Two-Parameter Topological Expansion of Helmholtz Problems with Inhomogeneity

Victor A. Kovtunenko

Abstract For forward and inverse Helmholtz problems with inhomogeneity in the 2d-setting, high-order topological analysis is provided based on singular perturbations and variational methods. When diminishing the inhomogeneity, the two-parameter asymptotic result is proved rigorously with respect to the size of inhomogeneity and its refractive index. In particular, for a fixed refractive index this implies the topological derivative. For identifying an unknown inhomogeneity put in a test domain, variation of a complex refractive index leads to the zero-order necessary optimality condition of minimum of the objective function. This condition is realized as an imaging function for finding center of the inhomogeneity.

Key words: forward and inverse Helmholtz problem, inhomogeneity, singular perturbation, two-parameter asymptotic analysis, high-order asymptotic expansion, variational method, shape and topology optimization, topological derivative

1 Introduction

Forward and inverse Helmholtz problems with inhomogeneity are considered in 2d-spatial setting of the problem. By this, the inhomogeneity is characterized by a complex refractive index, see [22, Chapter 6]. This formulation is motivated by applications to non-destructive testing of scattering media with acoustic, elastic, and electromagnetic waves. We provide rigorously two-parameter topological analysis based on singular perturbations and variational methods.

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The classic analysis of Helmholtz problems is based on the theory of potential operators, see [8, 10, 11, 22, 35]. Specifically for inverse scattering problems, the direct (non-iterative) methods of factorization [3], sampling and probe [30], enclosure [15, 16], MUSIC [32], as well as iterative methods [18] are established. For the general mathematical theory of inverse problems we refer to [22, 27].

Within shape and topology optimization (see e.g. [1, 13, 28]), the theory of identification of small inhomogeneities was developed in [7, 12]. Newly the concept of topological derivatives was introduced in [34] and adapted to inhomogeneous problems in e.g. [2, 5, 6, 9]. It proceeds with high-order topological expansions in [14, 23]. The topological analysis is realized by adapting methods of singular perturbations which we refer to [17, 19, 20, 29] and references therein.

Within topological analysis of Helmholtz problems in inhomogeneous medium, inhomogeneities are described by arbitrary geometries and variable refractive indexes. An admissible geometry implies the triple of implicit geometric variables of the shape ω, the center x_0, and the size ε. This geometric description is treated within variational theory. We provide the underlying forward and inverse Helmholtz problems with minimax variational principles according to the Fenchel–Legendre duality.

A positive refractive index α is determined in the field of complex numbers. Varying the refractive index describes two important limit cases. First, $|\alpha| \searrow +0$ describes the homogeneous Neumann problem, and, second, $|\alpha| \nearrow +\infty$ corresponds to the homogeneous Dirichlet problem in the perforated domain outside inhomogeneity. The former case implies the sound-hard obstacle, and the latter one corresponds to the sound-soft obstacle in acoustics, which were treated previously in [23].

For the perturbation analysis, asymptotic arguments of small size $\varepsilon \searrow +0$ are employed. The problem of singular perturbations is supported by construction of a boundary layer. It is described by the transmission problem for the Laplace operator stated in the exterior domain, where weighted Sobolev spaces are useful. As the result we get a two-parameter asymptotic expansion with respect to ε and α.

From the perspective of topology optimization, the topology variables $(\omega, x_0, \varepsilon, \alpha)$ enter the objective implicitly through the solution of a geometry-dependent Helmholtz problem. This fact makes difficult to find optimality conditions based on conventional directional derivatives of the objective.

Following [23, 24] in the present contribution we show that an unknown parameter of the boundary impedance is well suitable for the purpose of variation of trial geometries. Passing $\mathrm{Im}(\alpha) \nearrow +\infty$ in the two-parameter expansion gets a zero-order necessary optimality condition which is derivative-free. For the objective minimizing misfit from boundary measurements, this optimality condition gives an imaging function identifying the optimal center x_0.

2 Forward Helmholtz problem in inhomogeneous medium

Let $\Omega \subset \mathbf{R}^2$ be a reference domain with the Lipschitz boundary $\partial\Omega$ and the unit normal vector $n = (n_1, n_2)^\top$ which is outward to Ω, where the upper $^\top$ denotes transposition swapping columns and rows. We assume that the boundary $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_D}$ consists of two disjoint parts Γ_N and Γ_D associated to the Neumann and the Dirichlet conditions, respectively.

Let ω be a compact set in $\mathbf{R}^2$ with the Lipschitz boundary $\partial\omega$ and the normal vector $n = (n_1, n_2)^\top$ outward to ω. We assume that $0 \in \omega$ and $\omega \subset B_1(0)$ such that the unit ball $B_1(0)$ is the minimum enclosing ball centered at origin 0. Then the set G_ω of such shapes ω is invariant to translations and isotropic scaling.

The reason of such a construction is to separate the near field $B_1(0)$ containing ω from the far field $\mathbf{R}^2 \setminus \overline{B_1(0)}$.

Rescaling $\omega \in G_\omega$ by a size parameter $\varepsilon > 0$ produces the geometric object

$$\omega_\varepsilon(x_0) = \left\{x \in \mathbf{R}^2 : \tfrac{x - x_0}{\varepsilon} \in \omega\right\} \subset B_\varepsilon(x_0)$$

posed at a center $x_0 \in \mathbf{R}^2$. In the reference domain Ω, the set of admissible geometries $G := G_\omega \times G_\varepsilon \times G_x$ consists of triples: the shape $\omega \in G_\omega$, the size $\varepsilon \in G_\varepsilon \subset \mathbf{R}_+$, and the center $x_0 \in G_x \subset \Omega$ which satisfy the geometric condition:

$$\omega_\varepsilon(x_0) \subset B_\varepsilon(x_0) \subset \Omega.$$

We distinguish two faces of the interface $\partial\omega_\varepsilon(x_0)$, respectively: the boundary of the inhomogeneity $\omega_\varepsilon(x_0)$ called the negative face $\partial\omega_\varepsilon(x_0)^-$ (having the outward normal vector n), and the part of the boundary of the perforated domain $\Omega \setminus \overline{\omega_\varepsilon(x_0)}$ called the positive face $\partial\omega_\varepsilon(x_0)^+$ (having the inward normal vector n). The respective jump across $\partial\omega_\varepsilon(x_0)$ is denoted by

$$[\![u]\!] = u|_{\partial\omega_\varepsilon(x_0)^+} - u|_{\partial\omega_\varepsilon(x_0)^-}.$$

Let $\alpha \in \mathbf{C}_+$ be a complex refractive index $\alpha = \mathrm{Re}(\alpha) + \mathrm{i}\mathrm{Im}(\alpha)$ with positive the real $\mathrm{Re}(\alpha) \in \mathbf{R}_+$ and the imaginary $\mathrm{Im}(\alpha) \in \mathbf{R}_+$ parts. It characterizes the inhomogeneity $\omega_\varepsilon(x_0)$ with the help of the complex-valued function $\chi^\alpha_{\omega_\varepsilon(x_0)} : \mathbf{R}^2 \mapsto \mathbf{C}_+$:

$$\chi^\alpha_{\omega_\varepsilon(x_0)}(x) = \begin{cases} 1 & \text{for } x \in \mathbf{R}^2 \setminus \overline{\omega_\varepsilon(x_0)} \\ \alpha & \text{for } x \in \omega_\varepsilon(x_0) \end{cases}$$

which is piecewise-constant with the jump $[\![\chi^\alpha_{\omega_\varepsilon(x_0)}]\!] = 1 - \alpha$ across the interface $\partial\omega_\varepsilon(x_0)$. We note that $\alpha = 1$ implies homogeneity since $\chi^1_{\omega_\varepsilon(x_0)} \equiv 1$ in $\mathbf{R}^2$.

The topology variables $(\omega, \varepsilon, x_0, \alpha)$ will be used for the sake of variation of inhomogeneities for inverse problems in Section 3.

For forward problems in Section 2, the geometry $(\omega, \varepsilon, x_0) \in G$ of inhomogeneity and its refractive index $\alpha \in \mathbf{C}_+$ are fixed. In the following we will mark the dependence of functions on the size ε and the refractive index α for the reason of subsequent two-parameter asymptotic analysis.

Given the Neumann $g \in L^2(\Gamma_N;\mathbf{C})$ and the Dirichlet $h \in H^{1/2}(\Gamma_D;\mathbf{C})$ complex data and the real wave number $k \in \mathbf{R}_+$, the *Helmholtz problem in inhomogeneous medium* is stated for the wave potential $u^{(\varepsilon,\alpha)}(x)$ satisfying for $x \in \overline{\Omega}$ the system:

$$-[\Delta + k^2]u^{(\varepsilon,\alpha)} = 0 \text{ in } \Omega \setminus \overline{\omega_\varepsilon(x_0)}, \quad -[\mathrm{Re}(\alpha)\Delta + \alpha k^2]u^{(\varepsilon,\alpha)} = 0 \text{ in } \omega_\varepsilon(x_0), \quad (1)$$

$$[\![u^{(\varepsilon,\alpha)}]\!] = 0, \quad \tfrac{\partial u^{(\varepsilon,\alpha)}}{\partial n}\big|_{\partial\omega_\varepsilon(x_0)^+} - \mathrm{Re}(\alpha)\tfrac{\partial u^{(\varepsilon,\alpha)}}{\partial n}\big|_{\partial\omega_\varepsilon(x_0)^-} = 0 \quad \text{on } \partial\omega_\varepsilon(x_0), \quad (2)$$

$$\tfrac{\partial u^{(\varepsilon,\alpha)}}{\partial n} = g \quad \text{on } \Gamma_N, \quad (3)$$

$$u^{(\varepsilon,\alpha)} = h \quad \text{on } \Gamma_D. \quad (4)$$

Here and in what follows, Δ is the Laplace operator, the notation $\frac{\partial}{\partial n} := n \cdot \nabla = n^\top \nabla$ stands for the normal derivative at the boundary, the dot $\cdot$ means the inner product of vectors, and ∇ is the gradient.

For the strong solution of (1)–(4), a weak formulation can be derived by standard variational arguments: multiplying equations (1) with a smooth test function u and subsequent integration by parts over Ω due to transmission conditions (2) and boundary conditions (3) and (4). The weak solution to (1)–(4) is described by the following variational problem: Find $u^{(\varepsilon,\alpha)} \in H^1(\Omega;\mathbf{C})$ such that (4) holds and

$$\int_\Omega \big(\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \nabla u^{(\varepsilon,\alpha)} \cdot \nabla \overline{u} - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 u^{(\varepsilon,\alpha)} \overline{u}\big)\, dx = \int_{\Gamma_N} g \overline{u}\, dS_x$$

for all test-functions $u \in H^1(\Omega;\mathbf{C})$ such that $u = 0$ on Γ_D. (5)

Here the discontinuous piecewise-constant function $\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} = 1$ in $\mathbf{R}^2 \setminus \overline{\omega_\varepsilon(x_0)}$ and $\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} = \mathrm{Re}(\alpha)$ in $\omega_\varepsilon(x_0)$, the usual notation $\overline{u} = \mathrm{Re}(u) - \mathrm{i}\mathrm{Im}(u)$ implies the complex conjugate of $u = \mathrm{Re}(u) + \mathrm{i}\mathrm{Im}(u)$, and i stands for the imaginary unit.

Avoiding eigenvalues, the unique variational solution of (5) can be argued by the Fredholm alternative using Rellich's lemma, see [22, Theorem 6.9].

In return, Green's formulas hold for every function $\tilde{u} \in H^1(\Omega;\mathbf{C})$ such that $\Delta\tilde{u} \in L^2(\Omega;\mathbf{C})$ and all test-functions $u \in H^1(\Omega;\mathbf{C})$ such that $u = 0$ on Γ_D:

$$\int_{\Omega\setminus\omega_\varepsilon(x_0)} (\nabla\tilde{u} \cdot \nabla\overline{u} + \overline{u}\Delta\tilde{u})\, dx = \langle \tfrac{\partial\tilde{u}}{\partial n}, \overline{u}\rangle_{\Gamma_N} - \langle \tfrac{\partial\tilde{u}}{\partial n}, \overline{u}\rangle_{\partial\omega_\varepsilon(x_0)^+}, \quad (6)$$

$$\int_{\omega_\varepsilon(x_0)} (\nabla\tilde{u} \cdot \nabla\overline{u} + \overline{u}\Delta\tilde{u})\, dx = \langle \tfrac{\partial\tilde{u}}{\partial n}, \overline{u}\rangle_{\partial\omega_\varepsilon(x_0)^-}. \quad (7)$$

Here $\langle \frac{\partial\tilde{u}}{\partial n}, \overline{u}\rangle_{\partial\omega_\varepsilon(x_0)^\pm}$ implies the duality between $u \in H^{1/2}(\partial\omega_\varepsilon(x_0)^\pm;\mathbf{C})$ and $\frac{\partial\tilde{u}}{\partial n} \in H^{-1/2}(\partial\omega_\varepsilon(x_0)^\pm;\mathbf{C})$, while $\langle \frac{\partial\tilde{u}}{\partial n}, \overline{u}\rangle_{\Gamma_N}$ stands for the duality pairing between $u \in H^{1/2}_{00}(\Gamma_N;\mathbf{C})$ and $\frac{\partial\tilde{u}}{\partial n} \in H^{-1/2}(\Gamma_N;\mathbf{C})$ in the Lions–Magenes dual spaces, see e.g. [19, Section 1.4] for detail. Summing up (6) and (7) multiplied by $\mathrm{Re}(\alpha)$ provides the Green formula in inhomogeneous medium:

$$\int_{\Omega} \big(\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \nabla \tilde{u} \cdot \nabla \overline{u} + \chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \overline{u}\, \Delta \tilde{u}\big)\, dx = \langle \tfrac{\partial \tilde{u}}{\partial n}, \overline{u} \rangle_{\Gamma_N} - \langle [\![\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \tfrac{\partial \tilde{u}}{\partial n}]\!], \overline{u} \rangle_{\partial \omega_\varepsilon(x_0)}. \tag{8}$$

With the help of (8) with $\tilde{u} = u^{(\varepsilon,\alpha)}$, we get from (5):

$$\int_{\Omega} \big([\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \Delta + \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2] u^{(\varepsilon,\alpha)} \big) \overline{u}\, dx = \langle \tfrac{\partial u^{(\varepsilon,\alpha)}}{\partial n} - g, \overline{u} \rangle_{\Gamma_N} - \langle [\![\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \tfrac{\partial u^{(\varepsilon,\alpha)}}{\partial n}]\!], \overline{u} \rangle_{\partial \omega_\varepsilon(x_0)}$$

and derive (1)–(3) by the fundamental lemma of calculus of variations when varying the test function u such that first $u = 0$ on $\Gamma_N \cup \partial \omega_\varepsilon(x_0)$ and then $u = 0$ on Γ_N.

Moreover, the following variational principle holds.

Theorem 1. *The variational equation (5) implies the first-order necessary optimality condition for the minimax variational problem:*

$$\mathscr{P}(u^{(\varepsilon,\alpha)}) = \min_{\mathrm{Re}(v)} \max_{\mathrm{Im}(v)} \mathscr{P}(v) \text{ over } v \in H^1(\Omega; \mathbf{C}) \text{ such that } v = h \text{ on } \Gamma_D \tag{9}$$

with the following Lagrangian $\mathscr{P} : H^1(\Omega; \mathbf{C}) \mapsto \mathbf{R}$,

$$\mathscr{P}(v) = \mathrm{Re}\Big\{ \tfrac{1}{2} \int_{\Omega} \big(\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \nabla v \cdot \nabla v - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 v^2 \big)\, dx - \int_{\Gamma_N} g v\, dS_x \Big\}.$$

Proof. If we rewrite $\mathscr{P}$ component-wisely for $v = \mathrm{Re}(v) + \mathrm{i}\mathrm{Im}(v)$ as

$$\mathscr{P}(v) = \tfrac{1}{2} \int_{\Omega} \big\{ \chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \big(|\nabla(\mathrm{Re}(v))|^2 - |\nabla(\mathrm{Im}(v))|^2 \big) \\ - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 \big(\mathrm{Re}(v)^2 - \mathrm{Im}(v)^2 \big) \big\}\, dx - \int_{\Gamma_N} \big(\mathrm{Re}(g)\mathrm{Re}(v) - \mathrm{Im}(g)\mathrm{Im}(v) \big)\, dS_x$$

and differentiate it with respect to $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$, then the necessary optimality condition for (9) implies two variational inequalities

$$\big\langle \tfrac{\partial}{\partial \mathrm{Re}(v)} \mathscr{P}(u^{(\varepsilon,\alpha)}), \mathrm{Re}(v - u^{(\varepsilon,\alpha)}) \big\rangle \geq 0, \quad \big\langle \tfrac{\partial}{\partial \mathrm{Im}(v)} \mathscr{P}(u^{(\varepsilon,\alpha)}), \mathrm{Im}(v - u^{(\varepsilon,\alpha)}) \big\rangle \leq 0$$

holding for all $v \in H^1(\Omega; \mathbf{C})$ such that $v = h$ on Γ_D. Inserting here $v = u^{(\varepsilon,\alpha)} \pm \tilde{u}$ with $\tilde{u} \in H^1(\Omega; \mathbf{C})$ such that $\tilde{u} = 0$ on Γ_D results in two variational equations:

$$\int_{\Omega} \big(\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \nabla \mathrm{Re}(u^{(\varepsilon,\alpha)}) \cdot \nabla \mathrm{Re}(\tilde{u}) - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 \mathrm{Re}(u^{(\varepsilon,\alpha)}) \mathrm{Re}(\tilde{u}) \big)\, dx \\ = \int_{\Gamma_N} \mathrm{Re}(g) \mathrm{Re}(\tilde{u})\, dS_x,$$

$$\int_{\Omega} \big(\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \nabla \mathrm{Im}(u^{(\varepsilon,\alpha)}) \cdot \nabla \mathrm{Im}(\tilde{u}) - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 \mathrm{Im}(u^{(\varepsilon,\alpha)}) \mathrm{Im}(\tilde{u}) \big)\, dx \\ = \int_{\Gamma_N} \mathrm{Im}(g) \mathrm{Im}(\tilde{u})\, dS_x.$$

The summation of these equations for the test function $\tilde{u} = u$:

$$\int_{\Omega} \big\{ \chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \big(\nabla \mathrm{Re}(u^{(\varepsilon,\alpha)}) \cdot \nabla \mathrm{Re}(u) + \nabla \mathrm{Im}(u^{(\varepsilon,\alpha)}) \cdot \nabla \mathrm{Im}(u) \big) - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 \big(\mathrm{Re}(u^{(\varepsilon,\alpha)}) \mathrm{Re}(u) + \mathrm{Im}(u^{(\varepsilon,\alpha)}) \mathrm{Im}(u) \big) \big\}\, dx = \int_{\Gamma_N} \big(\mathrm{Re}(g)\mathrm{Re}(u) + \mathrm{Im}(g)\mathrm{Im}(u) \big)\, dS_x,$$

with arbitrary $u \in H^1(\Omega;\mathbf{C})$ such that $u = 0$ on Γ_D, and for $\tilde{u} = \mathrm{i}u$:

$$\int_{\Omega} \big\{ \chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \big(\nabla \mathrm{Im}(u^{(\varepsilon,\alpha)}) \cdot \nabla \mathrm{Re}(u) - \nabla \mathrm{Re}(u^{(\varepsilon,\alpha)}) \cdot \nabla \mathrm{Im}(u) \big) - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 \big(\mathrm{Im}(u^{(\varepsilon,\alpha)}) \mathrm{Re}(u) - \mathrm{Re}(u^{(\varepsilon,\alpha)}) \mathrm{Im}(u) \big) \big\}\, dx = \int_{\Gamma_N} \big(\mathrm{Im}(g)\mathrm{Re}(u) - \mathrm{Re}(g)\mathrm{Im}(u) \big)\, dS_x,$$

constitutes respectively the real and imaginary parts of (5) and completes the proof. □

Now we gain insight into the two-parameter dependence on (ε,α).

On the one hand, for fixed $\varepsilon \in \mathbf{R}_+$, if $\mathrm{Im}(\alpha) \searrow +0$ and $\mathrm{Re}(\alpha) \searrow +0$, then the Helmholtz problem (1)–(4) decouples into the mixed problem in the perforated domain with homogeneous Neumann condition at the interface (sound-hard):

$$-[\Delta + k^2] u^{(\varepsilon,0)} = 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon(x_0)}, \qquad \frac{\partial u^{(\varepsilon,0)}}{\partial n} = 0 \quad \text{on } \partial\omega_\varepsilon(x_0)^+, \qquad \frac{\partial u^{(\varepsilon,0)}}{\partial n} = g \quad \text{on } \Gamma_N, \quad u^{(\varepsilon,0)} = h \quad \text{on } \Gamma_D,$$

and the Dirichlet problem inside the inhomogeneity:

$$-[\Delta + k^2] u^{(\varepsilon,0)} = 0 \text{ in } \omega_\varepsilon(x_0), \qquad u^{(\varepsilon,0)}\big|_{\partial\omega_\varepsilon(x_0)^-} = u^{(\varepsilon,0)}\big|_{\partial\omega_\varepsilon(x_0)^+} \quad \text{on } \partial\omega_\varepsilon(x_0)^-.$$

If $\mathrm{Re}(\alpha) \nearrow +\infty$ or $\mathrm{Im}(\alpha) \nearrow +\infty$, then $u^{(\varepsilon,\infty)} \equiv 0$ trivially inside $\omega_\varepsilon(x_0)$, and (1)–(4) turns into the mixed problem in the perforated domain with homogeneous Dirichlet boundary condition at the interface (sound-soft):

$$\begin{cases} -[\Delta + k^2] u^{(\varepsilon,\infty)} = 0 \quad \text{in } \Omega \setminus \overline{\omega_\varepsilon(x_0)}, \\ u^{(\varepsilon,\infty)}\big|_{\partial\omega_\varepsilon(x_0)^+} = 0 \quad \text{on } \partial\omega_\varepsilon(x_0)^+, \\ \frac{\partial u^{(\varepsilon,\infty)}}{\partial n} = g \quad \text{on } \Gamma_N, \quad u^{(\varepsilon,\infty)} = h \quad \text{on } \Gamma_D. \end{cases} \tag{10}$$

These limit cases of boundary conditions were treated separately in [23].

On the other hand, if $\varepsilon \searrow +0$, then we derive the *background Helmholtz problem in homogeneous medium* for the wave potential $u^0(x)$:

$$-[\Delta + k^2] u^0 = 0 \quad \text{in } \Omega, \tag{11}$$

$$\tfrac{\partial u^0}{\partial n} = g \quad \text{on } \Gamma_N, \tag{12}$$

$$u^0 = h \quad \text{on } \Gamma_D. \tag{13}$$

In the weak form, the system (11)–(13) is described by the following variational problem: Find $u^0 \in H^1(\Omega;\mathbf{C})$ such that (13) holds and

$$\int_\Omega (\nabla u^0 \cdot \nabla \overline{u} - k^2 u^0 \overline{u})\, dx = \int_{\Gamma_N} g\overline{u}\, dS_x \text{ for all } u \in H^1(\Omega;\mathbf{C}) : u = 0 \text{ on } \Gamma_D. \tag{14}$$

For an inhomogeneity $\omega_\varepsilon(x_0)$, applying Green's formulas (6) and (7) to the variational equation (14) it can be restated equivalently in the inhomogeneous medium similarly to (5):

$$\int_\Omega \big(\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \nabla u^0 \cdot \nabla \overline{u} - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 u^0 \overline{u}\big)\, dx = \int_{\Gamma_N} g\overline{u}\, dS_x$$
$$-(1-\mathrm{Re}(\alpha)) \int_{\partial\omega_\varepsilon(x_0)} \tfrac{\partial u^0}{\partial n} \overline{u}\, dS_x - \mathrm{i}\mathrm{Im}(\alpha) \int_{\omega_\varepsilon(x_0)} k^2 u^0 \overline{u}\, dx. \tag{15}$$

In Section 2.1 we start with an inner asymptotic expansion of u^0 in the near-field, Section 2.2 proceeds with an outer asymptotic expansion in the far-filed, and in Section 2.3 an uniform asymptotic expansion of $u^{(\varepsilon,\alpha)} - u^0$ is matched in Ω.

2.1 Inner asymptotic expansion in the near-field

We start with some preliminaries of asymptotic analysis.

We introduce the local polar coordinate system associated to the center $x_0 \in \Omega$ and determined by the polar radius $\rho \in (0,R)$ and the polar angle $\theta \in (-\pi,\pi]$ such that $x - x_0 = \rho\widehat{x}$ with

$$\rho := |x-x_0|, \quad \widehat{x} := \tfrac{x-x_0}{|x-x_0|} = (\cos\theta, \sin\theta)^\top, \quad \widehat{x}' = (-\sin\theta, \cos\theta)^\top, \tag{16}$$

where $R > 0$ can be chosen such that the ball of radius R and center x_0 lies in the reference domain, i.e. $B_R(x_0) \subset \Omega$.

For indexes $m \geq 0$, Bessel functions of the first kind $J_m(k\rho)$ and the second kind $Y_m(k\rho)$ are two linearly independent solutions to the Bessel equation:

$$\big(\rho (u^0_m)'_\rho\big)'_\rho + \rho\big(k^2 - \tfrac{m^2}{\rho^2}\big) u^0_m = 0 \quad \text{for } \rho \mapsto u^0_m : \mathbf{R}_+ \mapsto \mathbf{R}. \tag{17}$$

The functions J_0, J_1, and Y_0 yield the following expansions as $k\rho \searrow +0$:

$$J_0(k\rho) = 1 + a_0(k\rho), \quad a_0(k\rho) = -\tfrac{(k\rho)^2}{4} + \mathrm{O}((k\rho)^4), \tag{18}$$

$$J_1(k\rho) = \tfrac{1}{2}(k\rho + a_1(k\rho)), \quad a_1(k\rho) = -\tfrac{(k\rho)^3}{8} + \mathrm{O}((k\rho)^5), \tag{19}$$

$$Y_0(k\rho) = \tfrac{2}{\pi}\big(\ln\tfrac{k\rho}{2} + \gamma\big) J_0(k\rho) + a_2(k\rho), \quad a_2(k\rho) = \mathrm{O}((k\rho)^2), \tag{20}$$

where $\gamma > 0$ is the Euler constant.

Using (16)–(20) we prove the following truncated Fourier series.

Theorem 2. *The solution u^0 of the background problem (14) yields the first-order asymptotic representation in the near-field $B_R(x_0) \subset \Omega$:*

$$u^0(x) = u^0(x_0)J_0(k\rho) + U_0^0(x), \tag{21}$$
$$U_0^0(x) = \tfrac{2}{k}J_1(k\rho)\nabla u^0(x_0)\cdot\widehat{x} + U_1^0(x), \tag{22}$$

with the residuals $U_0^0, U_1^0 \in H^1(B_R(x_0);\mathbf{C})$ such that

$$\int_{-\pi}^{\pi} U_0^0\, d\theta = 0, \qquad \int_{-\pi}^{\pi} U_1^0\, d\theta = \int_{-\pi}^{\pi} U_1^0 \widehat{x}\, d\theta = 0, \tag{23}$$
$$U_0^0 = \mathrm{O}(\rho), \qquad U_1^0 = \mathrm{O}(\rho^2). \tag{24}$$

Moreover, the following formula for the gradient holds in $B_R(x_0)$:

$$\nabla u^0(x) = \nabla u^0(x_0) + b_u^0(x) + \nabla U_1^0(x),$$
$$b_u^0(x) := \big(u^0(x_0)ka_0'(k\rho) + a_1'(k\rho)\nabla u^0(x_0)\cdot\widehat{x}\big)\widehat{x} + \tfrac{a_1(k\rho)}{k\rho}(\nabla u^0(x_0)\cdot\widehat{x}')\widehat{x}', \tag{25}$$
$$\nabla U_0^0 = \mathrm{O}(1), \quad \nabla U_1^0 = \mathrm{O}(\rho), \quad b_u^0 = \mathrm{O}(\rho). \tag{26}$$

Proof. In a ball $B_\delta(x_0)$ of radius $\delta \in (0,R)$ we decompose the solution u^0 as

$$u^0(x) = u_0^0(\rho) + U_0^0(x) \text{ with } \quad u_0^0(\rho) := \tfrac{1}{2\pi}\int_{-\pi}^{\pi} u^0\, d\theta, \quad U_0^0 := u^0 - u_0^0, \tag{27}$$

then the residual function U_0^0 has the zero average as written in (23).

Using this decomposition, we substitute a smooth cut-off function $\eta(\rho)$ supported in $B_\delta(x_0)$ as the test function $u = \eta$ in (14) and integrate it by parts providing

$$0 = \int_{B_\delta(x_0)} (\nabla u^0\cdot\nabla\eta - k^2u^0\eta)\, dx$$
$$= \int_0^\delta \Big\{ \tfrac{\partial}{\partial\rho}\Big(\int_{-\pi}^{\pi}(u_0^0 + U_0^0)\, d\theta\Big)\eta' - k^2\int_{-\pi}^{\pi}(u_0^0 + U_0^0)\, d\theta\, \eta\Big\}\rho d\rho$$
$$= 2\pi\int_0^\delta \big((u_0^0)'_\rho\eta' - k^2u_0^0\eta\big)\rho d\rho = -2\pi\int_0^\delta \Big(\big(\rho(u_0^0)'_\rho\big)'_\rho + \rho k^2 u_0^0\Big)\eta\, d\rho$$

for all η. This results in the Bessel equation (17) for u_0^0 as $m = 0$, which has the general solution of the form

$$u_0^0(\rho) = K_0^0 J_0(k\rho) + S_0^0 Y_0(k\rho) \quad \text{with} \quad K_0^0, S_0^0 \in \mathbf{C}.$$

But the Neumann function $Y_0(k\rho) = \mathrm{O}(|\ln\rho|)$ in (20) disagrees the fact that $u_0^0 \in H^1((0,\delta);\mathbf{C})$ in (27), hence parameter $S_0^0 = 0$ and the radial function is

$$u_0^0(\rho) = K_0^0 J_0(k\rho) \quad \text{with} \quad K_0^0 \in \mathbf{C}. \tag{28}$$

Next we decompose the residual U^0_0 in the manner of (27) as

$$U^0_0(x) = u^0_1(\rho)\cdot\widehat{x} + U^0_1(x), \quad u^0_1(\rho) := \tfrac{1}{\pi}\int_{-\pi}^{\pi} u^0\widehat{x}\,d\theta, \quad U^0_1 := U^0_0 - u^0_1\cdot\widehat{x}, \tag{29}$$

with the residual function U^0_1 satisfying the equality in (23).

Inserting $u^0 = u^0_0 + u^0_1\cdot\widehat{x} + U^0_1$ into (14) with the test vector-function $u = \widehat{x}\eta(\rho)$ supported in $B_\delta(x_0)$ and recalling the trigonometric calculus:

$$\widehat{x}' = (-\widehat{x}_2, \widehat{x}_1)^\top, \quad \int_{-\pi}^{\pi}\widehat{x}\,d\theta = 0, \quad \int_{-\pi}^{\pi}\widehat{x}_i\widehat{x}_j\,d\theta = \begin{cases}\pi & \text{for } i=j,\\ 0 & \text{otherwise},\end{cases} \quad i,j=1,2,$$

after integration by parts the variational equation succeeds in

$$\begin{aligned}0 = \int_{B_\delta(x_0)}\big(\nabla u^0\cdot\nabla(\widehat{x}\eta) - k^2u^0\widehat{x}\eta\big)\,dx = \int_0^\delta\Big\{\tfrac{\partial}{\partial\rho}\Big(\int_{-\pi}^{\pi}(u^0_0 + u^0_1\cdot\widehat{x} + U^0_1)\widehat{x}\,d\theta\Big)\eta' \\ + \tfrac{\eta}{\rho^2}\int_{-\pi}^{\pi}\Big(u^0_1\cdot\tfrac{\partial\widehat{x}}{\partial\theta} + \tfrac{\partial U^0_1}{\partial\theta}\Big)\tfrac{\partial\widehat{x}}{\partial\theta}\,d\theta - k^2\eta\int_{-\pi}^{\pi}(u^0_0 + u^0_1\cdot\widehat{x} + U^0_1)\,d\theta\Big\}\rho\,d\rho \\ = \pi\int_0^\delta\big((u^0_1)'_\rho\eta' + \tfrac{\eta}{\rho^2}u^0_1 - k^2u^0_1\eta\big)\rho\,d\rho = -\pi\int_0^\delta\Big(\big(\rho(u^0_1)'_\rho\big)'_\rho + \rho\big(k^2 - \tfrac{1}{\rho^2}\big)u^0_1\Big)\eta\,d\rho\end{aligned}$$

which follows the Bessel equation (17) for u^0_1 as $m=1$ possessing the solution

$$u^0_1(\rho) = K^0_1 J_1(k\rho) \quad \text{with} \quad K^0_1 \in \mathbf{C}^2 \tag{30}$$

since the Neumann function $Y_1(k\rho)$ is singular as $\rho \searrow +0$.

In order to justify the asymptotic rate in (24), we apply the following Wirtinger inequality hold for U^0_1 due to (23):

$$\int_{-\pi}^{\pi}|U^0_1|^2\,d\theta \le \tfrac{1}{4}\int_{-\pi}^{\pi}\big|\tfrac{\partial U^0_1}{\partial\theta}\big|^2\,d\theta. \tag{31}$$

Indeed, decomposing the residual in the Fourier series:

$$U^0_1 = c^1_0 + \sum_{m=1}^{\infty} c^1_m\cdot\widehat{x}^m, \quad c^1_0 := \tfrac{1}{2\pi}\int_{-\pi}^{\pi}U^0_1\,d\theta \in \mathbf{C}, \quad c^1_m := \tfrac{1}{\pi}\int_{-\pi}^{\pi}U^0_1\widehat{x}^m\,d\theta \in \mathbf{C}^2,$$

where $\widehat{x}^m := (\cos(m\theta), \sin(m\theta))^\top$, and its derivative

$$\tfrac{\partial U^0_1}{\partial\theta} = \sum_{m=1}^{\infty} c^1_m\cdot(\widehat{x}^m)', \quad \text{where } (\widehat{x}^m)' = m(-\sin(m\theta), \cos(m\theta))^\top,$$

this leads to the following Parseval identities

$$\tfrac{1}{2\pi}\int_{-\pi}^{\pi}|U^0_1|^2\,d\theta = |c^1_0|^2 + \tfrac{1}{2}\sum_{m=1}^{\infty}|c^1_m|^2, \quad \tfrac{1}{2\pi}\int_{-\pi}^{\pi}\big|\tfrac{\partial U^0_1}{\partial\theta}\big|^2\,d\theta = \tfrac{1}{2}\sum_{m=1}^{\infty}m^2|c^1_m|^2.$$

Conditions in (23) imply $c_0^1 = c_1^1 = 0$ that allows us to derive (31):

$$\int_{-\pi}^{\pi} |U_1^0|^2\, d\theta = \pi \sum_{m=2}^{\infty} |c_m^1|^2 \le \tfrac{\pi}{2^2} \sum_{m=2}^{\infty} m^2 |c_m^1|^2 = \tfrac{1}{4} \int_{-\pi}^{\pi} \big|\tfrac{\partial U_1^0}{\partial \theta}\big|^2\, d\theta.$$

Considering the residual in the energy norm, after integration by parts and using the Helmholtz equation $-[\Delta + k^2] U_1^0 = 0$ hold in $B_\delta(x_0)$, we have

$$0 \le I(\delta) := \int_{B_\delta(x_0)} |\nabla U_1^0|^2\, dx = \int_{B_\delta(x_0)} k^2 |U_1^0|^2\, dx + \int_{\partial B_\delta(x_0)} \tfrac{\partial U_1^0}{\partial \rho} \overline{U_1^0}\, dS_x. \qquad (32)$$

Due to Wirtinger's inequality (31) the boundary integral in $I(\delta)$ can be estimated as

$$\int_{\partial B_\delta(x_0)} \tfrac{\partial U_1^0}{\partial \rho} \overline{U_1^0}\, dS_x \le \int_{-\pi}^{\pi} \Big(\tfrac{\delta}{4} \big|\tfrac{\partial U_1^0}{\partial \rho}\big|^2 + \tfrac{1}{\delta} |U_1^0|^2 \Big) \delta d\theta$$
$$\le \int_{-\pi}^{\pi} \Big(\tfrac{\delta}{4} \big|\tfrac{\partial U_1^0}{\partial \rho}\big|^2 + \tfrac{1}{4\delta} \big|\tfrac{\partial U_1^0}{\partial \theta}\big|^2 \Big) \delta d\theta = \tfrac{\delta}{4} \int_{\partial B_\delta(x_0)} |\nabla U_1^0|^2\, dS_x.$$

Applying to (32) the Poincare inequality:

$$\int_{B_\delta(x_0)} |U_1^0|^2\, dx \le \tfrac{(2\delta)^2}{\pi^2} \int_{B_\delta(x_0)} |\nabla U_1^0|^2\, dx \qquad (33)$$

and the co-area formula:

$$\tfrac{d}{d\delta} \int_{B_\delta(x_0)} |\nabla U_1^0|^2\, dx = \int_{\partial B_\delta(x_0)} |\nabla U_1^0|^2\, dS_x,$$

we obtain the differential inequality

$$\big(1 - k^2 \tfrac{4\delta^2}{\pi^2}\big) I(\delta) \le \tfrac{\delta}{4} \tfrac{d}{d\delta} I(\delta).$$

Its integration over $\delta \in (r, R)$ leads to Grönwall's inequality

$$0 \le I(r) \le \big(\tfrac{r}{R}\big)^4 \exp\big(\tfrac{8k^2}{\pi^2}(R^2 - r^2)\big) I(R) = \mathrm{O}(r^4). \qquad (34)$$

Due to the fundamental theorem of calculus and using homogeneity arguments, the function oscillation in $\mathbf{R}^2$ can be estimated from above (see e.g. [21]) as

$$\sup_{x,y \in B_\delta(x_0)} |U_1^0(x) - U_1^0(y)|^2 \le C \int_{B_\delta(x_0)} (|\nabla U_1^0|^2 + \delta^2 |\Delta U_1^0|^2)\, dx$$
$$= C \int_{B_\delta(x_0)} \big(|\nabla U_1^0|^2 + \delta^2 k^2 |U_1^0|^2\big)\, dx \le C_1 I(\delta) \text{ with } C_1 := \big(1 + \tfrac{4k^2}{\pi^2}\big) C > 0, \qquad (35)$$

where we have used $-[\Delta + k^2] U_1^0 = 0$ in $B_r(x_0)$, the notation $I(\delta)$ from (32), and the Poincare inequality (33). For $x = x_0 + \rho \widehat{x}$ and $y = x_0$ we have $U_1^0(x_0) = 0$ due to the zero average in (23), then (35) with $\delta = \rho$ provides the point-wise estimate

$$|U_1^0(x_0+\rho\widehat{x})|^2 \le C_1 I(\rho) = \mathrm{O}(\rho^4),$$

which justifies the asymptotic rate in (24).

Due to (24), passing $\rho \searrow +0$ in (28) we specify the constant $K_0^0 = u^0(x_0)$. After differentiation of u^0 in (21) and (22) according to (16), (18), and (19), this provides formulas (25) and (26) for the gradient ∇u^0 in $B_R(x_0)$. Here the constant $K_1^0 = \frac{2}{k}\nabla u^0(x_0)$ in (30) is specified when $\rho \searrow +0$, and the asymptotic rate of $\nabla U_1^0 = \mathrm{O}(\rho)$ in (24) can be argued by (34). The proof is complete. □

In the next section we proceed with outer asymptotic expansion in the far-field with respect to the geometry ω expressed by a boundary layer.

2.2 *Outer asymptotic expansion in the far-field*

We introduce the weighted Sobolev space (see [4, 31, 33]):

$$H_\mu^1(\mathbf{R}^2;\mathbf{C}) = \bigl\{v:\ \tfrac{v}{\mu} \in L^2(\mathbf{R}^2;\mathbf{C}), \nabla v \in L^2(\mathbf{R}^2;\mathbf{C}^2)\bigr\} \text{ with the weight}$$
$$\mu(y) = \mathrm{O}(|y|\ln|y|) \quad \text{in } \mathbf{R}^2\setminus\overline{B_2(0)}, \quad \mu(y) = \mathrm{O}(1) \quad \text{in } B_2(0),$$

suggested by the weighted Poincare inequality in exterior domains

$$\int_{\mathbf{R}^2\setminus B_2(0)} \bigl(\tfrac{v}{|y|\ln|y|}\bigr)^2 dy \le 4\int_{\mathbf{R}^2\setminus B_2(0)} |\nabla v|^2 dy \quad \text{if } \int_{\partial B_2(0)} v\, dS_x = 0.$$

We note that constant functions are allowed in H_μ^1.

Excluding constants implying polynomials of degree zero $\mathbf{P}_0$, a boundary layer depending on $1-\mathrm{Re}(\alpha)$ is described by the real-valued *exterior transmission problem*: Find vector-function $w(y) = (w_1,w_2)^\top \in H_\mu^1(\mathbf{R}^2;\mathbf{R}^2)/\mathbf{P}_0^2$ such that

$$\int_{\mathbf{R}^2} \chi_\omega^{\mathrm{Re}(\alpha)} Dw\nabla v\, dy = (1-\mathrm{Re}(\alpha))\int_{\partial\omega} nv\, dS_y \quad \text{for all } v \in H_\mu^1(\mathbf{R}^2;\mathbf{R}), \tag{36}$$

where $(Dw)_{ij} = \frac{\partial w_i}{\partial y_j}$ for $i,j = 1,2$ denotes the derivative matrix, and recalling that $n = (n_1,n_2)^\top$ is the normal vector outward to ω.

From the result of [4] it follows existence of the solution to the variational equation (36) which implies the boundary value problem:

$$-\Delta w = 0 \quad \text{in } \mathbf{R}^2\setminus\partial\omega, \tag{37}$$
$$[\![w]\!] = 0, \quad (Dw)n|_{\partial\omega^+} - \mathrm{Re}(\alpha)(Dw)n|_{\partial\omega^-} = -(1-\mathrm{Re}(\alpha))n \quad \text{on } \partial\omega, \tag{38}$$
$$w = \mathrm{O}\bigl(\tfrac{1}{|y|}\bigr) \quad \text{as } |y| \nearrow \infty. \tag{39}$$

If $\mathrm{Re}(\alpha) > \alpha_0 > 0$, then from (36) it follows that $\|w\|_{H_\mu^1(\mathbf{R}^2;\mathbf{R}^2)} = \mathrm{O}(|1-\mathrm{Re}(\alpha)|)$.

Rescaling $y = \frac{x-x_0}{\varepsilon}$ we reduce the exterior problem to Ω next.

Theorem 3. *The rescaled solution* $w^\varepsilon(x) := w(\frac{x-x_0}{\varepsilon}) \in H^1(\Omega;\mathbf{R}^2)$ *of (36) satisfies the variational equation:*

$$\int_\Omega \chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} D w^\varepsilon \nabla u \, dx = \int_{\Gamma_N} (Dw^\varepsilon n) u \, dS_x + \frac{1-\mathrm{Re}(\alpha)}{\varepsilon} \int_{\partial\omega_\varepsilon(x_0)} n u \, dS_x$$
$$\text{for all test-functions } u \in H^1(\Omega;\mathbf{R}) \text{ such that } u = 0 \text{ on } \Gamma_D. \tag{40}$$

The solution has the far-field representation in the truncated Fourier series

$$w^\varepsilon(x) = \frac{\varepsilon}{\rho}\frac{1}{2\pi} A_{(\omega,\mathrm{Re}(\alpha))}\widehat{x} + W^\varepsilon(x) \quad \text{for } x \in \mathbf{R}^2 \setminus \overline{B_\varepsilon(x_0)}, \tag{41}$$

with the residual vector-function $W^\varepsilon = (W^\varepsilon_1, W^\varepsilon_2)^\top$ *such that*

$$\int_{-\pi}^{\pi} W^\varepsilon \, d\theta = \int_{-\pi}^{\pi} W^\varepsilon \widehat{x} \, d\theta = 0, \tag{42}$$

$$W^\varepsilon = \mathrm{O}\big((\tfrac{\varepsilon}{\rho})^2\big),\ DW^\varepsilon = \mathrm{O}\big(\tfrac{\varepsilon^2}{\rho^3}\big) \quad \text{for } \rho > \varepsilon,\ \theta \in (-\pi,\pi]. \tag{43}$$

Entries of the 2-by-2 real matrix $A_{(\omega,\mathrm{Re}(\alpha))}$ *have the implicit expression:*

$$(A_{(\omega,\mathrm{Re}(\alpha))})_{ij} = (1-\mathrm{Re}(\alpha))\big(\delta_{ij}|\omega| + \int_{\partial\omega} w_i n_j \, dS_y\big), \quad i,j = 1,2. \tag{44}$$

If $\mathrm{Re}(\alpha) \in [0,1)$ *and* $|\omega| > 0$, *then* $A_{(\omega,\mathrm{Re}(\alpha))} \in \mathrm{Spd}(\mathbf{R}^{2\times 2})$, *i.e. symmetric positive definite. For ellipsoidal shapes* ω *the matrix has the explicit expression*

$$A_{(\omega,\mathrm{Re}(\alpha))} = \Theta(\phi) A_{(\omega',\mathrm{Re}(\alpha))} \Theta(\phi)^\top, \quad \Theta(\phi) := \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \tag{45}$$

$$A_{(\omega',\mathrm{Re}(\alpha))} = \pi(a+b) \begin{pmatrix} \frac{(1-\mathrm{Re}(\alpha))ab}{a+b\mathrm{Re}(\alpha)} & 0 \\ 0 & \frac{(1-\mathrm{Re}(\alpha))ab}{a\mathrm{Re}(\alpha)+b} \end{pmatrix}, \tag{46}$$

with the ellipse major $a = 1$ *and minor* $b \in (0,1]$ *semi-axes, where the major axis has an angle of* $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *with the* y_1*-axis counted in the anti-clockwise direction.*

Proof. The proof is based on the techniques from [12, Lemma 3.2] and [26].

The local coordinate system (16) in the stretched variables y implies the polar radius $|y| \in \mathbf{R}_+$ and the polar angle $\theta \in (-\pi,\pi]$ such that

$$y = \frac{x-x_0}{\varepsilon} = |y|\widehat{x}, \quad |y| = \frac{\rho}{\varepsilon}, \quad \widehat{x} = (\cos\theta, \sin\theta)^\top. \tag{47}$$

We apply the coordinate transformation (47) and differential calculus:

$$\frac{\partial}{\partial y} = \varepsilon \frac{\partial}{\partial x}, \quad dy = \frac{1}{\varepsilon^2} dx, \quad dS_y = \frac{1}{\varepsilon} dS_x$$

to equations (37) and (38) and derive the following relations for $w^\varepsilon(x) = w(\frac{x-x_0}{\varepsilon})$:

$$-\Delta w^\varepsilon = 0 \quad \text{in } \mathbf{R}^2 \setminus \partial\omega_\varepsilon(x_0), \tag{48}$$

$$[\![w^\varepsilon]\!] = 0, \quad [\![\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} D w^\varepsilon n]\!] = -\tfrac{1-\mathrm{Re}(\alpha)}{\varepsilon} n \quad \text{on } \partial\omega_\varepsilon(x_0). \tag{49}$$

Using Green's formulas (6) and (7) it follows variational formulation (40) written in the bounded domain.

We split $y \in \mathbf{R}^2$ in the far-field $\mathbf{R}^2 \setminus \overline{B_1(0)}$ and the near-field $B_1(0)$.

In the far-field, using the notation $\widehat{x}^m = (\cos(m\theta), \sin(m\theta))^\top$, the harmonic vector-valued function w solving (37) and (39) admits the Fourier series as

$$w(y) = \sum_{m=1}^{\infty} \tfrac{1}{|y|^m} C_m \widehat{x}^m \quad \text{with coefficient matrices } C_m \in \mathbf{R}^{2\times 2} \text{ for } y \in \mathbf{R}^2 \setminus \overline{B_1(0)}.$$

This formula implies the outer asymptotic expansion with $A_{(\omega,\mathrm{Re}(\alpha))} := 2\pi C_1$:

$$w(y) = \tfrac{1}{2\pi} A_{(\omega,\mathrm{Re}(\alpha))} \tfrac{\widehat{x}}{|y|} + W(y) \quad \text{for } y \in \mathbf{R}^2 \setminus \overline{B_1(0)}, \tag{50}$$

$$\int_{-\pi}^{\pi} W\, d\theta = \int_{-\pi}^{\pi} W \widehat{x} d\theta = 0, \quad W = \mathrm{O}\big(\big(\tfrac{1}{|y|}\big)^2\big),\ DW = \mathrm{O}\big(\big(\tfrac{1}{|y|}\big)^3\big)$$

and turns into (41)–(43) for $W^\varepsilon(x) := W(\frac{x-x_0}{\varepsilon})$ after rescaling.

In the near-field, we apply the second Green formula hold for $i, j = 1, 2$:

$$0 = \int_{B_1(0)} \chi^{\mathrm{Re}(\alpha)}_{\omega_1(0)} \{\Delta w_i y_j - w_i \Delta y_j\}\, dy = \int_{\partial B_1(0)} \big\{ \tfrac{\partial w_i}{\partial |y|} y_j - w_i \tfrac{\partial y_j}{\partial |y|} \big\}\, dS_y$$
$$- \int_{\partial\omega} \big\{ \big[\tfrac{\partial w_i}{\partial n}|_{\partial\omega^+} - \mathrm{Re}(\alpha) \tfrac{\partial w_i}{\partial n}|_{\partial\omega^-} \big] y_j - (1-\mathrm{Re}(\alpha)) w_i \tfrac{\partial y_j}{\partial n} \big\}\, dS_y,$$

and substitute here the transmission condition (38) and $\frac{\partial y_j}{\partial n} = n_j$ to derive that

$$-\int_{\partial B_1(0)} \big\{ \tfrac{\partial w_i}{\partial |y|} - w_i \big\} \widehat{x}_j\, dS_y = (1-\mathrm{Re}(\alpha)) \int_{\partial\omega} \big\{ n_i y_j + w_i n_j \big\}\, dS_y. \tag{51}$$

We apply to (51) the divergence theorem with Kronecker's delta

$$\int_{\partial\omega} n_i y_j\, dS_y = \int_{\omega} \tfrac{\partial y_j}{\partial y_i}\, dy = \delta_{ij} |\omega|, \quad \text{where } |\omega| := \int_{\omega} dy, \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and substitute (50) to calculate the integral over $\partial B_1(0)$ as

$$-\int_{\partial B_1(0)} \big\{ \tfrac{\partial w_i}{\partial |y|} - w_i \big\} \widehat{x}_j\, dS_y = \tfrac{1}{\pi} \int_{-\pi}^{\pi} (A_{(\omega,\mathrm{Re}(\alpha))})_{ik} \widehat{x}_k \widehat{x}_j\, d\theta = (A_{(\omega,\mathrm{Re}(\alpha))})_{ij},$$

which together with (51) follows formula (44).

In order to prove the symmetry of $A_{(\omega,\mathrm{Re}(\alpha))}$, we insert $v = w_j$ into (36) written component-wisely for $i, j = 1, 2$ as

$$\int_{\mathbf{R}^2} \chi^{\mathrm{Re}(\alpha)}_{\omega} \nabla w_i^\top \nabla w_j\, dy = (1-\mathrm{Re}(\alpha)) \int_{\partial\omega} n_i w_j\, dS_y = (1-\mathrm{Re}(\alpha)) \int_{\partial\omega} n_j w_i\, dS_y.$$

Henceforth, $(A_{(\omega,\mathrm{Re}(\alpha))})_{ij} = (A_{(\omega,\mathrm{Re}(\alpha))})_{ji}$ follows due to the matrix entries expression by (44). For arbitrary $z \in \mathbf{R}^2$, from (36) we have

$$0 \leq \int_{\mathbf{R}^2} \chi_\omega^{\mathrm{Re}(\alpha)} |\nabla(z_1 w_1 + z_2 w_2)|^2 \, dy = (1 - \mathrm{Re}(\alpha)) \int_{\partial\omega} (n \cdot z)(w \cdot z) \, dS_y.$$

Multiplying (44) with $z_i z_j$ and summing the result over $i, j = 1, 2$ implies

$$z^\top A_{(\omega,\mathrm{Re}(\alpha))} z = (1 - \mathrm{Re}(\alpha)) \Big\{ |z|^2 |\omega| + \int_{\partial\omega} (n \cdot z)(w \cdot z) \, dS_y \Big\} \geq (1 - \mathrm{Re}(\alpha)) |z|^2 |\omega|,$$

which is positive, hence $A_{(\omega,\mathrm{Re}(\alpha))} \in \mathrm{Spd}(\mathbf{R}^{2\times 2})$, if $1 - \mathrm{Re}(\alpha) > 0$ and $|\omega| > 0$.

An explicit representation of $A_{(\omega,\mathrm{Re}(\alpha))}$ will be established for ellipses ω.

Let an ellipse $\omega' \subset B_1(0)$ have the major $a = 1$ and the minor $b \in (0, 1]$ semi-axes written with respect to y'-coordinates as

$$\omega' = \{ y' \in \mathbf{R}^2 : \, (\tfrac{y'_1}{a})^2 + (\tfrac{y'_2}{b})^2 < 1 \}, \quad a = 1.$$

Let the reference ellipse ω written in y-coordinates have an angle of $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ to the major y_1-axis counted in the anti-clockwise direction:

$$\omega = \big\{ y \in \mathbf{R}^2 : \, (\tfrac{y_1 \cos\phi + y_2 \sin\phi}{a})^2 + (\tfrac{-y_1 \sin\phi + y_2 \cos\phi}{b})^2 < 1 \big\}, \quad a = 1.$$

This implies that $y' \in \omega'$ when $\Theta^\top y \in \omega$ with the orthogonal matrix $\Theta(\phi)$ given in (45). Therefore, we prove first formula (46) for ω' and then transform $y' = \Theta^\top y$.

We introduce the elliptic coordinates $r \in \mathbf{R}_+$ and $\psi \in (-\pi, \pi]$ such that

$$y'_1 = c\cosh(r)\cos\psi, \quad y'_2 = c\sinh(r)\sin\psi, \quad c = \sqrt{a^2 - b^2}, \quad a = 1, \tag{52}$$

and c is the linear eccentricity. Setting the distance $r_0 \in \mathbf{R}_+$ implicitly by

$$a = c\cosh(r_0) \quad b = c\sinh(r_0), \tag{53}$$

the geometry ω' can be restated as

$$\omega' = \{ r < r_0, \psi \in (-\pi, \pi] \}, \quad \mathbf{R}^2 \setminus \overline{\omega'} = \{ r > r_0, \psi \in (-\pi, \pi] \}.$$

From (52) it follows the differential calculus in elliptic coordinates:

$$\begin{cases} \frac{\partial}{\partial y'_1} = \frac{1}{\kappa^2(r,\psi)} \big(c\cosh(r)\cos\psi \frac{\partial}{\partial r} - c\cosh(r)\sin\psi \frac{\partial}{\partial \psi} \big), \\ \frac{\partial}{\partial y'_2} = \frac{1}{\kappa^2(r,\psi)} \big(c\cosh(r)\sin\psi \frac{\partial}{\partial r} + c\sinh(r)\cos\psi \frac{\partial}{\partial \psi} \big), \end{cases}$$
$$dy' = \kappa^2(r,\psi) dr d\psi, \quad \kappa(r,\psi) = c\sqrt{\sinh^2(r) + \sin^2\psi}$$

involving the scale factor $\kappa(r,\psi)$. In particular, at the ellipse boundary $\partial\omega'$ as $r = r_0$, using (52) with constant ψ and (53), for the normal vector n' we get expressions

$$n' = \frac{1}{\kappa(r_0,\psi)}(b\cos\psi, a\sin\psi)^\top, \quad \frac{\partial}{\partial n'} = \frac{1}{\kappa(r_0,\psi)}\frac{\partial}{\partial r},$$
$$dS_{y'} = \kappa(r_0,\psi)d\psi, \quad \kappa(r_0,\psi) = \sqrt{a^2\cos^2\psi + b^2\sin^2\psi}.$$

Applying transformation (52) and the differential calculus, the exterior problem (37)–(39) for $w'(r,\psi)$ associated to ω' can be rewritten in elliptic coordinates as

$$-\frac{1}{\kappa^2(r,\psi)}\Big[\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \psi^2}\Big]w' = 0 \text{ for } r \neq r_0, \quad (54)$$
$$[\![w']\!] = 0,\ \frac{1}{\kappa(r_0,\psi)}\Big(\frac{\partial w'}{\partial r}\Big|_{r=r_0^+} - \mathrm{Re}(\alpha)\frac{\partial w'}{\partial r}\Big|_{r=r_0^-}\Big) = -(1-\mathrm{Re}(\alpha))n' \text{ as } r = r_0, \quad (55)$$
$$w' = \mathrm{o}(1) \text{ as } r \nearrow \infty, \quad (56)$$

where $r_0^\pm$ denotes the one-sided limit corresponding to $(\partial\omega')^\pm$, respectively. Due to (54) and (56) the harmonic vector-function w' admits the Fourier series:

$$w' = \sum_{m=1}^{\infty} C'_m R'_m \begin{pmatrix}\cos(m\psi)\\ \sin(m\psi)\end{pmatrix}, \quad R'_m = \begin{cases} e^{m(r_0-r)}I & \text{for } r > r_0,\\ \begin{pmatrix}\frac{\cosh(mr)}{\cosh(mr_0)} & 0\\ 0 & \frac{\sinh(mr)}{\sinh(mr_0)}\end{pmatrix} & \text{for } r < r_0,\end{cases}$$

where I stands for the identity 2-by-2 matrix. Then w' satisfies $w'(0,\psi) = w'(0,-\psi)$, $\frac{\partial w'}{\partial r}(0,\psi) = -\frac{\partial w'}{\partial r}(0,-\psi)$ as $r = 0$ for $\psi \in (-\pi,\pi]$, and the first jump condition $w'|_{r=r_0^+} - w'|_{r=r_0^-} = 0$ in (55). To find the unknown coefficient matrices $C'_m \in \mathbf{R}^{2\times 2}$ we substitute the Fourier series into the second jump condition in (55) implying that

$$\sum_{m=1}^{\infty} \frac{mC'_m}{\kappa(r_0,\psi)}\Big(I + \mathrm{Re}(\alpha)\begin{pmatrix}\tanh(mr_0) & 0\\ 0 & \frac{1}{\tanh(mr_0)}\end{pmatrix}\Big)\begin{pmatrix}\cos(m\psi)\\ \sin(m\psi)\end{pmatrix} = \frac{1-\mathrm{Re}(\alpha)}{\kappa(r_0,\psi)}\begin{pmatrix}b\cos\psi\\ a\sin\psi\end{pmatrix}.$$

Henceforth, $C'_m = 0$ for all $m \geq 2$ and $C'_1 = (1-\mathrm{Re}(\alpha))\begin{pmatrix}\frac{b}{1+\mathrm{Re}(\alpha)ba^{-1}} & 0\\ 0 & \frac{a}{1+\mathrm{Re}(\alpha)ab^{-1}}\end{pmatrix}$ due to $\tanh(r_0) = ba^{-1}$ resulting in the following analytic solution to (54)–(56):

$$w' = \begin{cases}\Big(\frac{(1-\mathrm{Re}(\alpha))ab}{a+b\mathrm{Re}(\alpha)}e^{r_0-r}\cos\psi, \frac{(1-\mathrm{Re}(\alpha))ab}{a\mathrm{Re}(\alpha)+b}e^{r_0-r}\sin\psi\Big)^\top & \text{for } r > r_0,\\ \Big(\frac{(1-\mathrm{Re}(\alpha))ab}{a+b\mathrm{Re}(\alpha)}\frac{\cosh(r)}{\cosh(r_0)}\cos\psi, \frac{(1-\mathrm{Re}(\alpha))ab}{a\mathrm{Re}(\alpha)+b}\frac{\sinh(r)}{\sinh(r_0)}\sin\psi\Big)^\top & \text{for } r < r_0.\end{cases} \quad (57)$$

The matrix $A_{(\omega',\mathrm{Re}(\alpha))}$ is calculated analytically after substitution of (57) in the representation formula (44) that implies the following two vectors for $j = 1,2$:

$$(A_{(\omega',\mathrm{Re}(\alpha))})_{(\cdot,j)} = (1-\mathrm{Re}(\alpha))\big(\delta_{(\cdot,j)}|\omega'| + I_{(\cdot,j)}\big), \quad \text{where}$$
$$I_{(\cdot,j)} := \int_{\partial\omega'} (w'_1 n'_j, w'_2 n'_j)^\top dS_{y'} = \begin{cases}\int_{-\pi}^{\pi}\Big(\frac{(1-\mathrm{Re}(\alpha))ab^2}{a+b\mathrm{Re}(\alpha)}\cos^2\psi, 0\Big)^\top d\psi & \text{for } j = 1,\\ \int_{-\pi}^{\pi}\Big(0, \frac{(1-\mathrm{Re}(\alpha))a^2b}{a\mathrm{Re}(\alpha)+b}\sin^2\psi\Big)^\top d\psi & \text{for } j = 2.\end{cases}$$

Using $\int_{-\pi}^{\pi} \cos^2\psi\, d\psi = \int_{-\pi}^{\pi} \sin^2\psi\, d\psi = \pi$ and $|\omega'| = \pi ab$ we arrive at (46).

The transformation formula (45) is justified by rotation $y = \Theta y'$ applied to the variational equation (36) which, after the left multiplication with $\Theta^\top$, results in

$$\int_{\mathbf{R}^2} \chi_{\omega'}^{\mathrm{Re}(\alpha)} D_{y'}(\Theta^\top w(\Theta y'))\nabla_{y'} v\, dy' = (1-\mathrm{Re}(\alpha)) \int_{\partial\omega'} \Theta^\top n v\, dS_{y'}\ \forall v \in H^1_\mu(\mathbf{R}^2;\mathbf{R}).$$

Since $\Theta^\top n = n'$ this proves $w'(y') = \Theta^\top w(\Theta y')$, then from (50) it follows

$$\tfrac{1}{2\pi} A_{(\omega',\mathrm{Re}(\alpha))} \tfrac{y'}{|y'|^2} + W' = w'(y') = \Theta^\top w(\Theta y') = \tfrac{\Theta^\top}{2\pi} A_{(\omega,\mathrm{Re}(\alpha))} \tfrac{\Theta y'}{|\Theta y'|^2} + \Theta^\top W(\Theta y'),$$

which implies $A_{(\omega',\mathrm{Re}(\alpha))} = \Theta^\top A_{(\omega,\mathrm{Re}(\alpha))}\Theta$, hence (45), and completes the proof. □

2.3 Uniform asymptotic expansion of the solution

The boundary layer w^ε depending on $1-\mathrm{Re}(\alpha)$ will express a leading asymptotic term in the uniform expansion over Ω. Moreover, for its refinement we need the auxiliary Helmholtz problem: Find $u^1 \in H^1(\Omega;\mathbf{C}^2)$ such that $u^1 = 0$ at Γ_D and

$$\int_\Omega \big(\chi_{\omega_\varepsilon(x_0)}^{\mathrm{Re}(\alpha)} Du^1 \nabla\overline{u} - \chi_{\omega_\varepsilon(x_0)}^{\alpha} k^2 u^1 \overline{u}\big)\, dx = \frac{k^2}{\varepsilon\sqrt{|\ln\varepsilon|}} \int_{\Omega\setminus B_\varepsilon(x_0)} w^\varepsilon \overline{u}\, dx \tag{58}$$

for all test-functions $u \in H^1(\Omega;\mathbf{C})$ such that $u = 0$ on Γ_D.

We show that the solution u^1 to problem (58) is of the order $\mathrm{O}(|1-\mathrm{Re}(\alpha)|)$ since its right-hand side can be estimated uniformly with respect to ε for fixed α. Indeed, we inscribe Ω in a ball $B_R(x_0)$ of radius $R>0$, then due to (41)– (43) it holds

$$\int_{\Omega\setminus B_\varepsilon(x_0)} |w^\varepsilon|^2\, dx \le \int_{B_R(x_0)\setminus B_\varepsilon(x_0)} |w^\varepsilon|^2\, dx = \int_{-\pi}^{\pi}\int_\varepsilon^R \big((\tfrac{\varepsilon}{\rho})^2 |\tfrac{1}{2\pi} A_{(\omega,\mathrm{Re}(\alpha))}\widehat{x}|^2 + |W^\varepsilon|^2\big)\,\rho\, d\rho\, d\theta = \mathrm{O}\big(|1-\mathrm{Re}(\alpha)|^2\varepsilon^2|\ln\varepsilon|\big),$$

because $A_{(\omega,\mathrm{Re}(\alpha))} = \mathrm{O}(|1-\mathrm{Re}(\alpha)|)$ due to expression (44).

Theorem 4. *The inhomogeneous solution $u^{(\varepsilon,\alpha)}$ of (5), the background solution u^0 of (14), the rescaled solution $w^\varepsilon(x) = w(\frac{x-x_0}{\varepsilon})$ of (36) together with its refinement u^1 from (58) compose the residual*

$$q^\varepsilon := u^{(\varepsilon,\alpha)} - u^0 - \varepsilon\nabla u^0(x_0)\cdot w^1, \quad w^1 := w^\varepsilon + \varepsilon\sqrt{|\ln\varepsilon|}u^1, \tag{59}$$

which satisfies the two-parameter asymptotic relation

$$\Big|\int_\Omega \big(\chi_{\omega_\varepsilon(x_0)}^{\mathrm{Re}(\alpha)} \nabla q^\varepsilon\cdot\nabla\overline{u} - \chi_{\omega_\varepsilon(x_0)}^{\alpha} k^2 q^\varepsilon \overline{u}\big)\, dx\Big| \le \underline{C}(\alpha,\varepsilon)\|u\|_{H^1(\Omega;\mathbf{C})} \tag{60}$$

where $0 \le \underline{C}(\alpha,\varepsilon) = \mathrm{O}\big(|1-\alpha|\varepsilon^2 + |\alpha||1-\mathrm{Re}(\alpha)|\varepsilon^3\big)$.

If for all $\varepsilon \in (0,\varepsilon_0)$ *with* $\varepsilon_0 > 0$, $\mathrm{Re}(\alpha) > \alpha_0 > 0$ *and* $|\alpha| < \alpha_1$ *with* $\alpha_1 > \alpha_0 > 0$, *and for* $u,v \in H^1(\Omega;\mathbf{C})$ *such that* $u = v = 0$ *on* Γ_D*, the inf-sup condition holds:*

$$\Big|\int_\Omega \big(\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \nabla v\cdot\nabla\overline{u} - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 v\overline{u}\big)\,dx\Big| \ge \underline{c}\|u\|_{H^1(\Omega;\mathbf{C})}\|v\|_{H^1(\Omega;\mathbf{C})}, \quad \underline{c} > 0, \tag{61}$$

then it follows from (60) and (61) the residual error estimate

$$\|q^\varepsilon\|_{H^1(\Omega;\mathbf{C})} \le \underline{c}^{-1}\underline{C}(\alpha,\varepsilon). \tag{62}$$

Proof. We subtract from (5) equations (15), (40) multiplied with $\varepsilon\nabla u^0(x_0)$, and (58) multiplied with $\varepsilon^2\sqrt{|\ln\varepsilon|}\nabla u^0(x_0)$. Using the expansion (25) of ∇u^0 at $\partial\omega_\varepsilon(x_0)$, the differential identity $\nabla(\nabla u^0(x_0)\cdot w^1) = \nabla u^0(x_0)^\top Dw^1$, and the notation of q^ε and w^1 introduced in (59) we obtain the variational equation

$$\begin{aligned}\int_\Omega \big(\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)}\nabla q^\varepsilon\cdot\nabla\overline{u} - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 q^\varepsilon\overline{u}\big)\,dx &= -\varepsilon\int_{\Gamma_N}(\nabla u^0(x_0)^\top Dw^\varepsilon n)\overline{u}\,dS_x \\ +(1-\mathrm{Re}(\alpha))\int_{\partial\omega_\varepsilon(x_0)} n\cdot(b^0_u + \nabla U^0_1)\,\overline{u}\,dS_x &+ \mathrm{i}\mathrm{Im}(\alpha)\int_{\omega_\varepsilon(x_0)} k^2 u^0\overline{u}\,dx \\ &+\varepsilon\int_{B_\varepsilon(x_0)}\chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2(\nabla u^0(x_0)\cdot w^\varepsilon)\overline{u}\,dx\end{aligned} \tag{63}$$

for all test-functions $u \in H^1(\Omega;\mathbf{C})$ such that $u = 0$ on Γ_D. Applying to the right-hand side of (63) the Cauchy–Schwarz inequality and trace theorems, with the help of the representations (21), (25), and (41), we get $w^\varepsilon = \mathrm{O}(|1-\alpha|\varepsilon)$ at Γ_N, while $w^\varepsilon = \mathrm{O}(|1-\alpha|)$ in $B_\varepsilon(x_0)$, and $b^0_u + \nabla U^0_1 = \mathrm{O}(\varepsilon)$ at $\partial\omega_\varepsilon(x_0)$, hence derive the upper bound in (60).

Now let the inf-sup condition (61) hold.

For a smooth cut-off function η_{Γ_D} supported in a neighborhood of the Dirichlet boundary Γ_D such that $\eta_{\Gamma_D} = 1$ at Γ_D, the lifting function is set

$$Q^\varepsilon := q^\varepsilon + R^\varepsilon, \quad R^\varepsilon := \varepsilon(\nabla u^0(x_0)\cdot w^\varepsilon)\eta_{\Gamma_D}, \quad Q^\varepsilon = 0 \text{ on } \Gamma_D,$$

and $R^\varepsilon = \mathrm{O}(|1-\mathrm{Re}(\alpha)|\varepsilon^2)$ can be estimated due to (41). Therefore, inserting $q^\varepsilon = Q^\varepsilon - R^\varepsilon$ into (60) and using (61) we derive (62). The proof is complete. □

As the consequence, from (63) we infer the boundary value problem for q^ε:

$$-\big[\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)}\Delta + \chi^{\alpha}_{\omega_\varepsilon(x_0)}k^2\big]q^\varepsilon = \begin{cases} 0 & \text{in } \Omega\setminus B_\varepsilon(x_0) \\ F := \varepsilon\chi^{\alpha}_{\omega_\varepsilon(x_0)}k^2\nabla u^0(x_0)\cdot w^\varepsilon & \text{in } B_\varepsilon(x_0)\setminus\omega_\varepsilon(x_0) \\ F + \mathrm{i}\mathrm{Im}(\alpha)k^2u^0 & \text{in } \omega_\varepsilon(x_0), \end{cases} \tag{64}$$

$$[\![q^\varepsilon]\!] = 0, \quad \Big[\!\!\Big[\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)}\frac{\partial q^\varepsilon}{\partial n}\Big]\!\!\Big] = -(1-\mathrm{Re}(\alpha))n\cdot(b^0_u + \nabla U^0_1) \quad \text{on } \partial\omega_\varepsilon(x_0), \tag{65}$$

$$\frac{\partial q^\varepsilon}{\partial n} = -\varepsilon\nabla u^0(x_0)^\top Dw^\varepsilon n \quad \text{on } \Gamma_N, \quad q^\varepsilon = -\varepsilon\nabla u^0(x_0)\cdot w^1 \quad \text{on } \Gamma_D. \tag{66}$$

Theorem 4 will be applied for the two-parameter asymptotic expansion of a geometry-dependent objective function as described in the following section.

3 Inverse Helmholtz problem in inhomogeneous medium

In the inverse setting of the Helmholtz problem in inhomogeneous medium, the geometry variables $(\omega^\star,\varepsilon^\star,x^\star)\in G$ and the refractive index $\alpha^\star\in\mathbf{C}_+$ of an unknown inhomogeneity $\omega^\star_{\varepsilon^\star}(x^\star)$ being tested in the reference domain Ω are to be identified and reconstructed from the known boundary measurement $u^\star\in L^2(\Gamma_N;\mathbf{C})$.

For variation of the topology, a trial inhomogeneity $\omega_\varepsilon(x_0)$ with admissible $(\omega,\varepsilon,x_0)\in G$ and $\alpha\in\mathbf{C}_+$ is put in Ω. For such trial variables we find a family of solutions $u^{(\varepsilon,\alpha)}$ to the forward Helmholtz problem (5) which determines the objective function of the misfit at the boundary

$$J: G\times\mathbf{C}_+\mapsto\mathbf{R}_+,\quad J(\omega,\varepsilon,x_0,\alpha):=\frac{1}{2}\int_{\Gamma_N}|u^{(\varepsilon,\alpha)}-u^\star|^2\,dS_x. \tag{67}$$

The objective (67) forces the state-constrained, *topology optimization problem*: Find $(\omega^\star,\varepsilon^\star,x^\star,\alpha^\star)\in G\times\mathbf{C}_+$ such that

$$J(\omega^\star,\varepsilon^\star,x^\star,\alpha^\star)=\min_{(\omega,\varepsilon,x_0,\alpha)\in G\times\mathbf{C}_+} J(\omega,\varepsilon,x_0,\alpha)\quad\text{subject to (5).} \tag{68}$$

If the test variables $(\omega^\star,\varepsilon^\star,x^\star,\alpha^\star)\in G\times\mathbf{C}_+$ are feasible, then the trivial minimum in (68) is attained at the solution $u^{(\varepsilon^\star,\alpha^\star)}$ of (5) because $u^{(\varepsilon^\star,\alpha^\star)}=u^\star$ at Γ_N in (67). Uniqueness of the minimum is open. Therefore,

Theorem 5. *For feasible measurement $u^\star=u^{(\varepsilon^\star,\alpha^\star)}$ at Γ_N, a solution to the inverse Helmholtz problem in inhomogeneous medium exists implying the trivial minimum in (68).*

While $u^{(\varepsilon,\alpha)}$ in (67) implies the primal state variable, a dual state variable $v^{(\varepsilon,\alpha)}$ associates a Fenchel–Legendre duality corresponding to the variational principle:

$$\mathscr{L}(u^{(\varepsilon,\alpha)},v^{(\varepsilon,\alpha)})=\min_{\mathrm{Re}(u),\mathrm{Re}(v)}\max_{\mathrm{Im}(u),\mathrm{Im}(v)}\mathscr{L}(u,v) \tag{69}$$
$$\text{over } u,v\in H^1(\Omega;\mathbf{C}) \text{ such that } u=h,\ v=0 \text{ on } \Gamma_D,$$

where the Lagrangian $\mathscr{L}$ has the form (compare with $\mathscr{P}$ in (9)):

$$\mathscr{L}(u,v):=\mathrm{Re}\Big\{\int_\Omega\big(\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)}\nabla u\cdot\nabla v-\chi^{\alpha}_{\omega_\varepsilon(x_0)}k^2uv\big)\,dx-\int_{\Gamma_N}gv\,dS_x + \frac{1}{2}\int_{\Gamma_N}(u-u^\star)^2\,dS_x\Big\}. \tag{70}$$

Theorem 6. *The first-order necessary optimality conditions for the minimax problem (69) imply the primal problem (5) and the dual Helmholtz problem in inhomogeneous medium: Find $v^{(\varepsilon,\alpha)} \in H^1(\Omega;\mathbf{C})$ such that $v^{(\varepsilon,\alpha)} = 0$ at Γ_D and*

$$\int_\Omega \big(\chi^{\mathrm{Re}(\alpha)}_{\omega_\varepsilon(x_0)} \nabla v^{(\varepsilon,\alpha)} \cdot \nabla \overline{u} - \chi^{\alpha}_{\omega_\varepsilon(x_0)} k^2 v^{(\varepsilon,\alpha)} \overline{u}\big)\, dx = -\int_{\Gamma_N} (u^{(\varepsilon,\alpha)} - u^\star)\overline{u}\, dS_x \qquad (71)$$

for all test-functions $u \in H^1(\Omega;\mathbf{C})$ such that $u = 0$ on Γ_D.

Proof. Applying variational calculus in the manner of Theorem 1, the first order optimality condition for (69) necessitates four variational inequalities:

$$\Big\langle \tfrac{\partial \mathscr{L}(u^{(\varepsilon,\alpha)}, v^{(\varepsilon,\alpha)})}{\partial \mathrm{Re}(v)}, \mathrm{Re}(v - v^{(\varepsilon,\alpha)}) \Big\rangle \geq 0, \quad \Big\langle \tfrac{\partial \mathscr{L}(u^{(\varepsilon,\alpha)}, v^{(\varepsilon,\alpha)})}{\partial \mathrm{Im}(v)}, \mathrm{Im}(v - v^{(\varepsilon,\alpha)}) \Big\rangle \leq 0,$$
$$\Big\langle \tfrac{\partial \mathscr{L}(u^{(\varepsilon,\alpha)}, v^{(\varepsilon,\alpha)})}{\partial \mathrm{Re}(u)}, \mathrm{Re}(u - u^{(\varepsilon,\alpha)}) \Big\rangle \geq 0, \quad \Big\langle \tfrac{\partial \mathscr{L}(u^{(\varepsilon,\alpha)}, v^{(\varepsilon,\alpha)})}{\partial \mathrm{Im}(u)}, \mathrm{Im}(u - u^{(\varepsilon,\alpha)}) \Big\rangle \leq 0,$$

holding for all $u, v \in H^1(\Omega;\mathbf{C})$ such that $u = h$, $v = 0$ on Γ_D. Inserting here $v = v^{(\varepsilon,\alpha)} \pm \widetilde{v}$ and $u = u^{(\varepsilon,\alpha)} \pm \widetilde{u}$ with $\widetilde{v}, \widetilde{u} \in H^1(\Omega;\mathbf{C})$ such that $\widetilde{v} = \widetilde{u} = 0$ on Γ_D we get four variational equations:

$$\Big\langle \tfrac{\partial \mathscr{L}(u^{(\varepsilon,\alpha)}, v^{(\varepsilon,\alpha)})}{\partial \mathrm{Re}(v)}, \mathrm{Re}(\widetilde{v}) \Big\rangle = 0, \quad \Big\langle \tfrac{\partial \mathscr{L}(u^{(\varepsilon,\alpha)}, v^{(\varepsilon,\alpha)})}{\partial \mathrm{Im}(v)}, \mathrm{Im}(\widetilde{v}) \Big\rangle = 0,$$
$$\Big\langle \tfrac{\partial \mathscr{L}(u^{(\varepsilon,\alpha)}, v^{(\varepsilon,\alpha)})}{\partial \mathrm{Re}(u)}, \mathrm{Re}(\widetilde{u}) \Big\rangle = 0, \quad \Big\langle \tfrac{\partial \mathscr{L}(u^{(\varepsilon,\alpha)}, v^{(\varepsilon,\alpha)})}{\partial \mathrm{Im}(u)}, \mathrm{Im}(\widetilde{u}) \Big\rangle = 0.$$

The summation of the first and the second equations for $\widetilde{v} = u$ and $\widetilde{v} = \mathrm{i}u$ constitutes the real and imaginary parts of (5), while the third and the fourth equations for $\widetilde{u} = u$ and $\widetilde{u} = \mathrm{i}u$ contribute to (71), respectively. This completes the proof. □

We note that the dual problem (71) is analogous to the primal problem (5) and differs by the boundary data given at $\partial\Omega$. Therefore, the asymptotic results stated for $u^{(\varepsilon,\alpha)}$ remains true also for $v^{(\varepsilon,\alpha)}$.

In particular, if $\varepsilon = 0$, then (71) turns into the dual background problem stated in the reference domain Ω as follows (compare to (14): Find $v^0 \in H^1(\Omega;\mathbf{C})$ such that $v^0 = 0$ at Γ_D and

$$\int_\Omega (\nabla v^0 \cdot \nabla \overline{u} - k^2 v^0 \overline{u})\, dx = -\int_{\Gamma_N} (u^0 - u^\star)\overline{u}\, dS_x \qquad (72)$$

for all test-functions $u \in H^1(\Omega;\mathbf{C})$ such that $u = 0$ on Γ_D.

which implies the weak solution to (cf. (11)–(13)):

$$-[\Delta + k^2] v^0 = 0 \quad \text{in } \Omega, \qquad (73)$$
$$\tfrac{\partial v^0}{\partial n} = -(u^0 - u^\star) \quad \text{on } \Gamma_N, \qquad (74)$$
$$v^0 = 0 \quad \text{on } \Gamma_D. \qquad (75)$$

Theorem 2 applied to the problem (73)–(75) provides the near-field representation in $B_R(x_0) \subset \Omega$:

$$v^0(x) = v^0(x_0)J_0(k\rho) + V_0^0(x), \quad V_0^0(x) = \tfrac{2}{k}J_1(k\rho)\nabla v^0(x_0)\cdot\widehat{x} + V_1^0(x), \tag{76}$$

with the residuals $V_0^0, V_1^0 \in H^1(B_R(x_0);\mathbf{C})$ such that

$$\int_{-\pi}^{\pi} V_0^0\,d\theta = 0, \qquad \int_{-\pi}^{\pi} V_1^0\,d\theta = \int_{-\pi}^{\pi} V_1^0 \widehat{x}\,d\theta = 0, \tag{77}$$

$$V_0^0 = \mathrm{O}(\rho), \qquad V_1^0 = \mathrm{O}(\rho^2), \tag{78}$$

and similar to (25) and (26) representation of the gradient in $B_R(x_0)$:

$$\nabla v^0(x) = \nabla v^0(x_0) + b_v^0(x) + \nabla V_1^0(x),$$

$$b_v^0(x) := \big(v^0(x_0)ka_0'(k\rho) + a_1'(k\rho)\nabla v^0(x_0)\cdot\widehat{x}\big)\widehat{x} + \tfrac{a_1(k\rho)}{k\rho}(\nabla v^0(x_0)\cdot\widehat{x}')\widehat{x}', \tag{79}$$

$$\nabla V_0^0 = \mathrm{O}(1), \quad \nabla V_1^0 = \mathrm{O}(\rho), \quad b_v^0 = \mathrm{O}(\rho). \tag{80}$$

Since the topological variables $(\omega, \varepsilon, x_0, \alpha)$ enter the objective J in fully implicit way through the solution $u^{(\varepsilon,\alpha)}$ of the state problem, further we get an explicit expansion of J applying asymptotic arguments as the size $\varepsilon \searrow +0$.

3.1 High-order topological expansion of the objective

We decompose $u^{(\varepsilon,\alpha)} - u^\star = u^{(\varepsilon,\alpha)} - u^0 + u^0 - u^\star$ and using the Neumann boundary condition (74) we express the objective in (67) equivalently as

$$J(\omega, \varepsilon, x_0, \alpha) = J_0 + \mathrm{Re}\big\{\mathscr{I}(u^{(\varepsilon,\alpha)} - u^0, v^0)\big\} + J_{(\varepsilon,\alpha)}, \tag{81}$$

with the following integral terms:

$$J_0 := \tfrac{1}{2}\int_{\Gamma_N} |u^0 - u^\star|^2\,dS_x, \quad J_{(\varepsilon,\alpha)} := \tfrac{1}{2}\int_{\Gamma_N} |u^{(\varepsilon,\alpha)} - u^0|^2\,dS_x \tag{82}$$

$$\mathscr{I}(u^{(\varepsilon,\alpha)} - u^0, v^0) := -\int_{\Gamma_N} (u^{(\varepsilon,\alpha)} - u^0)\tfrac{\partial \overline{v^0}}{\partial n}\,dS_x. \tag{83}$$

From (81)–(83) we infer the asymptotic result below.

Theorem 7. *The objective in (67) admits the two-parameter topological expansion*

$$J(\omega, \varepsilon, x_0, \alpha) = J_0 + \mathrm{Re}\big\{\varepsilon^2 J_1(\omega, x_0, \alpha) + J_2^{(\varepsilon,\alpha)} + J_3^{(\varepsilon,\alpha)} + J_4^{(\varepsilon,\alpha)}\big\}$$
$$+\mathrm{O}\big(|1-\mathrm{Re}(\alpha)|^2\varepsilon^4|\ln\varepsilon|\big), \tag{84}$$

with the first-order asymptotic term implying the topological derivative:

$$J_1(\omega, x_0, \alpha) := -\nabla u^0(x_0)^\top A_{(\omega,\mathrm{Re}(\alpha))}\nabla\overline{v^0}(x_0) + (1-\alpha)k^2|\omega|u^0(x_0)\overline{v^0}(x_0) \tag{85}$$

and the high-order asymptotic terms expressed by the following formulas:

$$J_2^{(\varepsilon,\alpha)} := \varepsilon^3\sqrt{|\ln\varepsilon|}\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big(\tfrac{\partial u^1}{\partial\rho}\overline{v^0}(x_0) - u^1(\nabla\overline{v^0}(x_0)\cdot\widehat{x})\big)\,d\theta = \mathrm{O}\big(|1-\mathrm{Re}(\alpha)|\varepsilon^3\sqrt{|\ln\varepsilon|}\big), \quad (86)$$

$$J_3^{(\varepsilon,\alpha)} := -\int_{\omega_\varepsilon(x_0)}\big((1-\mathrm{Re}(\alpha))\nabla q^\varepsilon\cdot\nabla\overline{v^0} - (1-\alpha)k^2 q^\varepsilon\overline{v^0}\big)\,dx - (1-\mathrm{Re}(\alpha))\int_{\partial\omega_\varepsilon(x_0)} n\cdot(b_u^0+\nabla U_1^0)\overline{V_0^0}\,dS_x - \mathrm{i}\,\mathrm{Im}(\alpha)\int_{\omega_\varepsilon(x_0)} k^2(U_0^0\overline{v^0}(x_0) + u^0(x_0)\overline{V_0^0})\,dx + \varepsilon^2\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big(\tfrac{\partial W^\varepsilon}{\partial\rho}\overline{V_1^0} - W^\varepsilon\tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)\,d\theta - \varepsilon\int_{B_\varepsilon(x_0)}\chi^\alpha_{\omega_\varepsilon(x_0)} k^2\nabla u^0(x_0)\cdot w^\varepsilon\overline{v^0}\,dx = \mathrm{O}\{\big(|1-\mathrm{Re}(\alpha)|(|1-\alpha|+\mathrm{Re}(\alpha))\big)\varepsilon^3\}, \quad (87)$$

$$J_4^{(\varepsilon,\alpha)} := \varepsilon^3\sqrt{|\ln\varepsilon|}\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big\{\tfrac{\partial u^1}{\partial\rho}\varepsilon(\nabla\overline{v^0}(x_0)\cdot\widehat{x}) - u^1\big(\overline{v^0}(x_0)ka_0' + \tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)\big\}\,d\theta = \mathrm{O}\big(|1-\mathrm{Re}(\alpha)|\varepsilon^4\sqrt{|\ln\varepsilon|}\big). \quad (88)$$

Proof. Due to $u^1 = \mathrm{O}(|1-\mathrm{Re}(\alpha)|)$ in (59), the asymptotic rate of $J_{(\varepsilon,\alpha)}$ in (82) is

$$J_{(\varepsilon,\alpha)} = \mathrm{O}\big(|1-\mathrm{Re}(\alpha)|^2\varepsilon^4|\ln\varepsilon|\big). \quad (89)$$

To verify (84) the boundary integral $\mathscr{I}$ in (83) should be expanded up to the order in (89). For this task we employ the second Green formula in $\Omega\setminus\overline{B_\varepsilon(x_0)}$ and rewrite

$$\mathscr{I}(u^{(\varepsilon,\alpha)}-u^0,v^0) = \int_{\Omega\setminus B_\varepsilon(x_0)}\big(\Delta(u^{(\varepsilon,\alpha)}-u^0)\overline{v^0} - (u^{(\varepsilon,\alpha)}-u^0)\Delta\overline{v^0}\big)\,dx + \int_{\partial B_\varepsilon(x_0)}\big(\tfrac{\partial(u^{(\varepsilon,\alpha)}-u^0)}{\partial\rho}\overline{v^0} - (u^{(\varepsilon,\alpha)}-u^0)\tfrac{\partial\overline{v^0}}{\partial\rho}\big)\,dS_x,$$

where the domain integral vanishes due to Helmholtz equations (1), (11), and (73).

We note that $\mathscr{I}$ is an invariant integral which can be written over arbitrary Lipschitz boundary $\partial\mathscr{O}$ of a domain $\mathscr{O}$ such that $\omega_\varepsilon(x_0)\subseteq\mathscr{O}\subset\Omega$. The integral over the circle $\partial B_\varepsilon(x_0)$ is advantageous when calculated analytically by substituting the uniform expansion (59) for $u^{(\varepsilon,\alpha)}-u^0$ and the inner expansion (76) for v^0.

The plan is to decompose $\mathscr{I}(u^{(\varepsilon,\alpha)}-u^0,v^0) = \mathscr{I}(q^\varepsilon,v^0) + \mathscr{I}(u^{(\varepsilon,\alpha)}-u^0-q^\varepsilon,v^0)$ with the residual q^ε from Theorem 4 and calculate two integrals separately.

First, applying to $\mathscr{I}(q^\varepsilon,v^0)$ the second Green formula in $B_\varepsilon(x_0)\setminus\overline{\omega_\varepsilon(x_0)}$ due to the Helmholtz equations (64) and the jump condition (65) for q^ε, we get

$$\mathscr{I}(q^\varepsilon,v^0) = \int_{\partial B_\varepsilon(x_0)}\big(\tfrac{\partial q^\varepsilon}{\partial\rho}\overline{v^0} - q^\varepsilon\tfrac{\partial\overline{v^0}}{\partial\rho}\big)\,dS_x = \int_{\partial\omega_\varepsilon(x_0)^+}\big(\tfrac{\partial q^\varepsilon}{\partial n}\overline{v^0} - q^\varepsilon\tfrac{\partial\overline{v^0}}{\partial n}\big)\,dS_x - \varepsilon\int_{B_\varepsilon(x_0)\setminus\omega_\varepsilon(x_0)} k^2(\nabla u^0(x_0)\cdot w^\varepsilon)\overline{v^0}\,dx = -\varepsilon\int_{B_\varepsilon(x_0)\setminus\omega_\varepsilon(x_0)} k^2(\nabla u^0(x_0)\cdot w^\varepsilon)\overline{v^0}\,dx + \int_{\partial\omega_\varepsilon(x_0)^-}\Big(\big(\mathrm{Re}(\alpha)\tfrac{\partial q^\varepsilon}{\partial n} - (1-\mathrm{Re}(\alpha))n\cdot(b_u^0+\nabla U_1^0)\big)\overline{v^0} - q^\varepsilon\tfrac{\partial\overline{v^0}}{\partial n}\Big)\,dS_x.$$

Further integration by part in $\omega_\varepsilon(x_0)$ of this expression gives

$$\begin{aligned}\mathscr{I}(q^\varepsilon,v^0)=-\int_{\omega_\varepsilon(x_0)}\big((1-\mathrm{Re}(\alpha))\nabla q^\varepsilon\cdot\nabla\overline{v^0}-(1-\alpha)k^2q^\varepsilon\overline{v^0}\big)\,dx\\ -\mathrm{i}\mathrm{Im}(\alpha)\int_{\omega_\varepsilon(x_0)}k^2u^0\overline{v^0}\,dx-(1-\mathrm{Re}(\alpha))\int_{\partial\omega_\varepsilon(x_0)}n\cdot(b_u^0+\nabla U_1^0)\overline{v^0}\,dS_x\\ -\varepsilon\int_{B_\varepsilon(x_0)}\chi^\alpha_{\omega_\varepsilon(x_0)}k^2(\nabla u^0(x_0)\cdot w^\varepsilon)\overline{v^0}\,dx.\end{aligned}$$

Expansions (21) for u^0 and (76) for v^0 in the second integral here proceed

$$\int_{\omega_\varepsilon(x_0)}u^0\overline{v^0}\,dx=\varepsilon^2|\omega|u^0(x_0)\overline{v^0}(x_0)+\int_{\omega_\varepsilon(x_0)}(U_0^0\overline{v^0}(x_0)+u^0(x_0)\overline{V_0^0}+U_0^0\overline{V_0^0})\,dx,$$

while in the third integral, applying the divergence theorem, this leads to

$$\begin{aligned}\int_{\partial\omega_\varepsilon(x_0)}n\cdot(b_u^0+\nabla U_1^0)\overline{v^0}\,dS_x=\int_{\partial\omega_\varepsilon(x_0)}n\cdot(b_u^0+\nabla U_1^0)\overline{V_0^0}\,dS_x\\ -\varepsilon^2k^2|\omega|u^0(x_0)\overline{v^0}(x_0)+\int_{\omega_\varepsilon(x_0)}\Big(\mathrm{div}\big(b_u^0+u^0(x_0)\tfrac{k^2\rho\widehat{x}}{2}\big)-k^2U_1^0\Big)\overline{v^0}\,dx\end{aligned}$$

due to $\mathrm{div}(\nabla U_1^0)=-k^2U_1^0$ and $\mathrm{div}(\frac{\rho\widehat{x}}{2})=1$, since $b_u^0=-u^0(x_0)\frac{k^2\rho\widehat{x}}{2}+\mathrm{O}(\rho^2)$ according to (18) and (25).

Gathering like terms in view of the following asymptotic relations:

$$\begin{aligned}U_0^0=V_0^0=b_u^0=\mathrm{O}(\varepsilon),\quad U_1^0=b_u^0+u^0(x_0)\tfrac{k^2\rho\widehat{x}}{2}=\mathrm{O}(\varepsilon^2)\quad\text{in } B_\varepsilon(x_0),\\ \|w^\varepsilon\|_{H^1(\omega_\varepsilon(x_0);\mathbf{C})}=\mathrm{O}(|1-\mathrm{Re}(\alpha)|),\quad \|q^\varepsilon\|_{H^1(\omega_\varepsilon(x_0);\mathbf{C})}=\mathrm{O}(|1-\alpha|\varepsilon^2),\end{aligned}$$

we get the two-parameter representation of the first integral

$$\mathscr{I}(q^\varepsilon,v^0)=I_1^q+I_2^q+I_3^q, \tag{90}$$

with the following asymptotic terms:

$$\begin{aligned}I_1^q:=(1-\alpha)\varepsilon^2k^2|\omega|u^0(x_0)\overline{v^0}(x_0),\\ I_2^q:=-\int_{\omega_\varepsilon(x_0)}\big((1-\mathrm{Re}(\alpha))\nabla q^\varepsilon\cdot\nabla\overline{v^0}-(1-\alpha)k^2q^\varepsilon\overline{v^0}\big)\,dx\\ -(1-\mathrm{Re}(\alpha))\int_{\partial\omega_\varepsilon(x_0)}n\cdot(b_u^0+\nabla U_1^0)\overline{V_0^0}\,dS_x-\mathrm{i}\mathrm{Im}(\alpha)\int_{\omega_\varepsilon(x_0)}k^2(U_0^0\overline{v^0}(x_0)\\ +u^0(x_0)\overline{V_0^0})\,dx-\varepsilon\int_{B_\varepsilon(x_0)}\chi^\alpha_{\omega_\varepsilon(x_0)}k^2(\nabla u^0(x_0)\cdot w^\varepsilon)\overline{v^0}\,dx\\ =\mathrm{O}\big\{\big(|1-\mathrm{Re}(\alpha)|(|1-\alpha|+\mathrm{Re}(\alpha))\big)\varepsilon^3\big\},\\ I_3^q:=-(1-\mathrm{Re}(\alpha))\int_{\omega_\varepsilon(x_0)}\Big(\mathrm{div}\big(b_u^0+u^0(x_0)\tfrac{k^2\rho\widehat{x}}{2}\big)-k^2U_1^0\Big)\overline{v^0}\,dx\end{aligned}$$

$$-\mathrm{i}\mathrm{Im}(\alpha)\int_{\omega_\varepsilon(x_0)} k^2 U_0^0 \overline{V_0^0}\,dx = \mathrm{O}\big(|1-\alpha|\varepsilon^4\big).$$

Second, inserting in $\mathscr{I}$ the asymptotic representations (41) for w^ε, (59) for q^ε, (76) and (79) for v^0 and ∇v^0, we have

$$\mathscr{I}(u^\varepsilon - u^0 - q^\varepsilon, v^0) = \int_{\partial B_\varepsilon(x_0)} \big(\tfrac{\partial(u^\varepsilon - u^0 - q^\varepsilon)}{\partial\rho}\overline{v^0} - (u^\varepsilon - u^0 - q^\varepsilon)\tfrac{\partial\overline{v^0}}{\partial\rho}\big)\,dS_x$$

$$= \varepsilon^2\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big\{\big(-\tfrac{A_{(\omega,\mathrm{Re}(\alpha))}\widehat{x}}{2\pi\varepsilon} + \tfrac{\partial W^\varepsilon}{\partial\rho} + \varepsilon\sqrt{|\ln\varepsilon|}\tfrac{\partial u^1}{\partial\rho}\big)\big(\overline{v^0}(x_0)(1+a_0) + (\varepsilon + \tfrac{a_1}{k})$$

$$\times\nabla\overline{v^0}(x_0)\cdot\widehat{x} + \overline{V_1^0}\big) - \big(\tfrac{A_{(\omega,\mathrm{Re}(\alpha))}\widehat{x}}{2\pi} + W^\varepsilon + \varepsilon\sqrt{|\ln\varepsilon|}u^1\big)(\nabla\overline{v^0}(x_0) + \overline{b_v^0} + \nabla\overline{V_1^0})\cdot\widehat{x}\big\}\,d\theta.$$

To calculate the terms here we use the orthogonality (42) for W^ε, (77) for V_1^0, and the following asymptotic relations due to (18)–(20), (43), (80), which hold at $\partial B_\varepsilon(x_0)$:

$$\overline{b_v^0}\cdot\widehat{x} = \overline{v^0}(x_0)ka_0' + a_1'\nabla\overline{v^0}(x_0)\cdot\widehat{x}, \quad u^1 = W^\varepsilon = \mathrm{O}(|1-\mathrm{Re}(\alpha)|),$$
$$a_0 = \mathrm{O}(\varepsilon^2), \quad a_0' = \mathrm{O}(\varepsilon), \quad a_1 = \mathrm{O}(\varepsilon^3), \quad a_1' = \mathrm{O}(\varepsilon^2).$$

Collecting like terms, as the result we get

$$\mathscr{I}(u^\varepsilon - u^0 - q^\varepsilon, v^0) = -I_1^w + I_2^w + I_3^w + I_4^w + I_5^w, \tag{91}$$

where the integral terms are

$$I_1^w := \varepsilon^2\int_{-\pi}^{\pi}\nabla u^0(x_0)^\top\tfrac{A_{(\omega,\mathrm{Re}(\alpha))}\widehat{x}}{\pi}(\nabla\overline{v^0}(x_0)\cdot\widehat{x})\,d\theta = \varepsilon^2\nabla u^0(x_0)^\top A_{(\omega,\mathrm{Re}(\alpha))}\nabla\overline{v^0}(x_0),$$

$$I_2^w := \varepsilon^3\sqrt{|\ln\varepsilon|}\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big(\tfrac{\partial u^1}{\partial\rho}\overline{v^0}(x_0) - u^1(\nabla\overline{v^0}(x_0)\cdot\widehat{x})\big)\,d\theta$$
$$= \mathrm{O}\big(|1-\mathrm{Re}(\alpha)|\varepsilon^3\sqrt{|\ln\varepsilon|}\big),$$

$$I_3^w := \varepsilon^2\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big(\tfrac{\partial W^\varepsilon}{\partial\rho}\overline{V_1^0} - W^\varepsilon\tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)d\theta = \mathrm{O}(|1-\mathrm{Re}(\alpha)|\varepsilon^3),$$

$$I_4^w := \varepsilon^3\sqrt{|\ln\varepsilon|}\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big\{\tfrac{\partial u^1}{\partial\rho}\varepsilon\nabla\overline{v^0}(x_0)\cdot\widehat{x} - u^1\big(\overline{v^0}(x_0)ka_0' + \tfrac{\partial\overline{V_1^0}}{\partial\rho}\big)\big\}d\theta$$
$$= \mathrm{O}\big(|1-\mathrm{Re}(\alpha)|\varepsilon^4\sqrt{|\ln\varepsilon|}\big),$$

$$I_5^w := \varepsilon^3\sqrt{|\ln\varepsilon|}\int_{-\pi}^{\pi}\nabla u^0(x_0)\cdot\big\{\tfrac{\partial u^1}{\partial\rho}\big(\overline{v^0}(x_0)a_0 + \tfrac{a_1}{k\varepsilon}\nabla\overline{v^0}(x_0)\cdot\widehat{x} + \overline{V_1^0}\big)$$
$$-u^1(a_1'\nabla\overline{v^0}(x_0)\cdot\widehat{x})\big\}d\theta + \tfrac{1}{2}(\tfrac{a_1}{\varepsilon k} + a_1')I_1^w = \mathrm{O}(|1-\mathrm{Re}(\alpha)|\varepsilon^4).$$

Finally, summing up $I_1^q - I_1^w = \varepsilon^2 J_1$, $I_2^w = J_2^{(\varepsilon,\alpha)}$, $I_3^q + I_3^w = J_3^{(\varepsilon,\alpha)}$, and $I_4^w = J_4^{(\varepsilon,\alpha)}$ in (89)–(91) gathers the asymptotic terms (84)–(88) and finishes the proof. □

We note that, for fixed α, from Theorem 7 it follows the topological derivative of the objective J following the terminology of [34]:

$$\mathrm{Re}\{J_1(\omega,x_0,\alpha)\} = \lim_{\varepsilon\searrow+0} \tfrac{1}{\varepsilon^2}\big(J(\omega,\varepsilon,x_0,\alpha)-J_0\big), \tag{92}$$

where $J_1(\omega,x_0,\alpha)$ is the first-order asymptotic term in (84). It is given explicitly by formula (85). For real $\alpha\in\mathbf{R}_+$, the topological derivative was found in [2, 5].

However, for fixed ε and complex $\alpha\in\mathbf{C}_+$, the leading term in (84) may change when varying the refractive index $|\alpha|\nearrow\infty$.

In fact, taking the limit as $\mathrm{Im}(\alpha)\nearrow\infty$, which corresponds to the sound-soft obstacle $\omega_\varepsilon(x_0)$ described by problem (10), in the next section we derive a derivative-free necessary optimality condition of the topology optimization (68).

3.2 Zero-order necessary optimality condition

Due to Theorem 5, the optimal objective $J(\omega^\star,\varepsilon^\star,x^\star,\alpha^\star)=0$ in (68) for all feasible test parameters $(\omega^\star,\varepsilon^\star,x^\star,\alpha^\star)\in G\times\mathbf{C}_+$. We rewrite formula (84) of J due to (85):

$$\begin{aligned} J(\omega^\star,\varepsilon^\star,x^\star,\alpha^\star) = J_0 + (\varepsilon^\star)^2\Big(\mathrm{Re}\big\{-\nabla u^0(x^\star)^\top A_{(\omega^\star,\mathrm{Re}(\alpha^\star))}\nabla\overline{v^0}(x^\star)\\ +(1-\mathrm{Re}(\alpha^\star))k^2|\omega^\star|u^0(x^\star)\overline{v^0}(x^\star)\big\} + \mathrm{Im}(\alpha^\star)k^2|\omega^\star|\mathrm{Im}\{u^0(x^\star)\overline{v^0}(x^\star)\}\Big)\\ +\mathrm{O}\big(|1-\mathrm{Re}(\alpha^\star)|(\varepsilon^\star)^3\sqrt{|\ln\varepsilon^\star|}\big). \end{aligned} \tag{93}$$

However, if $\mathrm{Im}(\alpha^\star)\nearrow\infty$, then zero optima of $J(\omega^\star,\varepsilon^\star,x^\star,\alpha^\star)$ can be preserved only when the complement $\mathrm{Im}\{u^0(x^\star)\overline{v^0}(x^\star)\}$ to $\mathrm{Im}(\alpha^\star)$ in (93) is zero.

This argument holds true also for finite optima $0\neq J(\omega^\star,\varepsilon^\star,x^\star,\alpha^\star)<\infty$ when the test parameters are infeasible. Thus, we have proved the following result.

Theorem 8. *For test geometries $(\omega^\star,\varepsilon^\star,x^\star)$ and refractive index $\mathrm{Im}(\alpha^\star)\nearrow\infty$, the zero-order necessary optimality condition of (68) implies*

$$\mathrm{Im}\{u^0(x^\star)\overline{v^0}(x^\star)\}=0 \tag{94}$$

expressed by the solution u^0 to the primal background problem (14) and the solution v^0 to the dual background problem (72).

Based on Theorem 8, in [24] the imaging function is introduced

$$f:\{u^\star\in G_u\}\mapsto C(\Omega;\mathbf{R}),\quad f_{u^\star}(x):=\mathrm{Im}\{u^0(x)\overline{v^0}(x)\} \tag{95}$$

over the set of feasible boundary measurements $u^\star\in G_u\subset L^2(\Gamma_N;\mathbf{C})$. Based on (95), the zero-level set $L_{=0}$ of $f_{u^\star}$ contains the test center $x^\star$:

$$x^\star\in L_{=0}(f_{u^\star}):=\{x\in\Omega:\quad f_{u^\star}(x)=0\}. \tag{96}$$

The level-set description (96) based on the zero-order optimality condition (95) was implemented numerically and reported in [24]. We remark the following.

- Implementation of the imaging function $f_{u^\star}$ in (95) has low computational costs since it needs to solve the background Helmholtz problems (14) and (72) in homogeneous medium.
- The center $x^\star$ can be detected as the intersection point of the zero-level sets

$$x^\star = \bigcap_{i=1}^{d} L_{=0}(f_{u_i^\star}) \tag{97}$$

from two or three different measurements $u_i^\star$ in 2d or 3d, respectively.
- For low wave numbers k, formula (97) holds for arbitrary geometries $(\omega^\star, \varepsilon^\star, x^\star)$.
- The numerical result by (97) is stable to discretization and noise errors.
- Although Theorem 8 is stated for $\mathrm{Im}(\alpha^\star) \nearrow \infty$, formulas (95)–(97) are applicable numerically as well to finite refractive indexes $\alpha^\star \in \mathbf{C}_+$.

Finally, we illustrate formulas (95) and (97) analytically in 1d and numerically in 2d.

3.3 The analytic solution in one dimension

Let $\Omega = (-R,R) \subset \mathbf{R}^1$ with $R > 0$, the size $\varepsilon > 0$ and the center x_0 be such that $|x_0| < R - \varepsilon$ implying the inhomogeneity $\omega_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon) \subset (-R,R)$ with the given refractive index $\alpha \in \mathbf{C}_+$.

Relying on the plane wave $u^0(x) = \exp(\mathrm{i}kx)$ we have the Neumann data $g(\pm R) = \pm \mathrm{i}k \exp(\pm \mathrm{i}kR)$. The Helmholtz problem (1)–(4) in this inhomogeneous medium turns into

$$\begin{aligned} -\Big[\tfrac{d^2}{dx^2} + k^2\Big] u^{(\varepsilon,\alpha)}(x) = 0 \quad \text{for } x \in (-R, x_0-\varepsilon) \cup (x_0+\varepsilon, R), \\ \tfrac{du^{(\varepsilon,\alpha)}}{d(-x)}(-R) = g(-R), \quad \tfrac{du^{(\varepsilon,\alpha)}}{dx}(R) = g(R), \end{aligned} \tag{98}$$

$$-\Big[\mathrm{Re}(\alpha)\tfrac{d^2}{dx^2} + \alpha k^2\Big] u^{(\varepsilon,\alpha)}(x) = 0 \quad \text{for } |x - x_0| < \varepsilon, \tag{99}$$

$$\begin{aligned} u^{(\varepsilon,\alpha)}((x_0-\varepsilon)^-) - u^{(\varepsilon,\alpha)}((x_0-\varepsilon)^+) = 0, \\ u^{(\varepsilon,\alpha)}((x_0+\varepsilon)^+) - u^{(\varepsilon,\alpha)}((x_0+\varepsilon)^-) = 0, \\ \tfrac{du^{(\varepsilon,\alpha)}}{d(-x)}((x_0-\varepsilon)^-) - \mathrm{Re}(\alpha)\tfrac{du^{(\varepsilon,\alpha)}}{d(-x)}((x_0-\varepsilon)^+) = 0, \\ \tfrac{du^{(\varepsilon,\alpha)}}{dx}((x_0+\varepsilon)^+) - \mathrm{Re}(\alpha)\tfrac{du^{(\varepsilon,\alpha)}}{dx}((x_0+\varepsilon)^-) = 0. \end{aligned} \tag{100}$$

The general solution to (98) and (99) can be expressed as

$$u^{(\varepsilon,\alpha)}(x) = \begin{cases} u^0(x) + c_1 \cosh(\mathrm{i}k(x+R)) & \text{for } x \in (-R, x_0-\varepsilon), \\ u^0(x) + c_2 \cosh(\mathrm{i}k(x-R)) & \text{for } x \in (x_0+\varepsilon, R), \\ c_3 \cosh(k\beta(x-x_0)) & \\ +c_4 \sinh(k\beta(x-x_0)) & \text{for } |x-x_0| < \varepsilon, \end{cases} \tag{101}$$

where $\beta \in \mathbf{C}$ is the complex root of the equation $\mathrm{Re}(\alpha)\beta^2 + \alpha = 0$ given by

$$\mathrm{Re}(\beta) = \sqrt{\tfrac{1}{2}\Big(\sqrt{1+\big(\tfrac{\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)}\big)^2}-1\Big)}, \quad \mathrm{Im}(\beta) = -\tfrac{2\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)\mathrm{Re}(\beta)},$$

and the coefficient vector $c \in \mathbf{C}^4$ is determined from the four jump conditions (100) as the solution of the following 4-by-4 matrix equations:

$$\begin{pmatrix} \cosh(\mathrm{i}k(x_0-\varepsilon+R)) & 0 & -\cosh(-k\beta\varepsilon) & -\sinh(-k\beta\varepsilon) \\ -\mathrm{i}\sinh(\mathrm{i}k(x_0-\varepsilon+R)) & 0 & \mathrm{Re}(\alpha)\beta\sinh(-k\beta\varepsilon) & \mathrm{Re}(\alpha)\beta\cosh(-k\beta\varepsilon) \end{pmatrix} c = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & \cosh(\mathrm{i}k(x_0+\varepsilon-R)) & -\cosh(k\beta\varepsilon) & -\sinh(k\beta\varepsilon) \\ 0 & \mathrm{i}\sinh(\mathrm{i}k(x_0+\varepsilon-R)) & -\mathrm{Re}(\alpha)\beta\sinh(k\beta\varepsilon) & -\mathrm{Re}(\alpha)\beta\cosh(k\beta\varepsilon) \end{pmatrix} c = \begin{pmatrix} b_3 \\ b_4 \end{pmatrix},$$

$$b = \big(-\exp(\mathrm{i}k(x_0-\varepsilon), \mathrm{i}\exp(\mathrm{i}k(x_0-\varepsilon), -\exp(\mathrm{i}k(x_0+\varepsilon), -\mathrm{i}\exp(\mathrm{i}k(x_0+\varepsilon)\big)^\top.$$

For the boundary measurement determined according to (101):

$$u^\star(-R) = u^0(-R) + c_1, \quad u^\star(R) = u^0(R) + c_2,$$

the corresponding dual problem (72) possesses the solution

$$v^0(x) = \tfrac{1}{\mathrm{i}k\sinh(\mathrm{i}2kR)}\big(c_1\cosh(\mathrm{i}k(x-R)) + c_2\cosh(\mathrm{i}k(x+R))\big). \tag{102}$$

Using (102) the imaging function $f_{u^\star}(x) := \mathrm{Im}\{u^0(x)\overline{v^0}(x)\}$ can be calculated.

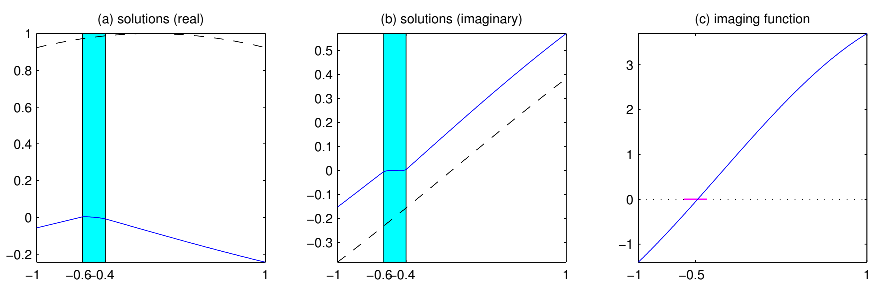

Fig. 1 The real part (a) and the imaginary part (b) of solutions of the Helmholtz problem in the homogeneous medium $u^0(x)$ (dashed line) and in the inhomogeneous medium $u^{(\varepsilon,\alpha)}(x)$ (solid line), and the imaging function $f_{u^\star}(x)$ (c) for the inhomogeneity $\omega_\varepsilon(x_0) = \{-0.6 < x < -0.4\}$ in $(-1,1)$.

Based on these analytic formulas, in Fig. 1 we plot a typical result of numerical calculation of the problem for $R = 1$, $k = \frac{\pi}{2}$, $x_0 = -0.5$, $\varepsilon = 0.1$, and $\alpha = 2 + \mathrm{i}10^4$. The real and the imaginary parts of the solutions u^0 and $u^{(\varepsilon,\alpha)}$ are depicted in the plots (a) and (b), respectively, while in plot (c) we observe the imaging function $f_{u^\star}$ crossing at $x^\star \approx -0.48$ the inhomogeneity drawn along the x-axis.

3.4 The numerical solution in two dimensions

In Figure 2 we show the numerical computation of the imaging function from (95) carried out in 2d on the uniform grid with the mesh size $h = 2^{-6}$ over the unit square $\Omega = (0,1)^2$. In this example, the shown inhomogeneity ω of size $\varepsilon = \sqrt{5}h$ centered at $x_0 = (3/8, 1/4)$ is sound-soft according to the particular case (10) of the forward Helmholtz problem.

Two boundary measurements $u_1^\star$ and $u_2^\star$ are synthesized when the inhomogeneity is illuminated by the plane wave $u^0(x) = e^{ik((x_1-1/2,x_2-1/2)\cdot(\cos\theta,\sin\theta))}$ in directions of $\theta_1 = \frac{\pi}{5}$ and $\theta_2 = \pi$, respectively. We note that the wave number should be chosen sufficiently small, in this example $k \leq \frac{\pi}{2}$. In plots (a) and (b) there are depicted the corresponding imaging functions $f_{u_1^\star}$ and $f_{u_2^\star}$ interpolated linearly over the grid, together with its discrete zero-level sets $L_{=0}(f_{u_1^\star})$ and $L_{=0}(f_{u_2^\star})$ found by a narrow band technique, see e.g. [25]. The zero-level sets form straight lines crossing $\omega_\varepsilon(x_0)$. They are depicted also in the test domain Ω in plot (c), where the intersection point $x^\star$ determined by (97) visually coincides with the test center x_0.

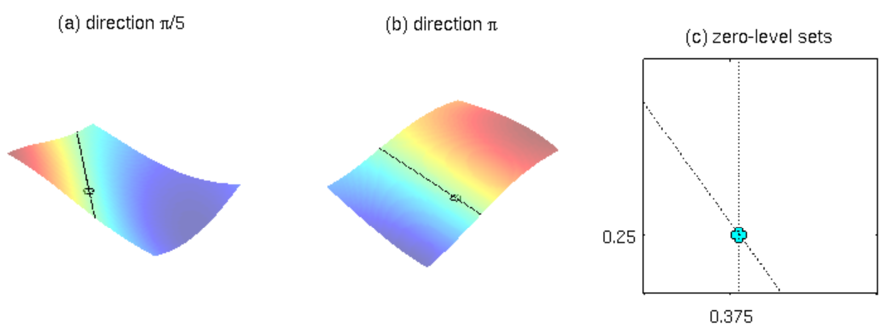

Fig. 2 Identification of the center x_0 of inhomogeneity from two boundary measurements in 2d.

Further development of the optimaliy based imaging theory and numerical methods with respect to identification of the center of inhomogeneities in 2d and 3d from far-field measurements is the subject of a forthcoming paper.

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