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Olena Atlasiuk, Arnaud Heibig, Adrien Petrov

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Abstract

This paper deals with a dynamic Gao beam of infinite length subjected to a moving concentrated Dirac mass. Under appropriate regularity assumptions on the initial data, the problem possesses a weak solution which is obtained as the limit of a sequence of solutions of regularized problems.

Key words. Gao beam equation, Dirac mass, moving load, existence result, Sobolev spaces.
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1 Description of the problem

The behavior of a beam plays a crucial role in various applications as in railway track design. The railway compagnies aim to enhance the train rolling speed to meet the increased demands in both passenger and freight transportation worldwide. Identifying the factors contributing to the occurrence of railway track defects is rather important to maintain the required track quality. The oscillations amplitudes in railway tracks due to the trains movement is intensively studied in scientific literature (see [12, 10]). These oscillations lead to undesirable consequences such as a premature wear and deformation of railway tracks. Understanding the impact of these oscillations on system reliability is essential to maintain the track quality necessary for traffic safety and passenger comfort.

Most of mathematical models consider rails as beams under the influence of moving loads and their properties are intensively studied. A considerable engineering and mathematical literature is devoted to study to numerous mathematical models for beams, such as the Euler-Bernoulli (see [8, 7, 1, 9]), Timoshenko (see [17, 6]) and Gao beams. The Euler-Bernoulli and Timoshenko beam theories are the oldest ones and they are widely employed nowadays in various structural analysis methodologies in engineering. The Euler-Bernoulli beam is commonly used to model the bending beams behavior, the axial and shear deformations are neglected compared to bending. Thus, the beam cross-section remains planar during loading. The Timoshenko beam is suitable to study the shear beams behavior, thereby providing an accurate representation of deformation within the cross-section of the beam.

The current study involves analysis of the horizontal motion of a vertical point-load on a metallic rail, considering scenarios where the load has mass or is massless, and examining the resulting oscillations of the system. The mathematical model of the Gao beam was originally introduced in [13]. However the Gao beam is also studied in different contexts (see [3, 2, 5, 15, 14, 16]). Notice also that semilinear beam is studied in [19, 18]. Comparisons between simulations

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of the Gao beam and the Euler-Bernoulli linear beam reveal significant differences, indicating that the linear beam is suitable only for small loads, whereas the Gao beam accommodates moderate loads (see [11]).

We consider in this paper a straight infinite Gao beam of thickness $2h$. A detailed derivation of the Gao beam model can be found in [13, 3]. An horizontal traction p, which is a time-depending function, is applied at one end. We are focused in this work on examining the sideways movement of a concentrated load $P(t)$, where the load may change over time. This load is applied to a mobile mass m positioned at $\zeta(t)$ with a horizontal velocity $\dot{\zeta}(t)$, induced by a horizontal applied force. The transverse displacement of the Gao beam $u(t,x)$ for $(t,x) \in (0,T) \times \mathbb{R}$ is governed by a following partial differential equation:

$$\begin{aligned} &\varrho u_{tt} + m\delta(x-\zeta(t))u_{tt} - m\delta'(x-\zeta(t))\dot{\zeta}(t)u_t + ku_{xxxx} - (eu_x^2 - \nu p)u_{xx} \\ &= \varrho f + \delta(x-\zeta(t))P(t), \end{aligned} \tag{1.1}$$

where $\delta(\cdot)$ is the Dirac function, ϱ is the material density, f is the applied mechanical loading, $k \overset{\text{def}}{=} \frac{2h^3E}{3(1-\bar{\nu}^2)}$, $\nu \overset{\text{def}}{=} (1+\bar{\nu})$ and $e \overset{\text{def}}{=} 3hE$ where E and $\bar{\nu}$ denote the Young modulus and the Poisson ratio, respectively. Here and below $(\cdot)_t \overset{\text{def}}{=} \partial_t(\cdot)$, $(\cdot)_x \overset{\text{def}}{=} \partial_x(\cdot)$ denote the partial derivatives with respect to t and x, respectively, while $\dot{(\cdot)}$ and $(\cdot)'$ denote the derivatives with respect to t and x, respectively. We prescribe also initial data

$$u(0,\cdot) = u_0 \quad \text{and} \quad u_t(0,\cdot) = u_1. \tag{1.2}$$

For further explanations on this model, the reader is referred to [11] and to the references therein. Equation (1.1) can be considered the limit of the equation treated in [11] when the gamma viscosity tends to zero, and is related to the usual viscous beam equation without Dirac measure in the time derivative. Nevertheless, this leads to an entirely different mathematical problem since three-order estimates cannot be obtained using the viscous term, and what is more, since the usual energy estimates fail. In particular, the non-linear term should be addressed somewhat indirectly since no L^∞ estimate is available immediately.

The positive constants ϱ, m, k and e play any role in the mathematical analysis carried out below. Consequently, without loss of generality, we set them equal to 1. Notice, however, that the case of non constant coefficients (for example $k = k(x)$) could be processed using approximate square roots for elliptic operators.

Throughout this paper, we assume that $\zeta \in \mathrm{C}^2([0,T])$, $P \in \mathrm{C}^2([0,T])$, $p \in \mathrm{C}^0([0,T];\mathrm{H}^2(\mathbb{R}))$ and $f \in \mathrm{C}^0([0,T];\mathrm{H}^2(\mathbb{R}))$, but these assumptions could be weakened. The main result of this paper can be stated as follows.

Theorem 1.1 *Let $T > 0$. Assume that $u_0 \in \mathrm{H}^2(\mathbb{R})$, $u_1 \in \mathrm{H}^1(\mathbb{R})$, $\zeta \in \mathrm{C}^2([0,T])$, $P \in \mathrm{C}^2([0,T])$ and $f, p \in \mathrm{C}^0([0,T];\mathrm{H}^2(\mathbb{R}))$. Then there exists a function $u \in \mathrm{C}^0([0,T);\mathrm{H}^{2-\alpha}_{\mathrm{loc}}(\mathbb{R})) \cap \mathrm{L}^\infty([0,T];\mathrm{H}^2(\mathbb{R}))$, for any $\alpha \in]0,2[$, and $u_t \in \mathrm{L}^\infty([0,T];\mathrm{L}^2(\mathbb{R}))$ such that for any $v \in \mathrm{C}^2([0,T];\mathrm{H}^2(\mathbb{R}))$ with a compact support in $[0,T[\times\mathbb{R}$, we have*

$$\begin{cases} \displaystyle\int_0^T\int_{\mathbb{R}} u_t v_t \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}} u_{xx}v_{xx}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{3}\int_0^T\int_{\mathbb{R}} u_x^3 v_x\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle -\int_0^T\int_{\mathbb{R}} \nu p u_{xx} v\,\mathrm{d}x\,\mathrm{d}t - u_0(\zeta(0))v_t(0,\zeta(0)) - \int_0^T u(t,\zeta(t))v_{tt}(t,\zeta(t))\,\mathrm{d}t \\ \displaystyle -\int_0^T \dot{\zeta}(t)u_x(t,\zeta(t))v_t(t,\zeta(t))\,\mathrm{d}t - \int_0^T \dot{\zeta}(t)u(t,\zeta(t))v_{xt}(t,\zeta(t))\,\mathrm{d}t \\ \displaystyle +\int_0^T\int_{\mathbb{R}} fv\,\mathrm{d}x\,\mathrm{d}t + \int_0^T P(t)v(t,\zeta(t))\,\mathrm{d}t + \int_{\mathbb{R}} u_1 v(0,\cdot)\,\mathrm{d}x \\ +u_1(\zeta(0))v(0,\zeta(0)) = 0. \end{cases} \tag{1.3}$$

Moreover, $u(0,\cdot) = u_0$ *holds.*

In this statement, we refrain from expressing the term $\int_0^T \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}(\delta(x-\xi(t))\partial_t u(t,\xi(t)))v(t,x)\,\mathrm{d}x\,\mathrm{d}t$ as $-\int_0^T \partial_t u(t,\xi(t))\partial_t v(t,\xi(t))\,\mathrm{d}t$. Indeed, the trace is not defined within our functional frame. To establish such an expression, higher-order regularity would be required. However, in the linear homogenous case, the formula $\partial_t^{(2)} u(t,x) + \partial_x^{(4)} u(t,x) = -\frac{\mathrm{d}}{\mathrm{d}t}(\delta(x-\zeta(t))\partial_t u(t,x))$ entails that $\partial_t^{(2)} u$ and $\partial_x^{(4)} u$ cannot both possess $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))$ regularity. The same formula suggests that $\partial_t^{(2)} u \notin \mathrm{C}^0([0,T];\mathrm{L}^2(\mathbb{R}))$ as the formal initial data $\partial_t^{(2)} u(0,x) = -(\partial_x^{(4)} u(0,x) - \frac{\mathrm{d}}{\mathrm{d}t}(\delta(x-\zeta(0))\partial_t u(0,x)))$ does not belong to $\mathrm{L}^2(\mathbb{R})$ for a smooth data $u_0(x) = u(0,x)$. Lastly, the usual energy estimate, obtained by multiplying (1.1) by $\partial_t u$ and by integrating the resulting result over $[0,T]\times\mathbb{R}$, can not be performed due to the term $\delta(x-\zeta(t))$.

The paper is organized as follows. An existence result for a linear mollified problem is presented in Section 2. Section 3 addresses a mollified nonlinear equation; some uniform a priori estimates are derived. The proof of Theorem 1.1 is provided in Section 4.

2 A preliminary existence result

Under some regularity assumptions on the data, we present an existence and uniqueness result to the system (2.2) in a suitable function space. To this aims, for any $t > 0$, we introduce the following sets:

$$\mathrm{E}_2(t) \stackrel{\text{def}}{=} \mathrm{C}^0([0,t];\mathrm{H}^2(\mathbb{R})), \tag{2.1a}$$

$$\mathrm{E}_1(t) \stackrel{\text{def}}{=} \mathrm{C}^0([0,t];\mathrm{H}^2(\mathbb{R})) \cap \mathrm{C}^1([0,t];\mathrm{L}^2(\mathbb{R})), \tag{2.1b}$$

$$\mathrm{E}_0(t) \stackrel{\text{def}}{=} \mathrm{C}^0([0,t];\mathrm{H}^4(\mathbb{R})) \cap \mathrm{C}^1([0,t];\mathrm{H}^2(\mathbb{R})) \cap \mathrm{C}^2([0,t];\mathrm{L}^2(\mathbb{R})). \tag{2.1c}$$

Hence, we have $\mathrm{E}_0(t) \hookrightarrow \mathrm{E}_1(t) \hookrightarrow \mathrm{E}_2(t)$. Next, let $\theta \in \mathcal{D}(\mathbb{R})$ be an even density of probability. For any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, we set

$$F(t,x) \stackrel{\text{def}}{=} \frac{1}{1+\theta(x-\zeta(t))} \quad \text{and} \quad G(t,x) \stackrel{\text{def}}{=} -\frac{\dot{\zeta}(t)\theta'(x-\zeta(t))}{1+\theta(x-\zeta(t))}.$$

We denote below by $C_\eta > 0$ a generic constant depending on η.

Proposition 2.1 *Let $T > 0$, $u_0 \in \mathrm{H}^4(\mathbb{R})$, $u_1 \in \mathrm{H}^2(\mathbb{R})$ and $g \in \mathrm{E}_2(T)$. Then, there exists a unique $u \in \mathrm{E}_0(T)$ solution to the following system:*

$$\begin{cases} u_{tt} + F u_{xxxx} + G u_t = g, & (2.2a) \\ u(0,\cdot) = u_0 \quad \text{and} \quad u_t(0,\cdot) = u_1. & (2.2b) \end{cases}$$

Moreover, u satisfies the following inequality:

$$\|u(t,\cdot)\|_{\mathrm{H}^2(\mathbb{R})} + \|u_t(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R})} \leq C_T\big(\|u_0\|_{\mathrm{H}^2(\Omega)} + \|u_1\|_{\mathrm{L}^2(\Omega)} + \|g\|_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}))}\big). \tag{2.3}$$

The proof is omitted. It follows from Proposition 2.1 that the application

$$\begin{aligned} \mathcal{A}_t : \mathrm{H}^4(\mathbb{R}) \times \mathrm{H}^2(\mathbb{R}) \times \mathrm{E}_2(t) &\to \mathrm{E}_0(t) \\ (u_0, u_1, g) &\mapsto u \end{aligned} \tag{2.4}$$

where u denotes the solution introduced in Proposition 2.1, can be continuously extended to a function, still denoted by $\mathcal{A}_t$:

$$\mathcal{A}_t : \mathrm{H}^2(\mathbb{R}) \times \mathrm{L}^2(\mathbb{R}) \times \mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R})) \to \mathrm{E}_1(t).$$

For $(u_0,u_1)\in \mathrm{H}^2(\mathbb{R})\times \mathrm{L}^2(\mathbb{R})$ fixed, we define

$$\mathcal{B}_t : \mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))\to \mathrm{E}_1(t)$$
$$g\mapsto \mathcal{A}_t(u_0,u_1,g).$$

If we denote by $\mathcal{B}_t(g)$ by u, then u still satisfies (2.3). In the sequel, we always assume that $(u_0,u_1)\in \mathrm{H}^2(\mathbb{R})\times \mathrm{L}^2(\mathbb{R})$.

3 The approximated problem

We mollify in system (2.2) the δ function, mollify and truncate the nonlinear term. Namely, we consider the following approximated problem:

$$\begin{cases} u_{tt}+Fu_{xxxx}+Gu_t-((\varphi^R(u\star\theta')-\nu p)(u\star\theta''))\star\theta+h)F=0, & (3.1a)\\ u(0,\cdot)=u_0 \quad\text{and}\quad u_t(0,\cdot)=u_1, & (3.1b)\end{cases}$$

where $\star$ denotes the convolution with respect to x, and $h\in \mathrm{C}^0([0,T];\mathrm{H}^2(\mathbb{R}))$ is a given function. Here, for $R>0$, $\varphi^R\in \mathrm{C}^\infty(\mathbb{R})$ is an even, increasing function on $\mathbb{R}_+$ such that $\varphi^R(x)\stackrel{\text{def}}{=}x^2$ for $|x|\le R$ and $\varphi^R(x)\stackrel{\text{def}}{=}(R+1)^2$ for $|x|\ge R+2$. Clearly, the equation (3.1a) can formally be rewritten as follows:

$$u_{tt}+Fu_{xxxx}+Gu_t=\mathcal{C}_{t,R}(u),$$

where $\mathcal{C}_{t,R}$ is defined by

$$\mathcal{C}_{t,R}:\mathrm{E}_1(t)\to \mathrm{E}_2(t),$$
$$v\mapsto \mathcal{C}_{t,R}(v)\stackrel{\text{def}}{=}\big([(\varphi^R(v\star\theta')-\nu p)(v\star\theta'')]\star\theta+h\big)F$$

The fact that $\mathcal{C}_{t,R}$ is well defined with values in $E_2(t)$ comes from usual convolution inequalities. Similarly, we have

Proposition 3.1 *Let $t\in[0,T]$. Then, for any $(v,\widetilde{v})\in \mathrm{E}_1(t)\times \mathrm{E}_1(t)$, the following inequality*

$$\begin{aligned}\|\mathcal{C}_{t,R}(v)-\mathcal{C}_{t,R}(\widetilde{v})\|_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}))}&\le C_{T,R}\big(\|v-\widetilde{v}\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}\\ &+\|v_t-\widetilde{v}_t\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}+\|\widetilde{v}\|_{\mathrm{C}^0([0,t];\mathrm{L}^2(\mathbb{R}))}\|v-\widetilde{v}\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}\big)\end{aligned}\qquad(3.2)$$

holds true.

Proof. Let $(v,\widetilde{v})\in \mathrm{E}_1(t)\times \mathrm{E}_1(t)$. Hence, we have

$$\begin{aligned}&\|([(\varphi^R(v\star\theta')-\nu p)(v\star\theta'')]\star\theta-[(\varphi^R(\widetilde{v}\star\theta')-\nu p)(\widetilde{v}\star\theta'')]\star\theta)F\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}\\ &\le C_{T,R}\big((\|\varphi^R(v\star\theta')-\nu p)(v-\widetilde{v})\star\theta''\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}\\ &+\|(\varphi^R(v\star\theta')-\varphi^R(\widetilde{v}\star\theta'))(\widetilde{v}\star\theta'')\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}\big).\end{aligned}$$

Since φ^R is bounded and globally Lipschitz and by convolution inequalities, it comes that

$$\begin{aligned}&\|([(\varphi^R(v\star\theta')-\nu p)(v\star\theta'')]\star\theta-[(\varphi^R(\widetilde{v}\star\theta')-\nu p)(\widetilde{v}\star\theta'')]\star\theta)F\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}\\ &\le C_{T,R}\big(\|v-\widetilde{v}\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}+\|\widetilde{v}\|_{\mathrm{C}^0([0,t];\mathrm{L}^2(\mathbb{R}))}\|v-\widetilde{v}\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}\big),\end{aligned}$$

which proves the result. □

We are now looking for the fixed points of application:

$$\mathcal{B}_T\mathcal{C}_{T,R}:\mathrm{E}_1(T)\to \mathrm{E}_1(T).$$

Proposition 3.2 *The application $\mathcal{B}_T\mathcal{C}_{T,R}$ admits a fixed point.*

Proof. We use the Picard fixed point theorem. Due to Proposition 3.1 *(i)*, we have mainly to bound $\|\widetilde{v}\|_{\mathrm{C}^0([0,t];\mathrm{L}^2(\mathbb{R}))}$.

(a) An invariant set.

First, let us assume that $v \in \mathrm{E}_1(T)$ with $(v(0,\cdot), v_t(0,\cdot)) = (u_0, u_1)$ and $\widetilde{v} = 0$. It follows from (2.3) and (3.2) that

$$\begin{aligned}
&\|(\mathcal{B}_T\mathcal{C}_{T,R}(v))(t)\|_{\mathrm{H}^2(\mathbb{R})} + \|\partial_t(\mathcal{B}_T\mathcal{C}_{T,R}(v))(t)\|_{\mathrm{L}^2(\mathbb{R})}\\
&\le C_{T,R}\big(\|u_0\|_{\mathrm{H}^2(\mathbb{R})} + \|u_1\|_{\mathrm{L}^2(\mathbb{R})} + \|\mathcal{C}_{T,R}(v)\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbb{R}))}\big)\\
&\le C_{T,R}\big(\|u_0\|_{\mathrm{H}^2(\mathbb{R})} + \|u_1\|_{\mathrm{L}^2(\mathbb{R})} + \|\mathcal{C}_{T,R}(0)\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbb{R}))}\\
&\quad + C_{T,R}\big(\|v\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))} + \|v_t\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}\big)\big).
\end{aligned} \tag{3.3}$$

For $f \in \mathrm{C}^0([0,t];\mathrm{L}^2(\mathbb{R}))$, we define

$$\|f\|_{\lambda,t} \stackrel{\text{def}}{=} \sup_{0\le s\le t}\big(\mathrm{e}^{-\lambda s}\|f(s)\|_{\mathrm{L}^2(\mathbb{R})}\big)$$

with $\lambda > 0$ and for $g \in \mathrm{E}_1(t)$, we define

$$\|g\|_{1,\lambda,t} \stackrel{\text{def}}{=} \sup_{0\le s\le t}\big(\mathrm{e}^{-\lambda s}\big(\|g(s)\|_{\mathrm{H}^2(\mathbb{R})} + \|g_t(s)\|_{\mathrm{L}^2(\mathbb{R})}\big)\big).$$

It comes from (3.3) that

$$\begin{aligned}
&\mathrm{e}^{-\lambda t}\big(\|(\mathcal{B}_T\mathcal{C}_{T,R}(v))(t)\|_{\mathrm{H}^2(\mathbb{R})} + \|\partial_t(\mathcal{B}_T\mathcal{C}_{T,R}(v))(t)\|_{\mathrm{L}^2(\mathbb{R})}\big)\\
&\le C_{T,R}\mathrm{e}^{-\lambda t}\big(\|u_0\|_{\mathrm{H}^2(\mathbb{R})} + \|u_1\|_{\mathrm{L}^2(\mathbb{R})} + \|\mathcal{C}_{T,R}(0)\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbb{R}))}\big)\\
&\quad + C_{T,R}\Big(\int_0^t \mathrm{e}^{-2\lambda t}\mathrm{e}^{2\lambda s}\mathrm{e}^{-2\lambda s}\big(\|v(s)\|^2_{\mathrm{L}^2(\mathbb{R})} + \|v_t(s)\|^2_{\mathrm{L}^2(\mathbb{R})}\big)\,\mathrm{d}s\Big)^{1/2}\\
&\le C_{T,R}\mathrm{e}^{-\lambda t}\big(\|u_0\|_{\mathrm{H}^2(\mathbb{R})} + \|u_1\|_{\mathrm{L}^2(\mathbb{R})} + \|\mathcal{C}_{T,R}(0)\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbb{R}))}\big)\\
&\quad + \frac{C_{T,R}}{\sqrt{\lambda}}\|v\|_{1,\lambda,t}.
\end{aligned} \tag{3.4}$$

Note that (3.4) can be rewritten for any $s \in [0,t]$ in place of t. Furthermore, in this new inequality, $\|\cdot\|_{1,\lambda,s}$ and $\|\cdot\|_{\mathrm{L}^2(0,s;\mathrm{L}^2(\mathbb{R}))}$ are respectively smaller that $\|\cdot\|_{1,\lambda,t}$ and $\|\cdot\|_{\mathrm{L}^2(0,t;\mathrm{L}^2(\mathbb{R}))}$. Hence, we get (3.4) with s in place of t in the left hand side. Taking $t = T$, we obtain

$$\|\mathcal{B}_T\mathcal{C}_{T,R}(v)\|_{1,\lambda,T} \le M_R + \frac{C_{T,R}}{\sqrt{\lambda}}\|v\|_{1,\lambda,T}, \tag{3.5}$$

where $M_R \stackrel{\text{def}}{=} C_{T,R}\big(\|u_0\|_{\mathrm{H}^2(\mathbb{R})} + \|u_1\|_{\mathrm{L}^2(\mathbb{R})} + \|\mathcal{C}_{T,R}(0)\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbb{R}))}\big)$. Let $\lambda = 4C^2_{T,R}$ and define $\beta(0,2M_R) \stackrel{\text{def}}{=} \{u \in \mathrm{E}_1(T) : \|u\|_{1,\lambda,T} \le 2M_R\}$. Then, we may deduce from (3.5) that $\mathcal{B}_T\mathcal{C}_{T,R}(\beta(0,2M_R)) \subset \beta(0,2M_R)$.

(b) Contraction of $\mathcal{B}_T\mathcal{C}_{T,R}$.

Let $(v,\widetilde{v}) \in \beta(0,2M_R)\times\beta(0,2M_R)$ with $v(0,\cdot) = \widetilde{v}(0,\cdot) = u_0$ and $v_t(0,\cdot) = \widetilde{v}_t(0,\cdot) = u_1$. Since $v - \widetilde{v}$ has 0 initial data, we have (see (2.3))

$$\begin{aligned}
&\|(\mathcal{B}_T\mathcal{C}_{T,R}(v))(t) - (\mathcal{B}_T\mathcal{C}_{T,R}(\widetilde{v}))(t)\|_{\mathrm{H}^2(\mathbb{R})} + \|\partial_t((\mathcal{B}_T\mathcal{C}_{T,R}(v))(t) - (\mathcal{B}_T\mathcal{C}_{T,R}(\widetilde{v}))(t))\|_{\mathrm{L}^2(\mathbb{R})}\\
&\le C_{T,R}\|\mathcal{C}_{T,R}(v) - \mathcal{C}_{T,R}(\widetilde{v})\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\mathbb{R}))}\\
&\le C_{T,R}\big(\|v-\widetilde{v}\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbb{R}))} + \|v_t - \widetilde{v}_t\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbb{R}))}\\
&\quad + \|\widetilde{v}\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\mathbb{R}))}\|v-\widetilde{v}\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbb{R}))}\big),
\end{aligned}$$

Since $\widetilde{v} \in \beta(0, 2M_R)$, we have $\|\widetilde{v}\|_{\mathrm{C}^0([0,t];\mathrm{L}^2(\mathbb{R}))} \le 2M_R \mathrm{e}^{\lambda T}$. It follows that

$$\|(\mathcal{B}_T\mathcal{C}_{T,R}(v))(t) - (\mathcal{B}_T\mathcal{C}_{T,R}(\widetilde{v}))(t)\|_{\mathrm{H}^2(\mathbb{R})} + \|\partial_t((\mathcal{B}_T\mathcal{C}_{T,R}(v))(t) - (\mathcal{B}_T\mathcal{C}_{T,R}(\widetilde{v}))(t))\|_{\mathrm{L}^2(\mathbb{R})}$$
$$\le C_{T,R}\big(\|v - \widetilde{v}\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbb{R}))} + \|v_t - \widetilde{v}_t\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbb{R}))}\big).$$

Introducing a suitable $\mu > 0$, we derive as before the following inequality

$$\|\mathcal{B}_T\mathcal{C}_{T,R}(v) - \mathcal{B}_T\mathcal{C}_{T,R}(\widetilde{v})\|_{1,\mu,T} \le \frac{1}{2}\|v - \widetilde{v}\|_{1,\mu,T},$$

which proves the Proposition. □

An immediate consequence of Proposition 3.1 is the following Corollary:

Corollary 3.3 *Assume that* $u_0 \in \mathrm{H}^4(\mathbb{R})$, $u_1 \in \mathrm{H}^2(\mathbb{R})$. *Let* $u^R \in \mathrm{E}_1(T)$ *such that* $\mathcal{B}_T\mathcal{C}_{T,R}(u^R) = u^R$. *Then,* $u^R \in \mathrm{E}_0(T)$. *Moreover,*

$$\begin{cases} u^R_{tt} + F u^R_{xxxx} + G u^R_t - F((\varphi^R(u^R \star \theta') - \nu p)(u^R \star \theta'')) \star \theta + h) = 0, & (3.6a) \\ u^R(0,\cdot) = u_0 \quad \text{and} \quad u^R_t(0,\cdot) = u_1. & (3.6b) \end{cases}$$

Proof. We have $\mathcal{C}_{T,R}(\mathrm{E}_1(T)) \subset \mathrm{E}_2(T)$. Hence, by (2.4), we get $\mathcal{B}_T\mathcal{C}_{T,R}(u^R) \in \mathrm{E}_0(T)$, which is $u^R \in \mathrm{E}_0(T)$. □

Let $\epsilon > 0$ and $\theta^\epsilon(x) \stackrel{\text{def}}{=} \frac{1}{\epsilon}\theta\big(\frac{x}{\epsilon}\big)$ with $x \in \mathbb{R}$. We assume now that $u_0 \in \mathrm{H}^2(\mathbb{R})$ and $u_1 \in \mathrm{L}^2(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$ and set $u_0^\epsilon \stackrel{\text{def}}{=} u_0 \star \theta^\epsilon$, $v_0^\epsilon \stackrel{\text{def}}{=} u_1 \star \theta^\epsilon$ and $h^\epsilon \stackrel{\text{def}}{=} -(f + \theta^\epsilon(\cdot - \zeta(t))P(t))$. Clearly, we have $u_0^\epsilon \in \mathrm{H}^4(\mathbb{R})$ and $u_1^\epsilon \in \mathrm{H}^2(\mathbb{R})$. We apply the Corollary 3.3 with $(u_0^\epsilon, u_1^\epsilon, \theta^\epsilon, h^\epsilon)$ in place of (u_0, u_1, θ, h). This provides a function $u^{R,\epsilon}$ solution of the following problem:

$$\begin{cases} u^{R,\epsilon}_{tt} + \dfrac{\mathrm{d}}{\mathrm{d}t}(\theta^\epsilon(\cdot - \zeta(t))u^{R,\epsilon}_t) + u^{R,\epsilon}_{xxxx} & (3.7a) \\ \quad - ((\varphi^R(u^{R,\epsilon} \star \theta^{\epsilon\prime}) - \nu p)(u^{R,\epsilon} \star \theta^{\epsilon\prime\prime})) \star \theta^\epsilon + h^\epsilon) = 0, & \\ u^{R,\epsilon}(0,\cdot) = u_0^\epsilon \quad \text{and} \quad u^{R,\epsilon}_t(0,\cdot) = u_1^\epsilon. & (3.7b) \end{cases}$$

In the sequel, we set $Q_t \stackrel{\text{def}}{=} [0,t] \times \mathbb{R}$ where $t \in [0,T]$. We have the following (uniform in R and ϵ) estimates:

Proposition 3.4 *Assume that* $u_0 \in \mathrm{H}^2(\mathbb{R})$, $u_1 \in \mathrm{L}^2(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$. *Then, there exists a constant* $C > 0$, *independent of* R *and* ϵ, *such that*

$$\sup_{0 \le t \le T}\big(\|u^{R,\epsilon}(t)\|_{\mathrm{H}^2(\mathbb{R})} + \|u^{R,\epsilon}_t(t)\|_{\mathrm{L}^2(\mathbb{R})}\big) \le C.$$

Proof. We define $\frac{\partial}{\partial\tau} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + \dot{\zeta}(t)\frac{\partial}{\partial x}$. Then, we multiply (3.7a) by $u^{R,\epsilon}_\tau$ and we integrate this result over $[0,t] \times \mathbb{R}$ with $t \in [0,T]$. Since $u^{R,\epsilon} \in \mathrm{E}_0(T)$, the boundary terms in $x = \pm\infty$ vanish. We observe that the term $u^{R,\epsilon}_{tt}$ in (3.7) gives

$$\begin{aligned} A_1(t) &\stackrel{\text{def}}{=} \frac{1}{2}\|u^{R,\epsilon}_t(t)\|^2_{\mathrm{L}^2(\mathbb{R})} - \frac{1}{2}\|u_1\|^2_{\mathrm{L}^2(\mathbb{R})} + \int_{Q_t} \dot{\zeta} u^{R,\epsilon}_{tt} u^{R,\epsilon}_x \,\mathrm{d}s\,\mathrm{d}x \\ &\ge \frac{1}{2}\|u^{R,\epsilon}_t(t)\|^2_{\mathrm{L}^2(\mathbb{R})} - \frac{1}{2}\|u_1\|^2_{\mathrm{L}^2(\mathbb{R})} - \frac{1}{2}\int_{Q_t} \partial_x(\dot{\zeta}(u^{R,\epsilon}_t)^2)\,\mathrm{d}s\,\mathrm{d}x \\ &\quad - \int_{Q_t} \ddot{\zeta} u^{R,\epsilon}_x u^{R,\epsilon}_t \,\mathrm{d}s\,\mathrm{d}x + \int_{\mathbb{R}} \big[\dot{\zeta} u^{R,\epsilon}_t u^{R,\epsilon}_x\big]_0^t \,\mathrm{d}x \\ &\ge \Big(\frac{1}{2} - \eta\Big)\|u^{R,\epsilon}_t(t)\|^2_{\mathrm{L}^2(\mathbb{R})} - C\int_0^t \big(\|u^{R,\epsilon}_t(s)\|^2_{\mathrm{L}^2(\mathbb{R})} + \|u^{R,\epsilon}_x(s)\|^2_{\mathrm{L}^2(\mathbb{R})}\big)\,\mathrm{d}s \\ &\quad - C_\eta\|u^{R,\epsilon}_x(t)\|^2_{\mathrm{L}^2(\mathbb{R})} - C\big(\|u_0\|^2_{\mathrm{H}^2(\mathbb{R})} + \|u_1\|^2_{\mathrm{L}^2(\mathbb{R})}\big). \end{aligned}$$

Concerning the term $\frac{\mathrm{d}}{\mathrm{d}t}(\theta_\epsilon(x-\zeta(t))u_t^{R,\epsilon})$, we have

$$\begin{aligned}
A_2(t) &\stackrel{\text{def}}{=} \int_{Q_t} \frac{\mathrm{d}}{\mathrm{d}s}(\theta^\epsilon(x-\zeta(s))u_t^{R,\epsilon})u_\tau^{R,\epsilon}\,\mathrm{d}s\,\mathrm{d}x \\
&= -\int_{Q_t} \theta^\epsilon(x-\zeta(s))u_t^{R,\epsilon}\partial_\tau\partial_t u^{R,\epsilon} + \ddot{\zeta}u_x^{R,\epsilon})\,\mathrm{d}s\,\mathrm{d}x + \int_{\mathbb{R}} \big[\theta^\epsilon(x-\zeta(s))u_t^{R,\epsilon}u_\tau^{R,\epsilon}\big]_0^t\,\mathrm{d}x \\
&= -\frac{1}{2}\int_{Q_t} \partial_\tau\big(\theta^\epsilon(x-\zeta(s))(u_t^{R,\epsilon})^2\big)\,\mathrm{d}s\,\mathrm{d}x - \int_{Q_t} \theta^\epsilon(x-\zeta(t))\ddot{\zeta}u_t^{R,\epsilon}u_x^{R,\epsilon}\,\mathrm{d}s\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \theta^\epsilon(x-\zeta(t))\big((u_t^{R,\epsilon})^2 + \dot{\zeta}(t)u_t^{R,\epsilon}(t,x)u_x^{R,\epsilon}(t,x)\big)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \theta^\epsilon(x-\zeta(0))\big((u_1^\epsilon)^2 + \dot{\zeta}(0)u_1^\epsilon u_0^{\epsilon\prime}\big)\,\mathrm{d}x \\
&\geq -\frac{1}{2}\int_{\mathbb{R}} \big(\theta^\epsilon(x-\zeta(t))(u_t^{R,\epsilon}(t,x))^2 - \theta^\epsilon(x-\zeta(0))(u_1^\epsilon)^2\big)\,\mathrm{d}x \\
&\quad - \int_{Q_t} \theta^\epsilon(x-\zeta(s))(u_t^{R,\epsilon})^2\,\mathrm{d}t\,\mathrm{d}x \\
&\quad - C\int_{Q_t} \theta^\epsilon(x-\zeta(s))\big(\|u^{R,\epsilon}(s)\|^2_{\mathrm{L}^2(\mathbb{R})} + \|u_{xx}^{R,\epsilon}(s)\|^2_{\mathrm{L}^2(\mathbb{R})}\big)\,\mathrm{d}s\,\mathrm{d}x + \int_{\mathbb{R}} \theta^\epsilon(x-\zeta(t))(u_t^{R,\epsilon}(t,x))^2\,\mathrm{d}x \\
&\quad - C_\eta\int_{\mathbb{R}} \theta^\epsilon(x-\zeta(t))(u_x^{R,\epsilon}(t,x))^2\,\mathrm{d}x - \eta\int_{\mathbb{R}} \theta^\epsilon(x-\zeta(t))(u_t^{R,\epsilon}(t,x))^2\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \theta^\epsilon(x-\zeta(0))(u_1^\epsilon)^2\,\mathrm{d}x - \int_{\mathbb{R}} \theta_\epsilon(x-\zeta(0))\dot{\zeta}(0)u_1^\epsilon u_0^{\epsilon\prime}\,\mathrm{d}x \\
&\geq \Big(\frac{1}{2}-\eta\Big)\int_{\mathbb{R}} \theta^\epsilon(x-\zeta(t))(u_t^{R,\epsilon}(t,x))^2\,\mathrm{d}x - \int_{Q_t} \theta^\epsilon(x-\zeta(s))(u_t^{R,\epsilon})^2\,\mathrm{d}s\,\mathrm{d}x \\
&\quad - C\int_0^t \|u^{R,\epsilon}(s)\|^2_{\mathrm{H}^2(\mathbb{R})}\,\mathrm{d}s - C_\eta\|u^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})} \\
&\quad - \eta\|u^{R,\epsilon}(t)\|^2_{\mathrm{H}^2(\mathbb{R})} - C\big(\|u_1\|^2_{\mathrm{L}^\infty(\mathbb{R})} + \|u_0\|^2_{\mathrm{H}^2(\mathbb{R})}\big)
\end{aligned}$$

The term $u_x^{R,\epsilon}$ provides

$$\begin{aligned}
A_3(t) &\stackrel{\text{def}}{=} \int_{Q_t} u_{xxxx}^{R,\epsilon}u_\tau^{R,\epsilon}\,\mathrm{d}s\,\mathrm{d}x \\
&\geq \frac{1}{2}\|u_{xx}^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})} - \frac{1}{2}\|u_0''\|^2_{\mathrm{L}^2} + \frac{1}{2}\int_0^t \dot{\zeta}(s)\int_{\mathbb{R}} \partial_x(u_{xx}^{R,\epsilon})^2\,\mathrm{d}s\,\mathrm{d}x \\
&= \frac{1}{2}\big(\|u_{xx}^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})} - \|u_0\|^2_{\mathrm{H}^2(\mathbb{R})}\big).
\end{aligned}$$

In the sequel, we define ψ^R and μ^R by $\psi^{R\prime} \stackrel{\text{def}}{=} \varphi^R$ and $\mu^{R\prime} \stackrel{\text{def}}{=} \psi^R$ such that $\mu^R(0) = \psi^R(0) = 0$. Since φ^R is even, so is μ^R. Furthermore, notice that $\mu^{R\prime\prime} = \varphi^R \geq 0$. Consequently, μ^R is convex, even with $\mu^R(0) = 0$. It follows that $\mu^R \geq 0$. Now, the term $-(\varphi^R(u_x^{R,\epsilon} \star \theta^\epsilon) - \nu p)(u_{xx}^{R,\epsilon} \star \theta^\epsilon)) \star \theta^\epsilon$ provides

$$\begin{aligned}
A_4(t) &\stackrel{\text{def}}{=} -\int_{Q_t} (\varphi^R(u_x^{R,\epsilon} \star \theta^\epsilon) - \nu p)(u_{xx}^{R,\epsilon} \star \theta_\epsilon))\partial_\tau(u^{R,\epsilon} \star \theta^\epsilon)\,\mathrm{d}s\,\mathrm{d}x \\
&= \int_{Q_t} (\psi_R(u_x^{R,\epsilon} \star \theta^\epsilon)\partial_\tau(u_x^{R,\epsilon} \star \theta^\epsilon)\,\mathrm{d}s\,\mathrm{d}x + \int_0^t \big[\psi^R(u_x^{R,\epsilon} \star \theta^\epsilon)\partial_\tau(u^{R,\epsilon} \star \theta^\epsilon)\big]_{-\infty}^{+\infty}\,\mathrm{d}s \\
&\quad + \int_{Q_t} \nu p(u_{xx}^{R,\epsilon} \star \theta^\epsilon)\frac{\partial}{\partial\tau}(u^{R,\epsilon} \star \theta^\epsilon)\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}$$

Since $|\psi^R(u^{R,\epsilon}\star\theta^{\epsilon\prime})|\le C|u^{R,\epsilon}\star\theta^{\epsilon\prime}|\le C_{R,\epsilon}$ and $\lim_{|x|\to+\infty}\left|\frac{\partial}{\partial\tau}(u^{R,\epsilon}\star\theta^\epsilon)\right|=0$, we may deduce that $\int_0^t\left[\psi^R(u_x^{R,\epsilon}\star\theta^\epsilon)\partial_\tau(u_x^{R,\epsilon}\star\theta^\epsilon)\right]_{-\infty}^{+\infty}\mathrm{d}t$ vanishes. Hence, we have

$$\begin{aligned}A_4(t)\ge&\int_{Q_t}\frac{\partial}{\partial\tau}(\mu^R(u_x^{R,\epsilon}\star\theta^\epsilon))\,\mathrm{d}s\,\mathrm{d}x-C\int_0^t\left(\|u^{R,\epsilon}(s)\|^2_{\mathrm{H}^2(\mathbb{R})}+\|u_t^{R,\epsilon}(s)\|^2_{\mathrm{L}^2(\mathbb{R})}\right)\mathrm{d}s\\ =&\int_{\mathbb{R}}\left(\mu^R(u_x^{R,\epsilon}(t)\star\theta^\epsilon(t))-\mu^R(u_x^{R,\epsilon}(0,x)\star\theta^\epsilon(0))\right)\mathrm{d}x\\ &-C\int_0^t\left(\|u^{R,\epsilon}(s)\|^2_{\mathrm{H}^2(\mathbb{R})}+\|u_t^{R,\epsilon}(s)\|^2_{\mathrm{L}^2(\mathbb{R})}\right)\mathrm{d}s.\end{aligned}$$

Since $\mu^R(u_x^{R,\epsilon}(t)\star\theta_\epsilon(t))\ge0$, it comes that

$$\begin{aligned}A_4(t)\ge&-C\int_{\mathbb{R}}|u_0^{\epsilon\prime}|^4\,\mathrm{d}x-C\int_0^t\left(\|u^{R,\epsilon}(s)\|^2_{\mathrm{H}^2(\mathbb{R})}+\|u_t^{R,\epsilon}(s)\|^2_{\mathrm{L}^2(\mathbb{R})}\right)\mathrm{d}s\\ \ge&-C\|u_0\|^4_{\mathrm{H}^2(\mathbb{R})}-C\int_0^t\left(\|u^{R,\epsilon}(s)\|^2_{\mathrm{H}^2(\mathbb{R})}+\|u_t^{R,\epsilon}(s)\|^2_{\mathrm{L}^2(\mathbb{R})}\right)\mathrm{d}s.\end{aligned}$$

The term h_ϵ provides

$$\begin{aligned}A_5(t)\overset{\text{def}}{=}&-\int_{Q_t}fu_\tau^{R,\epsilon}\,\mathrm{d}s\,\mathrm{d}x-\int_{Q_t}P(s)\theta_\epsilon(x-\zeta(t))u_\tau^{R,\epsilon}\,\mathrm{d}s\,\mathrm{d}x\\ \ge&-C\int_{Q_t}f^2\,\mathrm{d}s\,\mathrm{d}x-C\int_{Q_t}|u_t^{R,\epsilon}|^2\,\mathrm{d}s\,\mathrm{d}x-C\int_0^t\|u^{R,\epsilon}(s)\|^2_{\mathrm{H}^2(\mathbb{R})}\,\mathrm{d}s\\ &+\int_{Q_t}P(s)\partial_\tau(\theta^\epsilon(x-\zeta(s)))u^{R,\epsilon}(s,x)\,\mathrm{d}s\,\mathrm{d}x-\int_{\mathbb{R}}\left[P(s)\theta^\epsilon(x-\zeta(s))u^{R,\epsilon}(s,x)\right]_0^t\mathrm{d}x\\ \ge&-C\int_{Q_t}f^2-C\int_{Q_t}|u_t^{R,\epsilon}|^2\,\mathrm{d}s\,\mathrm{d}x-C\int_0^t\|u^{R,\epsilon}(s)\|^2_{\mathrm{H}^2(\mathbb{R})}\,\mathrm{d}s-C\|u_0\|^2_{\mathrm{H}^2(\mathbb{R})}\\ &-C_\eta\|u^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})}-\eta\|u^{R,\epsilon}(t)\|^2_{\mathrm{H}^2(\mathbb{R})}.\end{aligned}$$

Since $\sum_{k=1}^5A_k=0$, the previous estimates lead to

$$\begin{aligned}&\|u_t^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})}+\|u_{xx}^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})}+\int_{\mathbb{R}}\theta^\epsilon(x-\zeta(s))(u_t^{R,\epsilon}(t,x))^2\,\mathrm{d}x\\ &\le C_T\Big(\|u_0\|^2_{\mathrm{H}^2(\mathbb{R})}+\|u_0\|^4_{\mathrm{H}^2(\mathbb{R})}+\|u_1\|^2_{\mathrm{L}^2(\mathbb{R})\cap\mathrm{L}^\infty(\mathbb{R})}+\int_0^t\|u^{R,\epsilon}(s)\|^2_{\mathrm{H}^2(\mathbb{R})}\,\mathrm{d}s\\ &+\int_0^t\|u_t^{R,\epsilon}(s)\|^2_{\mathrm{L}^2(\mathbb{R})}\,\mathrm{d}s+\int_{Q_t}\theta^\epsilon(x-\zeta(s))(u_t^{R,\epsilon}(s,x))^2\,\mathrm{d}s\,\mathrm{d}x+1\Big)\\ &+2\eta\|u^{R,\epsilon}(t)\|^2_{\mathrm{H}^2(\mathbb{R})}+C_\eta\|u^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})}.\end{aligned}\tag{3.8}$$

In order to handle the terms $\eta\|u^{R,\epsilon}(t)\|^2_{\mathrm{H}^2(\mathbb{R})}$ and $C_\eta\|u^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})}$, we write

$$u^{R,\epsilon}(t,x)=u_0(x)+\int_0^tu_t^{R,\epsilon}(s,x)\,\mathrm{d}s,$$

which implies that

$$\|u^{R,\epsilon}(t)\|_{\mathrm{L}^2(\mathbb{R})}\le C\Big(\|u_0\|_{\mathrm{L}^2(\mathbb{R})}+\int_0^t\|u_t^{R,\epsilon}(s)\|_{\mathrm{L}^2(\mathbb{R})}\,\mathrm{d}s\Big),\tag{3.9}$$

and

$$\|u^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})} \leq C_T\Big(\|u_0\|^2_{\mathrm{L}^2(\mathbb{R})} + \int_0^t \|u_t^{R,\epsilon}(s)\|^2_{\mathrm{L}^2(\mathbb{R})}\,\mathrm{d}s\Big). \tag{3.10}$$

It follows from (3.10) that

$$\|u(t)\|^2_{\mathrm{H}^2(\mathbb{R})} \leq C_T\Big(\|u_0\|^2_{\mathrm{L}^2(\mathbb{R})} + \int_0^t \|u_t^{R,\epsilon}(s)\|^2_{\mathrm{L}^2(\mathbb{R})}\,\mathrm{d}s + \|u_{xx}^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})}\Big). \tag{3.11}$$

As a consequence of (3.10) and (3.11), (3.8) remains valid with $\|u^{R,\epsilon}(t)\|^2_{\mathrm{H}^2(\mathbb{R})}$ in place of $\|u_{xx}^{R,\epsilon}(t)\|^2_{\mathrm{L}^2(\mathbb{R})}$ in the left hand side, and $\eta = 0$ and $C_\eta = 0$ in the right hand side. Finally, the Grönwall's lemma gives the desired estimate. □

Now, from Proposition 3.4 and the Sobolev embeddings, there exists a constant $C > 0$, independent of R and ϵ, such that

$$\|u^{R,\epsilon}\|_{\mathrm{L}^\infty([0,T]\times\mathbb{R})} \leq C.$$

Hence, $\|u^{R,\epsilon} \star \theta^{\epsilon\prime}\|_{\mathrm{L}^\infty([0,T]\times\mathbb{R})} \leq \frac{C}{\epsilon}$. It follows that for $R = \frac{C}{\epsilon}$, we have $\varphi_R(u^{R,\epsilon} \star \theta^{\epsilon\prime}) = (u^{R,\epsilon} \star \theta^{\epsilon\prime})^2$. Consequently, we replace $\varphi^R(u^{R,\epsilon} \star \theta^{\epsilon\prime})$ by $(u^{R,\epsilon} \star \theta^{\epsilon\prime})^2$ in the identity (3.7a) and drop the index R in (3.7a). From (3.7), we induce the following weak formulation:

$$\begin{cases} \displaystyle\int_0^T \int_{\mathbb{R}} u_t^\epsilon v_t \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega u_{xx}^\epsilon v_{xx}\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle +\frac{1}{3}\int_0^T \int_{\mathbb{R}} ((u_x^\epsilon \star \theta^\epsilon)^3 (v_x \star \theta^\epsilon) + \nu p (u_x^\epsilon \star \theta^\epsilon)(v_x \star \theta^\epsilon) + (\nu p)_x (u_x^\epsilon \star \theta^\epsilon))\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle + \int_0^T \int_{\mathbb{R}} \theta^\epsilon(x-\xi(t)) u_t^\epsilon v_t\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}} f v\,\mathrm{d}x\,\mathrm{d}t + \int_0^T P(t) v(t,\xi(t))\,\mathrm{d}t \\ \displaystyle + \int_{\mathbb{R}} u_1^\epsilon(x) v(0,x)\,\mathrm{d}x + \int_{\mathbb{R}} \theta^\epsilon(x-\xi(0)) u_1^\epsilon(x) v(0,x)\,\mathrm{d}x = 0, \end{cases} \tag{3.12}$$

for any $v \in \mathrm{C}^2([0,T];\mathrm{H}^2(\Omega))$ with compact support in $[0,T) \times \mathbb{R}$.

4 Proof of Theorem 1.1

We derive from Proposition 3.4 the following convergences.

Corollary 4.1 *Assume that* $u_0 \in \mathrm{H}^2(\mathbb{R})$, $u_1 \in \mathrm{L}^2(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$. *Then, there exists a subsequence* $\{u_{\epsilon_k}\}_{k\in\mathbb{N}}$ *of* u_ϵ *such that*

(i) $u^{\epsilon_k}, u_t^{\epsilon_k}, u_{xx}^{\epsilon_k} \underset{k\to+\infty}{\rightharpoonup} u, u_t, u_{xx}$ *weakly in* $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))$.

(ii) For any $\alpha \in]0,2]$ *and any compact set* $K \subset \mathbb{R}$, *we have*

$$u^{\epsilon_k} \underset{k\to+\infty}{\rightarrow} u \text{ strongly in } \mathrm{C}^0([0,T];\mathrm{H}^{2-\alpha}(K)), \tag{4.1a}$$

$$u_x^{\epsilon_k} \underset{k\to+\infty}{\rightarrow} u_x \text{ strongly in } \mathrm{L}^\infty([0,T]\times K). \tag{4.1b}$$

(iii) For any $(a,b) \in \mathbb{R}^2$ *such that* $a < b$, $T > 0$, *we have*

$$u_x^{\epsilon_k} \star \theta^{\epsilon_k} \underset{k\to+\infty}{\rightarrow} u_x \text{ strongly in } \mathrm{L}^2([0,T]\times[a,b]), \tag{4.2a}$$

$$(u_x^{\epsilon_k} \star \theta^{\epsilon_k})^3 \underset{k\to+\infty}{\rightarrow} u_x^3 \text{ strongly in } \mathrm{L}^2([0,T]\times[a,b]). \tag{4.2b}$$

Proof. *(i)* follows from Proposition 3.4. Note that (4.1a) comes from the Sobolev embeddings and Aubin-Lions lemma while (4.1b) follows from (4.1a) and $\mathrm{H}^{1-\alpha}(K) \hookrightarrow \mathrm{L}^\infty(K)$ for $\alpha \in \left]0, \frac{1}{2}\right[$. It remains to prove *(iii)*. Let $t \in [0,T]$. We have

$$\begin{aligned}\|u_x^{\epsilon_k}(t)\star\theta^{\epsilon_k} - u_x(t)\|_{\mathrm{L}^2([a,b])} &\leq \|u_x^{\epsilon_k}(t)\star\theta^{\epsilon_k} - u_x(t)\star\theta^{\epsilon_k}\|_{\mathrm{L}^2([a,b])} + \|u_x(t)\star\theta^{\epsilon_k} - u_x(t)\|_{\mathrm{L}^2([a,b])}\\ &\leq \|u_x^{\epsilon_k}(t) - u_x(t)\|_{\mathrm{L}^2([a,b])} + \|u_x(t)\star\theta^{\epsilon_k} - u_x(t)\|_{\mathrm{L}^2([a,b])}.\end{aligned}$$

Since

$$u_x^{\epsilon_k} \underset{k\to+\infty}{\to} u_x \text{ strongly in } \mathrm{L}^\infty([0,T]\times[a,b]), \tag{4.3}$$

we have $\|u_x^{\epsilon_k}(t) - u_x(t)\|_{\mathrm{L}^2([a,b])} \underset{k\to+\infty}{\to} 0$. On the other hand, $\|u_x(t)\star\theta^{\epsilon_k} - u_x(t)\|_{\mathrm{L}^2([a,b])} \underset{k\to+\infty}{\to} 0$. Hence, we get $\|u_x^{\epsilon_k}(t)\star\theta^{\epsilon_k} - u_x(t)\|_{\mathrm{L}^2([a,b])} \underset{k\to+\infty}{\to} 0$. Besides, observe that

$$\|u_x^{\epsilon_k}(t)\star\theta^{\epsilon_k} - u_x(t)\|_{\mathrm{L}^2([a,b])} \leq \|u_x^{\epsilon_k}(t)\|_{\mathrm{L}^2([a,b])} + \|u_x(t)\|_{\mathrm{L}^2([a,b])} \overset{(4.3)}{\leq} C.$$

Finally, we deduce from the dominated convergence theorem that (4.2a) holds. It remains to prove (4.2b). To this aim, we may assume that $\{u_x^{\epsilon_k}\}_{k\in\mathbb{N}}$ is bounded on $\mathrm{L}^2([0,T]\times[a-1,b+1])$. Hence, we have

$$\begin{aligned}&\|(u_x^{\epsilon_k}\star\theta^{\epsilon_k})^3 - u_x^3\|_{\mathrm{L}^2([a,b])}\\ &\leq \|(u_x^{\epsilon_k}\star\theta^{\epsilon_k} - u_x)((u_x^{\epsilon_k}\star\theta^{\epsilon_k})^2 + (u_x^{\epsilon_k}\star\theta^{\epsilon_k})u_x + u_x^2)\|_{\mathrm{L}^2([a,b])}\\ &\overset{(4.3)}{\leq} C\|u_x^{\epsilon_k}\star\theta^{\epsilon_k} - u_x\|_{\mathrm{L}^2([a,b])}.\end{aligned}$$

The result follows from (4.2a). □

Proposition 4.2 *Assume that* $u_0 \in \mathrm{H}^2(\mathbb{R})$ *and* $u_1 \in \mathrm{H}^1(\mathbb{R})$. *Let* $v \in \mathrm{C}^2([0,T];\mathrm{H}^2(\mathbb{R}))$ *with a compact support* $[0,T[\,\times\mathbb{R}$. *Then, we have*

$$\int_{Q_T} u_t^{\epsilon_k} v_t \,\mathrm{d}t\,\mathrm{d}x \underset{k\to+\infty}{\to} \int_{Q_t} u_t v_t \,\mathrm{d}t\,\mathrm{d}x, \tag{4.4a}$$

$$\int_{Q_T} u_{xx}^{\epsilon_k} v_{xx} \,\mathrm{d}t\,\mathrm{d}x \underset{k\to+\infty}{\to} \int_{Q_t} u_{xx} v_{xx} \,\mathrm{d}t\,\mathrm{d}x, \tag{4.4b}$$

$$\begin{aligned}&\int_{Q_T} \Big(-\frac{1}{3}(u_x^{\epsilon_k}\star\theta^{\epsilon_k})^3(v_x\star\theta^{\epsilon_k}) + \nu p(u_x^{\epsilon_k}\star\theta^{\epsilon_k})(v_x\star\theta^{\epsilon_k})\\ &+ (\nu p)_x(u_x^{\epsilon_k}\star\theta^{\epsilon_k})(v\star\theta^{\epsilon_k})\Big)\,\mathrm{d}t\,\mathrm{d}x \underset{k\to+\infty}{\to} \int_{Q_T}\Big(-\frac{1}{3}u_x^3 v_x - \nu p u_{xx} v\Big)\,\mathrm{d}t\,\mathrm{d}x,\end{aligned} \tag{4.4c}$$

$$\int_{Q_T} (f + \theta^\epsilon(x-\zeta(t))P(t))v \,\mathrm{d}t\,\mathrm{d}x \underset{\epsilon\to0}{\to} \int_{Q_T} fv\,\mathrm{d}x\,\mathrm{d}t + \int_0^T P(t)v(t,\zeta(t))\,\mathrm{d}t, \tag{4.4d}$$

$$\begin{aligned}&\int_{\mathbb{R}} u_1^{\epsilon_k}(x)v(0,x)\,\mathrm{d}x + \int_{\mathbb{R}} \theta^{\epsilon_k}(x-\zeta(0))u_1^{\epsilon_k}(x)v(0,x)\,\mathrm{d}x\\ &\underset{k\to+\infty}{\to} \int_{\mathbb{R}} u_1(x)v(0,x)\,\mathrm{d}x + u_1(\zeta(0))v(0,\zeta(0)),\end{aligned} \tag{4.4e}$$

$$\begin{aligned}&\int_{Q_T} \theta^{\epsilon_k}(x-\zeta(t))u_t^{\epsilon_k} v_t\,\mathrm{d}t\,\mathrm{d}x \underset{k\to+\infty}{\to} -u_0(\zeta(0))v_t(0,\zeta(0))\\ &-\int_0^T u(t,\zeta(t))v_{tt}(t,\zeta(t))\,\mathrm{d}t\,\mathrm{d}x - \int_0^T \dot\zeta(t)u_x(t,\zeta(t))v_t(t,\zeta(t))\,\mathrm{d}t\\ &-\int_0^T \dot\zeta(t)u(t,\zeta(t)v_{xt}(t,\zeta(t))\,\mathrm{d}t.\end{aligned} \tag{4.4f}$$

Proof. We observe that (4.4a) and (4.4b) follow from Corollary 4.1 *(i)*. Notice that all the functions v_{xx}, $v\star\theta^{\epsilon_k}$ and $v_x\star\theta^{\epsilon_k}$ have their supports included in a fixed bounded product $[0,T]\times[a,b]$. Hence, we may deduce from (4.2) that

$$\int_{Q_T}\Big(-\frac{1}{3}(u_x^{\epsilon_k}\star\theta^{\epsilon_k})^3(v_x\star\theta^{\epsilon_k})+\nu p(u_x^{\epsilon_k}\star\theta^{\epsilon_k})(v_x\star\theta^{\epsilon_k})+(\nu p)_x(u_x^{\epsilon_k}\star\theta^{\epsilon_k})(v\star\theta^{\epsilon_k})\Big)\,\mathrm{d}t\,\mathrm{d}x$$
$$\underset{k\to+\infty}{\to}\int_{Q_T}\Big(-\frac{1}{3}u_x^3v_x+\nu pu_xv_x+(\nu pv)_xu_x\Big)\,\mathrm{d}t\,\mathrm{d}x=\int_{Q_T}\Big(-\frac{1}{3}u_x^3v_x-\nu pu_{xx}v\Big)\,\mathrm{d}t\,\mathrm{d}x,$$

which proves (4.4c). Property (4.4d) is straightforward. Next, $u_1\in\mathrm{H}^1(\mathbb{R})\subset(\mathrm{L}^2(\mathbb{R})\cap\mathrm{L}^\infty(\mathbb{R}))$ and $u_1^\epsilon=u_1\star\theta^\epsilon$. Hence, we have

$$u_1^\epsilon\underset{\epsilon\to0}{\to}u_1 \text{ strongly in } \mathrm{L}^2(\mathbb{R})\cap\mathrm{L}^\infty(\mathbb{R}),$$

and (4.4e) follows. In order to prove (4.4f), we first establish the following result:

Lemma 4.3 *Let $u_0\in\mathrm{H}^2(\mathbb{R})$ and $u_1\in\mathrm{H}^1(\mathbb{R})$. Then, for any $v\in\mathrm{C}^2([0,T];\mathrm{H}^2(\mathbb{R}))$ with a compact support in $[0,T[\times\mathbb{R}$, we have*

$$\int_{Q_T}\theta^{\epsilon_k}(x-\zeta(t))u_t^{\epsilon_k}v_t\,\mathrm{d}t\,\mathrm{d}x=-\int_{\mathbb{R}}\theta^{\epsilon_k}(x-\zeta(0))u_0^{\epsilon_k}v_t(0,x)\,\mathrm{d}x$$
$$-\int_{Q_T}\theta^{\epsilon_k}(x-\zeta(t))u^{\epsilon_k}v_{tt}\,\mathrm{d}t\,\mathrm{d}x-\int_{Q_T}\dot\zeta(t)\theta^{\epsilon_k}(x-\zeta(t))u_x^{\epsilon_k}v_t\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_{Q_T}\dot\zeta(t)\theta^{\epsilon_k}(x-\zeta(t))u^{\epsilon_k}v_{xt}\,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Observe that

$$\int_{Q_T}\theta^{\epsilon_k}(x-\zeta(t))u_t^{\epsilon_k}v_t\,\mathrm{d}t\,\mathrm{d}x$$
$$=-\int_{Q_T}(-\dot\zeta(t)\theta^{\epsilon_k\prime}(x-\zeta(t))v_t+\theta^{\epsilon_k}(x-\zeta(t))v_{tt})u^{\epsilon_k}\,\mathrm{d}t\,\mathrm{d}x$$
$$+\int_{\mathbb{R}}\big[\theta^{\epsilon_k}(x-\zeta(t))u^{\epsilon_k}v_t\big]_0^T\,\mathrm{d}x$$
$$=-\int_{\mathbb{R}}\theta^{\epsilon_k}(x-\zeta(0))u_0^{\epsilon_k}v_t(0,x)\,\mathrm{d}x-\int_{Q_T}\theta^{\epsilon_k}(x-\zeta(t))u^{\epsilon_k}v_{tt}\,\mathrm{d}t\,\mathrm{d}x$$
$$-\int_{Q_T}\dot\zeta(t)\theta^{\epsilon_k}(x-\zeta(t))\partial_x(u^{\epsilon_k}v_t)\,\mathrm{d}t\,\mathrm{d}x,$$

and the desired result follows. □

We now pass to the limit in the various terms appearing in the equality of Lemma 4.3.

End of the proof of (4.4f). We only treat the case of $\int_{Q_T}\dot\zeta(t)\theta^{\epsilon_k}(x-\zeta(t))u_x^{\epsilon_k}v_t\,\mathrm{d}t\,\mathrm{d}x$, which we replace by $\int_0^T\int_a^b\dot\zeta(t)\theta^{\epsilon_k}(x-\zeta(t))u_x^{\epsilon_k}v_t\,\mathrm{d}t\,\mathrm{d}x$. The other terms in Lemma 4.3 can be handled similarly. Notice that for $\alpha\in\left]0,\frac{1}{2}\right[$, $\mathrm{H}^{1-\alpha}([a,b])$ is a multiplicative algebra. Since $u_{\epsilon_k,x}\in\mathrm{C}^0([0,T];\mathrm{H}^{1-\alpha}([a,b]))$ and $v_t\in\mathrm{C}^1([0,T];\mathrm{H}^1([a,b]))\hookrightarrow\mathrm{C}^0([0,T];\mathrm{H}^{1-\alpha}([a,b]))$, it follows that $u_x^{\epsilon_k}v_t\in\mathrm{C}^0([0,T];\mathrm{H}^{1-\alpha}([a,b]))$ for any $\alpha\in,\left]0,\frac{1}{2}\right[$. Notice also, since $\{u_x^{\epsilon_k}\}_{n\in\mathbb{N}^\star}$ is bounded in $\mathrm{L}^\infty([0,T]\times[a,b])$ and $v_t\in\mathrm{C}^1([0,T];\mathrm{H}^1([a,b]))\hookrightarrow\mathrm{L}^\infty([0,T]\times[a,b])$, that $\{u_x^{\epsilon_k}v_t\}_{n\in\mathbb{N}^\star}$ is bounded in $\mathrm{L}^\infty([0,T]\times[a,b])$. Now, we write

$$\begin{aligned}&\int_{Q_T}\dot\zeta(t)\theta^{\epsilon_k}(x-\zeta(t))u_x^{\epsilon_k}v_t\,\mathrm{d}t\,\mathrm{d}x\\&=\int_{Q_T}\dot\zeta(t)\theta^{\epsilon_k}(x-\zeta(t))(u_x^{\epsilon_k}-u_x)v_t\,\mathrm{d}t\,\mathrm{d}x+\int_{Q_T}\dot\zeta(t)\theta^{\epsilon_k}(x-\zeta(t))u_xv_t\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}\tag{4.5}$$

Since $u_xv_t\in\mathrm{C}^0([0,T];\mathrm{H}^{1-\alpha}([a,b]))$ and $\mathrm{H}^{1-\alpha}([a,b])\hookrightarrow\mathrm{C}^0([a,b])$ with $\alpha\in\left]0,\frac12\right[$, we have

$$\int_{\mathbb{R}}\theta^{\epsilon_k}(x-\zeta(t))u_x(t,x)v_t(t,x)\,\mathrm{d}x\underset{k\to+\infty}{\to}u_x(t,\zeta(t))v_t(t,\zeta(t))$$

for any $t\in[0,T]$ and since

$$\left|\dot\zeta(t)\int_a^b\theta^{\epsilon_k}(x-\zeta(t))u_x(t,x)v_t(t,x)\,\mathrm{d}x\right|\le\|\dot\zeta\|_{\mathrm{L}^\infty(0,T)}\|u_xv_t\|_{\mathrm{L}^\infty([0,T]\times[a,b])},$$

it follows from the dominated convergence theorem that

$$\int_{Q_T}\dot\zeta\theta^{\epsilon_k}(x-\zeta(t))u_xv_t\,\mathrm{d}t\,\mathrm{d}x\underset{k\to+\infty}{\to}\int_0^T\dot\zeta(t)u_x(t,\zeta(t))v_t(t,\zeta(t)).$$

For the other term in (4.5), we have

$$\begin{aligned}&\Big|\int_{Q_T}\dot\zeta(t)\theta^{\epsilon_k}(x-\zeta(t))(u_x^{\epsilon_k}(t,x)-u_x(t,x))v_t(t,x)\,\mathrm{d}t\,\mathrm{d}x\Big|\\&\le C_T\|\dot\zeta\|_{\mathrm{L}^\infty(0,T)}\|u_x^{\epsilon_k}-u_x\|_{\mathrm{L}^\infty([0,T]\times[a,b])}\underset{k\to+\infty}{\to}0.\end{aligned}$$

Finally, we have

$$\int_{Q_T}\dot\zeta(t)\theta^{\epsilon_k}(x-\zeta(t))u_x^{\epsilon_k}(t,x)v_t(t,x)\,\mathrm{d}t\,\mathrm{d}x\underset{k\to+\infty}{\to}\int_0^T\dot\zeta(t)u_x(t,\zeta(t))v_t(t,\zeta(t))\,\mathrm{d}t.$$

□

Remark 4.4 *The right hand side of* (4.4f) *is formally equal to*

$$\begin{aligned}&-u_0(\zeta(0))v_t(0,\zeta(0))-\int_0^Tu(t,\zeta(t))v_{tt}(t,\zeta(t))\,\mathrm{d}t\,\mathrm{d}x\\&-\int_0^T\dot\zeta(t)u_x(t,\zeta(t))v_t(t,\zeta(t))\,\mathrm{d}t-\int_0^T\dot\zeta(t)u(t,\zeta(t)v_{xt}(t,\zeta(t))\,\mathrm{d}t\\&=-u_0(\zeta(0))v_t(0,\zeta(0))-\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}(u(t,\zeta(t))v_t(t,\zeta(t)))\,\mathrm{d}t\\&+\int_0^Tu_t(t,\zeta(t))v_t(t,\zeta(t))\,\mathrm{d}t=\int_0^Tu_t(t,\zeta(t))v_t(t,\zeta(t))\,\mathrm{d}t.\end{aligned}\tag{4.6}$$

Nevertheless, with regularity of Corollary 4.1, the trace $\int_0^Tu_t(t,\zeta(t))v_t(t,\zeta(t))\,\mathrm{d}t$ *is not well defined.*

End of the proof of of Theorem 1.1. The proof of (1.3) follows from (3.12) and Proposition 4.2. Details are omitted. It remains to show that $u(0,\cdot)=u_0$. To this aim, notice that by (4.1a), $u^{\epsilon_k}(0,\cdot)\underset{k\to+\infty}{\to}u(0,\cdot)$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R})$, and that $u^{\epsilon_k}(0,\cdot)=u_0\star\theta^{\epsilon_k}\underset{k\to+\infty}{\to}u_0$ in $\mathrm{L}^2(\mathbb{R})$. This proves the result. □

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