack path. Indeed, the direction in which an incipient crack will propagate is the subject for discussion in the mechanical literature; see [2,5,13,15, 41], and other works. Moreover, a role of interaction between the crack faces in the propagation of cracks is not well established yet. For example, in [36] there is supposed an heuristic hypothesis of known contact zones before and after the crack kinking. However, contact zones are unknown a priori and have to define after solving a free-boundary problem due to non-penetration. The proper variational theory for non-linear crack problems subject to non-penetration conditions was elaborated in [21].

When looking for geometric variables of an unknown crack path, the problem of crack propagation lies within the frame of structure optimization. An account of common approaches adopted in shape and topology optimization can be found in [1,10,18,27,28,40]. From the point of view of optimization, propagation of cracks can be stated as minimization of the total energy over all admissible extensions of the crack path; see [9]. Within the optimization approach, evolution of a crack with kink is given in [11,22], where the evolution process is described globally in time. Nevertheless, a local description of the energy at the time when a kink occurs is of especial interest for fracture. Thus, in the following we will show that the energy release rate gives rise to criteria of kinking.

The kink phenomenon implies arrest of the tangential movement along the direction of propagation and appearance of a new branch at the point of kink. In this sense, kinking is related to the phenomena of branching as well as appearance of elongated micro-defects in a continuum. From a geometric viewpoint, these features present topological changes. Examining between admissible geometries containing the crack before kink and the crack after kink is relevant to the topological methods developed in [14,43]. The respective formalism exploits expansion of an optimal value function (associated to the energy) with respect to diminishing holes (called "bubbles" in [14]). This approach was tested numerically for the problem of identifiability of cracks in [3,7]. In our case, we associated a hole with the crack branch and specify this topological conception for the problem of kink.

To gain insight into the matter, further we sketch the principal construction. It is regardless of concrete governing relations.

1. Problem of a crack kink

Let the reference domain $\Omega_0 \subset \mathbb{R}^2$ contain inside a pre-described crack Γ_0 of length $L > 0$ before kink. We introduce a finite branch $\gamma_{(1,\phi)}$ incipient at the crack tip O of Γ_0, as a rectilinear or slightly curved segment, and $|\gamma_{(1,\phi)}| > 0$. The parameter $\phi \in (-\pi, \pi)$ implies the angle of kink of $\gamma_{(1,\phi)}$ to the direction of Γ_0 at O. For simplicity of the following notation we associate O with the origin 0. We define a small crack branch as

$$\gamma_{(r,\phi)} := \{x \in \mathbb{R}^2\colon\ r^{-1}x \in \gamma_{(1,\phi)}\}, \tag{CB}$$

and refer to $r \geqslant 0$ as the "length" of the branch. Thus, $\gamma_{(r,\phi)}$ is identified by the two geometric parameters ϕ and $r \in [0, R]$. The bound $R > 0$ is assumed sufficiently small such that $\gamma_{(R,\phi)} \subset \Omega$. Therefore, we can define perturbed domains containing kinked cracks as $\Omega_{(r,\phi)} := \Omega_0 \setminus \gamma_{(r,\phi)}$. As $r \to 0$, the finite crack branches $\gamma_{(r,\phi)}$ collapse to the infinitesimal micro-crack at point O, thus the topological change occurs.

For fixed r and ϕ, let a scalar or vector-valued function u denote admissible displacements in $\Omega_{(r,\phi)}$. We suppose that unilateral constraints are imposed at the crack. They associate the admissible sets $K(\Omega_{(r,\phi)})$, which are assumed convex cones in Hilbert spaces $H(\Omega_{(r,\phi)})$. Let us consider a quadratic functional of the potential energy

$$\Pi : H(\Omega_{(r,\phi)}) \mapsto \mathbb{R}, \quad u \mapsto \Pi(u; \Omega_{(r,\phi)}),$$

assumed positive definite and bounded uniformly for all $\Omega_{(r,\phi)}$. Minimizing Π over admissible u yields the unique optimal solution $u^{(r,\phi)} \in K(\Omega_{(r,\phi)})$ such that

$$u^{(r,\phi)} = \operatorname{argmin} \Pi(u; \Omega_{(r,\phi)}) \quad \text{over } u \in H(\Omega_{(r,\phi)}) \text{ subject to } u \in K(\Omega_{(r,\phi)}).$$

Thus, we arrive at the optimal value function

$$\Pi : [0, R] \times (-\pi, \pi) \mapsto \mathbb{R}, \quad \Pi(r, \phi) := \Pi\big(u^{(r,\phi)}; \Omega_{(r,\phi)}\big)$$

which expresses minimal values of the potential energy Π with respect to the geometric parameters r and ϕ. Note that $\Omega_{(0,\phi)} = \Omega_0$, hence $u^{(0,\phi)} =: u^0$ and $\Pi(0, \phi) =: \Pi(0)$ do not depend on ϕ at $r = 0$.

For every fixed ϕ, our task is to find the limit (when it exists)

$$\lim_{r\to 0}\big\{r^{-1}\big(\Pi(r, \phi) - \Pi(0)\big)\big\} := \Pi'(0, \phi) \tag{TD}$$

with respect to the diminishing length $r \to 0$ of the crack branch. The reason is the following. From the point of view of fracture mechanics, $\Pi'(0, \phi)$ expresses the energy release rate at the tip of the reference crack Γ_0 in the direction ϕ of incipient kink. This quantity determines fracture criteria (FC1) and (FC2). Indeed, using the material parameter $G_c > 0$ of the density of surface energy of the (two) crack faces, the total potential energy T of the solid with the kinked crack reads

$$T(r, \phi) = \Pi(r, \phi) + (L + r)G_c.$$

The optimization approach to fracture claims that T attains its minimum during the crack propagation. The partial derivative of T with respect to r at $r = 0$ is equal to $\Pi'(0, \phi) + G_c := T'(0, \phi)$. Therefore, if $T'(0, \phi)$ is negative, hence

$$\Pi'(0, \phi) + G_c < 0, \tag{FC1}$$

then it implies exactly the Griffith criterion of crack propagation in the direction of ϕ. Moreover, minimizing $T'(0, \phi)$ over admissible angles ϕ gives the criterion of the optimal kink angle $\phi^\star$ such that

$$\phi^\star = \operatorname{argmin} \Pi'(0, \phi) \quad \text{over } \phi \in (-\pi, \pi). \tag{FC2}$$

Following [43], we refer to $\Pi'(0, \phi)$ in (TD) as the topological derivative. The reason is that it quantifies the topological change of the reference geometry Ω_0 before kink to $\Omega_{(r,\phi)}$ after kink with the infinitesimal crack branch as $r \to 0$. From the point of view of perturbation theory, the principal difficulty concerns the fact that (TD) is defined within singular perturbations (CB). This fact distinguishes topological changes from shape variations, while the latter imply regular perturbations. As the consequence, for $\phi \neq 0$ no diffeomorphic maps can be constructed between $\Omega_{(r,\phi)}$ and Ω_0 as it is used in the shape sensitivity analysis. Henceforth, the limit in (TD) cannot be calculated directly.

To endow (TD) with a form suitable for calculation, we note the following. While from $r = 0$ to $r > 0$ it implies the topological change, from $r > 0$ to $r + s > 0$ only changes of the shape of the reference geometry occur. Based on this observation, we suggest a two-step strategy. At the first step, for fixed $r > 0$ we calculate the shape derivative

$$\Pi'(r, \phi) := \lim_{s\to 0}\big\{s^{-1}\big(\Pi(r + s, \phi) - \Pi(r, \phi)\big)\big\}. \tag{SD}$$

The proper variational methods of the shape sensitivity analysis were developed for constrained crack problems in [21,24–26,30–32]. For the appropriate numerical methods we refer to [19,44].

Indeed, using diffeomorphic maps between $\Omega_{(r,\phi)}$ and $\Omega_{(r+s,\phi)}$ with a suitable kinematic velocity V, the limit in (SD) implies the directional derivative. Under proper assumptions, from [22] it follows the common structure

$$\Pi'(r, \phi) = r^{-1}\Pi_V^1\big(u^{(r,\phi)}, u^{(r,\phi)}; \Omega_{(r,\phi)}\big) \tag{SD$'$}$$

given by a bilinear form (see Proposition 3)

$$\Pi_V^1 : H(\Omega_{(r,\phi)}) \times H(\Omega_{(r,\phi)}) \mapsto \mathbb{R}, \quad u, v \mapsto \Pi_V^1(u, v; \Omega_{(r,\phi)}),$$

which we assume bounded and continuous uniformly for all $\Omega_{(r,\phi)}$. The factor r^{-1} appears due to (CB). We observe that (SD′) may become singular when $r \to 0$. Nevertheless, when $\Pi'(r,\phi)$ is bounded uniformly with respect to r (see Proposition 4), the optimal value function Π can be represented as

$$\Pi(r,\phi) = \Pi(0) + \Theta(r,\phi),$$
$$\Theta(r,\phi) := \int_0^r l^{-1}\Pi_V^1\bigl(u^{(l,\phi)}, u^{(l,\phi)}; \Omega_{(l,\phi)}\bigr)\,dl. \tag{AE1}$$

Note that according to the definition in (SD) and (TD), generally,

$$\Pi'(0,\phi) \neq \lim_{r\to 0} \Pi'(r,\phi)$$

in view of the interchanging of the limits of $r \to 0$ and $s \to 0$. Therefore, from (AE1) we restate (TD) as

$$\Pi'(0,\phi) = \lim_{r\to 0}\bigl\{r^{-1}\Theta(r,\phi)\bigr\}. \tag{TD′}$$

To find the topological derivative in (TD′), at the second step it needs to expand $\Theta(r,\phi)$ with respect to $r \to 0$. While the first step is elaborated for a rather general class of constrained crack problems, the second step is not well established. Below we look for sufficient conditions for (TD′).

Within variational methods, the convergence of the solutions $u^{(r,\phi)}$ to u^0 as $r \to 0$ can be justified; see [22]. This gives rise to the following decomposition. Denoting the increment $w^{(r,\phi)} := u^{(r,\phi)} - u^0$, let us decompose $u^{(r,\phi)} = u^0 + w^{(r,\phi)}$ and rewrite (SD′) as

$$\Pi'(r,\phi) = r^{-1}\Pi_V^1\bigl(u^{(r,\phi)}, u^{(r,\phi)}; \Omega_{(r,\phi)}\bigr) = r^{-1}\Pi_V^1\bigl(u^0, u^0; \Omega_{(r,\phi)}\bigr)$$
$$+ \Pi_V^1\bigl(2u^0, r^{-1}w^{(r,\phi)}; \Omega_{(r,\phi)}\bigr) + \Pi_V^1\bigl(w^{(r,\phi)}, r^{-1}w^{(r,\phi)}; \Omega_{(r,\phi)}\bigr).$$

Using the continuity property of Π_V^1, the following sufficient conditions (compare to Corollary 2 and Theorem 1)

$$\Pi_V^1\bigl(u^0, u^0; \Omega_{(r,\phi)}\bigr) = o(r), \tag{SC1}$$
$$\bigl\|w^{(r,\phi)}\bigr\|_{H(\Omega_{(r,\phi)})} = O(r), \tag{SC2}$$

provide the uniform bound of $\Pi'(r,\phi)$. Hence (see Theorem 2),

$$\Theta(r,\phi) = \Theta^1(r,\phi) + o(r),$$
$$\Theta^1(r,\phi) := \int_0^r \Pi_V^1\bigl(2u^0, l^{-1}w^{(l,\phi)}; \Omega_{(l,\phi)}\bigr)\,dl = O(r). \tag{AE2}$$

Based on (AE2) we represent the topological derivative as the limit

$$\lim_{r\to 0}\bigl\{r^{-1}\Theta^1(r,\phi)\bigr\} = \Pi'(0,\phi). \tag{TD″}$$

Note that $r^{-1}\Theta^1(r,\phi)$ may admit bounded oscillations when $r \to 0$. The oscillations would be prevented, if $r \mapsto \Pi'(r,\cdot)$ was monotone. Since the monotonicity fails, to ensure the topological derivative $\Pi'(0,\phi)$ in (TD″) it needs the analytic representation of $\Theta^1(r,\phi)$. Therefore, in addition to (SC2), a first-order expansion of $w^{(r,\phi)}$ over the varying domains $\Omega_{(r,\phi)}$ is required. Moreover, the asymptotic limit of $w^{(r,\phi)}$ is singular due to $\gamma_{(r,\phi)} \to \{0\}$ as $r \to 0$. Such expansions are not available in the general setting of the problem of kinking.

To focus the principal mathematical difficulties arising here, further we confine ourselves to the anti-plane setting of the crack problem, and to the rectilinear crack Γ_0 and crack branch $\gamma_{(r,\phi)}$. Moreover, to relate our problem with the 3-d effect of twisting around the direction of propagation, we model unilateral conditions of the non-penetration type, which are imposed at the crack faces. During the consideration we will discuss possible generalizations of our approach. In Section 2 we get a variational formulation of the model crack problem with kink. Following the two-step strategy, respective shape sensitivity analysis is presented in Section 3. Corollary 2 provides the sufficient condition (SC1). The rest of the paper is devoted to the topological sensitivity analysis.

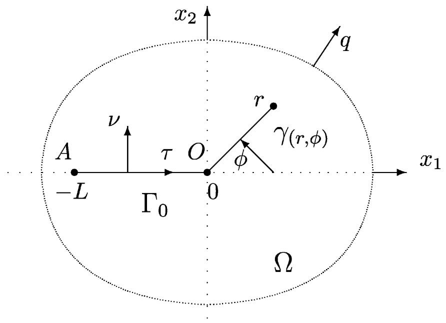

Fig. 1. Example configuration of the kinked crack $\Gamma_{(r,\phi)}$.

It is important to note a localization of the shape derivative (SD′) in subsets of $\Omega_{(r,\phi)}$ as stated in Corollary 1. As the consequence, only local representations of $w^{(r,\phi)}$ and u^0 are employed in (AE2). To expand the solutions, in Section 4 we start with a Fourier series analysis in varying domains. For this reason, we apply to our problem the asymptotic arguments from [20,33,37]. However, to evaluate the remainder terms it needs a Saint-Venant principle; see the related topic in [6,8]. The Saint-Venant principle establishes how solutions decay over diminishing domains. Following [23], for the kink problem we obtain a power-type decay of the solutions in the energy norm. These results provide the principal estimation of (SC2)-type established in Theorem 1. Hence, Theorem 2 concludes with the main expansion (AE2) in Section 5.

While (AE2) is derived for the unilaterally constrained crack problem, a further specification of the topological derivative is available in the linearized setting of the problem. Thus, abandoning the non-penetration condition in Section 6, the asymptotic expansions of u^0 and $w^{(r,\phi)}$ are refined in more details in Proposition 5 and Proposition 6, respectively. Henceforth, for the linearized crack problem, in Theorem 3 we express (TD″) in the terms of stress intensity factors, which are of the primary importance for engineers.

2. Formulation of anti-plane problem with kinking crack

In the following we specify the geometry of $\Omega_{(r,\phi)}$ due to a rectilinear crack Γ_0 and a rectilinear branch $\gamma_{(r,\phi)}$.

Let Ω be a bounded domain in $\mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega$ and the normal vector q given at $\partial\Omega$. We assume that the origin O of a Cartesian coordinate system $x = (x_1, x_2) \in \mathbb{R}^2$ is located strictly inside Ω. Denoting with B_δ a ball of radius $\delta > 0$ centered at O, this assumption ensures that there exists $R > 0$ such that $B_R \subset \Omega$. We consider the reference crack Γ_0 as a segment AO of length $L > 0$ posed in Ω along the x_1-axis. Its right end-point lying at the origin O is associated with the point of kink. Now we specify the geometric parameters $r \in [0, R]$ and $\phi \in (-\pi, \pi)$ describing the kinked cracks $\Gamma_{(r,\phi)}$. Let $\Gamma_{(r,\phi)}$ consist of two parts: the fixed one Γ_0 and varying branches $\gamma_{(r,\phi)}$. Every branch $\gamma_{(r,\phi)}$ is assumed to be a rectilinear segment of the length r starting from O with the kink of angle ϕ measured counter-clockwisely from the x_1-axis. An example configuration is illustrated in Fig. 1. The tangential vector τ and the normal vector ν at $\Gamma_{(r,\phi)}$ are given by

$$\begin{cases} \tau(0) = (1, 0), & \nu(0) = (0, 1) \quad \text{on } \Gamma_0, \\ \tau(\phi) = (\cos\phi, \sin\phi), & \nu(\phi) = (-\sin\phi, \cos\phi) \quad \text{on } \gamma_{(r,\phi)}. \end{cases}$$

We denote $\Omega_{(r,\phi)} := \Omega \setminus \Gamma_{(r,\phi)}$. For the further use we fix the radius $R \in (0, L)$ of the ball B_R inscribed in Ω which ensures that the left end-point A is located outside of B_R.

Let $\partial\Omega$ consist of two disjoint parts Γ_N and Γ_D, and $|\Gamma_D| > 0$. Let the volume force $f \in C^1(\overline{\Omega})$ and the boundary traction $g \in L^2(\Gamma_N)$ be given. We assume that $f = 0$ in B_{δ_f} with $\delta_f \in (0, R)$. Starting modeling, the geometric parameters r and ϕ are fixed. For points $x \in \overline{\Omega}_{(r,\phi)}$, we look for admissible displacements $u(x)$ which are zero at Γ_D. Moreover, along the crack we restrict $u(x)$ by the following unilateral conditions of the non-penetration type:

$$[\![u]\!] := u|_{\Gamma^+_{(r,\phi)}} - u|_{\Gamma^-_{(r,\phi)}} \geqslant 0 \quad \text{on } \Gamma_{(r,\phi)}. \tag{1}$$

The positive $\Gamma^+_{(r,\phi)}$ and the negative $\Gamma^-_{(r,\phi)}$ crack faces can be distinguished geometrically as the limit of points x going to $\Gamma_{(r,\phi)}$ “from above” and “from below”, respectively.

The potential energy of a solid is represented by the geometry-dependent functional

$$\Pi(u;\Omega_{(r,\phi)}) = \frac{1}{2}\int_{\Omega_{(r,\phi)}} |\nabla u|^2\,dx - \int_{\Omega_{(r,\phi)}} fu\,dx - \int_{\Gamma_N} gu\,dx \tag{2}$$

defined over the Sobolev space

$$H(\Omega_{(r,\phi)}) = \{u \in H^1(\Omega_{(r,\phi)}):\ u = 0 \text{ on } \Gamma_D\}. \tag{3}$$

It is equipped with the norm

$$\|u\|^2_{H(\Omega_{(r,\phi)})} = \int_{\Omega_{(r,\phi)}} |\nabla u|^2\,dx, \tag{4}$$

which is equivalent to the standard H^1-norm due to $u = 0$ at Γ_D. In view of (1), the set of admissible displacements reads

$$K(\Omega_{(r,\phi)}) = \{u \in H(\Omega_{(r,\phi)}):\ [\![u]\!] \geqslant 0 \text{ on } \Gamma_{(r,\phi)}\}, \tag{5}$$

which is a convex cone in $H(\Omega_{(r,\phi)})$.

We consider the following constrained minimization problem: Find $u^{(r,\phi)} \in K(\Omega_{(r,\phi)})$ such that

$$\Pi\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big) \leqslant \Pi(v;\Omega_{(r,\phi)}) \quad \text{for all } v \in K(\Omega_{(r,\phi)}). \tag{6}$$

The optimality condition for (6) yields the variational inequality

$$\int_{\Omega_{(r,\phi)}} \nabla u^{(r,\phi)} \cdot \nabla\big(v - u^{(r,\phi)}\big)\,dx \geqslant \int_{\Omega_{(r,\phi)}} f\big(v - u^{(r,\phi)}\big)\,dx + \int_{\Gamma_N} g\big(v - u^{(r,\phi)}\big)\,dx \quad \text{for all } v \in K(\Omega_{(r,\phi)}). \tag{7}$$

The coercivity and strictly positive definiteness properties of the quadratic functional Π guarantee the unique solvability of (6), hence (7). Variational inequality (7) describes a weak solution to the boundary-value problem:

$$-\Delta u^{(r,\phi)} = f \quad \text{in } \Omega_{(r,\phi)}, \tag{8a}$$

$$u^{(r,\phi)} = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial u^{(r,\phi)}}{\partial q} = g \quad \text{on } \Gamma_N, \tag{8b}$$

$$\begin{aligned} &\left[\!\!\left[\frac{\partial u^{(r,\phi)}}{\partial \nu}\right]\!\!\right] = 0, \qquad \frac{\partial u^{(r,\phi)}}{\partial \nu} \leqslant 0, \\ &[\![u^{(r,\phi)}]\!] \geqslant 0, \qquad \frac{\partial u^{(r,\phi)}}{\partial \nu}[\![u^{(r,\phi)}]\!] = 0 \quad \text{on } \Gamma_{(r,\phi)}. \end{aligned} \tag{8c}$$

To give an exact sense to the boundary terms in (8c), we introduce a Lions–Magenes space $H^{1/2}_{00}(\Gamma_{(r,\phi)})$, and with $H^{1/2}_{00}(\Gamma_{(r,\phi)})^\star$ we denote the formally dual space to $H^{1/2}_{00}(\Gamma_{(r,\phi)})$. Then the following proposition holds; see [21] for details.

Proposition 1. *The solution of* (7) *possesses the properties*:

$$\begin{aligned} &u^{(r,\phi)} \in K(\Omega_{(r,\phi)}), \qquad \Delta u^{(r,\phi)} \in L^2(\Omega_{(r,\phi)}), \\ &[\![u^{(r,\phi)}]\!] \in H^{1/2}_{00}(\Gamma_{(r,\phi)}), \qquad \frac{\partial u^{(r,\phi)}}{\partial \nu} \in H^{1/2}_{00}(\Gamma_{(r,\phi)})^\star, \end{aligned} \tag{9}$$

and $u^{(r,\phi)}$ is H^2-smooth up to $\Gamma^\pm_{(r,\phi)}$ away from kink and end points of the crack.

In fact, the normal derivative $\partial u^{(r,\phi)}/\partial \nu$ at the crack is defined in Proposition 1 by the Green formula, for $v \in H(\Omega_{(r,\phi)})$,

$$\left\langle \frac{\partial u^{(r,\phi)}}{\partial \nu}, [\![v]\!] \right\rangle_{\Gamma_{(r,\phi)}} = \int\limits_{\Omega_{(r,\phi)}} \left(-\nabla u^{(r,\phi)} \cdot \nabla v + f v\right) dx + \int\limits_{\Gamma_N} g v\, dx, \tag{10}$$

where $\langle \cdot,\cdot \rangle_{\Gamma_{(r,\phi)}}$ stands for the duality pairing between $H^{1/2}_{00}(\Gamma_{(r,\phi)})$ and $H^{1/2}_{00}(\Gamma_{(r,\phi)})^\star$. The variational inequality (7) together with (10) implies that

$$\left\langle \frac{\partial u^{(r,\phi)}}{\partial \nu}, [\![v - u^{(r,\phi)}]\!] \right\rangle_{\Gamma_{(r,\phi)}} \leqslant 0 \quad \text{for all } v \in K(\Omega_{(r,\phi)}). \tag{11}$$

Away from the kink and end points of the crack, where $u^{(r,\phi)}$ is smooth, from (11) it follows the boundary conditions (8c) pointwisely.

In what follows we keep the kink angle ϕ fixed and pass the length r of the crack branch to zero. At $r = 0$, data of the crack problem do not depend on ϕ. Therefore, we exclude ϕ in the following notation:

$$\Gamma_{(0,\phi)} =: \Gamma_0, \qquad \Omega_{(0,\phi)} =: \Omega_0, \qquad u^{(0,\phi)} =: u^0.$$

The reference solution $u^0 \in K(\Omega_0)$ is obtained from the minimization problem (6) such that

$$\Pi\left(u^0; \Omega_0\right) \leqslant \Pi(v; \Omega_0) \quad \text{for all } v \in K(\Omega_0), \tag{12}$$

or, equivalently, u^0 satisfies the variational inequality

$$\int\limits_{\Omega_0} \nabla u^0 \cdot \nabla\left(v - u^0\right) dx \geqslant \int\limits_{\Omega_0} f\left(v - u^0\right) dx + \int\limits_{\Gamma_N} g\left(v - u^0\right) dx \quad \text{for all } v \in K(\Omega_0). \tag{13}$$

It describes a weak solution to the reference boundary-value problem:

$$-\Delta u^0 = f \quad \text{in } \Omega_0, \tag{14a}$$

$$u^0 = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial u^0}{\partial q} = g \quad \text{on } \Gamma_N, \tag{14b}$$

$$\left[\!\!\left[\frac{\partial u^0}{\partial \nu} \right]\!\!\right] = 0, \qquad \frac{\partial u^0}{\partial \nu} \leqslant 0, \qquad [\![u^0]\!] \geqslant 0, \qquad \frac{\partial u^0}{\partial \nu} [\![u^0]\!] = 0 \quad \text{on } \Gamma_0. \tag{14c}$$

Similar to (10), the normal derivative $\partial u^0/\partial \nu$ is defined at $\Gamma_{(r,\phi)}$ by

$$\left\langle \frac{\partial u^0}{\partial \nu}, [\![v]\!] \right\rangle_{\Gamma_{(r,\phi)}} = \int\limits_{\Omega_{(r,\phi)}} \left(-\nabla u^0 \cdot \nabla v + f v\right) dx + \int\limits_{\Gamma_N} g v\, dx \tag{15}$$

for $v \in H(\Omega_{(r,\phi)})$. At the crack Γ_0, it fulfills the inequality

$$\left\langle \frac{\partial u^0}{\partial \nu}, [\![v - u^0]\!] \right\rangle_{\Gamma_{(r,\phi)}} \leqslant 0 \quad \text{for } v \in K(\Omega_0). \tag{16}$$

Away from the end points of crack Γ_0, the solution u^0 is smooth and satisfies boundary conditions (14c) pointwisely.

Following the streamline of Section 1, in the next section we find the shape derivative, which is defined at finite $r > 0$ as

$$\Pi'(r,\phi) = \lim_{s \to 0} \left\{ s^{-1}\left(\Pi\left(u^{(r+s,\phi)}; \Omega_{(r+s,\phi)}\right) - \Pi\left(u^{(r,\phi)}; \Omega_{(r,\phi)}\right)\right)\right\}. \tag{17}$$

With the help of (17), further we aim at the topological derivative

$$\lim_{r \to 0} \left(r^{-1} \int\limits_0^r \Pi'(l,\phi)\, dl \right) = \Pi'(0,\phi). \tag{18}$$

The value of $\Pi'(0,\phi)$ in (18) implies the energy release rate at the kink point in the incipient direction $\tau(\phi)$ of kinking.

3. Shape derivative of the energy functional at $r > 0$

For the shape sensitivity analysis of problem (6) at fixed $r \in (0, R)$ we apply the regular perturbation arguments. Indeed, for a kinematic parameter $t \in \mathbb{R}$, referred to as "time", a regular flow with a kinematic velocity V is constructed between the reference domain $\Omega_{(r,\phi)}$ and the perturbed domain $\Omega_{(r+s,\phi)}$, $r + s > 0$. In the following Proposition 2 and Proposition 3 we see that the length of perturbed branch $r + s$ and the kinematic parameter t are connected by the relation $r + s = re^t$.

We start with a proper kinematic description. Motivated by (CB) we introduce the time-independent velocity

$$V(x) = x\eta(x) \in W^{1,\infty}\big(\mathbb{R}^2\big)^2, \qquad V = 0 \quad \text{on } \partial\Omega, \tag{19}$$

where $x \mapsto \eta : \mathbb{R}^2 \to [0, 1]$ is a suitable cut-off function. We assume that η is supported in Ω such that $\eta \equiv 1$ in the ball B_δ of fixed radius $\delta \in (0, R)$, which we specify further, and $\eta \equiv 0$ outside B_R. The construction in (19) implies an extension of the local velocity $V(x) = x$ from $\Gamma_{(r,\phi)}$ in $\mathbb{R}^2$.

We consider the Cauchy problem for a non-linear ODE

$$\frac{d}{dt}\Phi(t, \cdot) = V\big(\Phi(t, \cdot)\big) \quad \text{for } t \neq 0, \qquad \Phi(0, x) = x. \tag{20}$$

The usual solvability arguments provide the unique solution

$$\Phi(t, x) \in C^1\big([-T, T]; W^{1,\infty}(\Omega)\big)^2, \quad T > 0. \tag{21}$$

Since (20) is an autonomous system, we obtain the identities

$$\Phi\big(-t, \Phi(t, x)\big) = \Phi\big(t, \Phi(-t, x)\big) = x \tag{22}$$

implying that $\Phi(-t, x)$ is an inverse function to $\Phi(t, x)$. In the ball B_δ where $\eta \equiv 1$, the solution to (20) can be calculated analytically as

$$\Phi(t, x) = xe^t \quad \text{when } xe^t \in B_\delta. \tag{23}$$

Relations (19)–(23) argue the following proposition; see [22] for details.

Proposition 2. *For $r \in (0, \delta)$ and $t \in (-\infty, \ln(\delta/r))$, the coordinate extension $y = \Phi(t, x)$ yields a bijective mapping between the domains $\Omega_{(r,\phi)}$ and $\Omega_{(re^t,\phi)}$, and between sets K in (5) in the following sense:*

$$\begin{aligned} &\text{if } u \in K(\Omega_{(r,\phi)}), \quad \text{then } u \circ \Phi(-t) \in K(\Omega_{(re^t,\phi)}); \\ &\text{if } u \in K(\Omega_{(re^t,\phi)}), \quad \text{then } u \circ \Phi(t) \in K(\Omega_{(r,\phi)}). \end{aligned} \tag{24}$$

Within the constructed flow, the shape derivative in (17) reads

$$\Pi'(r, \phi) = \lim_{t \to 0}\big\{\big(r\big(e^t - 1\big)\big)^{-1}\big(\Pi\big(u^{(re^t,\phi)}; \Omega_{(re^t,\phi)}\big) - \Pi\big(u^{(r,\phi)}; \Omega_{(r,\phi)}\big)\big)\big\}. \tag{25}$$

Due to Proposition 2 we can transform $\Pi(u^{(re^t,\phi)}; \Omega_{(re^t,\phi)})$ to the fixed domain $\Omega_{(r,\phi)}$, and then expand it with respect to small $t \to 0$. Indeed, for $u \in H(\Omega_{(re^t,\phi)})$, the potential energy functional (2) written over the perturbed domain $\Omega_{(re^t,\phi)}$ reads

$$\Pi(u; \Omega_{(re^t,\phi)}) = \frac{1}{2}\int_{\Omega_{(re^t,\phi)}} |\nabla u|^2\,dx - \int_{\Omega_{(re^t,\phi)}} fu\,dx - \int_{\Gamma_N} gu\,dx. \tag{26}$$

Applying the coordinate transformation $y = \Phi(t, x)$ to (26) we obtain

$$\Pi(u; \Omega_{(re^t,\phi)}) = \Pi \circ \Phi(t)\big(u \circ \Phi(t); \Omega_{(r,\phi)}\big) \quad \text{for } u \in H(\Omega_{(re^t,\phi)}). \tag{27}$$

The perturbed functional $\Pi \circ \Phi(t)$ is defined for $u \in H(\Omega_{(r,\phi)})$ by

$$\Pi\circ\Phi(t)(u;\Omega_{(r,\phi)}) := \frac{1}{2}\int_{\Omega_{(r,\phi)}} (\nabla u)^\top \frac{\partial\Phi^{-1}}{\partial x}(t)\left(\frac{\partial\Phi^{-1}}{\partial x}(t)\right)^\top \nabla u \left|\det\left(\frac{\partial\Phi}{\partial x}(t)\right)\right| dx - \int_{\Omega_{(r,\phi)}} f\circ\Phi(t)\,u\left|\det\left(\frac{\partial\Phi}{\partial x}(t)\right)\right| dx - \int_{\Gamma_N} gu\,dx. \tag{28}$$

To expand $\Phi(t)$, from (20) it follows the asymptotic representation

$$\Phi(t,x) = x + tV(x) + \mathrm{Res}_t, \quad \|\mathrm{Res}_t\|_{W^{1,\infty}(\Omega)^2} = o(t). \tag{29}$$

With "Res" we will denote respective residuals. Therefore, differentiating (29) with respect to x and substituting the result into (28), we derive the asymptotic expansion

$$\begin{aligned}&\Pi\circ\Phi(t)(u;\Omega_{(r,\phi)}) = \Pi(u;\Omega_{(r,\phi)}) + t\Pi_V^1(u,u,f;\Omega_{(r,\phi)}) + \mathrm{Res}_t(u),\\ &\left|\mathrm{Res}_t(u)\right| \leqslant \mathrm{c}(t)\big(\|u\|^2_{H(\Omega_{(r,\phi)})} + \mathrm{const}\big), \quad 0 \leqslant \mathrm{c}(t) = o(t).\end{aligned} \tag{30}$$

The first asymptotic term in (30) is associated to a quadratic form:

$$\Pi_V^1(u,v,f;\Omega_{(r,\phi)}) = \frac{1}{2}\int_{\Omega_{(r,\phi)}} \nabla u\cdot\left(\mathrm{div}(V)I - \frac{\partial V}{\partial x} - \frac{\partial V}{\partial x}^\top\right)\nabla v\,dx - \int_{\Omega_{(r,\phi)}} \mathrm{div}(Vf)v\,dx \quad \text{for } u,v\in H(\Omega_{(r,\phi)}). \tag{31}$$

Note that (31) is not symmetric between u and v in the linear term.

The conditions (24) and (30) are sufficient to prove the differentiability result stated in the following proposition.

Proposition 3. *For the problem* (6)*, the directional derivative in* (25) *exists,* $\Pi'(r,\phi)\leqslant 0$*, and it is expressed by the formula*

$$\Pi'(r,\phi) = r^{-1}\Pi_V^1\big(u^{(r,\phi)}, u^{(r,\phi)}, f;\Omega_{(r,\phi)}\big). \tag{32}$$

Proof. The perturbed problem (6) stated over $\Omega_{(re^t,\phi)}$ has the solution $u^{(re^t,\phi)}\in K(\Omega_{(re^t,\phi)})$ satisfying

$$\Pi\big(u^{(re^t,\phi)};\Omega_{(re^t,\phi)}\big) \leqslant \Pi(v;\Omega_{(re^t,\phi)}) \quad \text{for all } v\in K(\Omega_{(re^t,\phi)}). \tag{33}$$

Based on Proposition 2, from (33) it follows that $u^{(re^t,\phi)}\circ\Phi(t)\in K(\Omega_{(r,\phi)})$ is a unique minimizer of the perturbed functional $\Pi\circ\Phi(t)$:

$$\Pi\circ\Phi(t)\big(u^{(re^t,\phi)}\circ\Phi(t);\Omega_{(r,\phi)}\big) \leqslant \Pi\circ\Phi(t)(v;\Omega_{(r,\phi)}) \quad \text{for all } v\in K(\Omega_{(r,\phi)}). \tag{34}$$

Inserting $v = u^0$ in (34), due to (30) we derive the uniform estimate

$$\big\|u^{(re^t,\phi)}\circ\Phi(t)\big\|^2_{H(\Omega_{(r,\phi)})} \leqslant c_0 + c_1\big\|u^0\big\|^2_{H(\Omega_{(r,\phi)})} + O(t).$$

Hence, there exists a subsequence of $u^{(re^t,\phi)}\circ\Phi(t)$ which converges weakly to $u^{(r,\phi)}$ as $t\to 0$. By the usual arguments of monotone operators, from (34) and (6) we arrive at the strong convergence

$$u^{(re^t,\phi)}\circ\Phi(t) \to u^{(r,\phi)} \quad \text{strongly in } H(\Omega_{(r,\phi)}) \text{ as } t\to 0. \tag{35}$$

Using (27), (30), and (34) we evaluate the increment of energy from above:

$$\begin{aligned}&\Pi\big(u^{(re^t,\phi)};\Omega_{(re^t,\phi)}\big) - \Pi\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\\ &\quad= \Pi\circ\Phi(t)\big(u^{(re^t,\phi)}\circ\Phi(t);\Omega_{(r,\phi)}\big) - \Pi\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\\ &\quad\leqslant \Pi\circ\Phi(t)\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big) - \Pi\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\\ &\quad= t\Pi_V^1\big(u^{(r,\phi)}, u^{(r,\phi)}, f;\Omega_{(r,\phi)}\big) + \mathrm{Res}_t\big(u^{(r,\phi)}\big), \quad \mathrm{Res}_t\big(u^{(r,\phi)}\big) = o(t).\end{aligned} \tag{36}$$

On the other hand, using (6) we derive the estimation from below:

$$\begin{aligned}
&\Pi\big(u^{(re^t,\phi)};\Omega_{(re^t,\phi)}\big)-\Pi\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big)\\
&\quad\geqslant \Pi\circ\Phi(t)\big(u^{(re^t,\phi)}\circ\Phi(t);\Omega_{(r,\phi)}\big)-\Pi\big(u^{(re^t,\phi)}\circ\Phi(t);\Omega_{(r,\phi)}\big)\\
&\quad= t\Pi^1_V\big(u^{(re^t,\phi)}\circ\Phi(t),u^{(re^t,\phi)}\circ\Phi(t),f;\Omega_{(r,\phi)}\big)\\
&\qquad+\mathrm{Res}_t\big(u^{(re^t,\phi)}\circ\Phi(t)\big),\quad \mathrm{Res}_t\big(u^{(re^t,\phi)}\circ\Phi(t)\big)=o(t).
\end{aligned}\tag{37}$$

Dividing (36) and (37) with t, after passing $t\to 0$ due to (35), from (25) we infer (32).

The non-positive sign of $\Pi'(r,\phi)$ is provided by the fact that $\Pi(r,\phi)$ is a non-increasing function of the crack length r. Indeed, the minimum of Π over increasing sets $K(\Omega_{(r,\phi)})\subset K(\Omega_{(r+s,\phi)})$, for $s>0$, cannot increase from r to $r+s$. □

The first corollary of Proposition 3 concerns the important fact of localization of the shape derivative in subsets in $\Omega_{(r,\phi)}$.

Corollary 1. *Since $f=0$ in B_{δ_f}, the derivative in* (32) *is expressed equivalently by the domain integral over $\Omega_0\setminus B_\delta$ as*

$$\Pi'(r,\phi)=r^{-1}\Pi^1_V\big(u^{(r,\phi)},u^{(r,\phi)},f;\Omega_0\setminus B_\delta\big),\tag{38}$$

or, by the contour integral over ∂B_δ

$$\Pi'(r,\phi)=\delta r^{-1}\int\limits_{\partial B_\delta}\left\{\frac{1}{2}\big|\nabla u^{(r,\phi)}\big|^2-\left(\frac{\partial u^{(r,\phi)}}{\partial n}\right)^2\right\}dx\tag{39}$$

for $\delta\in(r,\delta_f)$. The notation uses the normal vector $n:=|x|^{-1}x$.

Proof. Let us rewrite (32) with the help of (31) explicitly as

$$\Pi'(r,\phi)=\frac{1}{r}\int\limits_{\Omega_{(r,\phi)}}\left\{\frac{1}{2}\nabla u^{(r,\phi)}\cdot\left(\mathrm{div}(V)I-\frac{\partial V}{\partial x}-\frac{\partial V}{\partial x}^{\top}\right)\nabla u^{(r,\phi)}-\mathrm{div}(Vf)u^{(r,\phi)}\right\}dx.\tag{40}$$

If $\eta\equiv 1$ in B_δ, thus $V(x)=x$, then the density of the integral in (40) is $-\mathrm{div}(xf)u^{(r,\phi)}$ due to $\mathrm{div}(x)I-\partial x/\partial x-\partial x/\partial x^\top=0$. Therefore, since $f=0$ in B_δ we obtain (38).

In $B_R\setminus B_\delta$ the solution $u^{(r,\phi)}$ is H^2-smooth according to Proposition 1. Hence we can differentiate the domain integral by parts, and due to $V=0$ in $\Omega_0\setminus B_R$ we derive that

$$\begin{aligned}
&\int\limits_{\Omega_0\setminus B_\delta}\left\{\frac{1}{2}\nabla u^{(r,\phi)}\cdot\left(\mathrm{div}(V)\,I-\frac{\partial V}{\partial x}-\frac{\partial V}{\partial x}^{\top}\right)\nabla u^{(r,\phi)}-\mathrm{div}(Vf)u^{(r,\phi)}\right\}dx\\
&\quad=\int\limits_{B_R\setminus B_\delta}\big(\Delta u^{(r,\phi)}+f\big)\big(V\cdot\nabla u^{(r,\phi)}\big)\,dx+\int\limits_{\partial B_\delta}\left\{\frac{1}{2}(n\cdot V)\big|\nabla u^{(r,\phi)}\big|^2-\frac{\partial u^{(r,\phi)}}{\partial n}\big(V\cdot\nabla u^{(r,\phi)}\big)\right\}dx\\
&\qquad-\int\limits_{\Gamma_0\cap(B_R\setminus B_\delta)}\left[\!\!\left[\frac{1}{2}(\nu\cdot V)\big|\nabla u^{(r,\phi)}\big|^2-\frac{\partial u^{(r,\phi)}}{\partial \nu}\big(V\cdot\nabla u^{(r,\phi)}\big)\right]\!\!\right]dx.
\end{aligned}$$

Using relations (8), the both integrals over $B_R\setminus B_\delta$ as well as over $\Gamma_0\cap(B_R\setminus B_\delta)$ are zero. At ∂B_δ it holds $V=x$. As the result, from (38) we arrive at formula (39). This contour integral is well known in fracture mechanics as the path-independent Cherepanov–Rice integral.

Note that, if the solution was extra smooth $u^{(r,\phi)}\in H^2(B_\delta\setminus\Gamma_{(r,\phi)})$, then integration by parts over B_R would result in $\Pi'(r,\phi)=0$. □

The second corollary of Proposition 3 addresses the special case of identical transformations.

Corollary 2. *Since $y = \Phi(t, x)$ maps Ω_0 and $K(\Omega_0)$ into themselves, then*

$$\Pi_V^1\big(u^0, u^0, f; \Omega_0\big) = 0. \tag{41}$$

Indeed, (41) is argued by the proof of Proposition 3 implying that

$$\begin{aligned} 0 &= \frac{d}{dt}\Pi\big(u^0; \Omega_0\big) = \lim_{t\to 0}\big\{t^{-1}\big(\Pi \circ \Phi(t)\big(u^0 \circ \Phi(t); \Omega_0\big) - \Pi\big(u^0; \Omega_0\big)\big)\big\} \\ &= \Pi_V^1\big(u^0, u^0, f; \Omega_0\big). \end{aligned}$$

We emphasize that such results of the shape sensitivity analysis are available for a rather general class of variational crack problems. In contrast, the topological sensitivity as $r \to 0$ needs asymptotic analysis of the solutions which we get in the following. Comparing with Section 1 we observe that Corollary 2 ensures the sufficient condition (SC1). Next we aim at the condition (SC2).

4. Convergence of the solutions as $r \to 0$

In this section, we evaluate with respect to $r \to 0$ the difference of the solutions of problems (7) and (13), which we denote by

$$w^{(r,\phi)} := u^{(r,\phi)} - u^0 \in H(\Omega_{(r,\phi)}). \tag{42}$$

Within variational methods, inserting the reference solution $u^0 \in K(\Omega_0) \subset K(\Omega_{(r,\phi)}) \subset K(\Omega_{(R,\phi)})$ of (13) in (7), we can infer the convergence $w^{(r,\phi)} \to 0$ as $r \to 0$ in the $H(\Omega_{(R,\phi)})$-norm over the fixed domain $\Omega_{(R,\phi)}$. With the help of Fourier series we establish the order of convergence over the singularly perturbed domains $\Omega_{(r,\phi)}$ as $r \to 0$. Thus, we will show in Theorem 1 the $\sqrt{r}$-order of the convergence $w^{(r,\phi)} \to 0$ in the $H(\Omega_{(r,\phi)})$-norm, and the r-order convergence with respect to the $H(\Omega_0 \setminus B_\delta)$-norm, for arbitrary fixed $\delta \in (0, R)$.

Subtracting (15) from (10) we get a variational formulation for $w^{(r,\phi)} \in H(\Omega_{(r,\phi)})$, which satisfies for all $v \in H(\Omega_{(r,\phi)})$ the equation

$$\int\limits_{\Omega_{(r,\phi)}} \nabla w^{(r,\phi)} \cdot \nabla v\, dx = -\left\langle \frac{\partial u^{(r,\phi)}}{\partial \nu} - \frac{\partial u^0}{\partial \nu}, [\![v]\!] \right\rangle_{\Gamma_{(r,\phi)}}. \tag{43}$$

It implies the following boundary-value problem:

$$-\Delta w^{(r,\phi)} = 0 \quad \text{in } \Omega_{(r,\phi)}, \tag{44a}$$

$$w^{(r,\phi)} = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial w^{(r,\phi)}}{\partial q} = 0 \quad \text{on } \Gamma_N, \tag{44b}$$

$$\frac{\partial w^{(r,\phi)}}{\partial \nu} = \frac{\partial u^{(r,\phi)}}{\partial \nu} - \frac{\partial u^0}{\partial \nu} \quad \text{on } \Gamma_{(r,\phi)}. \tag{44c}$$

We estimate the norm of $w^{(r,\phi)}$ from (43) as follows. Substituting $u^0 \in K(\Omega_0) \subset K(\Omega_{(r,\phi)})$ into (11) as a test function we get the inequality

$$-\left\langle \frac{\partial u^{(r,\phi)}}{\partial \nu}, \big[\!\big[w^{(r,\phi)}\big]\!\big] \right\rangle_{\Gamma_{(r,\phi)}} \leqslant 0. \tag{45}$$

In contrast, $u^{(r,\phi)}$ cannot be substituted into (16). By this reason, we partition $\Gamma_{(r,\phi)}$ with the help of a suitable cut-off function $\chi_r : \mathbb{R}^2 \mapsto [0, 1]$ supported in B_δ with $\delta \in (r, R)$ and satisfying

$$\chi_r(x) = 1 \quad \text{for } x \in \gamma_{(r,\phi)}, \qquad \chi_r(x) = 0 \quad \text{for } x \in \Gamma_0 \setminus B_r. \tag{46}$$

Consequently, $(1 - \chi_r)[\![u^{(r,\phi)}]\!] = 0$ at $\gamma_{(r,\phi)}$. Taking $v = (1 - \chi_r)u^{(r,\phi)} + \chi_r u^0$ in (16) provides us with the inequality

$$\left\langle \frac{\partial u^0}{\partial \nu}, \big[\!\big[w^{(r,\phi)}\big]\!\big] \right\rangle_{\Gamma_{(r,\phi)}} \leqslant \left\langle \frac{\partial u^0}{\partial \nu}, \chi_r \big[\!\big[w^{(r,\phi)}\big]\!\big] \right\rangle_{\Gamma_{(r,\phi)}}. \tag{47}$$

Therefore, substituting $w^{(r,\phi)}$ into (43), from (45) and (47) we infer that

$$\int\limits_{\Omega_{(r,\phi)}} \left|\nabla w^{(r,\phi)}\right|^2 dx \leqslant \left\langle \frac{\partial u^0}{\partial \nu}, \chi_r \left[\!\left[w^{(r,\phi)} \right]\!\right] \right\rangle_{\Gamma_{(r,\phi)} \cap B_r}, \tag{48}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_{(r,\phi)} \cap B_r}$ means the duality pairing between $H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)$ and its dual space $H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)^\star$.

In the further consideration we estimate (48) with respect to r. In fact, one can guess that the right-hand side of (48) decays as $r \to 0$. The task of rigorous evaluation needs the auxiliary results stated below.

We remark that, for $u \in H(\Omega_{(r,\phi)})$, continuity of the trace operator yields the following estimates in B_δ with arbitrary $\delta \in (r, R)$: for the jump

$$\begin{aligned} &\left\| [\![u]\!] \right\|^2_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)} \leqslant \frac{c}{\delta^2} \int\limits_{B_\delta \setminus \Gamma_{(r,\phi)}} |u|^2 \, dx + c \int\limits_{B_\delta \setminus \Gamma_{(r,\phi)}} |\nabla u|^2 \, dx, \\ &\quad \text{if } [\![u]\!] \in H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r); \end{aligned} \tag{49}$$

and for the normal derivative

$$\begin{aligned} &\left\| \frac{\partial u}{\partial \nu} \right\|^2_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)^\star} \leqslant c \int\limits_{B_\delta \setminus \Gamma_{(r,\phi)}} |\nabla u|^2 \, dx, \\ &\quad \text{if } \left[\!\left[\frac{\partial u}{\partial \nu} \right]\!\right] = 0 \text{ and } \Delta u = 0 \text{ in } B_\delta \setminus \Gamma_{(r,\phi)}. \end{aligned} \tag{50}$$

Moreover, a Poincaré inequality implies that

$$\frac{1}{\delta^2} \int\limits_{B_\delta \setminus \Gamma_{(r,\phi)}} |u|^2 \, dx \leqslant c \int\limits_{B_\delta \setminus \Gamma_{(r,\phi)}} |\nabla u|^2 \, dx, \quad \text{if} \int\limits_{B_\delta \setminus \Gamma_{(r,\phi)}} u \, dx = 0. \tag{51}$$

All involved constant c are independent of δ and r. The assertions (49)–(51) can be justified by homogeneity arguments.

In order to evaluate the right-hand side of (48), in Theorem 1 we use (49)–(51). To apply (51), however, one needs to exclude the mean values of the underlying functions. For this reason, we expand the solutions $w^{(r,\phi)}$ and u^0 with the arguments of Fourier series. In Lemma 1 and Lemma 2 we construct a zero-order expansions locally in B_R. Further, the representations of $w^{(r,\phi)}$ and u^0 allow us to apply the Poincaré–Wirtinger inequalities.

Following the radial structure of the kinked crack, we introduce a polar coordinate system at the origin 0 such that $x = \rho(\cos\theta, \sin\theta)$. The polar angle $\theta \in (-\pi, \pi)$ is measured counter-clockwisely from the x_1-axis. The polar radius $\rho := |x| = \sqrt{x_1^2 + x_2^2}$. Starting a procedure of the separation of variables within Fourier series, in $B_R \setminus \Gamma_{(r,\phi)}$ we decompose

$$w^{(r,\phi)} = \bar{w}^{(r,\phi)} + W^{(r,\phi)}, \quad \bar{w}^{(r,\phi)} := \frac{1}{2\pi} \left(\int\limits_{-\pi}^{\phi} + \int\limits_{\phi}^{\pi} \right) w^{(r,\phi)} \, d\theta. \tag{52}$$

Note that, generally, $\bar{w}^{(r,\phi)}$ depends on ρ in (52), and $W^{(r,\phi)}$ implies a residual. Firstly, integrating (52) over $\theta \in (-\pi, \phi) \cup (\phi, \pi)$ we conclude that

$$\left(\int\limits_{-\pi}^{\phi} + \int\limits_{\phi}^{\pi} \right) W^{(r,\phi)}(\rho, \theta) \, d\theta = 0 \quad \text{for all } \rho \in (0, R). \tag{53}$$

Second, we show that the mean value $\bar{w}^{(r,\phi)}$ is constant for all $\rho \in (0, R)$.

Lemma 1. *The function $x \mapsto \bar{w}^{(r,\phi)}$ is constant identically for $x \in B_R \setminus \Gamma_{(r,\phi)}$.*

Proof. We take a smooth cut-off function $\xi(\rho)$ supported in B_R and substitute $v=\xi$ into (43). In view of $[\![\xi]\!]=0$ at the crack, the right-hand side turns to be zero. For $\Delta w^{(r,\phi)} \in L^2(B_R \setminus \Gamma_{(r,\phi)})$ we can integrate by parts in $B_R \setminus \Gamma_{(r,\phi)}$. Thus, due to $\xi=0$ at ∂B_R, $[\![\xi]\!]=0$ and $[\![\partial w^{(r,\phi)}/\partial \nu]\!]=0$ at $\Gamma_{(R,\phi)}$ we obtain

$$0=\int\limits_{\Omega_{(r,\phi)}} \nabla w^{(r,\phi)}\cdot\nabla\xi\,dx=-\int\limits_{B_R\setminus\Gamma_{(r,\phi)}} \Delta w^{(r,\phi)}\xi\,dx$$
$$=-\int\limits_{-\pi}^{\pi}\int\limits_{0}^{R}\left\{\frac{\partial}{\partial\rho}\left(\rho\frac{\partial w^{(r,\phi)}}{\partial\rho}\right)+\frac{1}{\rho}\frac{\partial^2 w^{(r,\phi)}}{\partial\theta^2}\right\}\xi(\rho)\,d\rho\,d\theta=-2\pi\int\limits_0^R\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\bar{w}^{(r,\phi)}}{\partial\rho}\right)\xi\,d\rho.$$

Since ξ is arbitrary, this assures that

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\bar{w}^{(r,\phi)}}{\partial\rho}\right)=0\quad\text{for all }\rho\in(0,R).$$

A general solution to this differential equation implies $\bar{w}^{(r,\phi)}=c_1+c_2\ln\rho$. But the logarithmic term would contradict to the inclusion $w^{(r,\phi)}\in H^1(B_R\setminus\Gamma_{(r,\phi)})$. Indeed, if $c_2\neq 0$, then we can derive from (52) that

$$\int\limits_{B_R\setminus\Gamma_{(r,\phi)}}\left|\nabla w^{(r,\phi)}\right|^2dx\geqslant\int\limits_{-\pi}^{\pi}\int\limits_0^R\left(\frac{\partial w^{(r,\phi)}}{\partial\rho}\right)^2\rho\,d\rho\,d\theta$$
$$\geqslant 2\pi c_2^2\int\limits_0^R\frac{1}{\rho}\,d\rho+2c_2\int\limits_0^R\frac{\partial}{\partial\rho}\left(\int\limits_{-\pi}^{\pi}W^{(r,\phi)}\,d\theta\right)d\rho=+\infty$$

due to (53). Therefore, $c_2=0$ implies the assertion of lemma. □

The following lemma establishes a similar decomposition of u^0, provided that there is no forces applied in a neighborhood of the kink point.

Lemma 2. *Since $f=0$ in B_{δ_f}, $\delta_f\in(0,R)$, decomposing*

$$u^0=\bar{u}^0+U^0\quad\text{in } B_{\delta_f}\setminus\Gamma_0,\qquad \bar{u}^0:=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}u^0\,d\theta,\tag{54}$$

U^0 satisfies $\int_{-\pi}^{\pi}U^0\,d\theta=0$, and $\bar{u}^0(x)$ is constant identically for $x\in B_{\delta_f}\setminus\Gamma_0$.

Proof. We take a suitable cut-off function $\xi(\rho)$ supported in B_{δ_f} and insert $v=u^0\pm\xi$ into (13). In view of $f=0$ in B_{δ_f} and $\xi=0$ on Γ_N we obtain

$$\int\limits_{B_{\delta_f}\setminus\Gamma_0}\nabla u^0\cdot\nabla\xi\,dx=0.$$

Henceforth, repeating the arguments used in the proof of Lemma 1 implies that $\bar{u}^0$ is constant identically in $B_{\delta_f}\setminus\Gamma_0$. □

We note the following fact. Lemma 1 ensures that (52) admits the orthogonal decomposition of $w^{(r,\phi)}\in H(B_R\setminus\Gamma_{(r,\phi)})$ into the “rigid displacement” $\bar{w}^{(r,\phi)}\in\mathbb{R}$ and $W^{(r,\phi)}\in H(B_R\setminus\Gamma_{(r,\phi)})\setminus\mathbb{R}$. In fact, from (53) it follows that the $L^2(B_R\setminus\Gamma_{(r,\phi)})$-product of $W^{(r,\phi)}$ and arbitrary $c\in\mathbb{R}$ is zero. And $\bar{w}^{(r,\phi)}$ is the orthogonal projection of $w^{(r,\phi)}$ onto $\mathbb{R}$. Moreover, this decomposition is uniform for all $\rho\in(0,R)$. The similar assertion for u^0 follows from Lemma 2.

The orthogonal decompositions (52) and (54) are used to derive the next two lemmas. They state a Saint-Venant principle for our problem. The Saint-Venant principle establishes a decay of the solutions with respect to the diminishing disks $B_r \to \{0\}$ which collapse to the point of kink as $r \to 0$. The exact order of decay of $w^{(r,\phi)}$ and u^0 in the energy norm is given in the following Lemma 3 and Lemma 4.

Lemma 3. *Away from the kink point, for $r < \delta_0 \leqslant \delta < R$ the following estimate holds*

$$\int_{\Omega_0\setminus B_\delta} \left|\nabla w^{(r,\phi)}\right|^2 dx \leqslant \frac{\delta_0}{\delta} \int_{\Omega_0\setminus B_{\delta_0}} \left|\nabla w^{(r,\phi)}\right|^2 dx. \tag{55}$$

Proof. We start with the Green formula in $\Omega_0 \setminus B_\delta$

$$\int_{\Omega_0\setminus B_\delta} \nabla w^{(r,\phi)} \cdot \nabla v \, dx = -\int_{\partial B_\delta} \frac{\partial w^{(r,\phi)}}{\partial n} v \, dx - \int_{\Gamma_0\cap(B_R\setminus B_\delta)} \frac{\partial w^{(r,\phi)}}{\partial \nu} [\![v]\!] \, dx \tag{56}$$

written for $v \in H(\Omega_0 \setminus B_\delta)$ and $n = |x|^{-1}x$. Here we used $\Delta w^{(r,\phi)} = 0$ due to (44a). The boundary conditions (8c) and (14c) imply that

$$\frac{\partial w^{(r,\phi)}}{\partial \nu} \left[\!\left[w^{(r,\phi)}\right]\!\right] \geqslant 0 \quad \text{on } \Gamma_0 \cap (B_R \setminus B_r). \tag{57}$$

After substitution of $w^{(r,\phi)}$ into (56) and using (57) we arrive at the estimate

$$\int_{\Omega_0\setminus B_\delta} \left|\nabla w^{(r,\phi)}\right|^2 dx \leqslant -\int_{\partial B_\delta} \frac{\partial w^{(r,\phi)}}{\partial n} w^{(r,\phi)} \, dx. \tag{58}$$

Thanks to (52) and (53) we express the right-hand side of (58) as

$$\int_{\partial B_\delta} \frac{\partial w^{(r,\phi)}}{\partial n} w^{(r,\phi)} \, dx = \int_{-\pi}^{\pi} \frac{\partial W^{(r,\phi)}}{\partial \rho} W^{(r,\phi)} \delta \, d\theta + \left(\frac{\partial}{\partial \rho} \int_{-\pi}^{\pi} W^{(r,\phi)} \delta \, d\theta \right) \bar{w}^{(r,\phi)} = \int_{\partial B_\delta} \frac{\partial w^{(r,\phi)}}{\partial n} W^{(r,\phi)} \, dx.$$

Applying to $W^{(r,\phi)}$ the Wirtinger inequality along the circle

$$\frac{1}{4} \int_{-\pi}^{\pi} u^2 \, d\theta \leqslant \int_{-\pi}^{\pi} \left(\frac{\partial u}{\partial \theta} \right)^2 d\theta \quad \text{for } u \text{ such that } \int_{-\pi}^{\pi} u \, d\theta = 0, \tag{59}$$

we proceed the estimation

$$\begin{aligned} \left| \int_{\partial B_\delta} \frac{\partial w^{(r,\phi)}}{\partial n} w^{(r,\phi)} \, dx \right| &\leqslant \delta \int_{-\pi}^{\pi} \left\{ \delta \left(\frac{\partial w^{(r,\phi)}}{\partial \rho} \right)^2 + \frac{1}{4\delta} \left(W^{(r,\phi)}\right)^2 \right\} d\theta \\ &\leqslant \delta \int_{-\pi}^{\pi} \left\{ \delta \left(\frac{\partial w^{(r,\phi)}}{\partial \rho} \right)^2 + \frac{1}{\delta} \left(\frac{\partial W^{(r,\phi)}}{\partial \theta} \right)^2 \right\} d\theta = \delta \int_{\partial B_\delta} \left|\nabla w^{(r,\phi)}\right|^2 dx. \end{aligned} \tag{60}$$

On the other hand, the co-area formula yields

$$\frac{d}{d\delta} \int_{\Omega_0\setminus B_\delta} \left|\nabla w^{(r,\phi)}\right|^2 dx = -\int_{\partial B_\delta} \left|\nabla w^{(r,\phi)}\right|^2 dx. \tag{61}$$

Combining the estimates (58)–(61) results in the differential inequality

$$\int_{\Omega_0\setminus B_\delta} \left|\nabla w^{(r,\phi)}\right|^2 dx \leqslant -\delta \frac{d}{d\delta} \int_{\Omega_0\setminus B_\delta} \left|\nabla w^{(r,\phi)}\right|^2 dx.$$

Integrating this inequality with respect to δ we arrive at (55). □

For the reference solution u^0 of (13), upon application of Lemma 2 we derive the similar result in a neighborhood of the kink point.

Lemma 4. *Since $f = 0$ in B_{δ_f}, for $0 < \delta_0 \leqslant \delta < \delta_f$ the following estimate holds*

$$\int_{B_{\delta_0}\setminus\Gamma_0} |\nabla u^0|^2\,dx \leqslant \frac{\delta_0}{\delta} \int_{B_\delta\setminus\Gamma_0} |\nabla u^0|^2\,dx. \tag{62}$$

Based on Lemma 1–Lemma 4 we state the main result of this section.

Theorem 1. *The difference $w^{(r,\phi)} = u^{(r,\phi)} - u^0$ of the solutions of (6) and (12) converges to zero as $r \to 0$ strongly in $H(\Omega_{(\delta,\phi)})$ for arbitrary fixed $\delta \in (0, \delta_f)$ with the following uniform estimates:*

$$\|w^{(r,\phi)}\|_{H(\Omega_{(r,\phi)})} \leqslant c\sqrt{r}, \tag{63}$$

$$\|w^{(r,\phi)}\|_{H(\Omega_0\setminus B_\delta)} \leqslant cr. \tag{64}$$

Proof. We start evaluation of (48) with the Cauchy inequality

$$\int_{\Omega_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2\,dx \leqslant \left\| \frac{\partial u^0}{\partial \nu} \right\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)}\cap B_r)^\star} \|\chi_r [\![w^{(r,\phi)}]\!]\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)}\cap B_r)}.$$

Applying to u^0 estimate (50) with $\delta = 2r$ in B_{2r} we obtain

$$\|w^{(r,\phi)}\|^2_{H(\Omega_{(r,\phi)})} \leqslant c_1 \|u^0\|_{H(\Omega_0\setminus B_{2r})} \|\chi_r [\![w^{(r,\phi)}]\!]\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)}\cap B_r)} \tag{65}$$

with constant c_1 which does not depend on r.

We take the cut-off function χ_r in (65) satisfying relations (46) in B_r, and extend it in B_{2r} such that

$$0 \leqslant \chi_r(x) \leqslant 1, \qquad |\nabla\chi_r(x)| \leqslant r^{-1}c \quad \text{for } x \in B_{2r}. \tag{66}$$

In view of Lemma 1, constant $\bar{w}^{(r,\phi)}$ can be avoided from the right-hand side of (65) since $[\![\bar{w}^{(r,\phi)}]\!] = 0$. Therefore, applying to $\chi_r W^{(r,\phi)}$ estimate (49) with $\delta = 2r$ and using (66) we infer that

$$\begin{aligned} \|\chi_r [\![w^{(r,\phi)}]\!]\|^2_{H_{00}^{1/2}(\Gamma_{(r,\phi)}\cap B_r)} &\leqslant c_2 \int_{B_{2r}\setminus\Gamma_{(r,\phi)}} |\nabla(\chi_r W^{(r,\phi)})|^2\,dx + \frac{c_3}{r^2} \int_{B_{2r}\setminus\Gamma_{(r,\phi)}} |W^{(r,\phi)}|^2\,dx \\ &\leqslant c_4 \int_{B_{2r}\setminus\Gamma_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2\,dx \end{aligned} \tag{67}$$

due to $\nabla\bar{w}^{(r,\phi)} = 0$. Here we used the Poincaré inequality (51) with $\delta = 2r$. In view of Lemma 4 with $\delta_0 = 2r$ and $\delta \in (2r, \delta_f)$, from (65) and (67) we infer that

$$\begin{aligned} \int_{\Omega_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2\,dx &\leqslant c_5 \left(2r\delta^{-1} \int_{B_\delta\setminus\Gamma_{(r,\phi)}} |\nabla u^0|^2\,dx \int_{B_{2r}\setminus\Gamma_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2\,dx \right)^{1/2} \\ &\leqslant c_6\sqrt{r} \left(\int_{\Omega_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2\,dx \right)^{1/2}, \end{aligned}$$

thus (63) holds true. It implies the strong convergence $w^{(r,\phi)} \to 0$ as $r \to 0$.

To derive (64), we rewrite (63) as the sum of the two integrals

$$\int_{\Omega_0\setminus B_{2r}} |\nabla w^{(r,\phi)}|^2\,dx + \int_{B_{2r}\setminus\Gamma_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2\,dx \leqslant c_7 r. \tag{68}$$

Using Lemma 3 with $\delta_0 = 2r$, from (68) it follows the estimate

$$r^{-1} \int\limits_{\Omega_0 \setminus B_\delta} \left|\nabla w^{(r,\phi)}\right|^2 dx + \int\limits_{B_{2r} \setminus \Gamma_{(r,\phi)}} \left|\nabla w^{(r,\phi)}\right|^2 dx \leqslant cr,$$

hence (64). The theorem is proved. □

The estimates obtained in Theorem 1 are optimal. In fact, from the proof we observe that the powers of r in (63) and (64) appear according to the powers of δ_0/δ in the Saint-Venant principles (55) and (62). The latter are determined by the factor in upper bound (60) given in the proof of Lemma 3. For the Laplace equation in a disk this bound is optimal, since derived from the Wirtinger inequality. For the general case of elasticity equations such exact bounds, however, are not available analytically. This question is relevant to the optimal constant in estimates of the Friedrichs–Korn–Poincaré type; see [8,29].

5. Expansion of the energy functional at $r = 0$

Following Section 1, the estimates of the solution in Theorem 1 (compare with (SC2)) provide the respective expansion of the energy given below (compare with (AE2)).

Proposition 4. *The sequence $\{\Pi'(r,\phi)\}$ is bounded uniformly with respect to $r \to 0$.*

Proof. Due to $f = 0$ in B_{δ_f} we can apply Corollary 1 and rewrite $\Pi'(r,\phi)$ with the help of decomposition $u^{(r,\phi)} = u^0 + w^{(r,\phi)}$ as

$$\begin{aligned}\Pi'(r,\phi) = {} & \Pi_V^1\big(2u^0, r^{-1}w^{(r,\phi)}, f; \Omega_0 \setminus B_\delta\big) \\ & + \Pi_V^1\big(w^{(r,\phi)}, r^{-1}w^{(r,\phi)}, 0; \Omega_0 \setminus B_\delta\big) + r^{-1}\Pi_V^1\big(u^0, u^0, f; \Omega_0 \setminus B_\delta\big).\end{aligned} \tag{69}$$

Hence Corollary 2 and estimate (64) yield the assertion. □

For fixed $s > 0$, from definition (17) and Proposition 3 we get the expansion of $\Pi(s + r, \cdot)$ with respect to $r > 0$ in the following form

$$\Pi\big(u^{(s+r,\phi)}; \Omega_{(s+r,\phi)}\big) = \Pi\big(u^{(s,\phi)}; \Omega_{(s,\phi)}\big) + \int\limits_0^r \Pi'(s + l, \phi)\, dl.$$

Passing here to the limit as $s \to 0$, due to the strong convergence (35) and Proposition 4, the Lebesgue dominated convergence theorem ensures that

$$\Pi\big(u^{(r,\phi)}; \Omega_{(r,\phi)}\big) = \Pi\big(u^0; \Omega_0\big) + \int\limits_0^r \Pi'(l, \phi)\, dl. \tag{70}$$

Moreover, for $l \in (0, r)$ we apply decomposition (69) to $\Pi'(l,\phi)$ in (70) and conclude with the following result.

Theorem 2. *As $r \to 0$, the expansion of the potential energy holds*

$$\Pi(r,\phi) = \Pi(0) + \Theta^1(r,\phi) + O\big(r^2\big), \quad \Theta^1(r,\phi) = O(r), \tag{71}$$

where $\Theta^1(r,\phi) \leqslant 0$, and, for arbitrary fixed $\delta \in (0, \delta_f)$,

$$\Theta^1(r,\phi) = \int\limits_0^r \Pi_V^1\big(2u^0, l^{-1}w^{(l,\phi)}, f; \Omega_0 \setminus B_\delta\big)\, dl. \tag{72}$$

Therefore, according to (18), if the following limit exists

$$\lim_{r\to 0}\big(r^{-1}\Theta^1(r,\phi)\big) = \Pi'(0,\phi), \tag{73}$$

then $\Pi'(0,\phi)$ implies the topological derivative. In general, Theorem 2 guarantees existence of the limit superior and the limit inferior of the quotient in (73), thus allowing bounded oscillations as $r \to 0$.

To find the limit in (73) one needs to calculate Π_V^1 in (72). This task requires an asymptotic representation of the solutions u^0 and $w^{(r,\phi)}$ in the spatial variables, near ∂B_δ due to Corollary 2. For this aim, we note that the Fourier series (54) of u^0 can be extended with a first-order asymptotic term; see [23]. The respective extension of the series (52) for $w^{(r,\phi)}$, however, is not available within the governing equation (43). Indeed, its right-hand side is not known a priori. For comparison, avoiding the non-linearity in the problem, (43) turns into Eq. (87) with the right-hand side determined by u^0. In the latter case, the Fourier series of u^0 provides an expansion of the solution $w^{(r,\phi)}$ of (87), too.

Motivated by the above consideration, in the next section we avoid the unilateral constraint (1), thus, linearize the problem. This allows us to proceed the assertion of Theorem 2 with more details, which express (73) in the terms of coefficients of the extended Fourier series.

6. Specification of expansions for a linear problem

We restate the crack problem abandoning the constraint (1) from the variational formulation. As the consequence, all previous results obtained for the non-linear problem (6)–(8) remain true for the linear problem (74)–(76) below.

Indeed, resetting minimization in (6) over $H(\Omega_{(r,\phi)})$, there exists the unique solution $u^{(r,\phi)} \in H(\Omega_{(r,\phi)})$ such that

$$\Pi\big(u^{(r,\phi)};\Omega_{(r,\phi)}\big) \leqslant \Pi(v;\Omega_{(r,\phi)}) \quad \text{for all } v \in H(\Omega_{(r,\phi)}), \tag{74}$$

which is equivalent to the variational equation

$$\int_{\Omega_{(r,\phi)}} \nabla u^{(r,\phi)} \cdot \nabla v\,dx = \int_{\Omega_{(r,\phi)}} f v\,dx + \int_{\Gamma_N} g v\,dx \quad \text{for } v \in H(\Omega_{(r,\phi)}). \tag{75}$$

It describes a weak solution to the linear boundary-value problem:

$$-\Delta u^{(r,\phi)} = f \quad \text{in } \Omega_{(r,\phi)}, \tag{76a}$$

$$u^{(r,\phi)} = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial u^{(r,\phi)}}{\partial q} = g \quad \text{on } \Gamma_N, \tag{76b}$$

$$\frac{\partial u^{(r,\phi)}}{\partial \nu} = 0 \quad \text{on } \Gamma_{(r,\phi)}. \tag{76c}$$

At $r = 0$, the reference solution $u^0 \in H(\Omega_0)$ implies the minimum

$$\Pi\big(u^0;\Omega_0\big) \leqslant \Pi(v;\Omega_0) \quad \text{for all } v \in H(\Omega_0) \tag{77}$$

corresponding to the variational equation

$$\int_{\Omega_0} \nabla u^0 \cdot \nabla v\,dx = \int_{\Omega_0} f v\,dx + \int_{\Gamma_N} g v\,dx \quad \text{for all } v \in H(\Omega_0). \tag{78}$$

The reference boundary-value problem reads:

$$-\Delta u^0 = f \quad \text{in } \Omega_0, \tag{79a}$$

$$u^0 = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial u^0}{\partial q} = g \quad \text{on } \Gamma_N, \tag{79b}$$

$$\frac{\partial u^0}{\partial \nu} = 0 \quad \text{on } \Gamma_0. \tag{79c}$$

Firstly, we extend the Fourier series (54) of u^0 with a first-order asymptotic term, which is given in Proposition 5. To evaluate the remainder terms, we employ estimation of the Saint-Venant type. In this context, the formal technique is similar to that of Section 4. Second, we refine the Fourier series (52) for $w^{(r,\phi)}$ in Proposition 6. These expansions result in a refined representation of the topological derivative given in Theorem 3. We note that the derivative is expressed in the terms of coefficients in the first asymptotic terms in the expansions obtained for u^0 and $w^{(r,\phi)}$. We refer to these (dimensionless) coefficients as "stress intensity factors", since they associate respective material parameters as adopted in fracture mechanics.

We start with the reference solution $u^0 \in H(\Omega_0)$ of the unconstrained minimization problem (77).

Proposition 5. *Around the kink point, the following expansion holds*

$$u^0 = \bar{u}^0 + K\sqrt{\rho}\sin\frac{\theta}{2} + U_1^0 \quad \text{in } B_{\delta_f} \setminus \Gamma_0 \tag{80}$$

with the unique stress intensity factor $K \in \mathbb{R}$ given by

$$K = \frac{1}{\pi\sqrt{\rho}} \int_{-\pi}^{\pi} u^0(\rho,\theta)\sin\frac{\theta}{2}\,d\theta \quad \text{for arbitrary } \rho \in (0,\delta_f), \tag{81}$$

and the reminder term $U_1^0 \in H^1(B_{\delta_f} \setminus \Gamma_0)$ satisfies

$$\int_{-\pi}^{\pi} U_1^0(\rho,\theta)\,d\theta = \int_{-\pi}^{\pi} U_1^0(\rho,\theta)\sin\frac{\theta}{2}\,d\theta = 0 \quad \text{for any } \rho \in (0,\delta_f), \tag{82a}$$

$$\int_{B_{\delta_0}\setminus\Gamma_0} \left|\nabla U_1^0\right|^2 dx \leqslant \left(\frac{\delta_0}{\delta}\right)^2 \int_{B_\delta\setminus\Gamma_0} \left|\nabla U_1^0\right|^2 dx, \quad 0 < \delta_0 \leqslant \delta < \delta_f. \tag{82b}$$

Proof. Indeed, continuing decomposition (54) from Lemma 2, according to the Fourier series for the Laplace operator in $B_{\delta_f} \setminus \Gamma_0$, we define

$$a(\rho) := \frac{1}{\pi}\int_{-\pi}^{\pi} U^0 \sin\frac{\theta}{2}\,d\theta, \qquad U_1^0 := U^0 - a(\rho)\sin\frac{\theta}{2}. \tag{83}$$

Hence the property (82a) follows immediately from (54) and (83). To justify that $a(\rho) = K\sqrt{\rho}$ and to estimate the remainder term U_1^0 in the decomposition (80), we follow the lines in the proof of Lemma 2 and Lemma 4, respectively.

Repeating the arguments of Lemma 2 we take a smooth cut-off function $\xi(\rho)$ supported in B_{δ_f} and substitute $v = \xi \sin\theta/2$ into Eq. (78). Due to $f = 0$ in B_{δ_f}, using (82a) and (83) we calculate

$$\begin{aligned} 0 &= \int_{B_{\delta_f}\setminus\Gamma_0} \nabla u^0 \cdot \nabla\left(\xi \sin\frac{\theta}{2}\right) dx = \int_{B_{\delta_f}\setminus\Gamma_0} \nabla\left(a\sin\frac{\theta}{2}\right) \cdot \nabla\left(\xi\sin\frac{\theta}{2}\right) dx \\ &= \pi \int_0^{\delta_f} \left\{-\frac{\partial}{\partial\rho}\left(\rho\frac{\partial a}{\partial\rho}\right) + \frac{a}{4\rho}\right\}\xi\,d\rho + \pi\left(\rho\frac{\partial a}{\partial\rho}\xi\right)_{\rho=0}^{\rho=\delta_f}. \end{aligned}$$

Since ξ is arbitrary we derive the ordinary differential equation

$$-\frac{\partial}{\partial\rho}\left(\rho\frac{\partial a}{\partial\rho}\right) + \frac{a}{4\rho} = 0 \quad \text{for } \rho \in (0,\delta_f).$$

Its general solution implies

$$a(\rho) = K\sqrt{\rho} + c/\sqrt{\rho} \quad \text{with } K, c \in \mathbb{R}. \tag{84}$$

Using (84) we evaluate the norm of u^0 from below as

$$\int_{B_{\delta_f}\setminus\Gamma_0} |\nabla u^0|^2\,dx = \pi\int_0^{\delta_f}\left\{\rho\left(\frac{\partial a}{\partial\rho}\right)^2 + \frac{a^2}{4\rho}\right\}d\rho + \int_{B_{\delta_f}\setminus\Gamma_0} |\nabla U_1^0|^2\,dx$$

$$\geqslant \frac{\pi}{2}\int_0^{\delta_f}\left(K^2 + \frac{c^2}{\rho^2}\right)d\rho = +\infty.$$

This fact contradicts to $u^0 \in H^1(B_{\delta_f}\setminus\Gamma_0)$ and concludes that $c = 0$ necessarily. From (54), (83), and (84) we infer (80) and (81).

To derive (82b) we modify the arguments of Lemma 4 for U_1^0. For $\delta \in (0, \delta_f)$, using (80) and (82a) we calculate

$$\int_{B_\delta\setminus\Gamma_0} |\nabla u^0|^2\,dx = \frac{\pi}{2}K^2\delta + \int_{B_\delta\setminus\Gamma_0} |\nabla U_1^0|^2\,dx = \int_{\partial B_\delta} \frac{\partial u^0}{\partial n}u^0\,dx = \frac{\pi}{2}K^2\delta + \int_{\partial B_\delta}\frac{\partial U_1^0}{\partial n}U_1^0\,dx,$$

which implies the equality

$$\int_{B_\delta\setminus\Gamma_0} |\nabla U_1^0|^2\,dx = \int_{\partial B_\delta}\frac{\partial U_1^0}{\partial n}U_1^0\,dx. \tag{85}$$

The property (82a) for U_1^0 guarantees the following Wirtinger inequality

$$\int_{-\pi}^{\pi}(U_1^0)^2\,d\theta \leqslant \int_{-\pi}^{\pi}\left(\frac{\partial U_1^0}{\partial\theta}\right)^2 d\theta,$$

which is stronger than (59). Therefore, from (85) we estimate

$$\int_{B_\delta\setminus\Gamma_0} |\nabla U_1^0|^2\,dx \leqslant \frac{\delta}{2}\int_{-\pi}^{\pi}\left\{\delta\left(\frac{\partial U_1^0}{\partial\rho}\right)^2 + \frac{1}{\delta}(U_1^0)^2\right\}(\delta)\,d\theta$$

$$\leqslant \frac{\delta}{2}\int_{\partial B_\delta}|\nabla U_1^0|^2\,dx = \frac{\delta}{2}\frac{d}{d\delta}\int_{B_\delta\setminus\Gamma_0}|\nabla U_1^0|^2\,dx.$$

Integrating this inequality with respect to δ results in (82b). □

Note that the square-root singularity in (80) is well known; see [16,37].

Now we consider the difference $w^{(r,\phi)} := u^{(r,\phi)} - u^0$ of the solutions of (74) and (77), which satisfies the boundary-value problem:

$$-\Delta w^{(r,\phi)} = 0 \quad \text{in } \Omega_{(r,\phi)}, \tag{86a}$$

$$w^{(r,\phi)} = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial w^{(r,\phi)}}{\partial q} = 0 \quad \text{on } \Gamma_N, \tag{86b}$$

$$\frac{\partial w^{(r,\phi)}}{\partial\nu} = 0 \quad \text{on } \Gamma_0, \qquad \frac{\partial w^{(r,\phi)}}{\partial\nu} = -\frac{\partial u^0}{\partial\nu} \quad \text{on } \gamma_{(r,\phi)}. \tag{86c}$$

Moreover, $w^{(r,\phi)} \in H(\Omega_{(r,\phi)})$ solves the variational equation

$$\int_{\Omega_{(r,\phi)}} \nabla w^{(r,\phi)}\cdot\nabla v\,dx = \left\langle\frac{\partial u^0}{\partial\nu}, [\![v]\!]\right\rangle_{\Gamma_{(r,\phi)}} \qquad \text{for all } v \in H(\Omega_{(r,\phi)}). \tag{87}$$

To expand the solution of (87) we propose the following strategy.

With the help of representation (80) in Proposition 5 we can decompose the normal derivative as

$$\frac{\partial u^0}{\partial \nu} = -\frac{K}{2\sqrt{\rho}}\cos\frac{\phi}{2} + \frac{\partial U_1^0}{\partial \nu} \quad \text{on } \gamma_{(r,\phi)}, \qquad \frac{\partial u^0}{\partial \nu} = 0 \quad \text{on } \Gamma_0, \tag{88}$$

for $r < \delta_f$. Therefore, after substitution of (88) into the right-hand side of (87) we get the following decomposition

$$w^{(r,\phi)} = \frac{K}{2}\cos\frac{\phi}{2}\, h^{(r,\phi)} + Q_1, \tag{89}$$

where $h^{(r,\phi)} \in H(\Omega_{(r,\phi)})$ solves the problem

$$\int\limits_{\Omega_{(r,\phi)}} \nabla h^{(r,\phi)} \cdot \nabla v\, dx = -\left\langle \frac{1}{\sqrt{\rho}}\mathcal{H}_{\gamma_{(r,\phi)}}, [\![v]\!]\right\rangle_{\Gamma_{(r,\phi)}} \qquad \text{for } v \in H(\Omega_{(r,\phi)}),$$

$$\mathcal{H}_{\gamma_{(r,\phi)}} = 1 \quad \text{on } \gamma_{(r,\phi)}, \qquad \mathcal{H}_{\gamma_{(r,\phi)}} = 0 \quad \text{on } \Gamma_0, \tag{90}$$

and the remainder term $Q_1 \in H(\Omega_{(r,\phi)})$ satisfies the equation

$$\int\limits_{\Omega_{(r,\phi)}} \nabla Q_1 \cdot \nabla v\, dx = \left\langle \frac{\partial U_1^0}{\partial \nu}, [\![v]\!]\right\rangle_{\Gamma_{(r,\phi)}} \qquad \text{for all } v \in H(\Omega_{(r,\phi)}). \tag{91}$$

The variational setting (90) corresponds to the boundary-value problem

$$-\Delta h^{(r,\phi)} = 0 \quad \text{in } \Omega_{(r,\phi)}, \tag{92a}$$

$$h^{(r,\phi)} = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial h^{(r,\phi)}}{\partial q} = 0 \quad \text{on } \Gamma_N, \tag{92b}$$

$$\frac{\partial h^{(r,\phi)}}{\partial \nu} = 0 \quad \text{on } \Gamma_0, \qquad \frac{\partial h^{(r,\phi)}}{\partial \nu} = -\frac{1}{\sqrt{\rho}} \quad \text{on } \gamma_{(r,\phi)}. \tag{92c}$$

In Lemma 5 below we evaluate Q_1 from (91). In Lemma 6 we expand the solution $h^{(r,\phi)}$ of (90) in the Fourier series. Consequently, collecting the expansions yields the Fourier series for $w^{(r,\phi)}$ in Proposition 6.

We start with the estimation of the remainder term Q_1.

Lemma 5. *The solution of Eq.* (87) *admits the decomposition* (89) *with the following estimates of the remainder term*

$$\|Q_1\|_{H(\Omega_{(r,\phi)})} = O(r), \qquad \|Q_1\|_{H(\Omega_0\setminus B_\delta)} = O\big(r^{3/2}\big) \quad \textit{for } \delta \in (0, \delta_f). \tag{93}$$

Proof. Here we repeat the arguments used in the proof of Theorem 1.

Let us substitute $v = \chi_r Q_1 + (1 - \chi_r)Q_1$ into (90) with the cut-off function χ_r from (46). Using $\partial U_1^0/\partial\nu = \partial u^0/\partial\nu = 0$ at Γ_0 and the Cauchy inequality we derive the estimate

$$\int\limits_{\Omega_{(r,\phi)}} |\nabla Q_1|^2\, dx \leqslant \left\| \frac{\partial U_1^0}{\partial \nu}\right\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)}\cap B_r)^\star} \big\|\chi_r [\![Q_1]\!]\big\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)}\cap B_r)}.$$

In view of $\Delta(\sqrt{\rho}\sin\theta/2) = 0$, we have $\Delta U_1^0 = 0$ in B_{δ_f}, hence can apply (50) to U_1^0. Continuing the estimation, due to (82b) we infer that

$$\int\limits_{\Omega_{(r,\phi)}} |\nabla Q_1|^2\, dx \leqslant c_1\Bigg(\int\limits_{B_{2r}\setminus\Gamma_{(r,\phi)}} \big|\nabla U_1^0\big|^2\, dx \int\limits_{B_{2r}\setminus\Gamma_{(r,\phi)}} \big|\nabla Q_1\big|^2\, dx\Bigg)^{1/2}$$

$$\leqslant c_2 r\Bigg(\int\limits_{B_\delta\setminus\Gamma_{(r,\phi)}} \big|\nabla U_1^0\big|^2\, dx \int\limits_{B_{2r}\setminus\Gamma_{(r,\phi)}} |\nabla Q_1|^2\, dx\Bigg)^{1/2} \leqslant c_3 r\Bigg(\int\limits_{\Omega_{(r,\phi)}} |\nabla Q_1|^2\, dx\Bigg)^{1/2},$$

thus $\|Q_1\|_{H(\Omega_{(r,\phi)})} = O(r)$ holds true in (93).

The Green formula in $\Omega_0 \setminus B_\delta$ provides the following equality for Q_1

$$\int\limits_{\Omega_0\setminus B_\delta} |\nabla Q_1|^2\,dx = -\int\limits_{\partial B_\delta} \frac{\partial Q_1}{\partial n} Q_1\,dx,$$

similarly to (58). Therefore, we can apply Lemma 3 to Q_1 and obtain

$$\int\limits_{\Omega_0\setminus B_\delta} |\nabla Q_1|^2\,dx \leqslant \frac{\delta_0}{\delta} \int\limits_{\Omega_0\setminus B_{\delta_0}} |\nabla Q_1|^2\,dx, \quad r < \delta_0 \leqslant \delta < \delta_f.$$

Taking here $\delta_0 = 2r$ finishes the proof of $\|Q_1\|_{H(\Omega_0\setminus B_\delta)} = O(r^{3/2})$. $\square$

We proceed with an expansion of $h^{(r,\phi)}$ from (90) in the Fourier series, which is specified away from the kink point.

Lemma 6. *With a cut-off function χ supported in B_R such that $\chi \equiv 1$ in B_δ, $r < \delta < R$, the following expansion holds*

$$h^{(r,\phi)}(\rho,\theta) = \left(\bar{h}^{(r,\phi)} + b^{(r,\phi)}(\rho)\sin\frac{\theta}{2}\right)\chi(\rho) + Q_2(\rho,\theta) \quad \textit{in } \Omega_{(r,\phi)}, \tag{94}$$

where the mean value $\bar{h}^{(r,\phi)} := (2\pi)^{-1}\int_{-\pi}^{\pi} h^{(r,\phi)}\,d\theta = \mathrm{const}$ in B_R,

$$b^{(r,\phi)}(\rho) = C_{1/2}^{(r,\phi)}\sqrt{\rho} - C_{-1/2}^{(r,\phi)}/\sqrt{\rho} \quad \textit{for } \rho > r, \tag{95}$$

and the unique stress intensity factors $C_{\pm 1/2}^{(r,\phi)} \in \mathbb{R}$ are given by:

$$C_{1/2}^{(r,\phi)} = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \frac{\partial}{\partial\rho}\big(\sqrt{\rho}\,h^{(r,\phi)}\big)\sin\frac{\theta}{2}\,d\theta,$$

$$C_{-1/2}^{(r,\phi)} = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \rho^2\frac{\partial}{\partial\rho}\left(\frac{h^{(r,\phi)}}{\sqrt{\rho}}\right)\sin\frac{\theta}{2}\,d\theta \quad \textit{for } \rho \in (r,R). \tag{96}$$

The remainder term Q_2 possesses the estimate

$$\|Q_2\|_{H(\Omega_0\setminus B_\delta)} = O\big(r^{3/2}\big) \quad \textit{for fixed } \delta \in (r,R). \tag{97}$$

Proof. Since (90) is a particular case of the variational equation (87), we can apply to the solution $h^{(r,\phi)}$ of (90) the results which are valid for the solution $w^{(r,\phi)}$ of (87). In this way we justify the assertions of the lemma.

Upon application to $h^{(r,\phi)}$, Lemma 1 ensures the decomposition

$$h^{(r,\phi)} = \bar{h}^{(r,\phi)} + B^{(r,\phi)} \quad \text{in } B_R \setminus \Gamma_{(r,\phi)},$$

$$\bar{h}^{(r,\phi)} := \frac{1}{2\pi}\left(\int\limits_{-\pi}^{\phi} + \int\limits_{\phi}^{\pi}\right) h^{(r,\phi)}\,d\theta = \mathrm{const}, \qquad \left(\int\limits_{-\pi}^{\phi} + \int\limits_{\phi}^{\pi}\right) B^{(r,\phi)}\,d\theta = 0.$$

In $B_R \setminus \Gamma_{(r,\phi)}$ we expand the remainder term $B^{(r,\phi)}$ with

$$b^{(r,\phi)} := \frac{1}{\pi}\left(\int\limits_{-\pi}^{\phi} + \int\limits_{\phi}^{\pi}\right) h^{(r,\phi)}\sin\frac{\theta}{2}\,d\theta, \qquad B_1^{(r,\phi)} := B^{(r,\phi)} - b^{(r,\phi)}\sin\frac{\theta}{2}, \tag{98}$$

which implies that

$$\int\limits_{-\pi}^{\pi} B_1^{(r,\phi)}\,d\theta = \int\limits_{-\pi}^{\pi} B_1^{(r,\phi)}\sin\frac{\theta}{2}\,d\theta = 0 \quad \text{for } \rho \in (0,R). \tag{99}$$

For arbitrary cut-off function $\xi(\rho)$ supported in (r,R), inserting $v=\xi\sin\theta/2$ as a test function into (90) we derive

$$0=\int_{B_R\setminus B_r}\nabla h^{(r,\phi)}\cdot\nabla\left(\xi\sin\frac{\theta}{2}\right)dx=\pi\int_r^R\left\{-\frac{\partial}{\partial\rho}\left(\rho\frac{\partial b^{(r,\phi)}}{\partial\rho}\right)+\frac{b^{(r,\phi)}}{4\rho}\right\}\xi\,d\rho.$$

This proves the representation of $b^{(r,\phi)}$ in the form (95) which is similar to (84). From (95) and (98) we have the equality

$$C_{1/2}^{(r,\phi)}\sqrt{\rho}-C_{-1/2}^{(r,\phi)}\frac{1}{\sqrt{\rho}}=\frac{1}{\pi}\int_{-\pi}^{\pi}h^{(r,\phi)}\sin\frac{\theta}{2}\,d\theta\quad\text{for }\rho\in(r,R).$$

Differentiating it with respect to ρ results in (96).

With the help of a cut-off function χ supported in B_R and such that $\chi\equiv1$ in B_δ, $\delta\in(r,R)$, we extend $B_1^{(r,\phi)}$ to a function $Q_2\in H(\Omega_{(r,\phi)})$ defined by

$$Q_2:=h^{(r,\phi)}-\left(\bar h^{(r,\phi)}+b^{(r,\phi)}\sin\frac{\theta}{2}\right)\chi,\qquad Q_2=B_1^{(r,\phi)}\quad\text{in }B_\delta\setminus\Gamma_{(r,\phi)}.$$

Next we write the Green formula in $\Omega_0\setminus B_\delta$ for $v\in H(\Omega_0\setminus B_\delta)$ as

$$\begin{aligned}\int_{\Omega_0\setminus B_\delta}\nabla h^{(r,\phi)}\cdot\nabla v\,dx&=\int_{\Omega_0\setminus B_\delta}\nabla\left(Q_2+b^{(r,\phi)}\chi\sin\frac{\theta}{2}\right)\cdot\nabla v\,dx\\&=-\int_{\partial B_\delta}\frac{\partial h^{(r,\phi)}}{\partial n}v\,dx=-\int_{\partial B_\delta}\left\{\frac{\partial}{\partial n}\left(B_1^{(r,\phi)}+b^{(r,\phi)}\sin\frac{\theta}{2}\right)\right\}v\,dx.\end{aligned}$$

Inserting $v=Q_2$ as a test function here, it follows the equality for the norm of Q_2, which is similar to (85). We estimate it in the following way

$$\int_{\Omega_0\setminus B_\delta}|\nabla Q_2|^2\,dx=-\int_{\partial B_\delta}\frac{\partial B_1^{(r,\phi)}}{\partial n}B_1^{(r,\phi)}\,dx\leqslant\frac{\delta}{2}\int_{\partial B_\delta}\left|\nabla B_1^{(r,\phi)}\right|^2dx=-\frac{\delta}{2}\frac{d}{d\delta}\int_{\Omega_0\setminus B_\delta}|\nabla Q_2|^2\,dx$$

due to (99). Integrating this inequality provides the estimate

$$\int_{\Omega_0\setminus B_\delta}|\nabla Q_2|^2\,dx\leqslant\left(\frac{\delta_0}{\delta}\right)^2\int_{\Omega_0\setminus B_{\delta_0}}|\nabla Q_2|^2\,dx,\quad 0<\delta_0\leqslant\delta<R.\tag{100}$$

Taking $\delta_0=2r$, in view of Lemma 5 and Theorem 1 we proceed (100) with

$$\begin{aligned}\int_{\Omega_0\setminus B_\delta}|\nabla Q_2|^2\,dx&\leqslant c_1r^2\int_{\Omega_0\setminus B_{2r}}|\nabla Q_2|^2\,dx\leqslant c_2r^2\int_{\Omega_{(r,\phi)}}\left|\nabla h^{(r,\phi)}\right|^2dx\\&\leqslant c_3r^2\int_{\Omega_{(r,\phi)}}\left|\nabla w^{(r,\phi)}\right|^2dx+O\left(r^4\right)=O\left(r^3\right).\end{aligned}$$

The latter estimate implies (97) and ends the proof. □

From Lemma 5 and Lemma 6 we conclude with the following result.

Proposition 6. *Away from the kink point, the following representation holds*

$$\begin{aligned}&w^{(r,\phi)}(\rho,\theta)=\bar w^{(r,\phi)}+\frac{K}{2}\cos\frac{\phi}{2}\left(C_{1/2}^{(r,\phi)}\sqrt{\rho}-C_{-1/2}^{(r,\phi)}\frac{1}{\sqrt{\rho}}\right)\sin\frac{\theta}{2}+Q(\rho,\theta),\\&\|Q\|_{H(B_R\setminus B_\delta)}=O\left(r^{3/2}\right)\quad\textit{for fixed }\delta\in(r,\delta_f).\end{aligned}\tag{101}$$

Using Proposition 5 and Proposition 6, we state the main result of this section as follows.

Theorem 3. *For the linear crack problems* (74)–(79)*, the expansion of energy* (71) *takes the particular form*

$$\Pi(r,\phi)=\Pi(0)-\frac{\pi}{4}K^2\cos\frac{\phi}{2}\int_0^r l^{-1}C_{-1/2}^{(l,\phi)}\,dl+O\big(r^{3/2}\big),\tag{102}$$

with the stress intensity factors K and $C_{-1/2}^{(r,\phi)}$ defined in (81) *and* (96)*. The first asymptotic term in* (71) *reads as*

$$\Theta^1(r,\phi)=\int_0^r l^{-1}C_{-1/2}^{(l,\phi)}\,dl=O(r),\quad \Theta^1(r,\phi)\leqslant 0.\tag{103}$$

Proof. Inserting the representations (80) and (101) we calculate the integral $r^{-1}\Pi_V^1(2u^0,w^{(r,\phi)},f;\Omega_0\setminus B_\delta)$ in (72). In this way we derive the particular relations (102) and (103) from formulas (71)–(72) given in Theorem 2.

For the following calculation, we take the cut-off function $\eta(\rho)$ which determines the velocity V in (19) such that

$$\eta\equiv 1\quad\text{in } B_\delta,\qquad \operatorname{supp}(\eta)\subset B_{\delta_f},$$

and the cut-off function $\chi(\rho)$ in representation (94) satisfying

$$\chi\equiv 1\quad\text{in } B_{\delta_f},\qquad \operatorname{supp}(\chi)\subset B_R\quad\text{for } \delta<\delta_f<R.$$

Using Proposition 6 and the representations (89), (94) for $w^{(r,\phi)}$, we have

$$r^{-1}\Pi_V^1\big(2u^0,w^{(r,\phi)},f;\Omega_0\setminus B_\delta\big)=\frac{K}{2r}\cos\frac{\phi}{2}\cdot I+O(\sqrt{r}),\tag{104}$$

where I denotes the following integral

$$\begin{aligned}I&:=\Pi_V^1\left(2u^0,\chi\left(\bar h^{(r,\phi)}+b^{(r,\phi)}\sin\frac{\theta}{2}\right),f;\Omega_0\setminus B_\delta\right)\\&=\int_{B_{\delta_f}\setminus B_\delta}\left\{\nabla u^0\cdot\left(\operatorname{div}(V)I-2\frac{\partial V}{\partial x}\right)\nabla\left(b^{(r,\phi)}\sin\frac{\theta}{2}\right)-\operatorname{div}(Vf)\left(\bar h^{(r,\phi)}+b^{(r,\phi)}\sin\frac{\theta}{2}\right)\right\}dx.\end{aligned}$$

Integrating by parts in $B_{\delta_f}\setminus B_\delta$, where no singularity occurs and $\chi\equiv 1$, similarly to the calculation used in the proof of Corollary 1 we obtain

$$\begin{aligned}I=&\int_{B_{\delta_f}\setminus B_\delta}\left\{\big(\Delta u^0+f\big)\left(V\cdot\nabla\left(b^{(r,\phi)}\sin\frac{\theta}{2}\right)\right)+\Delta\left(b^{(r,\phi)}\sin\frac{\theta}{2}\right)\big(V\cdot\nabla u^0\big)\right\}dx\\&+\delta\int_{\partial B_\delta}\left\{\nabla u^0\cdot\nabla\left(b^{(r,\phi)}\sin\frac{\theta}{2}\right)-2\frac{\partial u^0}{\partial n}\frac{\partial}{\partial n}\left(b^{(r,\phi)}\sin\frac{\theta}{2}\right)\right\}dx\\&+\int_{\Gamma_0\cap(B_{\delta_f}\setminus B_\delta)}\left\{\frac{\partial u^0}{\partial\nu}\left[\!\left[\left(b^{(r,\phi)}\sin\frac{\theta}{2}\right)_{,1}\right]\!\right]+\frac{\partial}{\partial\nu}\left(b^{(r,\phi)}\sin\frac{\theta}{2}\right)\big[\!\big[u^0_{,1}\big]\!\big]\right\}dx.\end{aligned}$$

Therefore, in view of (79a), (79c), and relations

$$\Delta\left(b^{(r,\phi)}\sin\frac{\theta}{2}\right)=0\quad\text{in }\Omega_0,\qquad \frac{\partial}{\partial\nu}\left(b^{(r,\phi)}\sin\frac{\theta}{2}\right)=0\quad\text{on }\Gamma_0$$

provided by (95), we get

$$I = \delta \int_{\partial B_\delta} \left\{ \nabla u^0 \cdot \nabla \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) - 2 \frac{\partial u^0}{\partial n} \frac{\partial}{\partial n} \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) \right\} dx.$$

Next we apply Proposition 5 and substitute here the decomposition (80) of u^0. Due to the orthogonality property of the Fourier series, the calculation of I finishes with

$$I = \pi K \left\{ \frac{\sqrt{\delta}}{4} \left(\sqrt{\delta} C_{1/2}^{(r,\phi)} - \frac{C_{-1/2}^{(r,\phi)}}{\sqrt{\delta}} \right) - \frac{\delta^2}{2\sqrt{\delta}} \left(\frac{C_{1/2}^{(r,\phi)}}{2\sqrt{\delta}} + \frac{C_{-1/2}^{(r,\phi)}}{2\delta\sqrt{\delta}} \right) \right\} = -\frac{\pi}{2} K C_{-1/2}^{(r,\phi)}.$$

As the result of calculation, from (104) we arrive at the expression

$$r^{-1} \Pi_V^1 \left(2u^0, w^{(r,\phi)}, f; \Omega_0 \setminus B_\delta \right) = -\frac{\pi}{4r} K^2 \cos \frac{\phi}{2} C_{-1/2}^{(r,\phi)} + O(\sqrt{r}). \tag{105}$$

Finally, the assertion of Theorem 2 argues formulas (102) and (103). □

In the following remarks we comment Theorem 3 in relation to some known results in fracture mechanics.

If $K = 0$ in (81), then the solution u^0 is H^2-smooth around the kink point. Henceforth, (102) implies that $\Pi'(0, \phi) = 0$ and $\Pi(r, \phi) = \Pi(0) + O(r^{3/2})$.

If $\phi = 0$, then no kink occurs, and the crack propagates in a regular way. In this case, formula $\Pi'(0, 0) = -\pi/4 K^2$ is well known. Therefore, from (102) and (103) we infer that $C_{-1/2}^{(r,0)} = r + o(r)$.

For arbitrary fixed ϕ, a limit of $r^{-1} C_{-1/2}^{(r,\phi)}$ with respect to $r \to 0$ is given in [4]. Following the method of matched asymptotic expansions of [20,37], this limit uses an auxiliary asymptotic model stated at infinity. An asymptotic analysis of the energy with respect to the kink angle ϕ and r is given in [42]. In the cited reference, convergent series are constructed analytically for the solution of respective Cauchy–Riemann equations stated in the infinite domain $\mathbb{R}^2$ with a kinked crack; see the physical basics in [12].

In comparison, using the variational approach provides us with a rigorous estimation of the remainder terms, and it justifies the asymptotic expansions over bounded domains.

7. Conclusion

The asymptotic representations resulting our analysis are of practical meaning for engineers. In fact, the expansion of potential energy (102) is given via the stress intensity factors K and $C_{-1/2}^{(r,\phi)}$. They can be calculated as a path-independent integrals by the explicit formulas (81) and (96). The former constant K corresponds to the specific choice of data of the reference crack problem before kink, while the latter $C_{-1/2}^{(r,\phi)}$ are "universal" functions depending on the geometric parameters r and ϕ of the kinked domain $\Omega_{(r,\phi)}$. These implicit quantities are to be determined from a generic problem of the crack kinking (90) stated in the bounded domain. The latter problem is suitable for numerical computations.

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