VARIATIONAL INEQUALITY FOR A TIMOSHENKO PLATE CONTACTING AT THE BOUNDARY WITH AN INCLINED OBSTACLE

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Abstract. A class of variational inequalities describing equilibrium of elastic Timoshenko plates which boundary is in contact by the side surface with an inclined obstacle is considered. At the plate boundary, mixed conditions of Dirichlet type and a non-penetration condition of inequality type are imposed on displacements in the mid-plane. The novelty consists in modeling oblique interaction with the inclined obstacle which takes into account of shear deformation and rotation of transverse cross-sections in the plate.

For proposed problems of equilibrium of the plate contacting the inclined obstacle, unique solvability of the corresponding variational inequality is proved. Under the assumption that the variational solution is smooth enough, optimality conditions are obtained in the form of equilibrium equations and relations revealing the mechanical properties of integrated stresses, moments, and generalized displacements on the contact part of the boundary.

Accounting for complementarity type conditions due to the contact of the plate with the inclined obstacle, a primal-dual variational formulation of the obstacle problem is derived. A semi-smooth Newton method based on a generalized gradient is constructed and performed as a primal-dual active-set algorithm. It is advantageous for efficient numerical solution of the problem, provided by a super-linear estimate for the corresponding iterates in function spaces.

1. Introduction

Starting from the well known work of Fichera [10], the theory of contact problems for elastic bodies has been intensively developed. Inequality type boundary relations, named Signorini conditions to honour Fichera's teacher, describe a mutual non-penetration of surfaces (or curves) being in contact. This leads to nonlinearity of the corresponding mathematical models described by variational inequalities. Open questions in the contact mechanics concern non-smooth behavior, e.g., due to non-coercive [12, 17] and non-convex functions [23], nonlinear constitutive equations [8, 13, 22], time-discontinuous evolution [20], and geometry singularities [1].

Within the theory of plates and shells (see textbooks [6, 38, 42]), a subclass of non-penetration and crack problems was developed in the monographs [18, 19]. These problems are formulated as non-smooth minimization for lower weakly semi-continuous functionals over closed convex sets of admissible functions. It is worth noting that obstacle problems may be derived by limit passage from families of problems describing equilibrium of cracked bodies using fictitious domain method [37, 39]. For mathematical analysis of elastic plates we refer to contact problems with obstacles [5] and inclusions [9], to history-dependent models [3], analysis of thickness dependence [35], inverse coefficient problems [14], and to

2020 *Mathematics Subject Classification.* 35J86, 70C20, 74K20.

Key words and phrases. contact problem, Timoshenko plate, obstacle, variational inequality, semi-smooth Newton method.

the references therein. For the numerical solution of unilateral problems for plates we cite [7, 21].

The oblique interaction phenomena in Kirchhoff–Love plates were considered earlier in [24] for inclined cracks, in [30, 31] for contact with sharp edges of rigid inclusions, and in [34] for the plate that is in contact by its side edges with inclined obstacle surfaces. Solvability of the corresponding variational inequalities has been established. However, the Kirchhoff–Love hypothesis that straight lines normal to the mid-surface remain straight after deformation does not conform to an oblique geometry. Therefore, we employ Timoshenko's hypotheses [41] taking into account the rotational effects: shear deformation and bending rotation.

For the literature on the Timoshenko plate, especially from the mathematical point of view, we cite [4, 40, 43]. A generalization of the Timoshenko plate model was treated for problems of contact with elastic obstacles [11], shape sensitivity analysis [32], by constructing penalty [27] and fictitious domain [28, 29] methods. The studies with respect to numerical solution can be found in [26], and elasto-plastic behavior in [21, 25]. In the present work, we propose and analyze a new mathematical model that describes an equilibrium of a Timoshenko plate which comes into contact with a non-deformable obstacle, where the obstacle surfaces are not perpendicular to the mid-plane of the plate.

To introduce modeling let us first consider a beam of constant thickness $2h$ posed along x-axis:

$$x \geq x_0, \quad |z| \leq h,$$

with the left-side (x_0, z), $|z| \leq h$, and $x \leq x_1$ for some $x_1 > x_0$. Timoshenko's hypothesis suggests displacements (W^z, w^z) given in the form

$$W^z(x) = W(x) + z\psi(x), \quad w^z(x) = w(x), \quad |z| \leq h, \tag{1.1}$$

where W is the in-plane displacement, w is the deflection of mid-line in z-direction, and ψ is the angle of rotation of the normal to the mid-line $z = 0$. Let an obstacle be prescribed by the half-plane $x - x_0 \leq k(z - b)$ with the straight-line boundary

$$x - x_0 = k(z - b), \tag{1.2}$$

which is determined by an inverse slope k and a z-intercept b at $x = x_0$. The non-penetration through the obstacle of the beam left-side in the deformed state $(x_0 + W^z(x_0), z + w^z(x_0))$, $|z| \leq h$, implies that

$$W^z(x_0) \geq k\big(z + w^z(x_0) - b\big), \quad |z| \leq h,$$

which after inserting (1.1) is equivalent to

$$W(x_0) + z\psi(x_0) \geq k\big(z + w(x_0) - b\big), \quad |z| \leq h. \tag{1.3}$$

In particular, (1.3) holds at $z = h$ and $z = -h$:

$$k\big(b - w(x_0)\big) + W(x_0) \geq h\big(k - \psi(x_0)\big), \quad k\big(b - w(x_0)\big) + W(x_0) \geq -h\big(k - \psi(x_0)\big),$$

implying that

$$k\big(b - w(x_0)\big) + W(x_0) \geq h\big|k - \psi(x_0)\big|. \tag{1.4}$$

Conversely, (1.4) implies (1.3) due to the linearity in z.

For illustration we present two particular configurations satisfying the non-penetration condition (1.4). First, if $k - \psi(x_0) > 0$ and $b = h$, then (1.4) reads

$$W(x_0) + h\psi(x_0) \geq kw(x_0),$$

and contact may occur by the upper beam side as drawn for $\psi(x_0) = 0$ in the left plot of Fig. 1. Here ν shows the normal vector outward to the beam at the left side. Second, if

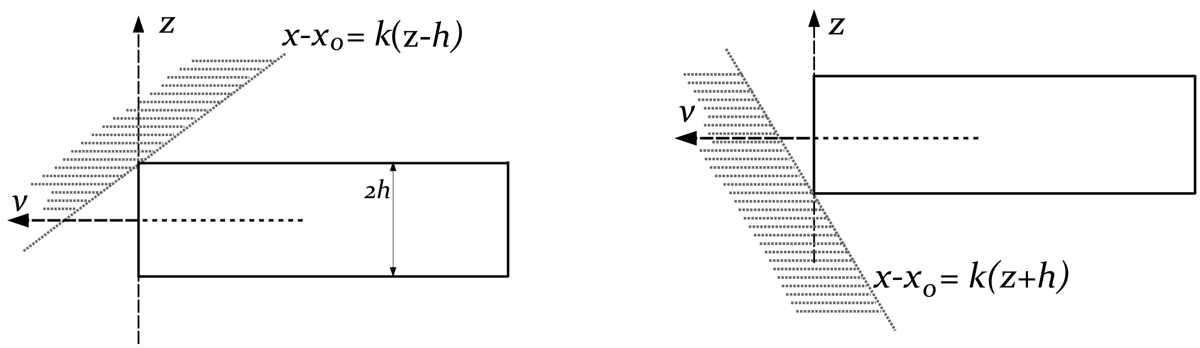

FIGURE 1. Beam contacting the inclined obstacle by the upper side (left plot), and the lower side (right plot).

$k - \psi(x_0) < 0$ and $b = -h$, that case corresponds to the inequality

$$W(x_0) - h\psi(x_0) \geq kw(x_0),$$

then possible contact by the lower beam side is sketched for $\psi(x_0) = 0$ in the right plot of Fig. 1.

It is important to note for consistency that, for the straight obstacle as $k = 0$, the non-penetration condition (1.4) turns into

$$W(x_0) \geq h\big|\psi(x_0)\big|,$$

which was considered in [27, 28, 29, 32, 33]. For Kirchhoff–Love plates the non-penetration reads

$$W(x_0) \geq h\Big|\frac{\partial w}{\partial x}(x_0)\Big|,$$

that was introduced earlier in [18, §3.5].

We will formulate condition that is analogous to (1.4) for the Timoshenko plate in §2. Existence and uniqueness of solution for the corresponding variational inequality is proven using the Weierstrass theorem in §3. In §4, a complete system of boundary conditions is found under the assumption of additional regularity for the variational solution. Complementarity boundary conditions describing contact with inclined obstacles are given in the form of equations and inequalities, which restrict deflection and rotation of transverse cross-sections of the plate.

The next question of our study concerns iterative solution of the complementarity boundary conditions and convergence of its iterates. It will be treated in §5 with the help of a semi-smooth Newton method using generalized differentiation of non-smooth functions. We apply a slant derivative to the minimum-based merit function and derive a super-linear error estimate in function spaces, following the approach in [15, 16] and the references therein. Numerically, the semi-smooth Newton method can be realized as a primal-dual active-set algorithm.

2. Variational problem formulation

We start with the description of the undeformed reference configuration for an isotropic plate of uniform thickness $2h$. Let the plate occupy the right cylinder $\Omega\times(-h,h)\subset\mathbf{R}^3$ with its mid-plane lying in the Ox_1x_2-plane, where Ω is a bounded simply connected domain with a Lipschitz boundary Γ. The unit external normal vector to Γ is denoted by $\nu=(\nu_1,\nu_2)(x)$, such that the tangential vector $\tau=(-\nu_2,\nu_1)$. The boundary Γ is decomposed into two disjoint parts Γ_0 and γ. The boundary part Γ_0 needs to have the one-dimensional Hausdorff measure, that is, $\mathcal{H}^1(\Gamma_0)>0$, to guarantee coercivity of the energy.

Let $(W,w)(x)$ denote the vector of displacements in the mid-plane $x=(x_1,x_2)\in\Omega$, such that $W=(w_1,w_2)$ are in-plane displacements, and w is the deflection along the axis z. The rotation of the mid-plane $x\in\Omega$ is denoted by $\psi=(\psi_1,\psi_2)(x)$, where ψ_1 and ψ_2 are rotations of the mid-plane normal about the x_1 and x_2-axes, respectively. Analogously to (1.1), in the Timoshenko plate the in-plane displacements depend linearly on z:

$$w_i^z(x)=w_i(x)+z\psi_i(x),\quad i=1,2,\quad |z|\le h, \tag{2.1}$$

whereas the deflection is independent of z, i.e.

$$w^z(x)=w(x),\quad |z|\le h. \tag{2.2}$$

Now we describe an inclined obstacle. Let the boundary γ can be parametrized by $t\in(0,T)$. We suppose that the obstacle surface is generated by a line (called generatrix) moved perpendicular to $\gamma(t)$. In this case, at every fixed point $x(t)\in\gamma$ the generatrix bounds the obstacle in the plane cross-section formed by the axis Oz and the normal vector $(\nu(x(t)),0)$. For example, a right cylinder surface is generated by $x-x(t)=0$. An inclined generatrix can be determined according to (1.2) as the straight line:

$$-(x-x(t))\cdot\nu(x(t))=k(t)\big(z-b(t)\big),\quad t\in(0,T), \tag{2.3}$$

prescribed by the functions of inverse slope $k\in C([0,T])$ and z-intercept $b\in C([0,T])$. Here and in what follows, the dot stands for the scalar product of vectors, e.g., $x\cdot\nu=x_1\nu_1+x_2\nu_2$. In every such cross-section, the non-penetration condition (1.4) holds within the mid-plane in the direction opposite to ν. Thus there holds the inequality

$$k(t)\big(b(t)-w(x(t))\big)-W(x(t))\cdot\nu(x(t))\ge h\big|k(t)+\psi(x(t))\cdot\nu(x(t))\big|,\quad t\in(0,T).$$

In short notation

$$k(b-w)-W\cdot\nu\ge h|k+\psi\cdot\nu|\quad\text{on }\gamma. \tag{2.4}$$

Since we prescribe zero condition (2.8) on the complementary part of the boundary Γ_0 we assume the compatibility condition $k=b=0$ at the intersection $\overline{\gamma}\cap\overline{\Gamma_0}$.

We proceed with governing relations for the Timoshenko model following [38]. The tensors $\varepsilon=\{\varepsilon_{ij}\}$ of bending and shear strains in the plate are given by

$$\varepsilon_{ij}(W)=\frac{1}{2}(w_{i,j}+w_{j,i}),\quad \varepsilon_{ij}(\psi)=\frac{1}{2}(\psi_{i,j}+\psi_{j,i}),\quad i,j=1,2\quad \Big(w_{,i}=\frac{\partial w}{\partial x_i}\Big).$$

The integrated stresses $\sigma=\{\sigma_{ij}\}$ and moments $m=\{m_{ij}\}$ read using summation over repeated indices as follows:

$$\sigma_{ij}(W)=\frac{3}{h^2}a_{ijkl}\varepsilon_{kl}(W),\quad m_{ij}(\psi)=a_{ijkl}\varepsilon_{kl}(\psi),\quad i,j,k,l=1,2. \tag{2.5}$$

Here nonzero components of the elastic modulus tensor $A = \{a_{ijkl}\}$ are

$$a_{iiii} = D, \quad a_{iijj} = D\kappa, \quad a_{ijij} = a_{ijji} = D\frac{1-\kappa}{2} \quad (i \neq j), \quad i,j = 1,2,$$

where the flexural rigidity $D = E(2h)^3/(12(1-\kappa^2))$ for the plate , $E > 0$ is Young's modulus, and κ is Poisson's ratio, $0 < \kappa < 1/2$. The shear forces $q = (q_1, q_2)(x)$ are given by

$$q_i(w,\psi) = S(w_{,i} + \psi_i), \quad i = 1,2, \tag{2.6}$$

where the transverse shear stiffness $S = 2hG\varkappa$ is determined by the thickness $2h$, shear modulus $G = E/(2(1+\kappa))$ and shear correction factor $\varkappa > 0$.

For prescribed external forces $F = (f_1, f_2, f_3, \mu_1, \mu_2) \in L^2(\Omega)^5$, the potential energy of the Timoshenko plate occupying the mid-plane Ω has the form

$$\Pi(\chi) = \frac{1}{2}B(\chi,\chi) - \int_\Omega F \cdot \chi \, dx,$$

defined by the symmetric bilinear form

$$B(\xi,\chi) = \int_\Omega \big(\sigma_{ij}(U)\varepsilon_{ij}(W) + q_i(u,\phi)(w_{,i} + \psi_i) + m_{ij}(\phi)\varepsilon_{ij}(\psi)\big)\, dx, \tag{2.7}$$

for $\chi = (W, w, \psi)(x) \in H^1(\Omega)^5$, and $\xi = (U, u, \phi)(x) \in H^1(\Omega)^5$ with $U = (u_1, u_2)$, $\phi = (\phi_1, \phi_2)$.

We introduce the Sobolev spaces

$$H(\Omega) = H^1_{\Gamma_0}(\Omega)^5, \quad H^1_{\Gamma_0}(\Omega) = \big\{w \in H^1(\Omega) \big| \quad w = 0 \text{ on } \Gamma_0\big\},$$

which includes the homogeneous Dirichlet boundary conditions

$$W = (0,0), \quad w = 0, \quad \psi = (0,0) \quad \text{on } \Gamma_0. \tag{2.8}$$

The set of admissible functions is specified by the non-penetration condition:

$$K = \big\{\chi \in H(\Omega) \big| \quad \chi \text{ satisfies (2.4)}\big\}. \tag{2.9}$$

Then the inclined obstacle problem reads in variational formulation:

$$\text{find } \xi \in K \text{ such that } \Pi(\xi) = \min_{\chi \in K} \Pi(\chi). \tag{2.10}$$

Solvability of the constrained minimization problem (2.10) is established in the next section.

3. Existence, uniqueness, and optimality condition

In order to prove the existence of solution to (2.10), we establish some auxiliary results. We recall the Korn and Poincaré inequalities: there exist constants $c_1 > 0$ and $c_2 > 0$ such that [18, §1.4]:

$$\int_\Omega \sigma_{ij}(W)\varepsilon_{ij}(W)\, dx \geq \frac{3}{h^2} c_1 \|W\|^2_{H^1(\Omega)^2}, \quad \int_\Omega m_{ij}(\psi)\varepsilon_{ij}(\psi)\, dx \geq c_1 \|\psi\|^2_{H^1(\Omega)^2} \tag{3.1}$$

according to (2.5), and

$$\|\nabla w\|_{L^2(\Omega)^2} \geq c_2 \|w\|_{H^1(\Omega)}, \tag{3.2}$$

which are valid for $\chi = (W, w, \psi) \in H(\Omega)$ due to the zero boundary conditions on Γ_0.

Lemma 3.1 (Coercivity of the bilinear form B). *For the bilinear form B defined in* (2.7)*, the lower estimate holds:*

$$B(\chi,\chi) \geq c_0 \|\chi\|^2_{H^1(\Omega)^5} \tag{3.3}$$

with a constant $c_0 > 0$ independent of $\chi \in H(\Omega)$.

Proof. We apply weighted Young's inequality with arbitrary $\epsilon > 0$ to the mixed term

$$2\Big|\int_\Omega w_{,i}\psi_i\,dx\Big| \leq \frac{1}{\epsilon}\|\nabla w\|^2_{L^2(\Omega)^2} + \epsilon\|\psi\|^2_{L^2(\Omega)^2}$$

in order to estimate the second term in the bilinear form B in (2.7) from below

$$\int_\Omega q_i(w,\psi)(w_{,i}+\psi_i)\,dx \geq S\big(1-\frac{1}{\epsilon}\big)\|\nabla w\|^2_{L^2(\Omega)^2} + S(1-\epsilon)\|\psi\|^2_{L^2(\Omega)^2}, \tag{3.4}$$

with $S > 0$ from (2.6). Therefore, from (3.1) and (3.4) it follows the estimate

$$B(\chi,\chi) \geq \frac{3c_1}{h^2}\|W\|^2_{H^1(\Omega)^2} + S\big(1-\frac{1}{\epsilon}\big)\|\nabla w\|^2_{L^2(\Omega)^2} + \big(c_1 + S(1-\epsilon)\big)\|\psi\|^2_{H^1(\Omega)^2}. \tag{3.5}$$

By choosing $1 < \epsilon < 1 + c_1/S$ all factors in the right-hand-side of (3.5) are positive. Taking into account the Poincaré inequality (3.2), this provides the uniform lower estimate (3.3). □

Lemma 3.2 (Convexity and weak closedness of the admissible set K). *The set K of admissible functions $\chi = (W, w, \psi)$ determined in* (2.9) *is convex and weakly closed.*

Proof. We can express the non-penetration condition (2.4) determining the admissible set K equivalently as two inequalities:

$$\begin{aligned}\Phi_1(\chi) &:= k(b-w) - W\cdot\nu - h(k+\psi\cdot\nu) \geq 0,\\ \Phi_2(\chi) &:= k(b-w) - W\cdot\nu + h(k+\psi\cdot\nu) \geq 0.\end{aligned} \tag{3.6}$$

Since the both functions Φ_1 and Φ_2 in (3.6) are linear with respect to $\chi = (W, w, \psi)$, from $\Phi_1(\chi^i) \geq 0$ and $\Phi_2(\chi^i) \geq 0$, $i = 1, 2$, it follows $\Phi_1(\alpha\chi^1 + (1-\alpha)\chi^2) \geq 0$ and $\Phi_2(\alpha\chi^1 + (1-\alpha)\chi^2) \geq 0$, $\alpha \in [0, 1]$, thus the convexity of K.

It is enough to show that K is (norm) closed, since every closed convex set is weakly closed. Let us assume a convergent sequence $\chi^n = (W^n, w^n, \psi^n) \in K$ such that

$$\chi^n \to \chi \quad \text{strongly in } L^2(\Gamma)^5 \text{ as } n \to \infty,$$

hence it converges almost everywhere on γ. Passing to the limit $n \to \infty$ in the inequalities

$$\Phi_1(\chi^n) \geq 0, \quad \Phi_2(\chi^n) \geq 0 \quad \text{a.e. on } \gamma,$$

we conclude with (3.6) and $\chi \in K$. This proves that K is weakly closed. □

With the help of Lemmas 3.1 and 3.2 we prove the existence theorem below.

Theorem 3.1 (Existence and uniqueness of the solution, and the optimality condition). *The constrained minimization* (2.10) *describing the inclined obstacle problem has the unique solution $\xi \in K$. Its necessary and sufficient optimality condition is given by the variational inequality*

$$B(\xi, \chi-\xi) \geq \int_\Omega F\cdot(\chi-\xi)\,dx \quad \textit{for all } \chi \in K. \tag{3.7}$$

Proof. We apply the Weierstrass theorem for lower weakly semi-continuous functions from [2]. The coercivity of Π follows directly from the estimate (3.3) for B. The functional Π is quadratic and coercive, hence lower weakly semi-continuous. Together with the convexity and weak closedness of K in the Hilbert space $H(\Omega)$, this allows us to claim existence of at least one solution $\xi \in K$ to the minimization problem (2.10).

The functional Π is quadratic and coercive, hence convex. Calculating its Gâteaux derivative (see [27] for details) necessitates the variational inequality (3.7) for the bilinear form B, which is sufficient to the minimization (2.10) because of convexity of Π. By the virtue of (3.3), B is coercive, hence restricts at most one solution. The theorem is proven. □

In the next section we use the variational inequality (3.7) in order to derive a boundary value formulation, supported by a complete system of boundary conditions fulfilled at γ.

4. The strong formulation

In the sequel we use the following Green's formulas for $\xi \in H^2(\Omega)^5$ and $\chi \in H(\Omega)$, see [18, §1.4]:

$$\int_\Omega \sigma_{ij}(U)\varepsilon_{ij}(W)\,dx = -\int_\Omega \sigma_{ij,j}(U)w_i\,dx + \int_\gamma \big(\sigma_\nu(U)(W\cdot\nu) + \sigma_\tau(U)(W\cdot\tau)\big)\,d\gamma, \tag{4.1}$$

by decomposing the stress vector at the boundary into the normal $\sigma_\nu(U) = \sigma_{ij}(U)\nu_i\nu_j$ and the tangential $\sigma_\tau(U) = \sigma_{ij}(U)\nu_j\tau_i$ components; then

$$\int_\Omega q_i(u,\phi)(w_{,i} + \psi_i)\,dx = \int_\Omega \big(q_i(u,\phi)\psi_i - q_{i,i}(u,\phi)w\big)\,dx + \int_\gamma q_\nu(u,\phi)w\,d\gamma, \tag{4.2}$$

with the normal shear force $q_\nu(u,\phi) = q_i(u,\phi)\nu_i$; and

$$\int_\Omega m_{ij}(\phi)\varepsilon_{ij}(\psi)\,dx = -\int_\Omega m_{ij,j}(\phi)\psi_i\,dx + \int_\gamma \big(m_\nu(\phi)(\psi\cdot\nu) + m_\tau(\phi)(\psi\cdot\tau)\big)\,d\gamma, \tag{4.3}$$

where $m_\nu(\phi) = m_{ij}(\phi)\nu_j\nu_i$ and $m_\tau(\phi) = m_{ij}(\phi)\nu_j\tau_i$ are the normal and tangential moments.

Based on (4.1)–(4.3) and (2.7), after the substitution in (3.7) test functions $\chi = \xi \pm \eta$, where $\eta \in C_0^\infty(\Omega)^5$ are compactly supported smooth functions, we derive the equilibrium equations

$$-\sigma_{ij,j}(U) = f_i \quad (i=1,2), \quad -q_{i,i}(u,\phi) = f_3, \quad -m_{ij,j}(\phi) + q_i(u,\phi) = \mu_i \quad (i=1,2), \tag{4.4}$$

which hold in terms of distribution in Ω. We assume that the solution $\xi \in H^2(\Omega)^5$. Then the equations (4.4) hold true a.e. in Ω, since $F \in L^2(\Omega)^5$. By the trace and Sobolev embedding theorems, the trace $\xi \in L^{2+\epsilon}(\Gamma)^5$ for arbitrary $\epsilon \geq 0$. To derive relations on γ, we observe that the normal stress $\sigma_\nu(U)$, shear force $q_\nu(u,\phi)$, and moment $m_\nu(\phi)$ are also $L^{2+\epsilon}$-functions defined a.e. on Γ.

Theorem 4.1 (Equilibrium equations and boundary conditions)**.** *For the H^2-smooth solution $\xi = (U,u,\phi) \in K$, the variational inequality* (3.7) *is equivalent to the following system of equilibrium equations:*

$$-\sigma_{ij,j}(U) = f_i, \quad i=1,2, \quad \text{in } \Omega, \tag{4.5}$$

$$-q_{i,i}(u,\phi) = f_3 \quad \text{in } \Omega, \tag{4.6}$$

$$-m_{ij,j}(\phi) + q_i(u,\phi) = \mu_i, \quad i=1,2, \quad \text{in } \Omega, \tag{4.7}$$

and the boundary conditions:

$$U=(0,0), \quad u=0, \quad \phi=(0,0) \quad \text{on } \Gamma_0, \tag{4.8}$$

together with natural conditions

$$\sigma_\tau(U)=0, \quad m_\tau(\phi)=0, \quad q_\nu(u,\phi)=k\sigma_\nu(U) \quad \text{on } \gamma, \tag{4.9}$$

and complementarity conditions stemming from the non-penetration:

$$k(b-u) - U\cdot\nu \geq h|k+\phi\cdot\nu|, \quad h\sigma_\nu(U) \leq -|m_\nu(\phi)| \quad \text{on } \gamma, \tag{4.10}$$

$$\sigma_\nu(U)\big(k\big(b-u\big) - U\cdot\nu\big) - m_\nu(\phi)(k+\phi\cdot\nu) = 0 \quad \text{on } \gamma. \tag{4.11}$$

Proof. In the virtue of (4.1)–(4.4), the variational inequality (3.7) reduces itself to the boundary

$$\int_\gamma \Big(\sigma_\nu(U)\big((W-U)\cdot\nu\big) + \sigma_\tau(U)\big((W-U)\cdot\tau\big) + q_\nu(u,\phi)(w-u) + m_\nu(\phi)\big((\psi-\phi)\cdot\nu\big) + m_\tau(\phi)\big((\psi-\phi)\cdot\tau\big)\Big)d\gamma \geq 0 \tag{4.12}$$

for all $\chi \in K$. If we test (4.12) with $(W\cdot\nu, w, \psi\cdot\nu) = (U\cdot\nu, u, \phi\cdot\nu)$ and arbitrary tangential components $(W\cdot\tau, \psi\cdot\tau)$, then we get the equations

$$\sigma_\tau(U)=0, \quad m_\tau(\phi)=0 \quad \text{on } \gamma, \tag{4.13}$$

and the variational inequality for all $\chi \in K$:

$$\int_\gamma \Big(\sigma_\nu(U)\big((W-U)\cdot\nu\big) + q_\nu(u,\phi)(w-u) + m_\nu(\phi)\big((\psi-\phi)\cdot\nu\big)\Big)d\gamma \geq 0. \tag{4.14}$$

Next we test (4.14) with $(W\cdot\nu, w, \psi\cdot\nu) = (U\cdot\nu - k\eta, u+\eta, \phi\cdot\nu)$ for an arbitrary function $\eta \in C_0^\infty(\gamma)$. In this case, the non-penetration condition (2.4) holds because of

$$k\big(b-w\big) - W\cdot\nu - h\big|k-\psi\cdot\nu\big| = k\big(b-u\big) - U\cdot\nu - h\big|k+\phi\cdot\nu\big|, \tag{4.15}$$

and $\xi \in K$. As the consequence we infer

$$\int_\gamma \big(-k\sigma_\nu(U) + q_\nu(u,\phi)\big)\eta \, d\gamma \geq 0$$

for all η, which implies the equality

$$q_\nu(u,\phi) = k\sigma_\nu(U) \quad \text{on } \gamma. \tag{4.16}$$

Taking into account (4.16) we can rewrite (4.14) in the form

$$\int_\gamma \Big(\sigma_\nu(U)\big((W-U)\cdot\nu + k(w-u)\big) + m_\nu(\phi)\big((\psi-\phi)\cdot\nu\big)\Big)d\gamma \geq 0 \tag{4.17}$$

for all $\chi \in K$. Now we test (4.17) with $(W\cdot\nu, w, \psi\cdot\nu)$ such that $W\cdot\nu + kw = U\cdot\nu + ku - h|\eta|$ and $\psi\cdot\nu = \phi\cdot\nu + \eta$ for $\eta \in C_0^\infty(\gamma)$ fulfills (4.15). This reduces (4.17) to

$$\int_\gamma \big(-h\sigma_\nu(U)|\eta| + m_\nu(\phi)\eta\big)\, d\gamma \geq 0,$$

leading to the inequality

$$h\sigma_\nu(U) \leq -|m_\nu(\phi)| \quad \text{on } \gamma. \tag{4.18}$$

Finally, we employ the half-sum and the half-difference of functions in (3.6):

$$\frac{1}{2}\big(\Phi_1(\chi) + \Phi_2(\chi)\big) = k(b-w) - W\cdot\nu, \quad \frac{1}{2}\big(\Phi_1(\chi) - \Phi_2(\chi)\big) = -h(k + \psi\cdot\nu),$$

such that (4.17) allows the equivalent representation by Φ_1 and Φ_2 as

$$\int_\gamma \Big(\frac{1}{2}\sigma_\nu(U)\big(\Phi_1(\xi) + \Phi_2(\xi) - \Phi_1(\chi) - \Phi_2(\chi)\big) + \frac{1}{2h} m_\nu(\phi)\big(\Phi_1(\xi) - \Phi_2(\xi) - \Phi_1(\chi) + \Phi_2(\chi)\big)\Big) d\gamma \geq 0$$

for all $\chi \in K$. Collecting similar terms implies

$$\frac{1}{2h}\int_\gamma \Big(\big(h\sigma_\nu(U) + m_\nu(\phi)\big)\big(\Phi_1(\xi) - \Phi_2(\chi)\big) + \big(h\sigma_\nu(U) - m_\nu(\phi)\big)\big(\Phi_2(\xi) - \Phi_2(\chi)\big)\Big) d\gamma \geq 0. \tag{4.19}$$

When inserting into (4.19) $(W\cdot\nu, w, \psi\cdot\nu) = (0, b, -k)$ such that $\Phi_1(\chi) = \Phi_2(\chi) = 0$ we get

$$\int_\gamma \Big(\big(h\sigma_\nu(U) + m_\nu(\phi)\big)\Phi_1(\xi) + \big(h\sigma_\nu(U) - m_\nu(\phi)\big)\Phi_2(\xi)\Big) d\gamma \geq 0,$$

and testing (4.19) with $(W\cdot\nu, w, \psi\cdot\nu) = (2U\cdot\nu, 2u-b, 2\phi\cdot\nu + k)$ such that $\Phi_1(\chi) = 2\Phi_1(\xi) \geq 0$ and $\Phi_2(\chi) = 2\Phi_2(\xi) \geq 0$ leads to the opposite inequality

$$\int_\gamma \Big(\big(h\sigma_\nu(U) + m_\nu(\phi)\big)\Phi_1(\xi) + \big(h\sigma_\nu(U) - m_\nu(\phi)\big)\Phi_2(\xi)\Big) d\gamma \leq 0,$$

thus providing together the equality

$$\int_\gamma \Big(\big(h\sigma_\nu(U) + m_\nu(\phi)\big)\Phi_1(\xi) + \big(h\sigma_\nu(U) - m_\nu(\phi)\big)\Phi_2(\xi)\Big) d\gamma = 0. \tag{4.20}$$

The substitution of Φ_1 and Φ_2 from (3.6) into (4.20) after canceling $2h$ yields the identity

$$\int_\gamma \Big(\sigma_\nu(U)\big(k\big(b-u\big) - U\cdot\nu\big) - m_\nu(\phi)(k + \phi\cdot\nu)\Big) d\gamma = 0. \tag{4.21}$$

Relations (4.4), (4.13), (4.16), (4.18), (4.21) together with the non-penetration condition (2.4) imply the boundary value formulation (4.5)–(4.11).

Conversely, we will show that a function $\xi = (U, u, \phi) \in K \cap H^2(\Omega)^5$ satisfying the relations (4.4), (4.13), (4.16), (4.18), (4.21) is the solution to the variational inequality (3.7).

For arbitrary $\chi = (W, w, \psi) \in H(\Omega)$, we multiply the five equilibrium equations (4.4) by $(w_1 - u_1, w_2 - u_2, w - u, \psi_1 - \phi_1, \psi_2 - \phi_2)$ and integrate over the domain Ω. After summation this gives

$$\int_\Omega \Big(\sigma_{ij,j}(U)(w_i - u_i) + q_{i,i}(u,\phi)(w-u) + \big(m_{ij,j}(\phi) - q_i(u,\phi)\big)(\psi_i - \phi_i)\Big)dx = -\int_\Omega \big(f_i(w_i - u_i) + f_3(w-u) + \mu_i(\psi_i - \phi_i)\big)\,dx. \tag{4.22}$$

Applying Green's formulas (4.1)–(4.3) to (4.22) and recalling the bilinear form B in (2.7), it follows

$$B(\xi, \chi - \xi) - \int_\gamma \Big(\sigma_\nu(U)\big((W-U)\cdot\nu\big) + \sigma_\tau(U)\big((W-U)\cdot\tau\big) + q_\nu(u,\phi)(w-u) + m_\nu(\phi)\big((\psi-\phi)\cdot\nu\big) + m_\tau(\phi)\big((\psi-\phi)\cdot\tau\big)\Big)d\gamma = \int_\Omega F\cdot(\chi - \xi)\,dx.$$

In virtue of the boundary conditions (4.13), (4.16) at γ and the identity (4.21) we proceed

$$B(\xi, \chi-\xi) - \int_\Omega F\cdot(\chi-\xi)\,dx = \int_\gamma \Big(\sigma_\nu(U)\big((W-U)\cdot\nu + k(w-u)\big) + m_\nu(\phi)\big((\psi-\phi)\cdot\nu\big)\Big)d\gamma = \int_\gamma \Big(\sigma_\nu(U)\big(W\cdot\nu + k(w-b)\big) + m_\nu(\phi)(k + \psi\cdot\nu)\Big)d\gamma,$$

which after assembling the terms yields

$$B(\xi, \chi - \xi) - \int_\Omega F\cdot(\chi-\xi)\,dx = \frac{1}{2h}\int_\gamma \Big(\big(h\sigma_\nu(U) + m_\nu(\phi)\big)\Phi_1(\chi) + \big(h\sigma_\nu(U) - m_\nu(\phi)\big)\Phi_2(\chi)\Big)d\gamma. \tag{4.23}$$

The boundary inequality (4.18) and the non-penetration condition (2.4) at γ together guarantee the non-negative sign in the right-hand side of (4.23), thus concluding with the variational inequality (3.7). This finishes the proof. □

In the following section we will study the numerical solution of the relations (4.5)–(4.11). Its main challenge concerns realization of the complementarity conditions (4.10) and (4.11).

5. Semi-smooth Newton method of solution

First we give a primal-dual formulation of the problem.

Lemma 5.1 (Primal-dual problem)**.** *The smooth solution $\xi \in K \cap H^2(\Omega)^5$ of the variational inequality* (3.7) *together with the Lagrange multiplier $\Lambda = (\lambda_1, \lambda_2) \in L^2(\Omega)^2$ determined by*

$$\lambda_1 = h\sigma_\nu(U) + m_\nu(\phi), \quad \lambda_2 = h\sigma_\nu(U) - m_\nu(\phi) \tag{5.1}$$

solve the following primal-dual system:

$$B(\xi,\chi)-\int\limits_\Omega F\cdot\chi\,dx=\frac{1}{2h}\int\limits_\gamma\Big(\lambda_1\big(\Phi_1(\chi)-k(b-h)\big)+\lambda_2\big(\Phi_2(\chi)-k(b+h)\big)\Big)d\gamma \tag{5.2}$$

for all $\chi\in H(\Omega)$*, where* Φ_1 *and* Φ_2 *are given in* (3.6)*, and*

$$\Phi_i(\xi)\geq 0,\quad \lambda_i\leq 0\quad (i=1,2),\quad \lambda_1\Phi_1(\xi)=0,\quad \lambda_2\Phi_2(\xi)=0\quad \textit{on } \gamma. \tag{5.3}$$

Conversely, the primal component of $(\xi,\Lambda)\in H(\Omega)\times L^2(\Omega)^2$ *solving the primal-dual system* (5.2) *and* (5.3) *satisfies the variational inequality* (3.7)*.*

Proof. Indeed, Theorem 4.1 holds true for the smooth solution ξ. In this case, the normal stress $\sigma_\nu(U)$, shear force $q_\nu(u,\phi)$, and moment $m_\nu(\phi)$ are well-defined a.e. on γ. With their help we can determine the Lagrange multiplier components in (5.1). From (4.23) tested with $\chi=0$, using (3.6) it follows the identity

$$-B(\xi,\xi)+\int\limits_\Omega F\cdot\xi\,dx=\frac{1}{2h}\int\limits_\gamma\big(\lambda_1 k(b-h)+\lambda_2 k(b+h)\big)\,d\gamma, \tag{5.4}$$

and the variational equation (5.2) holds for all test functions $\chi\in H(\Omega)$. The complementarity conditions (4.10) and (4.11) take the respective form (5.3).

Conversely, if a pair $(\xi,\Lambda)\in H(\Omega)\times L^2(\Omega)^2$ solves the primal-dual system (5.2) and (5.3), then ξ satisfies (5.4) and (4.23), hence the variational inequality (3.7). According to (5.1), the normal stress and moment can be found as

$$\sigma_\nu(U)=\frac{1}{2h}\big(\lambda_1+\lambda_2\big),\quad m_\nu(\phi)=\frac{1}{2}\big(\lambda_1-\lambda_2\big).$$

The proof is complete. □

As a consequence of Lemma 5.1, we note that the primal-dual solution (ξ,Λ) is unique, too.

Now we express the complementarity relations (5.3) by a nonlinear merit function $C:\mathbb{R}^2\mapsto\mathbb{R}$ arising as the minimum for arbitrarily fixed constant $r>0$:

$$C(\Phi_i(\xi),\lambda_i):=\min(\Phi_i(\xi),-r\lambda_i)=0,\quad i=1,2,\quad \text{on } \gamma. \tag{5.5}$$

Since the min-function is not smooth, in the lemma below we employ a concept of semi-smooth functions from [36] with the generalized gradient

$$C'_\Phi(\Phi_i(\xi),\lambda_i)=\mathbf{1}_{\mathcal{I}(\Phi_i(\xi),\lambda_i)},\quad C'_\lambda(\Phi_i(\xi),\lambda_i)=-r\mathbf{1}_{\mathcal{A}(\Phi_i(\xi),\lambda_i)}, \tag{5.6}$$

where $\mathbf{1}$ denotes the indicator function of the corresponding active set:

$$\mathcal{A}(\Phi_i(\xi),\lambda_i)=\{x\in\gamma|\quad \big(\Phi_i(\xi)+r\lambda_i\big)(x)>0\}, \tag{5.7}$$

and its complementary to the boundary γ inactive set:

$$\mathcal{I}(\Phi_i(\xi),\lambda_i)=\{x\in\gamma|\quad \big(\Phi_i(\xi)+r\lambda_i\big)(x)\leq 0\}. \tag{5.8}$$

Lemma 5.2 (Semi-smooth min-function)**.** *For functions* $(\xi,\Lambda),(\chi,M)\in H(\Omega)\times L^{2+\epsilon}(\Omega)^2$*,* $M=(\mu_1,\mu_2)$*, the min-function* $C:L^{2+\epsilon}(\gamma)^2\mapsto L^2(\gamma)$*,* $\epsilon>0$*, with the generalized gradient* (C'_Φ,C'_λ) *defined in* (5.6)*–*(5.8)*, is semi-smooth in the sense of the following estimate*

$$\|\widetilde{C}_{\Phi_i}\|_{L^2(\mathcal{I}(\Phi_i(\chi),\mu_i))}+\|\widetilde{C}_{\lambda_i}\|_{L^2(\mathcal{A}(\Phi_i(\chi),\mu_i))}=o\big(\|\delta_i\|_{L^{2+\epsilon}(\gamma)}\big) \tag{5.9}$$

hold for the 1st-order asymptotic approximations for every $i=1,2$ (there is no summation over i):

$$\begin{aligned}&\widetilde{C}_{\Phi_i} := C(\Phi_i(\chi),\mu_i) - C(\Phi_i(\xi),\lambda_i) - C'_\Phi(\Phi_i(\chi),\mu_i)\big(\Phi_i(\chi)-\Phi_i(\xi)\big),\\ &\widetilde{C}_{\lambda_i} := C(\Phi_i(\chi),\mu_i) - C(\Phi_i(\xi),\lambda_i) - C'_\lambda(\Phi_i(\chi),\mu_i)(\mu_i-\lambda_i),\end{aligned} \tag{5.10}$$

and using the notation for increment:

$$\delta_i := \Phi_i(\chi) - \Phi_i(\xi) + r(\mu_i-\lambda_i),\quad i=1,2.$$

Proof. If we calculate the quantities $\widetilde{C}_{\Phi_i}$ and $\widetilde{C}_{\lambda_i}$ in (5.10) using (5.6)–(5.8), then we find that

(5.11)

$$\begin{cases}\widetilde{C}_{\Phi_i} = -r(\mu_i-\lambda_i),\quad \widetilde{C}_{\lambda_i}=0 & \text{at } \mathcal{A}(\Phi_i(\chi),\mu_i)\cap\mathcal{A}(\Phi_i(\xi),\lambda_i)\\ \widetilde{C}_{\Phi_i} = -\Phi_i(\xi)-r\mu_i,\quad \widetilde{C}_{\lambda_i} = -\Phi_i(\xi)-r\lambda_i & \text{at } \mathcal{A}(\Phi_i(\chi),\mu_i)\cap\mathcal{I}(\Phi_i(\xi),\lambda_i) =: N_i^1\\ \widetilde{C}_{\Phi_i}=0,\quad \widetilde{C}_{\lambda_i} = \Phi_i(\chi)-\Phi_i(\xi) & \text{at } \mathcal{I}(\Phi_i(\chi),\mu_i)\cap\mathcal{I}(\Phi_i(\xi),\lambda_i)\\ \widetilde{C}_{\Phi_i} = \Phi_i(\xi)+r\lambda_i,\quad \widetilde{C}_{\lambda_i} = \Phi_i(\chi)+r\lambda_i & \text{at } \mathcal{I}(\Phi_i(\chi),\mu_i)\cap\mathcal{A}(\Phi_i(\xi),\lambda_i) =: N_i^2\end{cases},$$

and on the sets N_i^1 and N_i^2 it holds

$$0\le -\Phi_i(\xi)-r\lambda_i\le\delta_i\quad\text{on } N_i^1,\quad 0< -\Phi_i(\xi)-r\lambda_i\le-\delta_i\quad\text{on } N_i^2. \tag{5.12}$$

Therefore, from (5.11) and (5.12) we estimate the L^2-norm using the Hölder inequality as

$$\begin{aligned}&\|\widetilde{C}_{\Phi_i}\|_{L^2(\mathcal{I}(\Phi_i(\chi),\mu_i))} \le \|\delta_i\|_{L^2(N_i^2)} \le |N_i^2|^{\frac{\epsilon}{2(2+\epsilon)}}\|\delta_i\|_{L^{2+\epsilon}(N_i^2)},\\ &\|\widetilde{C}_{\lambda_i}\|_{L^2(\mathcal{A}(\Phi_i(\chi),\mu_i))} \le \|\delta_i\|_{L^2(N_i^1)} \le |N_i^1|^{\frac{\epsilon}{2(2+\epsilon)}}\|\delta_i\|_{L^{2+\epsilon}(N_i^1)}.\end{aligned} \tag{5.13}$$

Since the measures $|N_i^1|$ and $|N_i^2|$ of intersection of the active and inactive sets in (5.11) decrease to zero at $\chi=\xi$ and $M=\Lambda$, inequalities (5.13) lead to the assertion (5.9) and finish the proof. □

With the help of Lemmas 5.1 and 5.2 we prove the theorem on iterative solution of the primal-dual problem written in the form (5.2) and (5.5) by a semi-smooth Newton method.

Theorem 5.1 (Semi-smooth Newton method)**.** *Let the primal-dual system* (5.2) *and* (5.5) *describing the obstacle problem have the smooth solution $(\xi,\Lambda)\in H(\Omega)\times L^{2+\epsilon}(\Omega)^2$, $\epsilon>0$. Initializing (ξ^0,Λ^0), a semi-smooth Newton iteration reads: for $n\ge 0$ find $(\xi^{n+1},\Lambda^{n+1})\in H(\Omega)\times L^{2+\epsilon}(\Omega)^2$ such that*

$$B(\xi^{n+1},\chi) - \int_\Omega F\cdot\chi\,dx = \frac{1}{2h}\int_\gamma \Big(\lambda_1^{n+1}\big(\Phi_1(\chi)-k(b-h)\big) + \lambda_2^{n+1}\big(\Phi_2(\chi)-k(b+h)\big)\Big)d\gamma \tag{5.14}$$

for all $\chi\in H(\Omega)$, and for every $i=1,2$ (there is no summation over i):

$$\begin{aligned}&C'_\Phi(\Phi_i(\xi^n),\lambda_i^n)\big(\Phi_i(\xi^{n+1})-\Phi_i(\xi^n)\big) = -C(\Phi_i(\xi^n),\lambda_i^n),\\ &C'_\lambda(\Phi_i(\xi^n),\lambda_i^n)(\lambda_i^{n+1}-\lambda_i^n) = -C(\Phi_i(\xi^n),\lambda_i^n)\quad \textit{on } \gamma.\end{aligned} \tag{5.15}$$

It possesses the super-linear estimate

$$\|\xi^{n+1}-\xi\|_{H^1(\Omega)^5} + \|\Lambda^{n+1}-\Lambda\|_{L^2(\gamma)^2} = o\big(\|\delta^n\|_{L^{2+\epsilon}(\gamma)^2}\big), \tag{5.16}$$

where the increment:

$$\delta^n = (\delta_1^n,\delta_2^n),\quad \delta_i^n := \Phi_i(\xi^n)-\Phi_i(\xi)+r(\lambda_i^n-\lambda_i),\quad i=1,2. \tag{5.17}$$

The Newton iterate (5.15) *on the primal-dual active and inactive sets yields the linear equations:*

$$\Phi_i(\xi^{n+1}) = 0 \quad \text{on } \mathcal{I}(\Phi_i(\xi^n), \lambda_i^n), \quad \lambda_i^{n+1} = 0 \quad \text{on } \mathcal{A}(\Phi_i(\xi^n), \lambda_i^n), \quad i = 1, 2. \tag{5.18}$$

Proof. Subtracting $\Phi_i(\xi)$ and λ_i from the respective equations in (5.15), adding $C(\Phi_i(\xi), \lambda_i) = 0$, using $C'_\Phi(\Phi_i(\xi^n), \lambda_i^n) = 1$ on $\mathcal{I}(\Phi_i(\xi^n), \lambda_i^n)$, and $C'_\lambda(\Phi_i(\xi^n), \lambda_i^n) = -r$ on $\mathcal{A}(\Phi_i(\xi^n), \lambda_i^n)$ according to (5.6), we get on the inactive set $\mathcal{I}(\Phi_i(\xi^n), \lambda_i^n)$:

$$\Phi_i(\xi^{n+1}) - \Phi_i(\xi) = -\big[C(\Phi_i(\xi^n), \lambda_i^n) - C(\Phi_i(\xi), \lambda_i) - C'_\Phi(\Phi_i(\xi^n), \lambda_i^n)\big(\Phi_i(\xi^n) - \Phi_i(\xi)\big)\big],$$

and on the active set $\mathcal{A}(\Phi_i(\xi^n), \lambda_i^n)$:

$$\lambda_i^{n+1} - \lambda_i = \frac{1}{r}\big[C(\Phi_i(\xi^n), \lambda_i^n) - C(\Phi_i(\xi), \lambda_i) - C'_\lambda(\Phi_i(\xi^n), \lambda_i^n)(\lambda_i^n - \lambda_i)\big].$$

Applying to the right-hand side the estimate (5.9) with $\chi = \xi^n$ and $M = \lambda^n$ in (5.10) follows that

$$\|\Phi_i(\xi^{n+1}) - \Phi_i(\xi)\|_{L^2(\mathcal{I}(\Phi_i(\xi^n), \lambda_i^n))} + \|\lambda_i^{n+1} - \lambda_i\|_{L^2(\mathcal{A}(\Phi_i(\xi^n), \lambda_i^n))} = o\big(\|\delta_i^n\|_{L^{2+\epsilon}(\gamma)}\big), \tag{5.19}$$

where δ_i^n is defined in (5.17).

Next we subtract the variational equation (5.2) from (5.14) such that

$$B(\xi^{n+1} - \xi, \chi) = \frac{1}{2h} \int_\gamma \Big((\lambda_1^{n+1} - \lambda_1)\big(\Phi_1(\chi) - k(b-h)\big) + (\lambda_2^{n+1} - \lambda_2)\big(\Phi_2(\chi) - k(b+h)\big) \Big) d\gamma$$

for all $\chi \in H(\Omega)$. This equation allows to estimate the increment of the Lagrange multipliers $\Lambda^{n+1} - \Lambda$ on γ in the dual norm, using the trace theorem and boundedness of B, as

$$\|\Lambda^{n+1} - \Lambda\|_{L^2(\gamma)^2} \le c_4 \|\xi^{n+1} - \xi\|_{H^1(\Omega)^5}, \quad c_4 > 0. \tag{5.20}$$

Further testing it with $\chi = \xi^{n+1} - \xi$ and decomposing into active and inactive sets we have

$$B(\xi^{n+1} - \xi, \xi^{n+1} - \xi) = \frac{1}{2h} \int_{\mathcal{A}(\Phi_i(\xi^n), \lambda_i^n) \cup \mathcal{I}(\Phi_i(\xi^n), \lambda_i^n)} (\lambda_i^{n+1} - \lambda_i)\big(\Phi_i(\xi^{n+1}) - \Phi_i(\xi)\big)\, d\gamma. \tag{5.21}$$

Applying to (5.21) weighted Young's inequality, (3.3), (5.19) and (5.20) proves the estimate (5.16). Inserting (5.6)–(5.8) into the Newton iterate (5.15) yields (5.18) and finishes the proof. □

Provided by the estimate (5.16) in Theorem 4.1, after discretization of the primal-dual iterative problem (5.14) and (5.18) it follows the super-linear convergence in finite-dimensional spaces, when the initialization (ξ^0, Λ^0) is sufficiently close to the solution (ξ, Λ) with small δ^0 in (5.17).

6. Conclusion

We summarize our principal findings in the papers. The variational inequality describing Timoshenko plates that may come into contact by the side surface with inclined obstacles is studied. We start with the constrained minimization problem (2.10) over the set of admissible functions K, defined according to the newly proposed non-penetration condition (2.4). The existence and uniqueness of solution, and its optimality condition in the form of the variational inequality (3.7) is established. For the smooth solution, it yields the equilibrium equations (4.5)–(4.7), and mixed boundary conditions of the equality type (4.8)–(4.9) and the complementarity relations (4.10)–(4.11). The corresponding primal-dual variational formulation (5.2) and (5.3) of the obstacle problem is derived. Using a slant derivative for the minimum function, the semi-smooth Newton method is constructed and performed as the primal-dual active-set algorithm (5.14) and (5.18). The super-linear estimate for the Newton iterates is proven in function spaces.

Acknowledgments

The results of §5 were obtained with the support of the Ministry of Science and Higher Education of the Russian Federation (Grant No. FSRG-2023-0025). The results of §1–§4 were obtained with the support of the Ministry of Science and Higher Education of the Russian Federation, supplementary agreement no. 075-02-2023-947, 16 February 2023.

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