Variational and Hemivariational Inequalities in Mechanics of Elastoplastic, Granular Media, and Quasibrittle Cracks

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Abstract This contribution is devoted to the mathematical theory of elastoplastic and granular solids as well as the quasibrittle fracture of nonlinear cracks. Basic variational and hemivariational inequalities describing nonlinear phenomena due to plasticity, internal friction, interfacial interaction, and alike dissipative physics are outlined from the point of view of nonsmooth and nonconvex optimization. Primary results of the nonlinear theory and its application to solid mechanics are surveyed.

Key words: plasticity, granular solid, quasibrittle crack, (hemi)variational inequality, set valued - constrained - nonsmooth - nonconvex optimization

1 Introduction

The mathematical theory of elastoplastic and granular solids as well as their fracture is originated in the engineering sciences related to materials, geophysics, and biophysics. As it is marked in the literature, modern materials developed in the recent past exhibit essentially nonlinear properties. In particular, when the materials are undergoing by critical deformations. This motivates the actual research of nonlin-

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ear phenomena due to plasticity, internal friction, interfacial interaction, and other dissipative physics. For its mathematical modeling, variational and hemivariational inequalities are suitable well. In the present contribution we survey the principal issues of modeling in this respect.

Variational inequalities were applied to the problems of mechanics yet in 19th century by V.M. Ostrogradsky for the description of constrained motion of a material point. Significant development of the methods using inequalities in mechanics was contributed recently by (alphabetically) B.D. Annin, G. Fichera, A. Haar, J. Haslinger, T. Karman, A.S. Kravchuk, J.-L. Lions, P.D. Panagiotopoulos, A. Signorini, R. Temam, and others.

From the mathematical point of view, variational and hemivariational inequalities appear in the governing relations as the consequence of fundamental thermodynamics principles subject to one sided restrictions. In fact, inequality constraints imposed on geometric displacements lead to contact conditions, virtual stress that does not exceed the yield limit implies plasticity, while the material strength is expressed by restrictions on strains.

In the following sections we outline, respectively, the modeling of governing inequalities for elastoplastic and granular media, and in the theory of quasibrittle cracks.

2 Variational inequalities in elastoplastic theory

We start with the notation. For a reference solid occupying the domain Ω, spatial points $x \in \Omega$, and time $t \geq 0$, we refer the displacement vector $u(t,x)$, the strain $\varepsilon(u)$ which relies on the symmetric tensor $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^\top u)$ of linear deformations, and the stress tensor σ. In elastoplasticity, there is assumed a plastic strain ε^p.

Within the *Hencky law* it holds (see the books [1, 13]) the constitutive equation

$$\varepsilon(u) = \frac{\partial W}{\partial \sigma} + \varepsilon^p \tag{1}$$

supported with the following variational inequality

$$f(\sigma) \leq 0, \quad (\bar{\sigma} - \sigma) : \varepsilon^p \leq 0 \quad \text{for all } \bar{\sigma}\text{: } f(\bar{\sigma}) \leq 0. \tag{2}$$

In (1) the notation $W(\sigma)$ stands for the strain energy potential. In linear elasticity it is quadratic, $W(\sigma) = \frac{1}{2}\sigma : A : \sigma$ with the symmetric compliance tensor A, hence $\frac{\partial W}{\partial \sigma} = A : \sigma$. Inequalities (2) imply that the true stress σ lies inside the given yield surface $f(\sigma) \leq 0$, and the plastic deformation ε^p is orthogonal to this surface. Typical yield surfaces are the Tresca and von Mises ones. In the simplest case of scalar σ the yield surface is determined by $f(\sigma) = |\sigma| - \sigma^0$ with the yield limit σ^0. Together with the (quasi)static equilibrium equation

$$-\nabla \cdot \sigma = F \tag{3}$$

where F is the external force, governing relations (1)–(3) form a complete system.

Note that, if the one sided constraint $f(\sigma) \leq 0$ was skipped, then inequalities (2) would turn into the equality $\varepsilon^p = 0$. In this case, there is no plastic deformations, and governing relations (1)–(3) correspond to common (generally, nonlinear) elasticity.

From the nonlinear optimization viewpoint, the variational inequality (2) implies that ε^p is a (nonunique) element of the Clarke subdifferential

$$\varepsilon^p \in \underline{\partial}\chi_K(\sigma) := \{\varepsilon : \chi_K(\bar{\sigma}) - \chi_K(\sigma) \geq (\bar{\sigma} - \sigma) : \varepsilon \text{ for all } \bar{\sigma}\}. \tag{4}$$

In (4) the nonsmooth potential χ_K implies the indicator function of the convex set

$$K = \{\bar{\sigma} : f(\bar{\sigma}) \leq 0\} \tag{5}$$

that is $\chi_K(\sigma) = 0$ for $\sigma \in K$ if $f(\sigma) \leq 0$, otherwise $\chi_K(\sigma) = +\infty$. Details of this formalism are presented in [29] and endowed with dual arguments using the Legendre–Fenchel–Young transformation.

The *Prandtl–Reuss law* provides the following flow model (see [4, 30]):

$$\varepsilon(v) = \tfrac{\partial}{\partial t}\big(\tfrac{\partial W}{\partial \sigma}\big) + e^p \tag{6}$$

with the velocities $v := \dot{u}$ and $e^p := \dot{\varepsilon}^p$, the variational inequality

$$f(\sigma) \leq 0, \quad (\bar{\sigma} - \sigma) : e^p \leq 0 \quad \text{for all } \bar{\sigma}\text{: } f(\bar{\sigma}) \leq 0, \tag{7}$$

and the dynamic equation of motion

$$\rho \dot{v} - \nabla \cdot \sigma = F. \tag{8}$$

Following the approach of J. Mandel, discontinuous solutions to the dynamic elastoplastic problem (6)–(8), which are of the shock wave type, are analyzed in the monograph [30]. In this case, as far as in more general case of hardening materials, the system of governing equations can be written in the unified form

$$\Big(\tfrac{\partial \varphi(U)}{\partial t} - \sum_{k=1}^{n} \tfrac{\partial \psi_k(U)}{\partial x_k}\Big) \cdot (\bar{U} - U) \geq 0 \tag{9}$$

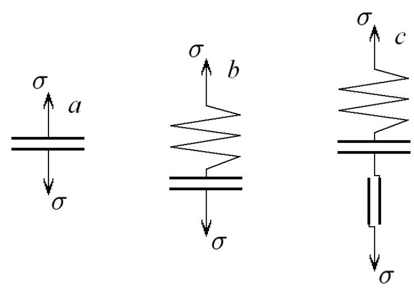
where n is the spacial dimension. In (9), the true variable U and its variation $\bar{U}$ are admissible vectors composing all unknown functions: velocities, stresses, and parameters of hardening. The vector functions $\varphi = \frac{\partial \Phi}{\partial U}$ and $\psi_k = \frac{\partial \Psi_k}{\partial U}$ are expressed in the terms of scalar generating potentials $\Phi(U)$ and $\Psi_k(U)$, $k = 1, \ldots, n$, which are quadratic functions in the case of physically linear processes.

The formulation in form (9), in particular, is numerically advantageous for the construction of algorithms of the Wilkins type, see [4]. This theoretical result is strengthened by the engineering applications presented in [2, 3].

3 Variational inequalities in theory of granular media

Granular solids exhibit complex behavior, in particular, different resistance under compression and tension. To take it into account, the classical rheological method is supplemented by a new element, namely, the rigid contact which can be expressed schematically as two plates being in contact. Combining this element with the traditional rheological elements: the elastic spring, viscous dashpot, and plastic hinge, models with a suitable level of complexity can be derived (see the examples in Fig. 1).

Fig. 1 Rheological schemes of granular material with rigid (*a*), elastic (*b*) and elastic-plastic (*c*) particles



The constitutive relationships expressing the rigid contact can be represented as the linear complementarity problem

$$\sigma \le 0, \quad \varepsilon \ge 0, \quad \sigma\varepsilon = 0 \tag{10}$$

for the scalar stress σ and strain ε. Indeed, the inequalities in (10) exclude arising tensile stresses and compressive strains in a perfect granular material composed of rigid particles. From the complementarity condition it follows that one of the quantities being considered (either stress or strain) must be zero. Therefore, (10) can be reduced to two variational inequalities:

$$\sigma \le 0, \quad (\sigma - \bar{\sigma})\varepsilon \le 0 \quad \text{for all } \bar{\sigma} \le 0, \tag{11}$$

$$\varepsilon \ge 0, \quad \sigma(\bar{\varepsilon} - \varepsilon) \le 0 \quad \text{for all } \bar{\varepsilon} \ge 0 \tag{12}$$

which are equivalent. This consideration admits extension to more complicated rheological models and higher spacial dimensions.

Within the rheological approach, phenomenological models of granular solids are generalized in the book [29]. In [28] the model of granular materials under finite strains is considered. The generic model of materials with different compressive and tensile strengths is analyzed in [23] where the modeling result is supported with the existence theorems, analysis of mechanical properties, and estimation of the critical equilibrium.

4 Hemivariational inequalities in nonlinear theory of cracks

Macro and micro cracks appear in a wide range of real world applications related to fracture in the material science, faults in the earth causing earthquakes and subsequent tsunami, modern biomedical methodologies, and alike. The actual problems of tribology and fracture need nonlinear modeling of cracking phenomena taking into account for dissipative interaction between the crack faces. This results in quasibrittle models of fracture in contrast to classic brittle hypothesis due to Griffith–Irwin–Rice.

The fundamentals of quasibrittle and dynamic fracture were developed by (alphabetically) G.I. Barenblatt, G.P. Cherepanov, S.A. Christianovich, D.S. Dugdale, R.V. Goldstein, M.Ya. Leonov, N.F. Morozov, V.V. Novozhilov, V.V. Panasyuk, Yu.N. Rabotnov, L. Truskinovsky, and others. From a physical point of view, the dissipative work of interaction phenomena due to contact with cohesion or friction at the crack is closely related to elastoplastic physics. From a mathematical viewpoint, its modeling results in hemivariational inequalities within set valued and nonconvex optimization context.

The basics of mathematical theory describing quasibrittle cracks are outlined below by following the results obtained in [11, 17, 19, 20]. In the nonlinear optimization framework, we suggest a class of hemivariational inequalities introduced as follows.

Let $\Gamma \subset \Omega$ be an interface. In the equilibrium equation (3), the total stress σ distribution admits the following representation (compare with (1)):

$$\sigma = \tfrac{\partial W}{\partial \varepsilon} + F^i \chi_\Gamma \tag{13}$$

with the stress energy potential $W(\varepsilon)$ and the interfacial traction F^i. It is added in Ω with the help of characteristic function χ_Γ of the interface Γ. In linear elasticity W is quadratic, $W(\varepsilon) = \frac{1}{2}\varepsilon : C : \varepsilon$ with the symmetric tensor C of elastic stiffness, hence $\frac{\partial W}{\partial \varepsilon} = C : \varepsilon$ in (13).

At Γ we suggest complementary contact conditions (compare with (10)):

$$\mathrm{tr}_\Gamma(u) \geq 0, \quad F^c \geq 0, \quad F^c\, \mathrm{tr}_\Gamma(u) = 0 \tag{14}$$

where the contact force F^c admits, generally, the decomposition as

$$F^c = -F^i + F^d. \tag{15}$$

In (15) the dissipative force F^d represents irreversible work caused by cohesion as well as friction at the interface Γ. In the context of cracks, F^d describes interaction force between two crack faces being in contact, and $\mathrm{tr}_\Gamma(u)$ implies the jump of the normal traces of u across the crack. For more issues of the modeling of nonpenetration conditions $\mathrm{tr}_\Gamma(u) \geq 0$, see [13, 16, 18].

With a generating potential g, which is typically a concave function, the cohesion force F^d in (15) can be expressed as a (nonunique) element of the superdifferential

$$F^d \in \overline{\partial} g(u) := \{F : g(\mathrm{tr}_\Gamma(\bar{u})) - g(\mathrm{tr}_\Gamma(u)) \le F\,\mathrm{tr}_\Gamma(\bar{u}-u) \text{ for all } \bar{u}\}. \tag{16}$$

Examples of the generating potential g are given in [17]. For the reference setup

$$g(\mathrm{tr}_\Gamma(u)) = \tfrac{\sigma^0}{l}\min\bigl(l, \mathrm{tr}_\Gamma(u)\bigr)$$

with the yield limit σ^0 and the characteristic length l.

After integration over the domain $\Omega \setminus \Gamma$, relations (13)–(15) can be summarized as the hemivariational inequality

$$\mathrm{tr}_\Gamma(u) \ge 0, \quad \int_{\Omega\setminus\Gamma} \bigl(\tfrac{\partial W}{\partial \varepsilon} - \sigma\bigr) : \varepsilon(\bar{u}-u)\,dx + \int_\Gamma F^d\,\mathrm{tr}_\Gamma(\bar{u}-u)\,dS_x \ge 0 \tag{17}$$
$$\text{for all } \bar{u}\text{: } \mathrm{tr}_\Gamma(\bar{u}) \ge 0.$$

In fact, inclusion (16) argues that (17) is the necessary optimality condition for minimization over admissible $\bar{u}$ of the nonsmooth functional of energy

$$\text{minimize}\Bigl\{\int_{\Omega\setminus\Gamma} (W-\sigma) : \varepsilon(\bar{u})\,dx + \int_\Gamma g(\mathrm{tr}_\Gamma(\bar{u}))\,dS_x\Bigr\} \text{ subject to } \mathrm{tr}_\Gamma(\bar{u}) \ge 0. \tag{18}$$

Moreover, the energy functional is nonconvex since g is concave.

The issues of nonsmoothness and nonconvexity are the principal difficulties for analysis of the constrained minimization problem (18) which is presented in the cited works.

While the hemivariational inequality (17) is necessary to (18), its sufficient optimality condition implies a saddle point minimax problem with respect to the pair of the primal variable u and the dual variable F^c (the Lagrange multiplier) associated to the constraint $\mathrm{tr}_\Gamma(u) \ge 0$ in accordance with (14). The saddle point problem reads

$$\min_u \max_{F^c}\Bigl\{\int_{\Omega\setminus\Gamma} (W-\sigma) : \varepsilon(\bar{u})\,dx + \int_\Gamma g(\mathrm{tr}_\Gamma(\bar{u}))\,dS_x - \int_\Gamma F^c\,\mathrm{tr}_\Gamma(\bar{u})\,dS_x\Bigr\} \tag{19}$$
$$\text{subject to } F^c \ge 0.$$

For nonunique solutions, the sufficient and necessary conditions do not coincide with each other. This fact is in contrast to the case of convex minimization.

For the numerical solution of (19), hence (18) and (17), a primal dual active set based (PDAS) strategy is suggested in [11, 19]. The PDAS strategy is associated to generalized Newton methods obeying locally superlinear as well as globally monotone convergence properties.

5 Conclusion

Here we outline the further development of the subject directed to variational modeling of fractional, damage, and geometrically singular phenomena in mechanics.

Starting with frictional contact due to the Coulomb law [7, 12], typically, tangential components of the shear are subject to restriction. Its generalization is developed in [6] for non-monotone friction laws, and further in [21] for a cohesive-frictional interaction restricting both the tangential as well as the normal shear components. The resulting hemivariational inequalities are argued as pseudo-monotone variational inequalities by the authors of [25].

The actual task concerns singular geometries, see [10, 14], arising in practice. Motivated by fracture of composites (and used also in inverse problems for stratified media [24]), geometrically heterogeneous models with nonlinear inclusions subject to cracks and anti-cracks were developed in [15]. This study aims at the shape-topological control to force either shielding or amplification of an incipient cracking. For the variational analysis Γ-convergence techniques are useful [26].

The other development consists in constituting variational models of damaged elastic, elastoplastic, and cracked materials. The damaged models are treated by using Γ-limits in [8] and within hysteresis formalism and rate-independent systems in [22].

In respect to numerical theory of the underlying optimization problems, we refer to [5] for saddle-point algorithms within nonconvex programming, to [9] for globalization strategies, and to [27] for parametric and dynamic optimization.

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