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Long-time behaviour of a porous medium model with degenerate hysteresis

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Hysteresis in the pressure-saturation relation in unsaturated porous media, owing to surface tension on the liquid–gas interface, exhibits strong degeneracy in the resulting mass balance equation. As an extension of previous existence and uniqueness results, we prove that under physically admissible initial conditions and without mass exchange with the exterior, the unique global solution of the fluid diffusion problem exists and asymptotically converges as time tends to infinity to a possibly non-homogeneous mass distribution and an *a priori* unknown constant pressure.

This article is part of the theme issue 'Non-smooth variational problems with applications in mechanics'.

1. Introduction

This article deals with the problem of existence, uniqueness and long-time stabilization of the solution to the degenerate PDE with hysteresis in a bounded space domain $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$ of class $C^{1,1}$

$$s_t - \Delta u = 0 \quad \text{for } (x,t) \in \Omega \times (0,\infty), \tag{1.1}$$

where $t > 0$ is the time variable, $u = u(x,t) \in \mathbb{R}$ represents the pressure, $s = s(x,t) \in (0,1)$ is the relative saturation of the fluid in the pores and Δ is the Laplacian in x. Hysteresis in the pressure-saturation relation is represented by a Preisach operator G in the form

$$s(x, t) = G[u](x, t)\,. \tag{1.2}$$

A detailed justification of why it is meaningful to consider Preisach hysteresis in porous media modelling can be found in Visintin [1] or in the introduction of Gavioli & Krejčí [2].

System equations (1.1) and (1.2) are considered with the boundary condition

$$-\nabla u(x, t) \cdot n = 0 \tag{1.3}$$

on $\partial\Omega$, and with a given initial condition that includes not only the initial pressure

$$u(x, 0) = u_0(x), \tag{1.4}$$

but also an initial Preisach memory distribution specified below in a rigorous formulation of the Preisach operator in definition 1.1. Let us only mention at this point that the time evolution described by equation (1.1) is doubly degenerate: in typical situations, the function $s(x, t) = G[u](x, t)$ is bounded independently of the evolution of u, so that no *a priori* lower bound for u_t is immediately available. Furthermore, at every point $x \in \Omega$ and every time t_0 where u_t changes sign (which the engineers call a *turning point*), we have

$$u_t(x, t_0 - \delta) \cdot u_t(x, t_0 + \delta) < 0 \;\; \forall \delta \in (0, \delta_0(x)) \implies \liminf_{\delta \to 0+} \frac{G[u]_t(x, t_0 + \delta)}{u_t(x, t_0 + \delta)} = 0, \tag{1.5}$$

so that the knowledge of $G[u]_t$ alone does not give complete information about u_t. In particular, if $\Delta u_0(x) \neq 0$ and the initial memory of G corresponds to a turning point, then even a local solution cannot be expected to exist. Hypothesis 1.4 is shown to avoid this pathological situation. A more detailed discussion about this issue can be found in the recent paper [2]. There, under suitable hypotheses on the data, the existence and uniqueness of a solution to equation (1.1) on an arbitrary time interval $t \in (0, T)$ has been proved in the case of Robin boundary conditions. Crucial assumptions to obtain the existence result are that the operator G is a so-called convexifiable Preisach operator, and its initial memory state is compatible with the initial condition u_0 in equation (1.4) (see §1). The present autonomous case with homogeneous Neumann boundary conditions makes it possible to prove a stronger existence and uniqueness statement (see the first paragraph after theorem 1.5) under the same hypotheses on the data, as well as asymptotic convergence to an *a priori* unknown equilibrium as time tends to infinity.

The structure of this paper is as follows: in §1, we list the definitions of the main concepts, including convexifiable Preisach operators, and state the main theorem 1.5. In §2, we propose a time discretization scheme with time step $\tau > 0$, and in §§3–5, we derive estimates independent of τ. In particular, estimate (6.24) improves the corresponding convexity estimate obtained in Gavioli & Krejčí [2]. By a compactness argument, we pass to the limit as $\tau \to 0$ and prove in §6 that the limit is the unique solution to our PDE problem with hysteresis. The long-time stabilization result is proved in §7.

2. Statement of the problem

The Preisach operator was originally introduced in Preisach [3]. For our purposes, it is more convenient to use the equivalent variational setting from Krejčí [4].

Definition 1.1. Let $\lambda \in L^\infty(\Omega \times (0, \infty))$ be a given function which we call the *initial memory distribution* and which has the following properties:

$$|\lambda(x, r_1) - \lambda(x, r_2)| \le |r_1 - r_2| \quad \text{a. e. } \forall r_1, r_2 \in (0, \infty), \tag{2.1}$$

$$\exists \Lambda > 0\colon \lambda(x, r) = 0 \;\text{ for } r \ge \Lambda \text{ and a. e. } x \in \Omega\,. \tag{2.2}$$

For a given $r > 0$, we call the play operator with threshold r and initial memory λ the mapping, which, with a given function $u \in L^p(\Omega; W^{1,1}(0,T))$ for $p \geq 1$, associates the solution $\xi^r \in L^p(\Omega; W^{1,1}(0,T))$ of the variational inequality

$$\left|u(x,t) - \xi^r(x,t)\right| \leq r, \quad \xi^r_t(x,t)(u(x,t) - \xi^r(x,t) - z) \geq 0 \text{ a. e. } \forall z \in [-r, r], \tag{2.3}$$

with a given initial memory distribution

$$\xi^r(x,0) = \lambda(x,r), \tag{2.4}$$

and we denote

$$\xi^r(x,t) = \mathfrak{p}_r[\lambda, u](x,t)\,. \tag{2.5}$$

Given a measurable function $\rho: \Omega \times (0,\infty) \times \mathbb{R} \to [0,\infty)$ and a constant $\bar{G} \in \mathbb{R}$, the Preisach operator G is defined as a mapping $G: L^p(\Omega; W^{1,1}(0,T)) \to L^p(\Omega; W^{1,1}(0,T))$ by the formula

$$G[u](x,t) = \bar{G} + \int_0^\infty \int_0^{\xi^r(x,t)} \rho(x,r,v)\,\mathrm{d}v\,\mathrm{d}r\,. \tag{2.6}$$

The Preisach operator is said to be *regular* if the density function ρ of G in equation (2.6) belongs to $L^1(\Omega \times (0,\infty) \times \mathbb{R}) \cap L^\infty(\Omega \times (0,\infty) \times \mathbb{R})$, and there exists a constant ρ_1 and a decreasing function $\rho_0: \mathbb{R} \to \mathbb{R}$ such that for all $U > 0$ we have

$$\rho_1 > \rho(x,r,v) > \rho_0(U) > 0 \text{ for a. e. } (x,r,v) \in \Omega \times (0,U) \times (-U,U)\,. \tag{2.7}$$

Let us mention the following classical result (see proposition II.3.11 in Krejčí [4]).

Proposition 1.2. *Let G be a regular Preisach operator in the sense of definition 1.1. Then it can be extended to a Lipschitz continuous mapping $G: L^p(\Omega; C[0,T]) \to L^p(\Omega; C[0,T])$ for every $p \in [1,\infty)$.*

The Preisach operator is rate-independent. Hence, for input functions $u(x,t)$ which are monotone in a time interval $t \in (a,b)$, a regular Preisach operator G can be represented by a superposition operator $G[u](x,t) = B(x, u(x,t))$ with an increasing function $u \mapsto B(x,u)$ called a *Preisach branch*. Indeed, the branches may be different at different points x and different intervals (a,b). The branches corresponding to increasing inputs are said to be *ascending* (the so-called wetting curves in the context of porous media), and the branches corresponding to decreasing inputs are said to be *descending* (drying curves).

Definition 1.3. Let $U > 0$ be given. A Preisach operator is said to be *uniformly counterclockwise convex on* $[-U,U]$ if for all inputs u such that $|u(x,t)| \leq U$ a.e., all ascending branches are uniformly convex and all descending branches are uniformly concave.

A regular Preisach operator is called *convexifiable* if for every $U > 0$ there exists a uniformly counterclockwise convex Preisach operator P on $[-U,U]$, positive constants $g_*(U), g^*(U), \bar{g}(U)$ and a twice continuously differentiable mapping $g: [-U,U] \to [-U,U]$ such that

$$g(0) = 0, \ \ 0 < g_*(U) \leq g'(u) \leq g^*(U), \ \ |g''(u)| \leq \bar{g}(U) \quad \forall u \in [-U,U], \tag{2.8}$$

and $G = P \circ g$.

A typical example of a uniformly counterclockwise convex operator is the so-called *Prandtl–Ishlinskii operator* characterized by positive density functions $\rho(x,r)$ independent of v (see Krejčí [4, §4.2]). Operators of the form $P \circ g$ with a Prandtl–Ishlinskii operator P and an increasing function g are often used in control engineering because of their explicit inversion formulas (see [5–7]). They are called the *generalized Prandtl–Ishlinskii operators* and represent an important subclass of Preisach operators. Note also that for every Preisach operator P and every Lipschitz continuous increasing function g, the superposition operator $G = P \circ g$ is also a Preisach operator, and there exists an explicit formula for its density (see proposition 2.3 in Krejčí [8]).

Another class of convexifiable Preisach operators is shown in proposition 1.3 of Gavioli & Krejčí [2].

As it has been mentioned in §1, even a local solution to equations (1.1)–(1.4) may fail to exist if for example $\lambda(x,r) \equiv 0$ and $\Delta u_0(x) \neq 0$. Then $t = 0$ is a turning point for all $x \in \Omega$, and there is no way to satisfy equation (1.1) in any sense. We therefore need an initial memory compatibility condition, which we state in the following way.

Hypothesis 1.4. The initial pressure u_0 belongs to $W^{2,2}(\Omega)$, $\Delta u_0 \in L^\infty(\Omega)$, and there exist constants $L > 0$, $\Lambda > 0$ and a function $r_0 \in L^\infty(\Omega)$ such that $\sup\operatorname{ess}_{x\in\Omega} |u_0(x)| \leq \Lambda$, equation (2.2) is satisfied and the following initial compatibility conditions hold:

$$\lambda(x,0) = u_0(x) \text{ a. e. in } \Omega, \tag{2.9}$$

$$\frac{1}{L}\sqrt{|\Delta u_0(x)|} \leq r_0(x) \leq \Lambda \text{ a. e. in } \Omega, \tag{2.10}$$

$$-\frac{\partial}{\partial r}\lambda(x,r) \in \operatorname{sign}(\Delta u_0(x)) \text{ a. e. in } \Omega \text{ for } r \in (0, r_0(x)), \tag{2.11}$$

$$-\nabla u_0(x)\cdot n = 0 \ \text{ a. e. on } \partial\Omega\,. \tag{2.12}$$

It was shown in proposition 2.2.16 of Brokate & Sprekels [9] that the solution to inequality (2.3) is such that for every $u \in L^p(\Omega; W^{1,1}(0,T))$ the property

$$\left|\xi^{r_1}(x,t) - \xi^{r_2}(x,t)\right| \leq \left|r_1 - r_2\right| \ \text{ a. e. } \forall r_1, r_2 \in (0,\infty) \tag{2.13}$$

is preserved during evolution. From relations (2.1)–(2.4) it follows that for a.e. $(x,t) \in \Omega \times (0,\infty)$ there exists a so-called *active memory level* $r^*(x,t) \geq 0$ such that $\left|\xi^r(x,t) - u(x,t)\right| = r$ and $\xi^r_t(x,t) = u_t(x,t)$ for $r < r^*(x,t)$, $|\xi^r(x,t) - u(x,t)| < r$ and $\xi^r_t(x,t) = 0$ for $r > r^*(x,t)$. In this sense, the value r_0 in conditions (2.10) and (2.11) represents the active memory level at time $t = 0$. The meaning of conditions (2.9) and (2.11) is that for large values of $\Delta u_0(x)$, the initial memory λ has to go deeper in the memory direction. We refer to Gavioli & Krejčí [2] for an explanation of how hypothesis 1.4 guarantees the existence of some previous admissible history of the process prior to the time $t = 0$ and ensures the existence of a continuation for $t > 0$.

Problem (1.1)–(1.3) is to be understood in variational form

$$\int_\Omega G[u]_t \varphi \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla\varphi \, \mathrm{d}x = 0 \tag{2.14}$$

for every test function $\varphi \in W^{1,2}(\Omega)$. Existence and uniqueness of a solution to equations (1.1) and (1.2) with the Robin boundary condition and admissible initial conditions has been proved in Gavioli & Krejčí [2]. We now state the main result of this paper.

Theorem 1.5. *Let hypothesis 1.4 be satisfied, and let G be a convexifiable Preisach operator in the sense of definition 1.3. Then problem* (2.14) *with initial condition* $u(x,0) = u_0(x)$, $\mathfrak{p}_r[\lambda, u](x,0) = \lambda(x,r)$ *for all* $r > 0$ *admits a unique global solution* u *on* $\Omega \times (0,\infty)$ *such that*

$$\begin{gathered} u \in L^\infty(\Omega\times(0,\infty)), \qquad |u(x,t)| \leq \Lambda \text{ a. e.}, \\ \nabla u \in L^\infty\big(0,\infty; L^2(\Omega;\mathbb{R}^N)\big) \cap L^2\big(\Omega\times(0,\infty);\mathbb{R}^N\big), \\ \Delta u, G_u \in L^2(\Omega\times(0,\infty)), \\ u_t \in L^Q(\Omega\times(0,\infty)), \end{gathered}$$

where $L^Q(\Omega \times (0,\infty))$ *is the Orlicz space generated by a function* $Q(z)$*, which behaves like* z^3 *for* $z \in (0,1)$ *and* z^p *with* $p = 1 + (2/N)$ *for* $z > 1$*; see equation* (6.20) *for* $N \geq 3$, equation (6.29) *for* $N = 2$ *and equation* (6.28) *for* $N = 1$. *Moreover, there exist a constant* $\bar{u} \in \mathbb{R}$

such that $\lim_{t\to\infty}\int_\Omega \left|u(x,t)-\bar u\right|^q \,\mathrm{d}x = 0$ *for all* $q \ge 1$, *and a function* $\bar\lambda \in L^\infty(\Omega\times[0,\infty))$ *such that* $\bar\lambda(x,0)=\bar u$, $\left|\bar\lambda(x,r)-\bar\lambda(x,s)\right| \le r-s$ *for a.e.* $x\in\Omega$ *and all* $r>s\ge 0$, *and such that* $\lim_{t\to\infty}\int_\Omega \left|\mathfrak{p}_r[\lambda,u](x,t)-\bar\lambda(x,r)\right|^q \,\mathrm{d}x = 0$ *for all* $q\ge 1$ *and* $r \ge 0$.

Note that the regularity of u_t in Gavioli & Krejčí [2] is only $L^{p-\varepsilon}$ for each $\varepsilon>0$ and on bounded time intervals. We propose here a refined estimation technique that allows us to derive a global in time Orlicz bound for u_t (for the theory of Orlicz spaces, refer to [10]) and full L^p bound on bounded time intervals.

Putting $\varphi=1$ in equation (2.14), we formally get the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s(x,t)\,\mathrm{d}x = 0,$$

which means that the total liquid mass is preserved during the evolution. The meaning of theorem 1.5 is that the pressure u is asymptotically uniformly distributed in Ω, but because of hysteresis, we cannot expect that the spatial distribution of the liquid mass will also be asymptotically uniform and that the constant limit pressure value $\bar u$ can be computed explicitly.

3. Time discretization

We proceed as in Gavioli & Krejčí [2], choose a sufficiently small time step $\tau>0$ and replace equation (2.14) with its time-discrete system for the unknowns $\left\{u_i : i\in\mathbb{N}\cup\{0\}\right\}\subset W^{1,2}(\Omega)$ of the form

$$\int_\Omega \left(\frac{1}{\tau}(G[u]_i - G[u]_{i-1})\varphi + \nabla u_i\cdot\nabla\varphi\right)\mathrm{d}x = 0 \tag{3.1}$$

for $i\in\mathbb{N}$ and for every test function $\varphi\in W^{1,2}(\Omega)$, where u_0 is the initial condition in equation (1.4). Here, the time-discrete Preisach operator $G[u]_i$ is defined by equation (2.6):

$$G[u]_i(x) = \bar G + \int_0^\infty\int_0^{\xi_i^r(x)} \rho(x,r,v)\,\mathrm{d}v\,\mathrm{d}r, \tag{3.2}$$

where ξ_i^r denotes the output of the time-discrete play operator

$$\xi_i^r(x) = \mathfrak{p}_r[\lambda,u]_i(x) \tag{3.3}$$

defined as the solution operator of the variational inequality

$$\left|u_i(x)-\xi_i^r(x)\right|\le r,\quad (\xi_i^r(x)-\xi_{i-1}^r(x))(u_i(x)-\xi_i^r(x)-z)\ge 0\quad \forall i\in\mathbb{N}\ \ \forall z\in[-r,r], \tag{3.4}$$

with a given initial memory curve

$$\xi_0^r(x) = \lambda(x,r)\ \text{ a. e.}, \tag{3.5}$$

in a similar manner to relations (2.3) and (2.4). Note that the discrete variational inequality (3.4) can be interpreted as weak formulation of inequality (2.3) for piecewise constant inputs in terms of the Kurzweil integral; details can be found in §2 of Eleuteri & Krejčí [11].

Arguing as in Gavioli & Krejčí [2, eqn. (35)], we obtain the two-sided estimate

$$\frac{1}{C}\left|G[u]_i(x)-G[u]_{i-1}(x)\right|^2 \le (u_i(x)-u_{i-1}(x))(G[u]_i(x)-G[u]_{i-1}(x)) \le C\left|u_i(x)-u_{i-1}(x)\right|^2, \tag{3.6}$$

with $C>0$ depending only on the constant ρ_1 from relation (2.7).

For each $i \in \mathbb{N}$, there is no hysteresis in the passage from u_{i-1} to u_i, so that equation (3.1) is a standard monotone semilinear elliptic equation that admits a unique solution $u_i \in W^{1,2}(\Omega)$ for every $i \in \mathbb{N} \cup \{0\}$.

4. Uniform upper bound

Following Hilpert [12], the idea is to test equation (3.1) by $\varphi = H_\varepsilon(u_i - \Lambda)$, with H_ε being a Lipschitz regularization of the Heaviside function $H(s) = 1$ for $s > 0$, $H(s) = 0$ for $s \leq 0$ and Λ from equation (2.2). We then let ε tend to 0. The elliptic term gives a non-negative contribution, and we get for all $i \in \mathbb{N}$ that

$$\int_\Omega (G[u]_i - G[u]_{i-1}) H(u_i - \Lambda)\, \mathrm{d}x \leq 0. \tag{4.1}$$

We define the functions

$$\psi(x, r, \xi) := \int_0^\xi \rho(x, r, v)\, \mathrm{d}v, \quad \Psi(x, r, \xi) := \int_0^\xi \psi(x, r, v)\, \mathrm{d}v\,. \tag{4.2}$$

In terms of the sequence $\xi_i^r(x) = \mathfrak{p}_r[\lambda, u]_i$, we have

$$G[u]_i(x) = \bar{G} + \int_0^\infty \psi(x, r, \xi_i^r(x))\, \mathrm{d}r\,. \tag{4.3}$$

Choosing $z = \Lambda - (\Lambda - r)^+ = \min\{\Lambda, r\}$ in inequality (3.4) and using the fact that ψ is an increasing function of ξ, we get for all $i \in \mathbb{N}$, all $r > 0$ and a.e. $x \in \Omega$ that

$$\big(\psi(x, r, \xi_i^r(x)) - \psi(x, r, \xi_{i-1}^r(x))\big)\big((u_i(x) - \Lambda) - (\xi_i^r(x) - (\Lambda - r)^+)\big) \geq 0.$$

The Heaviside function H is non-decreasing, hence,

$$\big(\psi(x, r, \xi_i^r(x)) - \psi(x, r, \xi_{i-1}^r(x))\big)\big(H(u_i(x) - \Lambda) - H(\xi_i^r(x) - (\Lambda - r)^+)\big) \geq 0.$$

From inequality (4.1), it follows that

$$\int_\Omega \int_0^\infty \big(\psi(x, r, \xi_i^r(x)) - \psi(x, r, \xi_{i-1}^r(x))\big) H(\xi_i^r(x) - (\Lambda - r)^+)\, \mathrm{d}r\, \mathrm{d}x \leq 0. \tag{4.4}$$

We have by relations (2.1) and (2.2) that $\xi_0^r(x) = \lambda(x, r) \leq (\Lambda - r)^+$ a.e. We now proceed by induction assuming that $\xi_{i-1}^r(x) \leq (\Lambda - r)^+$ a.e. for some $i \in \mathbb{N}$. By relation (4.4), we have

$$\int_\Omega \int_0^\infty \big(\psi(x, r, \xi_i^r(x)) - \psi(x, r, \xi_{i-1}^r(x))\big)\big(H(\xi_i^r(x) - (\Lambda - r)^+) - H(\xi_{i-1}^r(x) - (\Lambda - r)^+)\big)\, \mathrm{d}r\, \mathrm{d}x \leq 0. \tag{4.5}$$

The expression under the integral sign in relation (4.5) is non-negative almost everywhere; hence, it vanishes almost everywhere, and we conclude that $\xi_i^r(x) \leq (\Lambda - r)^+$ a.e. for all $r \geq 0$ and $i \in \mathbb{N}$. Similarly, putting $z = (\Lambda - r)^+ - \Lambda = -\min\{\Lambda, r\}$ in inequality (3.4), we get $\xi_i^r(x) \geq -(\Lambda - r)^+$, so that

$$|u_i(x)| \leq \Lambda, \quad |\xi_i^r(x)| \leq (\Lambda - r)^+ \tag{4.6}$$

for a.e. $x \in \Omega$ and all $r \geq 0$ and $i \in \mathbb{N}$. In particular, this implies that in equation (4.3), we can actually write

$$G[u]_i(x) = \bar{G} + \int_0^{\Lambda} \psi(x, r, \xi_i^r(x))\, dr\,. \tag{4.7}$$

5. Estimates of the pressure

We first test equation (3.1) by $\varphi = u_i - u_{i-1}$ and get

$$\frac{1}{\tau}\int_\Omega (G[u]_i - G[u]_{i-1})(u_i - u_{i-1})\, dx + \int_\Omega \nabla u_i \cdot \nabla(u_i - u_{i-1})\, dx = 0 \tag{5.1}$$

for all $i \in \mathbb{N}$. Using the elementary inequality $\nabla u_i \cdot \nabla(u_i - u_{i-1}) \geq \frac{1}{2}\left(|\nabla u_i|^2 - |\nabla u_{i-1}|^2\right)$ and putting $V_i = \frac{1}{2}\int_\Omega \left|\nabla u_i\right|^2 dx$, we obtain

$$\frac{1}{\tau}\int_\Omega (G[u]_i - G[u]_{i-1})(u_i - u_{i-1})\, dx + V_i - V_{i-1} \leq 0. \tag{5.2}$$

By comparison, in equation (3.1) (i.e. testing by a function $\varphi \in W^{1,2}(\Omega)$ with compact support in Ω and using the fact that such functions form a dense subset of $L^2(\Omega)$) and by using estimate (3.6), we get for all $i \in \mathbb{N}$ that

$$\int_\Omega |\Delta u_i|^2\, dx = \frac{1}{\tau^2}\int_\Omega |G[u]_i - G[u]_{i-1}|^2\, dx \leq \frac{c}{\tau^2}\int_\Omega (G[u]_i - G[u]_{i-1})(u_i - u_{i-1})\, dx, \tag{5.3}$$

with a constant $c > 0$ independent of i. Coming back to inequality (5.2), for all $i \in \mathbb{N}$, we thus have

$$\int_\Omega \left|\Delta u_i\right|^2 dx + \frac{c}{\tau}(V_i - V_{i-1}) \leq 0. \tag{5.4}$$

Consider the complete orthonormal basis $\{e_k : k \in \mathbb{N}\}$ in $L^2(\Omega)$ of eigenfunctions of the operator

$$-\Delta e_k = \mu_k e_k \ \text{ in } \Omega, \qquad -\nabla e_k \cdot n = 0 \text{ on } \partial\Omega, \tag{5.5}$$

with eigenvalues $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots$. The Fourier expansion of u_i in terms of the basis $\{e_k\}$ is of the form

$$u_i(x) = \sum_{k=1}^{\infty} u_k^i e_k(x) \tag{5.6}$$

for all $i \in \mathbb{N} \cup \{0\}$. Using the orthonormality of the system $\{e_k\}$, we rewrite inequality (5.4) in the form

$$\sum_{k=1}^{\infty} \mu_k^2 (u_k^i)^2 + \frac{c}{\tau}\sum_{k=1}^{\infty} \mu_k\left((u_k^i)^2 - (u_k^{i-1})^2\right) \leq 0. \tag{5.7}$$

The sequence

$$V_i = \int_\Omega \left|\nabla u_i\right|^2 dx = \sum_{k=1}^{\infty} \mu_k (u_k^i)^2$$

thus satisfies the inequality

$$\mu_1 V_i + \frac{c}{\tau}(V_i - V_{i-1}) \leq 0,$$

and we conclude that

$$V_i \leq V_0\left(1 + \tfrac{\mu_1 \tau}{c}\right)^{-i}, \tag{5.8}$$

that is, V_i decays exponentially as $i \to \infty$.

6. Convexity estimate

By relation (4.6), the functions u_i do not leave the interval $[-\Lambda, \Lambda]$. Since G is convexifiable in the sense of definition 1.3, there exist positive numbers $g_*, g^*, \bar{g}$ and a twice continuously differentiable mapping $g : [-\Lambda, \Lambda] \to [-\Lambda, \Lambda]$ such that $g(0) = 0$, $0 < g_* \leq g'(u) \leq g^* < \infty$, $|g''(u)| \leq \bar{g}$ for all $u \in [-\Lambda, \Lambda]$, and G is of the form

$$G = P \circ g, \tag{6.1}$$

where P is a uniformly counterclockwise convex Preisach operator on $[-\Lambda, \Lambda]$. The following result is a straightforward consequence of Gavioli & Krejčí [2, proposition 3.6].

Proposition 5.1. *Let P be uniformly counterclockwise convex on $[-\Lambda, \Lambda]$, and let f be an odd increasing function such that $f(0) = 0$. Then there exists $\beta > 0$ such that for every sequence $\{w_i : i \in \mathbb{N} \cup \{-1, 0\}\}$ in $[-\Lambda, \Lambda]$, we have*

$$\begin{aligned}\sum_{i=0}^{\infty} (P[w]_{i+1} - 2P[w]_i + P[w]_{i-1}) f(w_{i+1} - w_i) + \frac{P[w]_0 - P[w]_{-1}}{w_0 - w_{-1}} F(w_0 - w_{-1}) \\ \geq \frac{\beta}{2} \sum_{i=0}^{\infty} \Gamma(w_{i+1} - w_i),\end{aligned} \tag{6.2}$$

where we set for $w \in \mathbb{R}$

$$F(w) := \int_0^w f(v)\, \mathrm{d}v, \qquad \Gamma(w) := |w| (w f(w) - F(w)) = |w| \int_0^w v f'(v)\, \mathrm{d}v. \tag{6.3}$$

We need to define a backward step u_{-1} satisfying the strong formulation of equation (3.1) for $i = 0$, that is,

$$\frac{1}{\tau}(G[u]_0(x) - G[u]_{-1}(x)) = \Delta u_0(x) \ \text{ in } \Omega, \tag{6.4}$$

with homogeneous Neumann boundary conditions. Repeating the argument of Gavioli & Krejčí [2, proposition 3.3], we use hypotheses (2.7) and (2.10) to find for each $0 < \tau < \rho_0(\Lambda)/2L^2$ functions u_{-1} and $G[u]_{-1}$ satisfying equation (6.4) as well as the estimate

$$\tfrac{1}{\tau} |u_0(x) - u_{-1}(x)| \leq C \tag{6.5}$$

with a constant $C > 0$ independent of τ and x. The discrete equation (3.1) extended to $i = 0$ has the form

$$\int_\Omega \left(\frac{1}{\tau}(P[w]_i - P[w]_{i-1})\varphi + \nabla u_i \cdot \nabla \varphi\right) \mathrm{d}x = 0 \tag{6.6}$$

with $w_i = g(u_i)$, for $i \in \mathbb{N} \cup \{0\}$ and for an arbitrary test function $\varphi \in W^{1,2}(\Omega)$. We proceed as in Gavioli & Krejčí [2] and test the difference of equation (6.6) taken at discrete times $i + 1$ and i

$$\int_\Omega \left(\frac{1}{\tau}(P[w]_{i+1} - 2P[w]_i + P[w]_{i-1})\varphi + \nabla(u_{i+1} - u_i) \cdot \nabla \varphi\right) \mathrm{d}x = 0 \tag{6.7}$$

by $\varphi = \tau^\alpha f(w_{i+1} - w_i)$ with a suitably chosen odd increasing absolutely continuous function f and exponent $\alpha \in \mathbb{R}$, which will be specified later.

We estimate the initial time increment in the following way. By relations (2.8) and (3.6), we have

$$\begin{aligned}\tau^{\alpha-1}\frac{P[w]_0-P[w]_{-1}}{w_0-w_{-1}}F(w_0-w_{-1}) &\le \tau^{\alpha-1}\frac{|P[w]_0-P[w]_{-1}|}{g_*|u_0-u_{-1}|}F(w_0-w_{-1})\\ &\le C\tau^{\alpha-1}|F(w_0-w_{-1})| =: \widehat{F}_0(\tau).\end{aligned}\tag{6.8}$$

Using proposition 5.1, we have the inequality

$$\tau^{\alpha-1}\sum_{i=0}^{\infty}(P[w]_{i+1}-2P[w]_i+P[w]_{i-1})f(w_{i+1}-w_i) \ge \frac{\beta\tau^{\alpha-1}}{2}\sum_{i=0}^{\infty}\Gamma(w_{i+1}-w_i)-\widehat{F}_0(\tau)\tag{6.9}$$

with $\widehat{F}_0(\tau)$ defined in relation (6.8). From equation (6.7) and relation (6.9), we obtain

$$\frac{\beta\tau^{\alpha-1}}{2}\sum_{i=0}^{\infty}\int_\Omega \Gamma(w_{i+1}-w_i)\,\mathrm{d}x+\tau^\alpha\sum_{i=0}^{\infty}\int_\Omega \nabla(u_{i+1}-u_i)\cdot\nabla f(w_{i+1}-w_i)\,\mathrm{d}x \le \widehat{F}_0(\tau).\tag{6.10}$$

Note that

$$\begin{aligned}\nabla(u_{i+1}-u_i)\cdot\nabla f(w_{i+1}-w_i) &= f'(w_{i+1}-w_i)\nabla(u_{i+1}-u_i)\cdot\nabla(g(u_{i+1})-g(u_i))\\ &= f'(w_{i+1}-w_i)\Big(g'(u_{i+1})\,|\nabla(u_{i+1}-u_i)|^2+(g'(u_{i+1})-g'(u_i))\nabla(u_{i+1}-u_i)\cdot\nabla u_i\Big).\end{aligned}$$

The properties of g stated in definition 1.3 yield

$$\begin{aligned}&\nabla(u_{i+1}-u_i)\cdot\nabla f(w_{i+1}-w_i)\\ &\quad\ge f'(w_{i+1}-w_i)\Big(g_*|\nabla(u_{i+1}-u_i)|^2-\bar{g}\,|u_{i+1}-u_i|\;|\nabla(u_{i+1}-u_i)|\;|\nabla u_i|\Big)\\ &\quad\ge \frac{g_*}{2}f'(w_{i+1}-w_i)|\nabla(u_{i+1}-u_i)|^2-Cf'(w_{i+1}-w_i)|u_{i+1}-u_i|^2|\nabla u_i|^2.\end{aligned}\tag{6.11}$$

We have

$$g_*\,|u_{i+1}-u_i| \le |w_{i+1}-w_i| \le g^*\,|u_{i+1}-u_i|\tag{6.12}$$

for all i, and we conclude from relation (6.10) that there exists a constant $C>0$ independent of τ such that

$$\tau^{\alpha-1}\sum_{i=0}^{\infty}\int_\Omega \Gamma(w_{i+1}-w_i)\,\mathrm{d}x \le C\left(\widehat{F}_0(\tau)+\tau^\alpha\sum_{i=0}^{\infty}\int_\Omega f'(w_{i+1}-w_i)\left|w_{i+1}-w_i\right|^2\left|\nabla u_i\right|^2\mathrm{d}x\right).\tag{6.13}$$

For a suitable $p>1$, we estimate the integral on the right-hand side of relation (6.13) using Hölder's inequality as

$$\begin{aligned}&\tau^\alpha\sum_{i=0}^{\infty}\int_\Omega f'(w_{i+1}-w_i)|w_{i+1}-w_i|^2|\nabla u_i|^2\,\mathrm{d}x\\ &\quad\le\left(\tau^{1-(1-\alpha)p'}\sum_{i=0}^{\infty}\int_\Omega\left(f'(w_{i+1}-w_i)|w_{i+1}-w_i|^2\right)^{p'}\mathrm{d}x\right)^{1/p'}\left(\tau\sum_{i=0}^{\infty}\int_\Omega|\nabla u_i|^{2p}\,\mathrm{d}x\right)^{1/p},\end{aligned}\tag{6.14}$$

where $p'=\frac{p}{p-1}$ is the conjugate exponent to p. We now refer to the Gagliardo–Nirenberg inequality in the form

$$\int_\Omega\left|\nabla u_i\right|^{2p}\mathrm{d}x \le C\left(\left(\int_\Omega\left|\nabla u_i\right|^2\mathrm{d}x\right)^p+\left(\int_\Omega\left|\Delta u_i\right|^2\mathrm{d}x\right)^{p\kappa}\left(\int_\Omega\left|\nabla u_i\right|^2\mathrm{d}x\right)^{p(1-\kappa)}\right)$$

with $\kappa=\frac{N}{2p'}$ and a constant $C>0$ independent of τ. We first specify the choice of p, namely

$$p=1+\frac{2}{N}.\tag{6.15}$$

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Then $p\varkappa = 1$, and we obtain

$$\tau\sum_{i=0}^{\infty}\int_{\Omega}\left|\nabla u_i\right|^{2p}\mathrm{d}x \le C\left(\tau\sum_{i=0}^{\infty}\left(\int_{\Omega}\left|\nabla u_i\right|^{2}\mathrm{d}x\right)^{p} + \tau\sum_{i=0}^{\infty}\int_{\Omega}\left|\Delta u_i\right|^{2}\mathrm{d}x\left(\sup_{j\in\mathbb{N}}\int_{\Omega}\left|\nabla u_j\right|^{2}\mathrm{d}x\right)^{p-1}\right). \tag{6.16}$$

It follows from inequalities (5.4) and (5.8) that the right-hand side of inequality (6.16) is bounded by a constant $C>0$ independent of τ. Using relations (6.13), (6.14) and (6.16), we thus obtain the inequality

$$\begin{aligned}&\tau^{\alpha-1}\sum_{i=0}^{\infty}\int_{\Omega}\Gamma(w_{i+1}-w_i)\,\mathrm{d}x\\ &\le C\left(\widehat{F}_0F_0(\tau) + \left(\tau^{1-(1-\alpha)p'}\sum_{i=0}^{\infty}\int_{\Omega}\left(f'(w_{i+1}-w_i)\,|\,w_{i+1}-w_i\,|^2\right)^{p'}\mathrm{d}x\right)^{1/p'}\right).\end{aligned} \tag{6.17}$$

We now consider separately the cases $N\ge 3$ and $N<3$. Assume first that $N\ge 3$. For p from equation (6.15) and $v\in\mathbb{R}$, we define the functions

$$f'(v) = (\tau + |\,v\,|)^{p-3}, \tag{6.18}$$

$$\Phi(v) = \int_0^v sf'(s)\,\mathrm{d}s = \tfrac{1}{p-1}\left((\tau+|\,v\,|)^{p-1}-\tau^{p-1}\right) - \tfrac{\tau}{2-p}\left(\tau^{p-2}-(\tau+|\,v\,|)^{p-2}\right). \tag{6.19}$$

Putting for $z\ge 0$

$$Q(z) = z\int_0^z s(1+s)^{p-3}\,\mathrm{d}s = \tfrac{z}{p-1}\left((1+z)^{p-1}-1\right) - \tfrac{z}{2-p}\left(1-(1+z)^{p-2}\right), \tag{6.20}$$

$$M(z) = \left(z^2(1+z)^{p-3}\right)^{p'} = z^{2p/(p-1)}(1+z)^{p(p-3)/(p-1)}, \tag{6.21}$$

and recalling the definition of Γ in relation (6.3), we have for $v\in\mathbb{R}$

$$\Gamma(v) = \left|\,v\,\right|\Phi(v) = \tau^pQ\left(\tfrac{|v|}{\tau}\right),\quad \left(v^2f'(v)\right)^{p'} = \tau^pM\left(\tfrac{|v|}{\tau}\right). \tag{6.22}$$

Choosing $1-\alpha = p-1$, we have $1-(1-\alpha)p' = \alpha-1 = 1-p$. We use relations (6.5), (6.8) and (6.12) and the identity

$$F(v) = \tfrac{\tau^{p-1}}{2-p}\left(\tfrac{|v|}{\tau} - \tfrac{1}{p-1}\left(\left(1+\tfrac{|v|}{\tau}\right)^{p-1}-1\right)\right) \tag{6.23}$$

to check that $\widehat{F}_0(\tau)$ is bounded by a constant C independent of τ, and from relations (6.17) and (6.22), we obtain the estimate

$$\tau\sum_{i=0}^{\infty}\int_{\Omega}Q\left(\tfrac{|w_{i+1}-w_i|}{\tau}\right)\mathrm{d}x \le C\left(1+\left(\tau\sum_{i=0}^{\infty}\int_{\Omega}M\left(\tfrac{|w_{i+1}-w_i|}{\tau}\right)\right)^{1/p'}\mathrm{d}x\right) \tag{6.24}$$

with a constant $C>0$ independent of τ. We have $2p/(p-1) = N+2$, and equations (6.20) and (6.21) imply that there exist positive constants c_1, c_2, c_3, c_4 such that

$$\lim_{z\to 0}\tfrac{Q(z)}{z^3}\ge c_1,\quad \lim_{z\to 0}\tfrac{M(z)}{z^{N+2}}\le c_2,\quad \lim_{z\to\infty}\tfrac{Q(z)}{z^p}\ge c_3,\quad \lim_{z\to\infty}\tfrac{M(z)}{z^p}\le c_4\,. \tag{6.25}$$

In particular, there exists a constant $K>0$ such that

$$M(z)\le KQ(z)\quad \forall z>0. \tag{6.26}$$

Estimate (6.24) thus remains valid if we replace M with Q. We conclude that

$$\tau\sum_{i=0}^{\infty}\int_{\Omega}Q\left(\tfrac{|w_{i+1}-w_i|}{\tau}\right)\mathrm{d}x \le C \tag{6.27}$$

with a constant $C > 0$ independent of τ.

We proceed similarly in the case of dimensions $N < 3$. For $N = 1$ and $p = 3$, we put $f'(v) = 1$ and get the same formulas with

$$Q(z) = \tfrac{1}{2}z^3, \quad M(z) = z^3, \quad F(v) = \tfrac{1}{2}\left|v\right|^2; \tag{6.28}$$

for $N = 2$ and $p = 2$, we put $f'(v) = (\tau + |v|)^{-1}$ and a similar computation gives

$$Q(z) = z^2 - z\log(1+z), \quad M(z) = z^4/(1+z)^2, \quad F(v) = \tau\Big(\Big(1 + \tfrac{|v|}{\tau}\Big)\log\Big(1 + \tfrac{|v|}{\tau}\Big) - \tfrac{|v|}{\tau}\Big), \tag{6.29}$$

with the same conclusion as estimate (6.27). In all cases, Q is convex with superlinear growth, so that it generates an Orlicz norm on Ω. Estimate (6.27) as the main result of §5 will play a crucial role in the next sections, where we let the discretization parameter τ tend to 0 and prove the existence and uniqueness of solutions as well as the asymptotic stabilization result.

7. Limit as $\tau \longrightarrow 0$

We define for $x \in \Omega$ and $t \in [(i-1)\tau, i\tau)$, $i \in \mathbb{N}$, piecewise linear and piecewise constant interpolations

$$u^{(\tau)}(x,t) = u_{i-1}(x) + \tfrac{t-(i-1)\tau}{\tau}(u_i(x) - u_{i-1}(x)), \quad \bar{u}^{(\tau)}(x,t) = u_i(x), \tag{7.1}$$

$$G^{(\tau)}(x,t) = G[u]_{i-1}(x) + \tfrac{t-(i-1)\tau}{\tau}(G[u]_i(x) - G[u]_{i-1}(x)). \tag{7.2}$$

Repeating the argument of the proof of Gavioli & Krejčí [2, theorem 1.6], we let τ tend to 0. On every bounded time interval $(0,T)$, we have bounds independent of τ for $u_t^{(\tau)} \in L^p(\Omega \times (0,T))$ by estimate (6.27) and $\nabla u^{(\tau)} \in L^{2p}(\Omega \times (0,T))$ by relation (6.16). The sequence $u^{(\tau)}$ is thus compact in $L^p(\Omega; C[0,T])$, and there exists $u \in L^p(\Omega; C[0,T])$ and a subsequence of $u^{(\tau)}$ such that $u^{(\tau)} \to u$ and, by proposition 1.2, $G[u^{(\tau)}] \to G[u]$ and $G^{(\tau)} \to G[u]$ in $L^p(\Omega; C[0,T])$ strongly for all $T > 0$, $u_t^{(\tau)} \to u_t$ weakly-star in $L^Q(\Omega \times (0,\infty))$ for Q given by equation (6.20) (or, if $N < 3$, by equations (6.28) or (6.29)), $u^{(\tau)} \to u$ and $\bar{u}^{(\tau)} \to u$ weakly-star in $L^\infty(0,\infty; W^{1,2}(\Omega))$, and u is the unique solution of equation (2.14) satisfying the conditions of theorem 1.5. Moreover, using relations (4.6), (5.4), (5.8) and (6.24), we find constants $\mu > 0$, $C > 0$ such that

$$\sup\operatorname{ess}\{|u(x,t)| : (x,t) \in \Omega \times (0,\infty)\} \le \Lambda, \tag{7.3}$$

$$\int_\Omega \Big|\nabla u(x,t)\Big|^2 \,\mathrm{d}x \le C\mathrm{e}^{-\mu t} \quad \text{for all } t > 0, \tag{7.4}$$

$$\int_0^\infty \int_\Omega \Big|\Delta u(x,t)\Big|^2 \,\mathrm{d}x\,\mathrm{d}t \le C, \tag{7.5}$$

$$\int_0^\infty \int_\Omega Q\Big(\Big|u_t(x,t)\Big|\Big) \,\mathrm{d}x\,\mathrm{d}t \le C \tag{7.6}$$

for Q given by equation (6.20) (or, if $N < 3$, by equations (6.28) or (6.29) with $p = 1 + \frac{2}{N}$. Note that on each bounded time interval $(0,T)$ we can use Hölder's inequality to get from relations (6.25) and (7.6) an L^p-bound for u_t in the form

$$\int_0^T \int_\Omega \Big|u_t(x,t)\Big|^p \,\mathrm{d}x\,\mathrm{d}t \le C(1 + |\Omega|\, T). \tag{7.7}$$

Finally, still from the fact that relation (4.6) is preserved in the limit $\tau \to 0$, that is,

$$\left|\xi^r(x,t)\right| \le (\Lambda - r)^+ \ \text{for a. e. } x \in \Omega, \text{ all } t \ge 0, \text{ and all } r \ge 0, \tag{7.8}$$

we obtain that $\xi^r(x,t) = 0$ for all $r \ge \Lambda$. Hence, similarly as in equation (4.7), we actually have

$$G[u](x,t) = \bar{G} + \int_0^\Lambda \psi(x, r, \xi^r(x,t))\,\mathrm{d}r\,. \tag{7.9}$$

8. Asymptotics as $t \longrightarrow \infty$

Notice first that by choosing in relation (2.3) $z = r\,\mathrm{sign}(\xi^r_t)$, we get $r\,|\,\xi^r_t\,| \le \xi^r_t(u - \xi^r) \le r\,|\,\xi^r_t\,|$ a.e., hence, $\xi^r_t(u - \xi^r) = r\,|\,\xi^r_t\,|$ a.e. With the notation of functions (4.2), we have

$$\begin{aligned}\rho(x,r,\xi^r)r\,|\,\xi^r_t\,| &= \rho(x,r,\xi^r)\xi^r_t(u - \xi^r) = (u - \xi^r)\frac{\partial}{\partial t}\psi(x,r,\xi^r)\\ &= u\frac{\partial}{\partial t}\psi(x,r,\xi^r) - \frac{\partial}{\partial t}(\psi(x,r,\xi_r)\xi_r - \Psi(x,r,\xi_r)) \ \text{ a. e.}\end{aligned}$$

which gives the hysteresis energy balance equation

$$G[u]_t u = \tfrac{\partial}{\partial t}\int_0^\Lambda \left(\psi(x,r,\xi^r)\xi^r - \Psi(x,r,\xi^r)\right)\mathrm{d}r + \int_0^\Lambda \rho(x,r,\xi^r)r\left|\xi^r_t\right|\,\mathrm{d}r, \tag{8.1}$$

with $G[u]$ as in equation (7.9). To prove the convergence of ξ^r (and, consequently, the convergence of $G[u]$) as $t \to \infty$, we test equation (2.14) by $\varphi = u$ and use equation (8.1) to obtain (omitting the arguments x and r of ρ, ψ and Ψ for simplicity)

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_0^\Lambda \left(\psi(\xi^r)\xi^r - \Psi(\xi^r)\right)\mathrm{d}r\,\mathrm{d}x + \int_\Omega\int_0^\Lambda \rho(\xi^r)r\left|\xi^r_t\right|\,\mathrm{d}r\,\mathrm{d}x + \int_\Omega \left|\nabla u\right|^2\,\mathrm{d}x = 0. \tag{8.2}$$

By relation (2.7), we have $\psi(\xi)\xi - \Psi(\xi) = \int_0^\xi s\rho(s)\,\mathrm{d}s \ge \frac{1}{2}\rho_0(\Lambda)\left|\xi\right|^2 \ge 0$ for every $|\,\xi\,| \le \Lambda$, which is our case, thanks to (7.8). Hence, integrating (8.2) over $t \in (0,\infty)$, we get

$$\int_0^\infty\int_\Omega\int_0^\Lambda r\left|\xi^r_t\right|\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}t \le C \tag{8.3}$$

with a constant $C > 0$ depending only on the initial condition. For every sequence $0 = t_0 < t_1 < t_2 < \ldots$, we thus have

$$\sum_{j=1}^\infty \int_\Omega\int_0^\Lambda r\left|\xi^r(x,t_j) - \xi^r(x,t_{j-1})\right|\,\mathrm{d}r\,\mathrm{d}x \le C\,. \tag{8.4}$$

The sequence $\left\{\xi^r(x,t_j)\right\}$ is therefore a fundamental (Cauchy) sequence in the space $L^1(\Omega\times(0,\Lambda))$ endowed with the weighted norm $\left\|\zeta\right\| = \int_\Omega\int_0^\Lambda r\left|\zeta(x,r)\right|\,\mathrm{d}x\,\mathrm{d}r$. This is indeed a Banach space, and we conclude that there exists $\bar\lambda \in L^\infty(\Omega\times(0,\Lambda))$, $\bar\lambda(x,r) = 0$ for $r > \Lambda$, $\left|\bar\lambda(x,r) - \bar\lambda(x,s)\right| \le r - s$ for all $r > s \ge 0$ such that

$$\bar\omega(t) := \int_\Omega\int_0^\Lambda r\left|\xi^r(x,t) - \bar\lambda(x,r)\right|\,\mathrm{d}r\,\mathrm{d}x \to 0 \quad \text{as } t \to \infty\,. \tag{8.5}$$

Thanks to the above-given convergence, we can choose $T > 0$ such that $\bar\omega(t) \le (\Lambda/3)^3$ for $t \ge T$. For $r \in [0,\Lambda]$ and $t \ge T$, put

$$\omega(r,t) := \int_\Omega \left| \xi^r(x,t) - \bar{\lambda}(x,r) \right| \, \mathrm{d}x,$$

with the intention to prove that $\omega(r,t) \to 0$ as $t \to \infty$ uniformly with respect to r, and that the convergence rate can be estimated in terms of $\bar{\omega}(t)$ given by relation (8.5) . To this aim, for $t \geq T$, we introduce the sets

$$A(t) := \left\{ r \geq \bar{\omega}(t)^{1/3} : \omega(r,t) \geq \bar{\omega}(t)^{1/3} \right\}.$$

Using relation (8.5), we get

$$\bar{\omega}(t) = \int_0^\Lambda r\omega(r,t)\, \mathrm{d}r \geq \int_{A(t)} r\omega(r,t)\, \mathrm{d}r \geq \bar{\omega}(t)^{2/3} \left| A(t) \right|,$$

which yields the upper bound for the Lebesgue measure of $A(t)$ in the form

$$\left| A(t) \right| \leq \bar{\omega}(t)^{1/3}. \tag{8.6}$$

Let now $r \in (0, \Lambda)$ be arbitrary. In the 'good' case $r \in [\bar{\omega}(t)^{1/3}, \Lambda] \setminus A(t)$, we immediately have by definition of $A(t)$ the desired bound $\omega(r,t) < \bar{\omega}(t)^{1/3}$. Instead, in the 'bad' cases $r \in (0, \bar{\omega}(t)^{1/3})$ or $r \in A(t)$, by inequality (8.6), we find a 'good' $s \in [\bar{\omega}(t)^{1/3}, \Lambda] \setminus A(t)$ such that $\left| r - s \right| \leq 2\bar{\omega}(t)^{1/3}$, and thanks to relation (2.13) we estimate

$$\omega(r,t) \leq \omega(s,t) + 2\left|\Omega\right| \left| r - s \right| \leq (1 + 4\left|\Omega\right|)\bar{\omega}(t)^{1/3} \quad \text{for } t \geq T.$$

This shows that $\xi^r(x,t) \to \bar{\lambda}(x,r)$ strongly in $L^1(\Omega)$ and uniformly with respect to r as $t \to \infty$. By the Lebesgue-dominated convergence theorem, we also obtain the strong convergence in $L^q(\Omega)$ for all $q \geq 1$, still uniformly in r.

To prove the convergence of u, we define the mean value of $u(x,t)$

$$U(t) := \frac{1}{|\Omega|} \int_\Omega u(x,t)\, \mathrm{d}x. \tag{8.7}$$

For all $t > 0$, we have $|U(t)| \leq \Lambda$ by virtue of relation (7.3), and

$$\int_\Omega \left| u(x,t) - U(t) \right|^2 \mathrm{d}x \leq C \int_\Omega \left| \nabla u(x,t) \right|^2 \mathrm{d}x \tag{8.8}$$

with some constant $C > 0$ by virtue of the classical Poincaré–Wirtinger inequality. From relation (7.4), it follows that

$$\int_\Omega \left| u(x,t) - U(t) \right|^2 \mathrm{d}x \leq C\mathrm{e}^{-\mu t} \tag{8.9}$$

with some constants $\mu > 0$ and $C > 0$. To conclude the proof of theorem 1.5, it suffices to check that there exists a constant $\bar{u} \in \mathbb{R}$ such that

$$\lim_{t \to \infty} U(t) = \bar{u}. \tag{8.10}$$

The fact that $\bar{\lambda}(x,0) = \bar{u}$ then follows from the uniform (with respect to r) convergence $\xi^r(x,t) \to \bar{\lambda}(x,r)$.

To prove equation (8.10), we proceed by contradiction. Assume that equation (8.10) does not hold. In this case, we would have

$$\liminf_{t \to \infty} U(t) = A < B = \limsup_{t \to \infty} U(t). \tag{8.11}$$

Using relation (8.9), we find an increasing sequence $\{t_j\}$, $t_j \to \infty$ as $j \to \infty$, such that

$$\int_{\Omega} \left| u(x,t_j) - U(t_j) \right|^2 \mathrm{d}x \le \mathrm{e}^{-2j} \quad \forall j \in \mathbb{N}, \tag{8.12}$$

$$U(t_{2i}) \to A,\ U(t_{2i+1}) \to B \ \text{ as } i \to \infty . \tag{8.13}$$

We define the sets $\Omega_j = \{x \in \Omega : |u(x,t_j) - U(t_j)|^2 > \mathrm{e}^{-j}\}$. By relation (8.12), we have $|\Omega_j| \le \mathrm{e}^{-j}$. Consider now $j \ge n$ for $n \in \mathbb{N}$, and put

$$\Omega^{(n)} := \bigcup_{j=n}^{\infty} \Omega_j .$$

Then $|\Omega^{(n)}| \le \frac{\mathrm{e}}{\mathrm{e}-1}\mathrm{e}^{-n}$, and by relation (8.13), we may assume, choosing n sufficiently large, that for some $A < a < b < B$, for a.e. $x \in \Omega \setminus \Omega^{(n)}$, and for $i = 1, 2, \ldots$, we have

$$u(x,t_{2i}) \le a, \quad u(x,t_{2i+1}) \ge b . \tag{8.14}$$

From the elementary inequalities

$$\xi^r(x,t_{2i}) \le u(x,t_{2i}) + r, \qquad \xi^r(x,t_{2i+1}) \ge u(x,t_{2i+1}) - r,$$

and from relation (8.14) we infer that

$$\xi^r(x,t_{2i+1}) - \xi^r(x,t_{2i}) \ge b - a - 2r > 0 \tag{8.15}$$

for $r \in (0, (b-a)/2)$ and $x \in \Omega \setminus \Omega^{(n)}$. Hence, for $r < (b-a)/2$, we get

$$\int_{\Omega} |\xi^r(x,t_{2i+1}) - \xi^r(x,t_{2i})| \,\mathrm{d}x \ge \int_{\Omega\setminus\Omega^{(n)}} |\xi^r(x,t_{2i+1}) - \xi^r(x,t_{2i})| \,\mathrm{d}x \ge \left(|\Omega| - |\Omega^{(n)}|\right)(b-a-2r).$$

We thus have

$$\begin{aligned}\int_{\Omega}\int_0^{\Lambda} r\,|\xi^r(x,t_{2i+1}) - \xi^r(x,t_{2i})| \,\mathrm{d}x\,\mathrm{d}r &\ge \left(|\Omega| - |\Omega^{(n)}|\right)\int_0^{min\{\Lambda,(b-a)/2\}} r(b-a-2r)\,\mathrm{d}r \\ &\ge \left(|\Omega| - |\Omega^{(n)}|\right)(min\{\Lambda,(b-a)/2\})^2\,\frac{b-a}{6} > 0\end{aligned}$$

for all $i \ge n$, which contradicts relation (8.4). We thus conclude that equation (8.10) holds, which completes the proof of theorem 1.5.

Data accessibility. This article has no additional data.

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

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